

ON MODULAR COMPUTATION OF GRÖBNER BASES WITH INTEGER COEFFICIENTS

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§1. Introduction. Let $A = \mathbb{Z}[X]$, $X = (x_1, \dots, x_n)$. We consider A as a subset of $\mathbb{Q}[X]$, so, if I is an ideal of A , then $\mathbb{Q}I$ is the ideal of $\mathbb{Q}[X]$ generated by I . Given a ring R , we denote the natural mapping $A \rightarrow A \otimes R = R[X]$ by ι_R . This note is devoted to the following algorithmic problem (see [1] for a definition and properties of Gröbner bases of ideals in polynomial rings over \mathbb{Z}).

Problem (P). *Suppose that we have an infinite sequence f_1, f_2, \dots of elements of A (a blackbox producing them one by one). Let $I = (f_1, f_2, \dots)$ be the ideal generated by all the f_i 's. Compute the Gröbner base of I under the assumption that the Gröbner base of $\mathbb{Q}I$ and the Gröbner bases of $\iota_{\mathbb{Z}/m\mathbb{Z}}(I)$, $m \in \mathbb{Z}$, are known.*

This problem arises in [5], [6] (see §2 for more details). Our solution is based on a computation of the exponent of the torsion part of A/I (see [3]), and Main Lemma in §3 below.

§2. Motivation. More precisely, the algorithmic problem appeared in [5], [6] is the following. Let F be a finite rank free A -module, M_0 a submodule of F (given by a finite generating set G_0), $\tau \in \text{Hom}_A(F, A)$, and $\rho_1, \dots, \rho_p \in \text{End}_A(F)$. Compute (a Gröbner base of) $\tau(M)$ where M is the minimal submodule of F such that $M_0 \subset M$ and $\rho_i(M) \subset M$ for any $i = 1, \dots, p$. Theoretically, this problem has an evident solution, namely, we could compute one by one the Gröbner bases of the modules M_j defined recursively by $M_{j+1} = \sum_{i=1}^p \rho_i(M_j)$ till the moment when $M_{j+1} = M_j$. We set at this moment $M = M_j$ and $I = \tau(M)$.

However, if the rank of F is very big, then the computation of Gröbner bases of M_j 's may be so long that it cannot be performed in practice whereas the Gröbner bases of $(\mathbb{Z}/m\mathbb{Z})[X]$ -modules $M_j \otimes (\mathbb{Z}/m\mathbb{Z})$ can be computed much faster. When we know them, we can compute the Gröbner bases of $\mathbb{Q}[X]$ -modules $M_j \otimes \mathbb{Q}$ (see [2] and §5 below). Therefore we can compute the modules $M \otimes (\mathbb{Z}/m\mathbb{Z})$ and $M \otimes \mathbb{Q}$, and hence the ideals $\iota_{\mathbb{Z}/m\mathbb{Z}}(I)$ and $\mathbb{Q}I$. Also we can successively compute the generators of I of the form $\tau \rho_{i_1} \rho_{i_2} \dots \rho_{i_k}(g)$, $g \in G_0$. So, the problem (P) naturally arises.

§3. Solution to Problem (P). Let the notation be as in Introduction.

Main Lemma. *Let I and J be ideals in A such that $J \subset I$. Let m be the exponent of the torsion part of the abelian group A/J , i. e., m is the minimal positive integer such that $m(\mathbb{Q}J \cap A) \subset J$. Let m_1, \dots, m_t be pairwise coprime integers whose product is equal to m . Suppose that*

- (i) $\mathbb{Q}J = \mathbb{Q}I$, i. e., I and J generate the same ideal in $\mathbb{Q}[X]$,
- (ii) $\iota_{\mathbb{Z}/m_i\mathbb{Z}}(J) = \iota_{\mathbb{Z}/m_i\mathbb{Z}}(I)$, $i = 1, \dots, t$;

Then $I = J$.

Proof. Let $c_i = m/m_i = \prod_{j \neq i} m_j$, $i = 1, \dots, t$. Since m_i 's are pairwise coprime, there exist $b_1, \dots, b_t \in \mathbb{Z}$ such that

$$b_1 c_1 + \dots + b_t c_t = 1 \quad (1)$$

We have to show that $I \subset J$. Let $f \in I$. By Condition (ii), for any $i = 1, \dots, t$, we have $f = f_i + m_i h_i$ with $f_i \in J$ and $h_i \in A$. By Condition (i), we have $kf \in J$ for some $k \in \mathbb{Z}$, hence, for any i , we have $km_i h_i = kf - kf_i \in J$, i. e., h_i represents an element of the torsion part of the group A/J . By definition of m , this implies $mh_i \in J$, hence $c_i f = c_i f_i + c_i m_i h_i = c_i f_i + m h_i \in J$. Combined with (1), this yields

$$f = b_1(c_1 f) + \dots + b_t(c_t f) \in J \quad \square$$

Thus we obtain the following solution to Problem (\mathcal{P}). We fix any term order in A . We try one by one all ideals J of the form (f_1, \dots, f_k) , $k = 1, 2, \dots$ and for each of them do the following:

- (1) Compute the reduced Gröbner bases of the ideals $\mathbb{Q}I$ and $\mathbb{Q}J$ of the ring $\mathbb{Q}[X]$. If they do not coincide, then we pass to the next J .
- (2) Compute m as in Main Lemma using one of the algorithms given in [3] (see also §4). Let $m = p_1^{\alpha_1} \dots p_t^{\alpha_t}$ be the factorization of m into prime factors and let $m_i = p_i^{\alpha_i}$.
- (3) For each $i = 1, \dots, t$, compute the reduced Gröbner bases of $\iota_{\mathbb{Z}/m_i\mathbb{Z}}(I)$ and $\iota_{\mathbb{Z}/m_i\mathbb{Z}}(J)$. If they do not coincide, then we pass to the next J . If they do coincide, then we compute the Gröbner base of J and stop the computation.

§4. Computation of m . Several algorithms are discussed in [3] to compute m (defined in Main Lemma). For the reader's convenience, we reproduce here the algorithm which (we quote [3]) 'seems better suited for practical computations'. It consists in the following steps:

- (1) Fix any term ordering and compute a Gröbner base G of J . Let s be the least common multiple of the leading coefficients of the elements of G . Then $\mathbb{Q}J \cap A = \mathbb{Z}[1/s]J \cap A$.
- (2) Let Y be a new indeterminate. Then $\mathbb{Z}[1/s]J \cap A = (I, sY - 1) \cap A$. A Gröbner base G_s of this ideal can be computed using an elimination term ordering.
- (3) Compute m such that $mg \in J$ for any $g \in G_s$.

§5. On the modular computation of Gröbner bases over \mathbb{Q} . In our notation, Elizabeth Arnold's theorem can be formulated as follows (similar theorem holds also for modules).

Theorem. [2; Th. 7.1]. *Let I be a homogeneous ideal in $\mathbb{Z}[X]$ and $G \subset \mathbb{Z}[X]$. Let $\mathbb{Q}I$ be the ideal in $\mathbb{Q}[X]$ generated by I . Let p be a prime number. We set $I_p = \iota_{\mathbb{F}_p}(I)$, $G_p = \iota_{\mathbb{F}_p}(G)$. Suppose that:*

- (1) G_p is a Gröbner base of I_p ,
- (2) G is a Gröbner base of the ideal $\langle G \rangle_{\mathbb{Q}}$ that it generates in $\mathbb{Q}[X]$,
- (3) $\mathbb{Q}I \subset \langle G \rangle_{\mathbb{Q}}$,
- (4) $\text{LM}(G_p) = \text{LM}(G)$ where "LM" denotes the set of leading monomials.

Then G is a Gröbner base of $\mathbb{Q}I$.

It is claimed in [4; Th. 2.4] that the same statement holds for non-homogeneous ideals. However, it is wrong as one can see in the simplest example $I = (px + 1)$, $G = \{1\}$.

When the ideal is not homogeneous, it can be homogenized by adding a new variable and after that one can apply E. Arnold's theorem. For instance, if we homogenize the above counter-example, i. e., if we set $I = (px + y)$, $G = \{y\}$, then the condition (3) no longer holds true, so, the contradiction disappears.

Acknowledgment. I am grateful to Vladimir Gerdt for useful discussions.

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