MARKOV TRACES ON THE FUNAR ALGEBRA

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С меня при цифре 37 в момент слетает хмель.
(В. Высоцкий)

1. Introduction

Let B_n be the braid group with n strings and $\sigma_1,\dots,\sigma_{n-1}$ its standard generators. Let k be a commutative ring with $1\ne 0$. Given $\alpha,\beta\in k$, we define the k-algebra $K_n=K_n(\alpha,\beta)=K_n(\alpha,\beta;k)$ as the quotient of the group algebra kB_n by the relations

$$\sigma_1^3-\alpha\sigma_1^2+\beta\sigma_1-1=0 \tag{1}$$

and

$$\begin{aligned}y\bar xy =&2\alpha-\beta^2-(x+y)-(\alpha^2-\beta)(\bar x+\bar y)+\beta(xy+yx)+\alpha(x\bar y+y\bar x+\bar xy+\bar yx)\\ &+(\alpha\beta-1)(\bar x\bar y+\bar y\bar x)-\alpha xyx-(\bar xyx+x\bar yx+xy\bar x)-\beta(\bar x\bar yx+x\bar y\bar x)\\ &+(\alpha-\beta^2)\bar x\bar y\bar x,\end{aligned} \tag{2}$$

where x, $\bar x$, y, $\bar y$ in (2) stand for σ_1, σ_1^{-1}, σ_2, σ_2^{-1} respectively. Up to a change of the sign of β (for the sake of symmetricity), our definition of K_n is equivalent to the definition given by Bellingeri and Funar in [1]. Our relation (2) is much shorter than the corresponding relation in [1] (see [1; (2) and Table 1]) because we use σ_i^{-1} instead of σ_i^2. Multiplying (2) by σ_1 from the left or from the right, and simplifying the result using (1) and the braid group relations, we obtain

$$\begin{aligned}\bar yx\bar y =&2\beta-\alpha^2-(\bar x+\bar y)-(\beta^2-\alpha)(x+y)+\alpha(\bar x\bar y+\bar y\bar x)+\beta(\bar xy+\bar yx+x\bar y+y\bar x)\\ &+(\alpha\beta-1)(xy+yx)-\beta\bar x\bar y\bar x-(x\bar y\bar x+\bar xy\bar x+\bar x\bar yx)-\alpha(xy\bar x+\bar xyx)\\ &+(\beta-\alpha^2)xyx.\end{aligned} \tag{3}$$

Note that (3) is obtained from (2) by swapping $x\leftrightarrow\bar x$, $y\leftrightarrow\bar y$, $\alpha\leftrightarrow\beta$.

Using (1) – (3) together with the braid relations, it is easy to see that K_n are finitely generated k-modules. Following [2], we denote the image of σ_i in K_n by s_i.

Set $K_\infty=\lim K_n$. Contrary to the case of Hecke or BMW algebras, the morphisms $K_n\to K_{n+1}$ induced by the standard embeddings $B_n\subset B_{n+1}$ are not injective in general. We say that $t:K_\infty\otimes k[u,v]\to M$ is a *Markov trace* on K_∞ if M is a $k[u,v]$-module and t is a morphism of $k[u,v]$-modules such that $t(xy)=t(yx)$, $t(xs_n)=ut(x)$, $t(xs_n^{-1})=vt(x)$, $x,y\in K_n$, $n=1,2,\dots$.

It is claimed in [3] and [1] that a nontrivial Markov trace is constructed on K_n. About 2004–2005 I indicated a gap in the proof of its well-definedness (see

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Remark 2.11 below). As it is explained in [2], the gap was really serious: formally, the main result of [3] is wrong in the form it is stated. However, we show in this paper that the main idea in [1, 3] is correct: to construct a Markov trace on K_n, it suffices to check a finite number of identities though the number of them is much bigger than in [1, 3] and the algorithm of computation is much more complicated. Theoretically, this approach allows to compute the universal Markov trace on K_∞, i. e., the projection of $K_\infty(\alpha,\beta;\mathbb{Z}[\alpha,\beta,u,v])$ onto its quotient by the submodule $\bar{R}$ generated by

$$xy-yx,\ \ xs_n-ux,\ \ xs_n^{-1}-vx,\quad x,y\in K_n,\ n=1,2,\dots \tag{4}$$

The volume of computations is huge, so we performed them only in some cases. In particular, we found $K_\infty(0,0;A)/\bar{R}=A/I$ where $A=\mathbb{Z}[u,v]$, $I=(16,4u^2+4v,4v^2+4u,u^3+v^3+uv-3)$. Note, that it was checked in [2] that $K_5(0,0;A)/\bar{R}_5=A/I$ where $\bar{R}_5$ is the submodule generated by the elements of K_5 of the form (4).

In a sense, the results of the present paper can be divided into two independent parts: the "theoretical part" (Theorem 2.4 which provides an algorithm for computing the ideal I) and the "computational part" (Corollaries 2.5 and 2.6 which present the results of computer-aided computations according to this algorithm, and Section 4 where we discuss some properties of the obtained link invariants). The explicit form of the coefficients in the right hand side of (2) and (3) is not really used in the "theoretical part". Theorem 2.4 can be applied to a quotient of kB_∞ by (1) together with any two relations of the form

$$\sigma_2\sigma_1^{-1}\sigma_2=\sum_{i=1}^{21}\gamma_iX_i,\qquad \sigma_2^{-1}\sigma_1\sigma_2^{-1}=\sum_{i=1}^{21}\gamma_i'X_i,\qquad \gamma_i,\gamma_i'\in k, \tag{5}$$

where $X_1,\dots,X_{21}$ are all the reduced words in $\sigma_1^{\pm1}$, $\sigma_2^{\pm1}$ (including the empty word 1) that do not contain any subword of the form $\sigma_i^{\pm2}$ or $\sigma_2^{\pm1}\sigma_1^{\pm1}\sigma_2^{\pm1}$. So, if we consider α,β and all the γ_i, γ_i' in (5) as indeterminates and compute the ideal I described in Theorem 2.4, then we obtain the universal Markov trace on a cubic Hecke algebra that can be specialized to both Funar and BMW algebras. However, the required computations seem to exceed the capacity of any computer. On the other hand, if one chooses a random specialization of the coefficients γ_i, γ_i' in (5), then the quotient A/I might well be trivial. So, the explicit form of (2) and (3) is important for the "computational part" of the present paper.

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2. Definitions and statement of results

2.1. ***K*-reductions.** Let F_n^+ be the free monoid on generators $x_1^{\pm1},\dots,x_{n-1}^{\pm1}$ (the set of all not necessarily reduced words in $x_i^{\pm1}$) and $F_\infty^+=\bigcup F_n^+$. We denote the empty word by 1. Let kF_n^+ and kF_∞^+ be the corresponding free associative algebras over k (as k-modules, they are freely generated by F_n^+ and by F_∞^+ respectively).

We call *basic replacements* the pairs (U,V) with $U\in F_\infty^+$, $V\in kF_\infty^+$ (which we denote by $U\to V$) from the following list:

(*i*) $x_ix_i^{-1}\longrightarrow 1,\ x_i^{-1}x_i\longrightarrow 1,\ i\ge 1;$

(ii) $x_i^2 \longrightarrow \alpha x_i - \beta + x_i^{-1}$, $i \geq 1$;

(iii) $x_i^{-2} \longrightarrow \beta x_i^{-1} - \alpha + x_i$, $i \geq 1$;

(iv) $x_{i+1}^{\varepsilon_1} x_i^{\varepsilon_2} x_{i+1}^{\varepsilon_3} \longrightarrow x_i^{\varepsilon_3} x_{i+1}^{\varepsilon_2} x_i^{\varepsilon_1}$, $\varepsilon_2 \in \{\varepsilon_1, \varepsilon_3\} \subset \{-1, 1\}$, $i \geq 1$;

(v) $x_{i+1} x_i^{-1} x_{i+1} \longrightarrow$ (the right hand side of (2) with $x = x_i$, $y = x_{i+1}$), $i \geq 1$;

(vi) $x_{i+1}^{-1} x_i x_{i+1}^{-1} \longrightarrow$ (the right hand side of (3) with $x = x_i$, $y = x_{i+1}$), $i \geq 1$;

(vii) $x_{i+1}^{\varepsilon_1} x_i^{\varepsilon_2} W x_{i+1}^{\varepsilon_3} \longrightarrow VW$ where $x_{i+1}^{\varepsilon_1} x_i^{\varepsilon_2} x_{i+1}^{\varepsilon_3} \longrightarrow V$ is one of (iv)–(vi) and W is a word in $x_1^{\pm 1}, \dots, x_{i-1}^{\pm 1}$;

(viii) $x_j^{\varepsilon_1} x_i^{\varepsilon_2} \longrightarrow x_i^{\varepsilon_2} x_j^{\varepsilon_1}$, $\{\varepsilon_1, \varepsilon_2\} \subset \{-1, 1\}$, $j - 1 > i \geq 1$;

An *elementary K-reduction of a monomial* is $AUB \to AVB$ where $AUB \in F_\infty^+$ and $U \to V$ is a basic replacement. An *elementary K-reduction of an element of* kF_∞^+ is $\sum_{j=1}^m c_j W_j \to c_1 W_1' + \sum_{j=2}^m c_j W_j$ where $c_1, \dots, c_m \in k$, $W_1, \dots W_m$ are *pairwise distinct* elements of F_∞^+, and $W_1 \to W_1'$ is an elementary K-reduction of a monomial.

An element of F_∞^+ (resp. of kF_∞^+) is *K-reduced* if no K-reduction can be applied to it. We denote the set of such elements by $F_\infty^{\rm red}$ (resp. $kF_\infty^{\rm red}$). We set also $F_n^{\rm red} = F_n^+ \cap F_\infty^{\rm red}$ and $kF_n^{\rm red} = kF_n^+ \cap kF_\infty^{\rm red}$. Then $kF_\infty^{\rm red}$ is a submodule (not a subalgebra) of kF_∞^+. We denote $\pi : kF_\infty^+ \to K_\infty$ and $\pi_n : kF_n^+ \to K_n$ the morphisms of k-algebras induced by $x_i \mapsto s_i$.

We say that an element X of F_∞^+ is *almost K-reduced* if there exists a sequence $X = X_1 \to X_2 \to \dots \to X_m$ of elementary K-reductions of type (viii) such that X_m is K-reduced.

For $X = x_{i_1}^{\varepsilon_1} \dots x_{i_m}^{\varepsilon_m} \in F_\infty^+$, $\varepsilon_j = \pm 1$, we define the *weight* $\operatorname{wt} X = \sum_j i_j$ and the *auxiliary weight* $\operatorname{wt}' X = \sum_j j i_j$. It is clear that the set of all monomials of a given weight is finite. For $X \in kF_\infty^+$ we set $\operatorname{wt} X = \max_i \operatorname{wt} X_i$ if $X = \sum_i c_i X_i$ with $c_i \in k$, $c_i \neq 0$, and $X_1, X_2, \dots$ pairwise distinct elements of F_∞^+.

The following statement is easy and we omit its proof.

Proposition 2.1.

a). If $X \to X'$ is an elementary K-reduction, then $\pi(X) = \pi(X')$ and $\operatorname{wt} X \geq \operatorname{wt} X'$. If, moreover, X is a monomial, then $\operatorname{wt} X = \operatorname{wt} X'$ if and only if $X \to X'$ is a K-reduction of type (viii) and in this case we have $\operatorname{wt}'(X) < \operatorname{wt}'(X')$.

b). $\pi(F_\infty^{\rm red})$ generates K_∞ as a k-module.

c). $kF_\infty^{\rm red}$ is a free k-module and $F_\infty^{\rm red}$ is a free base of $kF_\infty^{\rm red}$.

d). $F_\infty^{\rm red}$ is the set of all words $X_1 X_2 \dots X_m$ where $X_\nu = x_{i_\nu}^{\pm 1} x_{i_\nu - 1}^{\pm 1} \dots x_{j_\nu}^{\pm 1}$, $i_\nu \geq j_\nu$ $(1 \leq \nu \leq m)$, $i_1 < \dots < i_m$, and all the signs are mutually independent.

e). (Proven in [3]) π_3 is an isomorphism of k-modules $kF_3^{\rm red}$ and K_3.

Remark 2.2. Let

$$S_{i,j} = \{x_i^{\pm 1} x_{i-1}^{\pm 1} \dots x_j^{\pm 1}\} \qquad \text{and} \qquad S_i = \{1\} \cup S_{i,i} \cup S_{i,i-1} \cup \dots \cup S_{i,1}. \tag{6}$$

Then Part (d) of Proposition 2.1 can be stated as follows: each element of $F_n^{\rm red}$ can be represented in a unique way as a product $X_1 X_2 \dots X_{n-1}$ with $X_i \in S_i$. Since $|S_i| = 1 + 2 + \dots + 2^i = 2^{i+1} - 1$, we obtain $|F_n^{\rm red}| = \prod_{i=1}^n (2^i - 1)$, in particular,

$$|F_2^{\rm red}| = 3, \quad |F_3^{\rm red}| = 3 \cdot 7 = 21, \quad |F_4^{\rm red}| = 3 \cdot 7 \cdot 15 = 315, \quad |F_5^{\rm red}| = 3 \cdot 7 \cdot 15 \cdot 31 = 9765.$$

Remark 2.3. In basic replacements (vii), it is enough to consider only words W belonging to S_{i-1} (see (6) for the definition of S_{i-1}).

We define a k-linear mapping $\mathbf{r} : kF_\infty^+ \to kF_\infty^{\rm red}$ as follows. For each $X \in F_\infty^+$ we fix an arbitrary sequence of elementary K-reductions $X = X_1 \to X_2 \to \dots \to X_m \in kF_\infty^{\rm red}$ and we set $\mathbf{r}(X) = X_m$. Then we extend the mapping to kF_∞^+ by linearity.

2.2. Markov traces. Let $A = k[u,v]$ and $AK_n = K_n(\alpha,\beta;A)$. Let $M = M(\alpha,\beta;k)$ be the quotient of AK_∞ by the relations (4) and let $t : AK_\infty \to M$ be the quotient map. We call t *the universal Markov trace* on K_∞ over k. It is indeed universal in the sense that any Markov trace on $K_\infty(\alpha,\beta;A)$ with values in an A-module M' is $f\circ t$ for some $f \in \mathrm{Hom}_A(M,M')$.

We define A-linear mappings $\tau_n : AF_n^+ \to AF_{n-1}^{\rm red}$ called *Markov reductions* as follows. By Proposition 2.1(d), we have $F_n^{\rm red} \subset F_{n-1}^{\rm red} \cup (F_{n-1}^{\rm red} x_{n-1} F_{n-1}^{\rm red}) \cup (F_{n-1}^{\rm red} x_{n-1}^{-1} F_{n-1}^{\rm red})$. So, we set $\tau_n(X) = X$, $\tau_n(X x_{n-1} Y) = u\mathbf{r}(XY)$, and $\tau_n(X x_{n-1}^{-1} Y) = v\mathbf{r}(XY)$ for $X, Y \in F_{n-1}^+$ and then we extend τ_n to $AF_n^{\rm red}$ by linearity and to AF_n^+ by setting $\tau_n(X) = \tau_n(\mathbf{r}(X))$. Finally, we define $\tau : F_\infty^+ \to AF_1^+ = A$ by setting $\tau(X) = \tau_2 \circ \dots \circ \tau_n(X)$ for $X \in AF_n^+$.

By definition of t and τ, we have $t(\pi(X)) = t(\pi(\tau(X)))$, thus $M = t(K_\infty)$ is generated by $t(1)$. Let $I = I(\alpha,\beta;k)$ be the annihilator of M. Thus we have $M \cong A/I$.

2.3. Statement of the main result. Let $\mathrm{sh}^n : AF_\infty^+ \to AF_\infty^+$, $n \in \mathbb{Z}$, be the A-algebra endomorphism (the *n-shift*) induced by

$$\mathrm{sh}^n x_i = \begin{cases} x_{i+n}, & i+n > 0, \\ 0, & i+n \le 0. \end{cases}$$

We set $\mathrm{sh} = \mathrm{sh}^1$.

For $X \in F_5^+$, we define $\rho_X \in \mathrm{End}_A(AF_4^{\rm red})$ by setting $\rho_X(Y) = \tau_5(X \,\mathrm{sh}\, Y)$. Let $J_4 = J_4(\alpha,\beta;k)$ be the minimal submodule of $AF_4^{\rm red}$ satisfying the following properties (recall that the sets $S_{i,j}$ and S_i are defined in (6)):

- (J1) $\mathbf{r}(\mathbf{r}(X_3X_2)X_1) - \mathbf{r}(X_3\mathbf{r}(X_2X_1)) \in J_4$ for any $X_j \in \mathrm{sh}^{3-j} S_j \setminus \{1\}$, $j = 1,2,3$;
- (J2) $\rho_X(J_4) \subset J_4$ for any $X \in S_4$.

In a similar way we define a module L. Let $N = AF_2^{\rm red} \otimes_A AF_2^{\rm red}$. We define A-linear mappings $\tau_N : N \to A$ and $\rho_\delta : N \to N$, $\delta = (\delta_1,\delta_2) \in \{-1,0,1\}^2$, by setting for any $Y = x_1^{\varepsilon_1} \otimes x_1^{\varepsilon_2}$ $(\varepsilon_1,\varepsilon_2 \in \{-1,0,1\})$

$$\tau_N(Y) = \tau(x_1^{\varepsilon_1} x_1^{\varepsilon_2}), \qquad \rho_\delta(Y) = x_1^{\delta_1} \otimes \tau_3(x_2^{\varepsilon_1} x_1^{\delta_2} x_2^{\varepsilon_2})$$

and we define L as the minimal submodule of N satisfying the conditions:

- (L1) $\tau_3(x_2^{\varepsilon_1} x_1^{\varepsilon_2} x_2^{\varepsilon_3}) \otimes x_1^{\varepsilon_4} - x_1^{\varepsilon_2} \otimes \tau_3(x_2^{\varepsilon_3} x_1^{\varepsilon_4} x_2^{\varepsilon_1}) \in L$ for any $\varepsilon_1, \varepsilon_3 \in \{-1,1\}$ and for any $\varepsilon_2, \varepsilon_4 \in \{-1,0,1\}$;
- (L2) $\rho_\delta(L) \subset L$ for any $\delta \in \{-1,0,1\}^2$.

Theorem 2.4. (Main Theorem). $I = \tau(J_4) + \tau_N(L)$.

The theorem is proved in §3 (see Proposition 3.2 for the inclusion "$\supset$" and Proposition 3.7 for the reverse inclusion).

This result allows (at least theoretically) to compute I. Indeed, we start with the A-module $J_4^{(0)}$ generated by the elements in (J1) and and we set $J_4^{(i+1)} = \sum_{X\in S_4} \rho_X(J_4^{(i)})$. Then the Gröbner bases $G^{(i)}$ of the modules $J_4^{(i)}$ can be computed recursively using the fact that $J_4^{(i+1)}$ is the module generated by $\bigcup_{X\in S_4} \rho_X(G^{(i)})$. So, we construct an increasing sequence of submodules $J_4^{(0)} \subset J_4^{(1)} \subset \dots$. Since the subring of A generated by α, β, u, v is noetherian, there exists m_0 such that $J_4^{(m_0)} = J_4^{(m_0+1)} = \dots$ (m_0 is determined by the condition $G^{(m_0)} = G^{(m_0+1)}$). The module L can be computed in a similar way as the limit of $L^{(0)} \subset L^{(1)} \subset \dots$ where $L^{(0)}$ is generated by the elements in (L1) and $L^{(i+1)} = \sum_\delta \rho_\delta(L^{(i)})$.

Performing in practice this computation for $\alpha = \beta = 0$, $k = \mathbb{Z}$ (the case considered in [3] and [2]) and in some other special cases, we obtain the following results. To compute the Gröbner bases, we used `Singular 3-1-3` and `Macaulay2` software; see details on the web page [6].

Corollary 2.5. $I(0,0;\mathbb{Z}) = (16,\ 4u^2+4v,\ 4v^2+4u,\ u^3+v^3+uv-3)$.

Corollary 2.6. *Let $k = \mathbf{k}[\alpha]$ for a ring $\mathbf{k}$ specified below and let $I = I(\alpha, 0, k)$. Let $\mathcal{G}$ be the reduced Gröbner base of I with respect to the lexicographic order such that $v > u > \alpha$ (the Gröbner bases in (e) – (g) also correspond to this order). Let*

$$\begin{aligned}
f_1 &= \gamma_1\gamma_2\gamma_3, \qquad \textit{where}\ \ \gamma_1 = \alpha^3+8,\ \gamma_2 = 2\alpha^3+1,\ \gamma_3 = 3\alpha^3+8,\\
f_2 &= \gamma_1\gamma_3(u-\alpha),\\
f_3 &= \gamma_3(6u^3 - 3\alpha^2 u + \alpha^3 + 2),\\
f_4 &= 336u^4 - 792\alpha u^3 + 12(15\alpha^3+106)\alpha^2u^2 + 6(141\alpha^3+544)u\\
&\qquad - 114\alpha^7 - 1405\alpha^4 - 3152\alpha,\\
f_5 &= 288v + 336\alpha^2u^3 + 72(3\alpha^3+28)u^2 - 48(9\alpha^3+44)\alpha u - 6a^8 + 53a^5 + 472a^2.
\end{aligned}$$

a). If $\mathbf{k} = \mathbb{F}_2$, then $\mathcal{G} = \{\alpha^4,\ \alpha^2(u^3+1+\alpha u^2),\ \alpha^2(v+u^2),\ v^3+u^3+uv+1+\alpha(uv^2+v+\alpha u)\}$, hence $\dim_{\mathbf{k}}(A/I) = \infty$ and the base of A/I over $\mathbf{k}$ is $\{\alpha^l u^m v^n \mid l \le 1, n \le 2\} \cup \{\alpha^l u^m \mid 2 \le l \le 3, m \le 2\}$.

b). If $\mathbf{k} = \mathbb{F}_3$, then $\mathcal{G} = \{\alpha^3-1,\ (u^2-\alpha^2)(u^2-\alpha u-\alpha^2),\ v+u^2\}$, hence $\dim_{\mathbf{k}}(A/I) = 12$.

c). If $\mathbf{k} = \mathbb{Q}$ or $\mathbf{k} = \mathbb{F}_p$ for a prime p in the range $5 \le p \le 599$, $p \ne 37$ (conjecturally, for any prime $p \notin \{2,3,37\}$), *then $I = (f_1,\dots,f_5)$. We have $\mathcal{G} = \{f_1,\dots,f_5\}$ and hence* $\dim_{\mathbf{k}}(A/I) = 24$ *except the case $\mathbf{k} = \mathbb{F}_7$ where we have $\mathcal{G} = \{f_1, f_2, u^3 + 2\alpha u(\gamma_1 u + \gamma_3\alpha) + 3\alpha^6 - 3\alpha^3, f_5\}$ and hence* $\dim_{\mathbf{k}}(A/I) = 21$.

d). If $\mathbf{k} = \mathbb{F}_{37}$, then we have $I = (f_1, f_2, f_3, (u+7\alpha)f_4, f_5 - 14\alpha f_4)$ and $\mathcal{G} = \{f_1, f_2, f_3, f_{4,37}, f_{5,37}\}$ where

$$\begin{aligned}
f_{4,37} &= u^5 + 2\alpha u^4 + 7\alpha^2u^3 - 9(\alpha^3-1)u^2 + (6\alpha^3+2)\alpha u - 12a^8 + 9a^5,\\
f_{5,37} &= v - 4\alpha u^4 + 15\alpha^2u^3 - (14\alpha^3+16)u^2 - (\alpha^3+18)\alpha u - 4\alpha^8 + 2\alpha^5 - 2\alpha^2
\end{aligned}$$

and hence $\dim_{\mathbf{k}}(A/I) = 27$.

e). If $\mathbf{k} = \mathbb{Z}/32\mathbb{Z}$, then $\{8\alpha^3,\ 4\alpha^4,\ \alpha^5+8\alpha^2,\ 16\alpha u^3+16\alpha^2u^2+4\alpha^3u+2\alpha^4,\ 8\alpha^2u^3-12\alpha^3u^2+8\alpha^2,\ 2\alpha^3u^3+16u^3+16\alpha^2u+6\alpha^3+16,\ \alpha^4u^3+8\alpha u^3+4\alpha^5u^2+16\alpha^2u^2+\alpha^4+$

8α, $16u^4-8\alpha u^3+8\alpha^2u^2-6\alpha^3u+\alpha^4+16\alpha$, $4\alpha^2u^4+7\alpha^3u^3-8u^3-2\alpha^4u^2-4\alpha^2u-9\alpha^3-8$, $8\alpha u^5+16u^3+\alpha^4u^2+8\alpha^2u+6\alpha^3$, $\alpha^3u^5-8u^5-8\alpha u^4-4\alpha^2u^3-5\alpha^3u^2+8u^2-13\alpha^4u+16\alpha u-4\alpha^2$, $4\alpha u^6-9\alpha^3u^4+8u^4+3\alpha^3u+8u-3\alpha^4+4\alpha$, $16v+8\alpha u^4+12\alpha^2u^3+16u^2+3\alpha^4u-12\alpha^2$, $8\alpha v+10\alpha^4u^2-8\alpha u^2+16\alpha^2u+2\alpha^3+16$, $4\alpha^2v-12\alpha^2u^2+12\alpha^3u-13\alpha^4-8\alpha$, $\alpha^3v-8v+\alpha^3u^2-8u^2$, $2\alpha^2uv+8v+4\alpha u^4-12\alpha^2u^3-4\alpha^3u^2-8u^2+13\alpha^4u+12\alpha u-2\alpha^2$, $4\alpha u^3v+12\alpha v-12\alpha u^5+13\alpha^3u^3+8u^3-10\alpha^4u^2+12\alpha u^2+3\alpha^3-8$, $4v^2-2\alpha uv+3\alpha^2v+\alpha^2u^2+4u-2\alpha$, $2\alpha v^2+15\alpha^2uv-4v+15\alpha^3u^2-4u^2+2\alpha u+15\alpha^2$, $\alpha^2v^2+4uv-2\alpha v-4\alpha u^5+15\alpha^4u^2-2\alpha u^2-7\alpha^2u+13a^3-4$, $v^3-15\alpha uv^2-3uv-7\alpha v+u^3+6\alpha u^2+\alpha^2u+2\alpha^3-15\}$ *is a Gröbner base of* I.

f). If $\mathbf{k}=\mathbb{Z}/3^r\mathbb{Z}$ *for* $r\le 6$ (conjecturally, for any r), *then* $I=(\mathcal{F})$ *for* $\mathcal{F}=\{f_1,f_2,f_3,\,2u^4+\alpha u^3-\alpha^2u^2+2u-\alpha,\,v+u^2\}$. *If, moreover,* $r\ge 2$, *then* $\mathcal{F}\cup\{3^{r-1}(\alpha^3-1)\}$ *is a Gröbner base of* I, *hence* $|A/I|=3^{12r}$. *By (b), this implies that* A/I *is a free* $\mathbf{k}$*-module of rank* 12.

g). If $\mathbf{k}=\mathbb{Z}/37p\mathbb{Z}$ *for a prime* $p\le 67$, $p\notin\{2,3,7\}$ (conjecturally, for any prime $p\notin\{2,3,7\}$), *then* $\{f_1,f_2,f_3,37f_4,(u+7\alpha)f_4,f_5-14\alpha f_4\}$ *is a Gröbner base of* I.

Remark 2.8. The Markov trace t over k defines an invariant of oriented links $P(L)=P_{\alpha,\beta,k}(L)=u^{(1-n-e)/2}v^{(1-n+e)/2}t(b)\in k[u^{\pm1/2},v^{\pm1/2}]/I(\alpha,\beta;k)$ where b is a representation of a link L by a braid with n strings and e is the sum of exponents of b. It is shown in [2] that $P_{0,0;\mathbb{Z}/4\mathbb{Z}}$ is determined by the HOMFLY polynomial (see Section 4.5 below). In the first arxiv version of this paper (arxiv:1206.0765v1) it was claimed that $P_{\alpha,0;\mathbb{Q}[\alpha]}$ and $P_{\alpha,0;\mathbb{F}_3[\alpha]}$ detect the chirality of the knot 10_{71}. Unfortunately, this is not so. However, $P_{\alpha,0;\mathbb{F}_{37}[\alpha]}$ detects the chirality of the knots 10_{48}, 10_{91} and it distinguishes many other pairs of knots with equal HOMFLY polynomials. In the cases computed so far, the invariants $P_{\alpha,0;k}$ do not distinguish any pair of knots up to 11 crossings with equal Kauffman polynomials. It is shown in Proposition 4.1(f) below that $P_{\alpha,\beta;k}$ never distinguishes mutant links.

Remark 2.9. According to [4; Theorem 1.5], we have $I(\alpha,\beta;\mathbb{Q}(\alpha,\beta))=(1)$. This is equivalent to say that $I(\alpha,\beta;\mathbb{Q}[\alpha,\beta])$ contains a nonzero polynomial in α and β (observe that the same phenomenon takes place in Corollaries 2.5 and 2.6).

Remark 2.10. Our main motivation for studying quotients of cubic Hecke algebras and their Markov traces stems from the Markov-type theorem for transversal links proved in [7]. If u,v were zero divisors in A/I then one might possibly obtain transversal links invariants which distinguish isotopic links with the same Bennequin invariants. In the cases computed so far, however, u and v are not zero divisors.

We say that $t\in\mathrm{Hom}_{k[u]}(K_\infty\otimes k[u],M)$ is a *semi-Markov trace* on K_∞ if $t(xy)=t(yx)$ and $t(xs_n)=ut(x)$ for $x,y\in K_n$, $n>0$. Due to [7], any semi-Markov trace provides an invariant of transversal links. In [5], the methods of the present paper are adapted for studying the universal semi-Markov trace on K_n.

Remark 2.11. The main error in [3] (which was repeated also in [1]) is that the modules $J_4^{(0)}$ and $L^{(0)}$ were considered instead of J_4 and L.

3. Proof of Main Theorem

The idea of the proof is as follows. Given $X\in AF_n^+$, the element of A representing $\tau(X)$ is computed by successive reductions (i)–(viii), Markov relations, and cyclic permutations. We have to find the minimal possible ideal I such that the result does not depend on the order of these operations. It is easy to observe (though not so easy to formalize this observation) that the main sources of the ambiguity are as follows. First, the reduction of a subword $W = x_3^{\varepsilon_1}x_2^{\varepsilon_2}x_1^{\varepsilon_3}x_3^{\varepsilon_4}x_2^{\varepsilon_5}x_3^{\varepsilon_6}$ can be started either with $x_3^{\varepsilon_4}x_2^{\varepsilon_5}x_3^{\varepsilon_6}$ or (after commuting $x_1^{\varepsilon_3}$ and $x_3^{\varepsilon_4}$) with $x_3^{\varepsilon_1}x_2^{\varepsilon_2}x_3^{\varepsilon_4}$. Thus $\tau(J_4^{(0)})$ should be included in I. Second, the reduction of a word $x_2^{\varepsilon_1}x_1^{\varepsilon_2}x_2^{\varepsilon_3}x_1^{\varepsilon_4}$ can be started either by $x_2^{\varepsilon_1}x_1^{\varepsilon_2}x_2^{\varepsilon_3}$ or (after a cyclic permutation) by $x_2^{\varepsilon_3}x_1^{\varepsilon_4}x_2^{\varepsilon_1}$. Thus $\tau_N(L^{(0)})$ should be included into I.

Let us focus on the first case. So, let Y be an element of $J_4^{(0)}$. Then, for any X and any n, we should have $\tau(X\,\mathrm{sh}^n\,Y)\in I$. In particular, for any $X\in F_5^+$, we should have $\tau(\rho_X(Y))\in I$, i. e., $\tau(J_4^{(1)})\subset I$. By iterating this process, we conclude that $\tau(J_4)\subset I$. Similarly, $\tau_N(L)\subset I$. This is the easy part of the proof which is formally exposed in Section 3.1.

The difficult part of the proof (formally exposed in Section 3.2) consists in checking that any choice of the reduction process leads to the same result modulo $I'=\tau(J_4)+\tau_N(L)$. We use induction on the weight (see the definition of the weight function wt in Section 2.1). As we pointed out above, there are two main sources of the ambiguity. Again, we discuss here only the first one. So, we have to prove that $\tau(X\,\mathrm{sh}^n\,Y)\in I'$ for any X when $Y\in J_4$. By additivity, we may assume that X is a monomial. If any reduction can be applied to X, then we do it and we use the induction hypothesis. So, we may assume that $X\in AF_{n+4}^+$. If $X=X_1X_2$ where X_2 commutes with $\mathrm{sh}^n\,Y$, then we replace $X\,\mathrm{sh}^n\,Y$ by $X_2X_1\,\mathrm{sh}^n\,Y$. Thus we arrive to the case when $X=X'\,\mathrm{sh}^{n-1}\,X_1$ with $X'\in F_{n+3}^+$, $X_1\in F_5^+$, and we apply the induction hypothesis to $X'\,\mathrm{sh}^{n-1}\,Y'$ where $Y'=\rho_{X_1}(Y)$.

3.1. Easy part: $\tau(J_4)+\tau_N(L)\subset I$.

Let $J_4^{(0)}\subset J_4^{(1)}\subset\dots$ and $L^{(0)}\subset L^{(1)}\subset\dots$ be as defined in §2.3.

For $n\ge 4$ and $a\in AK_n$, we define $t_{n,a}\in\mathrm{Hom}_A(F_4^{\mathrm{red}},A)$ by setting $t_{n,a}(X)=t(a\,\pi(\mathrm{sh}^{n-4}\,X))$. Similarly, for $n\ge 1$ and $a,b\in AK_n$, we define $t_{n,a,b}\in\mathrm{Hom}_A(N,A)$ by setting $t_{n,a,b}(X\otimes Y)=t\big(\pi(\mathrm{sh}^{n-1}\,X)\,a\,\pi(\mathrm{sh}^{n-1}\,Y)\,b\big)$.

Lemma 3.1.

a). $J_4\subset\ker t_{n,a}$ for any $n\ge 4$ and any $a\in K_n$.

b). $L\subset\ker t_{n,a,b}$ for any $n\ge 1$ and any $a,b\in K_n$.

Proof. We prove by induction that a) $J_4^{(i)}\subset\ker t_{n,a}$ and b) $L^{(i)}\subset\ker t_{n,a,b}$. For $i=0$, the statement is evident. Suppose that it is true for $i-1$ and let us prove it for i. Note that we have

$$t\big(a\,\pi(\mathrm{sh}^p\,\tau_{n-p}(X))\,b\big)=t\big(a\,\pi(\mathrm{sh}^p\,X)\,b\big)\qquad\text{for}\quad a,b\in K_{n-1},\ X\in AF_n^+ \tag{7}$$

a). It is enough to check that $\rho_X(Y)\in\ker t_{n,a}$ for any $Y\in J_4^{(i-1)}$, $X\in S_4$. $n\ge 4$, $a\in K_n$. Indeed,

$$\begin{aligned}
t_{n,a}(\rho_X(Y)) &= t\big(a\,\pi(\mathrm{sh}^{n-4}\,\rho_X(Y))\big) && \text{by definition of } t_{n,a}\\
&= t\big(a\,\pi(\mathrm{sh}^{n-4}\,\tau_5(X\,\mathrm{sh}\,Y))\big) && \text{by definition of } \rho_X\\
&= t\Big(a\,\pi\big((\mathrm{sh}^{n-4}\,X)(\mathrm{sh}^{n-3}\,Y)\big)\Big) && \text{by (7)}\\
&= t_{n+1,a'}(Y) && \text{for } a'=a\,\pi(\mathrm{sh}^{n-4}\,X)\in K_{n+1}\\
&= 0 && \text{by the induction hypothesis}
\end{aligned}$$

b). It is enough to check that $\rho_\delta(Y)\in\ker t_{n,a,b}$ for any $Y\in L^{(i-1)}$, $\delta=(\delta_1,\delta_2)\in\{-1,0,1\}^2$, $n\ge 1$, $a,b\in K_n$. Indeed, let $Y=\sum_j c_j x_1^{\varepsilon_1(j)}\otimes x_1^{\varepsilon_2(j)}$. Then

$$\begin{aligned}
t_{n,a,b}(\rho_\delta(Y)) &= t_{n,a,b}\Big(\sum c_j x_1^{\delta_1}\otimes\tau_3(x_2^{\varepsilon_1(j)}x_1^{\delta_2}x_2^{\varepsilon_2(j)})\Big) && \text{def. of } \rho_\delta\\
&=\sum c_j t\Big(\pi(\operatorname{sh}^{n-1}x_1^{\delta_1})\,a\,\pi\big(\operatorname{sh}^{n-1}\tau_3(x_2^{\varepsilon_1(j)}x_1^{\delta_2}x_2^{\varepsilon_2(j)})\big)\,b\Big) && \text{def. of } t_{n,a,b}\\
&=\sum c_j t\big(s_n^{\delta_1}\,a\,s_{n+1}^{\varepsilon_1(j)}s_n^{\delta_2}s_{n+1}^{\varepsilon_2(j)}\,b\big) && \text{by (7)}\\
&=\sum c_j t\big(s_{n+1}^{\varepsilon_1(j)}s_n^{\delta_2}s_{n+1}^{\varepsilon_2(j)}\,b\,s_n^{\delta_1}\,a\big) && t(xy)=t(yx)\\
&=\sum c_j t\Big(\pi\big(\operatorname{sh}^n x_1^{\varepsilon_1(j)}\big)\,s_n^{\delta_2}\pi\big(\operatorname{sh}^n x_1^{\varepsilon_2(j)}\big)\,b\,s_n^{\delta_1}\,a\Big)\\
&= t_{n+1,a',b'}(Y) && a'=s_n^{\delta_2},\ b'=bs_n^{\delta_1}a\\
&=0 && \text{by induction hypothesis}
\end{aligned}$$

Proposition 3.2. *$\tau(J_4)+\tau_N(L)\subset I$.*

Proof. Indeed, by Lemma 3.1, we have $t(\tau(X))=t_{4,1}(X)=0$ for any $X\in J_4$ and $t(\tau_N(X))=t_{1,1,1}(X)=0$ for any $X\in L$. Thus $\tau(J_4)+\tau_N(L)\subset\ker(t|_A)=I$.

3.2. Difficult part: $I\subset\tau(J_4)+\tau_N(L)$.

Let, as above, $\bar R$ be the submodule of K_∞ generated by the elements (4). Set $R=\pi^{-1}(\bar R)$. Then we have $I=A\cap\bar R=A\cap R$. Let $\operatorname{wt}:AF_\infty^+\to\mathbb{Z}_{\ge0}$ be the weight function defined in §2.1. It defines a filtration on AF_∞^+, namely, $A=AF_{[0]}^+\subset AF_{[1]}^+\subset AF_{[2]}^+\subset\dots$ where $AF_{[w]}^+=\{X\in AF_\infty^+\mid \operatorname{wt}X\le w\}$.

We shall work with the following set of generators $\mathcal R=\mathcal R_T\cup\mathcal R_M\cup\mathcal R_N\cup\mathcal R_H$ of R as an A-module (we set here $u_+=u$, $u_-=v$):

$$\begin{aligned}
\mathcal R_T&=\{XY-YX\mid X,Y\in F_\infty^+\}, && \text{trace relations;}\\
\mathcal R_M&=\{x_n^{\pm1}X-u_\pm X\mid X\in F_n^+, n\ge1\}, && \text{Markov relations;}\\
\mathcal R_N&=\{UX-VX\mid X,U\in F_\infty^+, U\overset{(i)\text{–}(vi)}{\longrightarrow}V\}, && \text{nonhomogeneous } K\text{-relations;}\\
\mathcal R_H&=\{UX-VX\mid X,U\in F_\infty^+, U\overset{(viii)}{\longrightarrow}V\}, && \text{homogeneous } K\text{-relations.}
\end{aligned}$$

Let $\mathcal R_{[w]}=\mathcal R\cap AF_{[w]}^+$, let $R_{[w]}$ be the A-submodule of R generated by $\mathcal R_{[w]}$, and let H be the submodule generated by $\mathcal R_T\cup\mathcal R_H$ (the elements of $H\cap R_{[w]}$ are wt-homogeneous for any w). Note, that by Proposition 2.1(a) we have

$$X\equiv\mathbf r(X)\equiv\tau_n(X)\equiv\tau(X)\mod R_{[\operatorname{wt}X]}\qquad\text{for } X\in AF_n^+. \tag{8}$$

In what follows, a notation like $X_1\equiv X_2\equiv X_3\equiv\dots$ means that $X_i\equiv X_{i+1}$ mod $R_{[\operatorname{wt}X_i]}$ and $\operatorname{wt}X_{i+1}\ge\operatorname{wt}X_i$, in particular, in this case we always have $X_1\equiv X_2\equiv X_3\equiv\dots\mod R_{[\operatorname{wt}X_1]}$.

Lemma 3.3. *Let $Z=X\operatorname{sh}^{n-4}Y$ for $X\in AF_\infty^+$, $Y\in J_4\cap\operatorname{sh}^{4-n}AF_\infty^+$, $n\ge1$. Then $Z\in R_{[w]}+\tau(J_4)$ where $w=\operatorname{wt}Z$.*

Proof. We denote $\operatorname{sh}^{n-4}Y$ by Y_n. If $X\in AF_m^+$ with $m>n$, then

$$XY_n\equiv\tau_m(X)Y_n\equiv\tau_{m-1}(\tau_m(X))Y_n\equiv\dots\equiv\tau_{n+1}\circ\dots\circ\tau_{m-1}\circ\tau_m(X)Y_n,$$

hence it is enough to prove the statement of the lemma under the additional hypothesis $X \in AF_n^+$. We prove it by induction.

If $n = 1$, then $X \in AF_1^+ = A$ and $Y \in J_4 \cap \operatorname{sh}^3 AF_\infty^+ = J_4 \cap A \subset \tau(J_4)$, so, the statement is trivial.

Suppose that $n \geq 2$, the statement is true for $n-1$, and let us prove it for n. By linearity, it is enough to consider the case when $X \in F_n^+$ and since $X \equiv \mathbf{r}(X)$, we may assume that $X \in F_n^{\mathrm{red}}$. Let $X = X_1X_2 \dots X_{n-1}$, $X_i \in S_i$ (see Remark 2.2). We have $X_{n-1} = (\operatorname{sh}^{n-5} X_4')X_{n-5}''$ with $X_4' \in S_4 \cap \operatorname{sh}^{5-n} AF_\infty^+$ and $X_{n-5}'' \in S_{n-5}$ (we assume here that $S_i = \{1\}$ when $i \leq 0$). Note that Y_n may involve only $x_{n-4+i}^{\pm 1}$, $i = 1,2,3$, whereas X_{n-5}'' may involve only $x_i^{\pm 1}$, $i \leq n-5$, hence they commute. Therefore, denoting $X_1 \dots X_{n-2}$ by X_{n-2}''', we obtain

$$\begin{aligned} Z &= X_{n-2}'''(\operatorname{sh}^{n-5} X_4')X_{n-5}''(\operatorname{sh}^{n-4} Y) \equiv X_{n-2}'''(\operatorname{sh}^{n-5} X_4')(\operatorname{sh}^{n-4} Y)X_{n-5}'' \\ &\equiv X_{n-5}''X_{n-2}'''(\operatorname{sh}^{n-5} X_4')(\operatorname{sh}^{n-4} Y) = X_{n-5}''X_{n-2}''' \operatorname{sh}^{n-5}(X_4' \operatorname{sh} Y) \\ &\equiv X_{n-5}''X_{n-2}''' \operatorname{sh}^{n-5}\big(\tau_5(X_4' \operatorname{sh} Y)\big) = X' \operatorname{sh}^{n-5} Y' \end{aligned}$$

where $X' = X_{n-5}''X_{n-2}''' \in AF_{n-1}^+$ and $Y' = \tau_5(X_4' \operatorname{sh} Y) = \rho_{X_4'}(Y) \in J_4$.

To complete the proof, it remains to check that $Y' \in \operatorname{sh}^{5-n} AF_\infty^+$. Indeed, we have $X_4' \in \operatorname{sh}^{5-n} AF_\infty^+$, $Y \in \operatorname{sh}^{4-n} AF_\infty^+$, hence $\operatorname{sh} Y \in \operatorname{sh}^{5-n} AF_\infty^+$ and we obtain $X_4' \operatorname{sh} Y \in \operatorname{sh}^{5-n} AF_\infty^+$ whence $Y' = \tau_5(X_4' \operatorname{sh} Y) \in \operatorname{sh}^{5-n} AF_\infty^+$. □

The next lemma is similar. For $n \geq 1$ and $X_1, X_2 \in AF_n^+$ we define $\varphi_{n,X_1,X_2} \in \operatorname{Hom}_A(N, AF_{n+1}^+)$ by setting $\varphi_{n,X_1,X_2}(Y_1 \otimes Y_2) = X_1(\operatorname{sh}^{n-1} Y_1)X_2(\operatorname{sh}^{n-1} Y_2)$.

Lemma 3.4. *Let $Z = \varphi_{n,X_1,X_2}(Y)$ for $n \geq 1$, $X_1, X_2 \in AF_n^+$, $Y \in L$. Then $Z \in R_{[w]} + \tau_N(L)$ where $w = \operatorname{wt} Z$.*

Proof. It is enough to consider the case when $X_1, X_2 \in F_n^{\mathrm{red}}$. Then there exist $X_i', X_i'' \in F_{n-1}^{\mathrm{red}}$ and $\delta_i \in \{-1,0,1\}$ such that $X_i = X_i' x_{n-1}^{\delta_i} X_i''$ $(i = 1,2)$. Let

$$Y = \sum_j c_j x_1^{\varepsilon_1(j)} \otimes x_1^{\varepsilon_2(j)}. \tag{9}$$

Then we have

$$\begin{aligned} Z &= \sum c_j\, X_1 x_n^{\varepsilon_1(j)} X_2 x_n^{\varepsilon_2(j)} = \sum c_j\, X_1' x_{n-1}^{\delta_1} X_1'' x_n^{\varepsilon_1(j)} X_2' x_{n-1}^{\delta_2} X_2'' x_n^{\varepsilon_2(j)} \\ &\equiv \sum c_j\, X_2'' X_1' x_{n-1}^{\delta_1} X_1'' X_2' x_n^{\varepsilon_1(j)} x_{n-1}^{\delta_2} x_n^{\varepsilon_2(j)} \\ &= \sum c_j\, X_2'' X_1' x_{n-1}^{\delta_1} X_1'' X_2' \operatorname{sh}^{n-2}\big(x_2^{\varepsilon_1(j)} x_1^{\delta_2} x_2^{\varepsilon_2(j)}\big) \\ &\equiv \sum c_j\, X_2'' X_1' x_{n-1}^{\delta_1} X_1'' X_2' \operatorname{sh}^{n-2} \tau_3\big(x_2^{\varepsilon_1(j)} x_1^{\delta_2} x_2^{\varepsilon_2(j)}\big) = \varphi_{n-1,\bar X_1,\bar X_2}(\bar Y) \end{aligned}$$

where $\bar X_1 = X_2'' X_1'$, $\bar X_2 = X_1'' X_2'$, $\bar Y = \rho_\delta(Y)$. So, we have $Z \equiv \bar Z = \varphi_{n-1,\bar X_1,\bar X_2}(\bar Y)$ where $\bar X_1, \bar X_2 \in AF_{n-1}^+$, $\bar Y \in L$.

Thus, by induction we reduce the problem to the case $n = 1$. In this case we have $X_1, X_2 \in AF_1^+ = A$, hence, for Y as in (9), we have $Z = \varphi_{1,X_1,X_2}(Y) = \sum c_j x_1^{\varepsilon_1(j)} x_1^{\varepsilon_2(j)}$, hence $Z \equiv \tau_2(Z) = \tau_N(Y) \in \tau_N(L)$. □

The next statement can be considered as an improvement of the Pentagon Lemma from [3].

Lemma 3.5 (Pentagon Lemma). *Let* $Z_1, Z_2 \in \mathcal{R}_N \cup \mathcal{R}_M$ *and* $Z_1 - Z_2 \in H + AF^+_{[w-1]}$ *where* $w = \operatorname{wt} Z_1 = \operatorname{wt} Z_2$*. Then* $Z_1 - Z_2 \in H + R_{[w-1]} + \tau(J_4) + \tau_N(L)$*.*

Proof. Let $X_i \in F^+_\infty$ be the leading monomial of Z_i, $i = 1, 2$, i. e., $\operatorname{wt} X_i = \operatorname{wt} Z_i$ and $\operatorname{wt}(Z_i - X_i) \le w - 1$. Then $X_1 - X_2 \in H$, hence there exists a sequence of words $X_1 = W_1, \dots, W_m = X_2$ such that W_{i+1} is obtained from W_i either by a cyclic permutation or by exchanging two consecutive commuting letters. By definition of $\mathcal{R}_M$ and $\mathcal{R}_N$ we have $X_i = U_i X'_i$ and $Z_i = (U_i - V_i)X'_i$, $i = 1, 2$, where $U_i \to V_i$ is an elementary K-reduction of types (i)–(vi) if $Z_i \in \mathcal{R}_N$ and $U_i = x_n^{\pm1}$, $V_i = u_\pm$, $X'_i \in F^+_n$ if $Z_i \in \mathcal{R}_M$.

Following [3] and [1], we represent such sequences $W_1, \dots, W_m$ by diagrams. A *diagram* is a union of mutually transversal curves in the cylinder $S^1 \times [0, 1]$, each curve being labeled by a letter $x_i^{\pm1}$. In pictures we represent the cylinder by a rectangle whose vertical sides are supposed to be identified, so, the fibers of the projection $\mathrm{pr}_2 : S^1 \times [0, 1] \to [0, 1]$ will be called *horizontal circles.* Each curve is *monotone*, i. e., its projection onto $[0, 1]$ is bijective. We say that a diagram is *admissible* if two curves labeled by $x_i^{\pm1}$ and $x_j^{\pm1}$ may cross only if $|i - j| \ge 2$. The words W_i (up to cyclic permutation) are read on horizontal circles.

We say that curves $\Gamma_1, \dots, \Gamma_m$ form a *bunch of parallel curves* or just a *bunch* if they are pairwise disjoint and all the crossings lying on $\bigcup \Gamma_i$ can be covered by disks whose intersections with the diagram are as in Figure 1 up to symmetry.

In our case, the first and the last word of the sequence are X_1 and X_2. So, on the boundary of the cylinder we indicate (by a bold line) segments corresponding to U_1 and U_2. As in [3] and [1], a diagram is called *interactive* if it contains a curve which joins the bold segments. We also say that a curve is *active* if it meets at least one bold segment.

Step 1. *If all active curves form a single bunch all whose ends are on the bold segments, then* $Z_1 - Z_2 \in H$*.*

In this case we have $U_1 = U_2$. Let $V_1 = \mathbf{r}(U_1) = \sum c_j W_j$, $c_j \in A$, $W_j \in F^+_\infty$. For each j we consider the diagram obtained from the initial diagram by replacing the bunch of active curves by a bunch of curves labeled by W_j. If a curve crosses the bunch, its label commutes with all letters occurring in U_1, hence it commutes with all letters in W_j, i. e., the new diagram is admissible and it defines a congruence $W_j X'_1 \equiv W_j X'_2 \mod H$. Hence (recall that $X_1 - X_2 \in H$) we have $Z_1 - Z_2 = (X_1 - V_1 X'_1) - (X_2 - V_1 X'_2) \equiv V_1 X'_2 - V_1 X'_1 = \sum c_j W_j (X'_2 - X'_1) \equiv 0 \mod H$.

Step 2. *If* $Z_1, Z_2 \in \mathcal{R}_M$*, then* $Z_1 - Z_2 \in H$*.*

In this case there is only one active curve, so we apply the result of Step 1.

Step 3. *If the diagram is non-interactive, then* $Z_1 - Z_2 \in H + R_{[w-1]}$*.*

Due to Step 2, we may suppose that $Z_1 \in \mathcal{R}_N$. Then $U_1 = x_n^{\varepsilon_1} x_{n-1}^{\varepsilon_2} x_n^{\varepsilon_3}$ with $\varepsilon_1, \varepsilon_3 \in \{-1, 1\}$ and $\varepsilon_2 \in \{-1, 0, 1\}$.

Let A and B be the points on the lower bold segment that correspond to the letters $x_n^{\varepsilon_1}$ and $x_n^{\varepsilon_3}$ of U_1 and let AD and BC be the corresponding active curves (see Figure 2). They cut the cylinder into two halves. Let Q be that half whose side AB is contained in the bold segment (the quadrangle $ABCD$ in Figure 2).

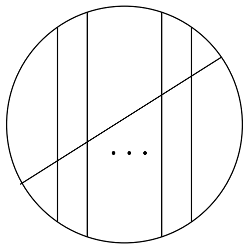


FIGURE 1

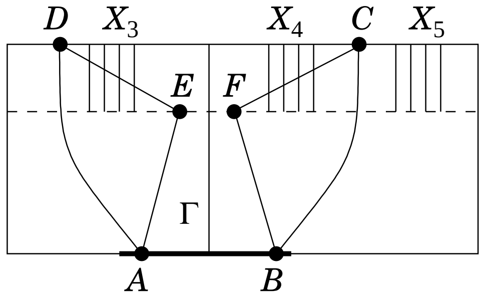


FIGURE 2

Let Γ be the curve outcoming from U_1 and labeled by $x_{n-1}^{\varepsilon_2}$ if $\varepsilon_2 \neq 0$ or a generic monotone curve in Q if $\varepsilon_2 = 0$. Let us choose a horizontal circle (the dashed line in Figure 2) so that all crossings are below it and let us choose points E and F on it so that the segment EF which crosses Γ has no other intersections with the diagram. We may suppose that the intersection of the diagram with the upper half-cylinder (above EF) is a union of segments of vertical lines.

Let Δ be the diagram obtained by replacing AD and BC with monotone curves AED and BFC where ED, FC are straight line segments and AE, BF are curves in Q which are chosen so close to Γ that the active curves outcoming from U_1 form a bunch in the lower half-cylinder (below EF). The label of any curve $\Gamma' \neq \Gamma$ entering Q is not $x_i^{\pm 1}$ with $|n-i| \le 1$ (indeed, since Γ' attains the lower boundary outside the bold segment, it crosses AD or BC). Hence Δ is admissible.

Let Y be the word read from Δ along the circle EF. The bunch of active curves in the lower half-cylinder ensures that $Y = U_1 Y'$ and the result of Step 1 yields

$$Z_1 \equiv (U_1 - \mathbf{r}(U_1))Y' \mod H. \tag{10}$$

Now, let us study the upper part of Δ (above EF). All the possible crossings in this part are on ED and FC. Hence, up to cyclic permutation, we have $X_2 = x_n^{\varepsilon_1} X_3 x_{n-1}^{\varepsilon_2} X_4 x_n^{\varepsilon_3} X_5$ and $Y = U_1 Y' = U_1 X_4 X_5 X_3$ (see Figure 2). Since the diagram is not interactive, U_2 is a subword of one of X_3, X_4, X_5, hence the active curves outcoming from U_2 form a bunch and $Y' = Y_1 U_2 Y_2$, i. e., $Y = U_1 Y_1 U_2 Y_2$, $Y_1, Y_2 \in F_\infty^+$. Hence, by Step 1, we have

$$Z_2 \equiv U_1 Y_1 (U_2 - \mathbf{r}(U_2)) Y_2 \mod H. \tag{11}$$

We have also

$$U_1 Y_1 \mathbf{r}(U_2) Y_2 \equiv \mathbf{r}(U_1) Y_1 \mathbf{r}(U_2) Y_2 \equiv \mathbf{r}(U_1) Y_1 U_2 Y_2 \mod R_{[w-1]}.$$

Combining this with (10) and (11), we obtain

$$Z_1 \equiv (U_1 - \mathbf{r}(U_1)) Y_1 U_2 Y_2 \equiv U_1 Y_1 (U_2 - \mathbf{r}(U_2)) Y_2 \equiv Z_2 \mod H + R_{[w-1]}.$$

Step 4. *Consider the open intervals obtained after removing all endpoints of all active curves. If at least one of the words corresponding to these intervals is not almost K-reduced* (*see the definition in §2.1*), *then* $Z_1 - Z_2 \in H + R_{[w-1]}$.

Suppose that the word which is not almost K-reduced is a subword Y of X_2. Since it is disjoint from the active curves, we can write $X_2 = U_2 X_3 Y X_4$. The fact that Y is not almost K-reduced means that there exists a sequence $Y = Y_0 \to Y_1 \to \cdots \to Y'U_3Y''$ of exchanges of commuting letters such that U_3 is the left hand side of an elementary replacement of type (i)–(vi). The fact that Y does not meet any active curve means that the diagrams corresponding to the both chains

$$X_1 \to \cdots \to X_2 = U_2X'Y_0X'' \to U_2X'Y_1X'' \to \cdots \to U_2X'(Y'U_3Y'')X'',$$
$$X_2 = U_2X'Y_0X'' \to U_2X'Y_1X'' \to \cdots \to U_2X'(Y'U_3Y'')X''$$

are non-interactive. By Step 3 this implies $Z_1 \equiv Z_3 \equiv Z_2 \mod H + R_{[w-1]}$ where $Z_3 = U_2X'Y'(U_3 - \mathbf{r}(U_3))Y''X''$.

Step 5. *Suppose that $Z_1 \in \mathcal{R}_N$ and the diagram is interactive but not as in Step 1. Then the active curves are arranged up to symmetry either as in Figure 3.1 or as in Figure 3.2 where each of the dashed lines may or may not be included into the diagram, $n \geq 1$.*

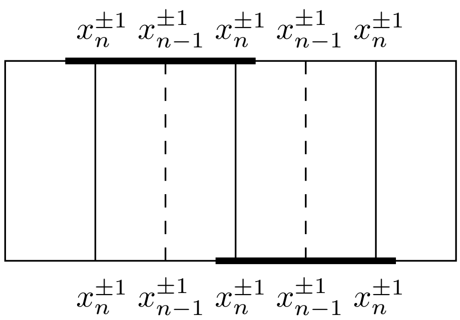


FIGURE 3.1

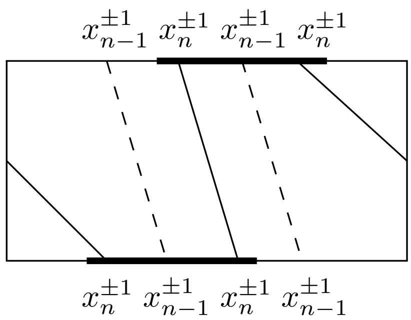


FIGURE 3.2

Indeed, we draw the curves adjacent to one of the bold segments and we try all the possibilities to complete the picture to an admissible diagram. It is easy to see that the pictures that could arise this way are the two pictures from the statement of Step 5.

Step 6. *If the active curves are as in Figure 3.1, then $Z_1 - Z_2 \in H + R_{[w-1]} + \tau(J_4)$.*

Suppose that the active curves are as in Figure 3.1 (the bottom boundary corresponds to X_1). Then $U_1 = x_n^{\varepsilon_4}x_{n-1}^{\varepsilon_5}x_n^{\varepsilon_6}$, $U_2 = x_n^{\varepsilon_1}x_{n-1}^{\varepsilon_2}x_n^{\varepsilon_4}$, $X_2 = U_2Yx_{n-1}^{\varepsilon_5}X_3x_n^{\varepsilon_6}X_4$ where $\varepsilon_1, \varepsilon_4, \varepsilon_6 = \pm 1$ and $\varepsilon_2, \varepsilon_5 \in \{-1, 0, 1\}$.

We begin as in Step 3. Let Q be the curvilinear quadrangle adjacent to the lower bold segment and bounded by the active x_n-curves outcoming from U_1. Let C_1 be a horizontal circle such that the part of the diagram above C_1 is a union of segments of vertical lines. Let Γ be either the (x_{n-1})-curve outcoming from U_1 (if it exists) or just a generic monotone curve in Q. Then we push the x_n-curves inside the domain Q from its boundary so that they form (together with Γ) a bunch below C_1, and so that the portions of the pushed curves above C_1 are segments of straight lines (see Figure 4.1).

Since all curves outcoming from Y cross an x_n-curve, Y does not contain $x_i^{\pm1}$ for $n-1 \leq i \leq n+1$. By Step 4, we may suppose that Y is almost K-reduced, hence Y has at most one occurrence of $x_{n-2}^{\pm1}$, i. e., $Y = Y_1x_{n-2}^{\varepsilon_3}Y_2$ with $\varepsilon_3 \in \{-1, 0, 1\}$ and Y_1, Y_2 do not contain $x_i^{\pm1}$ for $n-2 \leq i \leq n+1$.

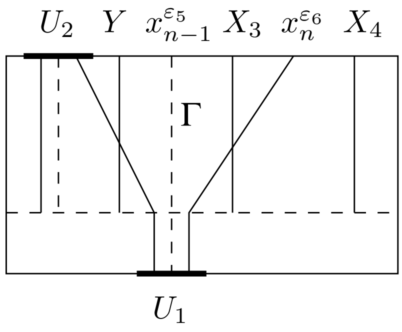


FIGURE 4.1

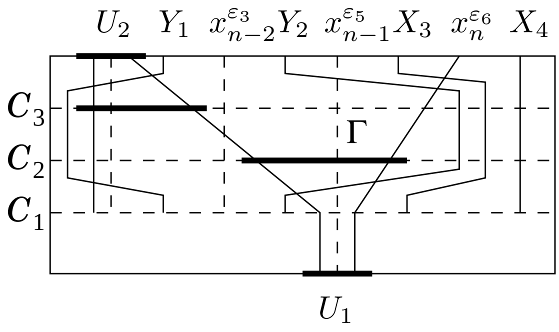


FIGURE 4.2

We choose horizontal circles C_2 and C_3 so that the intersection point of the x_n-curve and the (x_{n-2})-curve (if it exists) is between them and we modify the diagram as it is shown in Figure 4.2. If we apply the result of Step 1 to the part of the diagram which is below C_2 and to that which is above C_3, we obtain:

$$Z_1 \equiv Y_1 x_n^{\varepsilon_1} x_{n-1}^{\varepsilon_2} x_{n-2}^{\varepsilon_3}\big(U_1 - \mathbf{r}(U_1)\big) Y_2 X_3 X_4 \mod H \qquad (\text{below } C_2),$$
$$Z_2 \equiv Y_1\big(U_2 - \mathbf{r}(U_2)\big) x_{n-2}^{\varepsilon_3} x_{n-1}^{\varepsilon_5} x_n^{\varepsilon_6} Y_2 X_3 X_4 \mod H. \qquad (\text{above } C_3).$$

Hence $Z_1 - Z_2 \equiv X' \operatorname{sh}^{n-3} Y' \mod H$ where $X' = Y_2 X_3 X_4 Y_1$ and

$$Y' = \mathbf{r}(x_3^{\varepsilon_1} x_2^{\varepsilon_2} x_3^{\varepsilon_4})\, x_1^{\varepsilon_3} x_2^{\varepsilon_5} x_3^{\varepsilon_6} - x_3^{\varepsilon_1} x_2^{\varepsilon_2} x_1^{\varepsilon_3}\, \mathbf{r}(x_3^{\varepsilon_4} x_2^{\varepsilon_5} x_3^{\varepsilon_6}).$$

If $\varepsilon_2 \neq 0$, then $\mathbf{r}(Y') \in J_4$ by Condition (J1) of the definition of J_4. Thus, using Lemma 3.3 and observing that $\operatorname{wt}(X' \operatorname{sh}^{n-3} Y') < w$, we obtain

$$Z_1 - Z_2 \equiv X' \operatorname{sh}^{n-3} Y' \equiv X' \operatorname{sh}^{n-3} \mathbf{r}(Y') \equiv 0 \mod H + R_{[w-1]} + \tau(J_4).$$

If $\varepsilon_2 = 0$, then $X' \operatorname{sh}^{n-3} Y' \equiv X' \operatorname{sh}^{n-3}(x_1^{\varepsilon_3} Y'') \mod H$ where $\mathbf{r}(Y'') \in J_4$, thus

$$Z_1 - Z_2 \equiv X' \operatorname{sh}^{n-3}(x_1^{\varepsilon_3} Y'') \equiv X' x_{n-2}^{\varepsilon_3} \operatorname{sh}^{n-3} \mathbf{r}(Y'') \equiv 0 \mod H + R_{[w-1]} + \tau(J_4).$$

Step 7. *If the active curves are as in Figure 3.2, then* $Z_1 - Z_2 \in H + R_{[w-1]} + \tau_N(L)$.

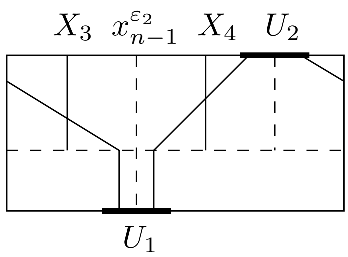


FIGURE 5

Again, as in the beginning of Steps 3 and 6, we transform the diagram as in Figure 5 and we obtain

$$Z_1 \equiv X_3(U_1 - \mathbf{r}(U_1)) X_4 x_{n-1}^{\varepsilon_4} \quad \text{and} \quad Z_2 \equiv X_3 x_{n-1}^{\varepsilon_2} X_4 (U_2 - \mathbf{r}(U_2)) \mod H$$

where $U_1 = x_n^{\varepsilon_1} x_{n-1}^{\varepsilon_2} x_n^{\varepsilon_3}$, $U_2 = x_n^{\varepsilon_3} x_{n-1}^{\varepsilon_4} x_n^{\varepsilon_1}$, $\varepsilon_1, \varepsilon_3 = \pm 1$, $\varepsilon_2, \varepsilon_4 \in \{-1, 0, 1\}$. Hence

$$Z_1 - Z_2 \equiv X_3 x_{n-1}^{\varepsilon_2} X_4 \mathbf{r}(U_2) - X_3 \mathbf{r}(U_1) X_4 x_{n-1}^{\varepsilon_4} \mod H \tag{12}$$

Note that $x_i^{\pm 1}$ for $n-1 \le i \le n+1$ does not occur in X_j, $j = 3, 4$. Indeed, if it does, then the diagram $x_i^{\pm 1}$-curve starting at a point from the upper circle corresponding to the letter $x_i^{\pm 1}$ in X_j cannot attain the opposite side of the cylinder outside the bold segment because it cannot cross the x_n-curves. Thus,

$$X_3 x_{n-1}^{\varepsilon_2} X_4 \mathbf{r}(U_2) \equiv X_3' x_{n-1}^{\varepsilon_2} X_4' \mathbf{r}(U_2) X_5 \equiv X_3' x_{n-1}^{\varepsilon_2} X_4' \mathbf{r}(U_2) \tau(X_5) \mod H + R_{[w-1]}$$

where $X_3', X_4' \in F_{n-1}^+$ and $X_5 \in \operatorname{sh}^{n+1} F_\infty^+$ (the same for other term in (12)). So, replacing, if necessary, X_j by X_j' ($j = 3, 4$), we may assume that $X_3, X_4 \in F_{n-1}^+$. Then we can pass from (12) to

$$\begin{aligned} Z_1 - Z_2 &\equiv X_3 x_{n-1}^{\varepsilon_2} X_4 \tau_{n+1}(U_2) - X_3 \tau_{n+1}(U_1) X_4 x_{n-1}^{\varepsilon_4} \mod H + R_{[w-1]} \\ &= \varphi_{n-1, X_3, X_4}(Y) \end{aligned}$$

where $Y = x_1^{\varepsilon_2} \otimes \tau_3(x_2^{\varepsilon_3} x_1^{\varepsilon_4} x_2^{\varepsilon_1}) - \tau_3(x_2^{\varepsilon_1} x_1^{\varepsilon_2} x_2^{\varepsilon_3}) \otimes x_1^{\varepsilon_4} \in L$. Thus if $n > 1$, then the result follows from Lemma 3.4. If $n = 1$, then $\varepsilon_2 = \varepsilon_4 = 0$, $X_3 = X_4 = 1$, and (12) yields $Z_1 - Z_2 \equiv 0$. □

Lemma 3.6. $R_{[w]} \cap AF_{[w-1]}^+ \subset H + R_{[w-1]} + \tau(J_4) + \tau_N(L)$

Proof. For $Z \in R_{[w]} \cap AF_{[w-1]}^+$, let $m = m(Z)$ be the minimal number such that $Z \equiv c_1 Z_1 + \dots + c_m Z_m \mod H + R_{[w-1]}$ with $c_i \in A$, $Z_i \in \mathcal{R}_N \cup \mathcal{R}_M$, $\operatorname{wt} Z_i = w$. To prove that $Z \in H + R_{[w-1]} + \tau(J_4) + \tau_N(L)$, we use induction on m. The statement is trivial for $m = 0$.

Suppose that $m > 0$ and the statement is true for all smaller values of m. Let X_i be the leading monomial of Z_i, i. e., $X_i \in F_\infty^+$ and $\operatorname{wt}(Z_i - X_i) < w$. Then $\sum c_i X_i \equiv 0 \mod H$. The term $c_m X_m$ of this congruence should be cancelled by other terms. Hence there exists $j < m$ such that $X_m - X_j \in H$. Then $c_m(Z_m - Z_j) \in H + R_{[w-1]} + \tau(J_4) + \tau_N(L)$ by Lemma 3.5 and $Z - c_m(Z_m - Z_j) \in H + R_{[w-1]} + \tau(J_4) + \tau_N(L)$ by the induction hypothesis. □

Proposition 3.7. $I \subset \tau(J_4) + \tau_N(L)$.

Proof. Let $I' = \tau(J_4) + \tau_N(L)$. Since $I = R \cap A$ and $R = \bigcup_w R_{[w]}$, it is enough to prove that $R_{[w]} \cap A \subset I'$ for any w. For $w = 0$ we have $R_{[0]} = 0$, hence $R_{[0]} \cap A \subset I'$. Suppose that $R_{[w-1]} \cap A \subset I'$. Let $Z \in R_{[w]} \cap A$. Since $R_{[w]} \cap A \subset R_{[w]} \cap AF_{[w-1]}^+$, by Lemma 3.6 we have $Z \in H + R_{[w-1]} + I'$, i. e., $Z = Z_H + Z' + Z_0$ with $Z_H \in H$, $Z' \in R_{[w-1]}$, $Z_0 \in I'$. Since $Z_H = Z - Z' - Z_0 \in AF_{[w-1]}^+$ and H is homogeneous, we have $Z_H \in R_{[w-1]}$, thus $Z = Z'' + Z_0$ with $Z'' = Z' + Z_H \in R_{[w-1]}$ and $Z_0 \in I'$. Since $Z'' = Z - Z_0 \in A$, we have $Z'' \in R_{[w-1]} \cap A$ and by the induction hypothesis we obtain $Z'' \in I'$ whence $Z = Z'' + Z_0 \in I'$. Thus $R_{[w]} \cap A \subset I'$. □

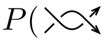
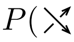
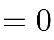
Our main Theorem follows from Propositions 3.2 and 3.7.

4. The link invariant $P_{\alpha,\beta;k}$

4.1. Some general properties. As we mentioned in Remark 2.8, the Markov trace on AK_∞ defines an invariant of oriented links $P = P_{\alpha,\beta;k}$ which takes its values in $\tilde{A}/I$ where $\tilde{A} = k[u^{\pm 1/2}, v^{\pm 1/2}]$. If u and v are not zero divisors (this is so in all the cases computed so far; see Remark 2.10), then A embeds into $\tilde{A}$.

Proposition 4.1. *a). $P(\text{unknot}) = 1$.*

b). P satisfies the skein relation coming from (1):

$u^{3/2}P($  $) - \alpha u v^{1/2} P($  $) + \beta u^{1/2} v P($  $) - v^{3/2} P($  $) = 0$ (13)

Similarly, (2) and (3) yield skein relations for tangles with 6 endpoints.

c). $P(L_1 \sqcup L_2) = (uv)^{-1/2} P(L_1) P(L_2)$ (disjoint union).

d). $P(L_1 \# L_2) = P(L_1)P(L_2)$ (connected sum).

e). If $\bar{L}$ is the mirror image of L, then $P_{\alpha,\beta}(\bar{L})(u,v) = P_{\beta,\alpha}(L)(v,u)$.

f). If L_2 is obtained from L_1 by a mutation (i. e., a rotation of a tangle with 4 endpoints by 180° such that no endpoint is fixed), then $P(L_1) = P(L_2)$.

g). $P(L) \equiv 1 \mod \big(u-1,\, v-1,\, \alpha-\beta,\, 2(\alpha-2)^2\big)$.

h). If L can be represented by an n-braid, then $P(L)$ can be represented by a Laurent polynomial $f \in \tilde{A}$ such that $1 - n \le \deg_{u,v}(T) \le 0$ for any monomial T of f.

Proof. a) – e) and h). Immediate from the definition of P.

f). Mutant links L_1 and L_2 can be represented by braids $Z_1 = X\,\mathrm{sh}^{n-2}\,Y$ and $Z_2 = X\,\mathrm{sh}^{n-2}(\sigma_1 Y \sigma_1^{-1})$ respectively where $X \in B_n$, $Y \in B_m$. Hence we have $t(Z_1) = t(\tau_{n+1} \circ \tau_{n+2} \circ \cdots \circ \tau_{n+m-2}(Z_1)) = t(X\,\mathrm{sh}^{n-2}\,Y_2)$ where $Y_2 = \tau_3 \circ \cdots \circ \tau_m(Y) \in AB_2$ and, similarly, $t(Z_2) = t(X\,\mathrm{sh}^{n-2}(\sigma_1 Y_2 \sigma_1^{-1}))$. It remains to note that $Y_2 = \sigma_1 Y_2 \sigma_1^{-1}$ because the group B_2 is abelian.

g). If we set $x = y = \bar{x} = \bar{y} = 1$ in (1) and (2), then we obtain identities modulo the ideal $(\alpha - \beta,\, 2(\alpha-2)^2)$. □

From now on we assume that $k = \mathbf{k}[\alpha, \beta]$ where $\mathbf{k}$ is a commutative ring and each of α and β is either zero or transcendent over $\mathbf{k}$. For a monomial in α, β, u, v we define its degree modulo 3 (denoted by $\deg_3$) by setting $\deg_3(u) = \deg_3(\alpha) = 1$ and $\deg_3(v) = \deg_3(\beta) = -1$. We denote the $\deg_3$-homogeneous component of A (resp. of $\tilde{A}$) of degree d by A_d (resp. by $\tilde{A}_d$). So, we have $A = A_0 \oplus A_1 \oplus A_2$ and $\tilde{A} = \tilde{A}_0 \oplus \tilde{A}_{1/2} \oplus \cdots \oplus \tilde{A}_{5/2}$. The following fact follows immediately from the definitions.

Proposition 4.2. *The ideal I is $\deg_3$-homogeneous. $P(L) \in \tilde{A}_0/I$ for any L.* □

4.2. Normal form of elements of $k[u^{\pm 1}, v^{\pm 1}]/I$. In fact, the square roots of u and v in the definition of the invariant P are needed only for writing the skein relation (13) in a nice form. Otherwise, for a μ-component link L represented by an n-braid X, we can set

$$P(L) = u^{(p-n-e)/2} v^{(p-n+e)/2} t(X) \quad \text{where} \quad p = \begin{cases} 1, & \mu \text{ is odd}, \\ 0, & \mu \text{ is even} \end{cases} \tag{14}$$

which ensures that $P(L)$ belongs to $k[u^{\pm1},v^{\pm1}]/I$ for any link L. Note that Proposition 4.2 still holds for $P(L)$ defined by (14). So, from now on we forget about fractional powers of u and v and we discuss the normal form of elements of the ring $k[u^{\pm1},v^{\pm1}]$.

Assume that $k=\mathbf{k}[\alpha,\beta]$ as in the previous subsection. We have a natural identification of $k[u^{\pm1},v^{\pm1}]$ with $\bar A/\bar I$ where $\bar A=A[\bar u,\bar v]$ ($\bar u$ and $\bar v$ are new independents variables) and $\bar I=I+(u\bar u-1,v\bar v-1)$. So, to define a normal form in $k[u^{\pm1},v^{\pm1}]$, it is enough to compute a Gröbner base $\bar{\mathcal G}$ of $\bar I$.

We assume that the monomial order is chosen so that $\bar u$ and $\bar v$ are greater than any monomial in α,β,u,v. In this case, the following conditions are equivalent:

(1) A/I embeds into $\bar A/\bar I\cong k[u^{\pm1},v^{\pm1}]/I$;
(2) u and v are not zero divisors in A/I;
(3) $\bar{\mathcal G}$ contains a Gröbner base of I.

Condition (3) holds in all the cases computed so far (see Remark 2.10).

If $\mathbf{k}$ is a field, then the normal form in $\bar A/\bar I$ defined by $\bar{\mathcal G}$ is evident: it is just a $\mathbf{k}$-linear combination of monomials which are not divisible by the leading terms of elements of $\bar{\mathcal G}$. In the case when $\mathbf{k}$ is $\mathbb Z$ or $\mathbb Z/m\mathbb Z$, the situation is more delicate. For any monomial T, we have to fix canonical representatives in $\mathbf{k}$ for elements of $\mathbf{k}/I(T)$ where $I(T)$ is the ideal of $\mathbf{k}$ consisting of the leading coefficients of elements of $\bar A$ whose leading monomial is T. When $I(T)\ne0$, we choose them in $\{0,\dots,m(T)-1\}$ where $m(T)$ is the positive generator of $I(T)$. Thus the normal form is a $\mathbf{k}$-linear combination of monomials T with coefficients belonging to $\{0,\dots,m(T)-1\}$. Note that $m(T)$ is the gcd of the leading coefficients of those elements $\bar{\mathcal G}$ whose leading monomials divide T.

4.3. The case $\mathbf{k}=\mathbb Q$ or $\mathbb F_p$ and $\beta=0$.

Proposition 4.3. *Let the notation be as in the respective parts of Corollary 2.6 and let $\bar{\mathcal G}$ be the reduced Gröbner base of $\bar I$ with respect to the lexicographic order such that $\bar v>\bar u>v>u>\alpha$. Then:*

a). ($\mathbf{k}=\mathbb F_2$). $\bar{\mathcal G}=\mathcal G\cup\{u\bar u+1,\ v\bar v+1,\ \alpha^2(\bar u+u^2+\alpha u),\ \alpha^2(\bar v+u+\alpha),\ u^3\bar v+v^2+u+\alpha(uv+1+\alpha u^2+\alpha^2u),\ \bar u\bar v+\bar uv^2+1+\alpha(\bar u+v+\alpha u+\alpha^2)\}$.

b). ($\mathbf{k}=\mathbb F_3$). $\bar{\mathcal G}=\mathcal G\cup\{\bar u+\alpha^2u^3-u^2+\alpha u+\alpha^2,\ \bar v+\alpha u^3+\alpha^2u^2-u\}$.

c). ($\mathbf{k}=\mathbb Q$ or $\mathbb F_p$, $p\notin\{2,3,37\}$). $\bar{\mathcal G}=\mathcal G\cup\{f_6,f_7\}$ *where*

$$f_6=\bar u+\tfrac{7}{12}\alpha^2u^3+(\tfrac78\alpha^3+6)u^2-(\tfrac98\alpha^3+\tfrac{20}{3})\alpha u+\tfrac1{48}\alpha^8+\tfrac{163}{288}\alpha^5+\tfrac{28}{9}\alpha^2,$$

$$f_7=\bar v+\tfrac{35}{6}\alpha u^3-(\tfrac{69}{32}\alpha^3+\tfrac{71}{4})\alpha^2u^2-(\tfrac{75}{8}\alpha^3+\tfrac{152}{3})u+\tfrac{155}{96}\alpha^7+\tfrac{2867}{144}\alpha^4+\tfrac{433}{9}\alpha.$$

d). ($\mathbf{k}=\mathbb F_{37}$). $\bar{\mathcal G}=\mathcal G\cup\{f_{6,37},f_{7,37}\}$ *where*

$$f_{6,37}=\bar u-3\alpha u^4-6\alpha^2u^3+(6\alpha^3-2)u^2-10(\alpha^3-1)\alpha u-2\alpha^8+15\alpha^5+14\alpha^2,$$

$$f_{7,37}=\bar v+14u^4+16\alpha u^3-(12\alpha^3u^2+11)\alpha^2u^2+(12\alpha^3-1)u-12\alpha^7-6\alpha^4-\alpha.$$

Thus, in the setting of Corollary 2.6(c) ($\mathbf{k}=\mathbb Q$ or $\mathbb F_p$, $p\notin\{2,3,37\}$), the normal form of elements of $\bar A/\bar I$ always belongs to A and we have $\bar A/\bar I\cong A/I$, in particular, $\dim_{\mathbf k}\bar A/\bar I=24$ (or 21 if $\mathbf{k}=\mathbb F_7$). By Proposition 4.2, the invariant $P(L)$ takes its values in $\tilde A_0/I$. So, its normal form is a linear combination of the eight monomials indicated in the header line of Table 1 (without u^3 in the case $\mathbf{k}=\mathbb F_7$). The values of

$P_{0,0,\mathbf{k}[\alpha]}$ for knots up to 9 crossings are presented in Table 2 (the choice between the "right" knots $3_1, 5_1, 5_2, \dots$ and their mirror images $\bar{3}_1, \bar{5}_1, \bar{5}_2 \dots$ is done according to the database "The Knot Atlas" `http://katlas.org`).

The most interesting case is $\mathbf{k} = \mathbb{F}_p$ for $p = 37$. Up to now this is the only case when the invariant P distinguishes knots with equal HOMFLY polynomials. In this case the normal form has one more monomial: $\alpha^2 u^4$.

4.4. The case $\mathbf{k} = \mathbb{Z}$, $\alpha = \beta = 0$.

Proposition 4.4. *Let $\mathbf{k} = \mathbb{Z}$ and $\alpha = \beta = 0$. We introduce the monomial order on $\bar{A}$ by saying that $u^{a_1} v^{b_1} \bar{u}^{c_1} \bar{v}^{d_1} > u^{a_2} v^{b_2} \bar{u}^{c_2} \bar{v}^{d_2}$ if and only if one of the following conditions holds:*

- *either $d_1 > d_2$, or $d_1 = d_2$ and $c_1 > c_2$,*
- *$(d_1, c_1) = (d_2, c_2)$ and $a_1 + b_1 > a_2 + b_2$,*
- *$(d_1, c_1, a_1 + b_1) = (d_2, c_2, a_2 + b_2)$ and $b_1 > b_2$.*

Let $\mathcal{G} = \{16, 4u^2 + 4v,\ 4v^2 + 4u,\ 4uv - 4,\ v^3 + uv + u^3 - 3\}$ and

$$\bar{\mathcal{G}} = \mathcal{G} \cup \{4\bar{u} - 4v,\ 4\bar{v} - 4u,\ u\bar{u} - 1,\ v\bar{v} - 1,\ u^3\bar{v} - 3\bar{v} + v^2 + u,\ \bar{u}\bar{v} + u^2\bar{v} + \bar{u}v^2 - 3\}.$$

Then $\mathcal{G}$ and $\bar{\mathcal{G}}$ are Gröbner bases of I and $\bar{I}$ respectively.

Remark 4.5 The monomial order used in Proposition 4.4 can be defined by saying that this is the lexicographic order with $\bar{v} > \bar{u} > w > v' > u'$ under the change of variables $u = uw'$, $v = vw'$.

Thus, in the normal form of an element of $\bar{A}/\bar{I}$, the coefficients of the monomials 1, u, v range in $\{0, \dots, 15\}$ (note that u and v do not appear in $P(L)$ by Proposition 4.2), the coefficients of

$$u^{n+1},\ u^n v,\ u^{n-1}v^2,\ \bar{u}^n,\ v\bar{u}^n,\ v^2\bar{u}^n,\ \bar{v}^n,\ u\bar{v}^n,\ u^2\bar{v}^n, \quad n \geq 1,$$

range in $\{0, 1, 2, 3\}$, and all the other coefficients vanish. Due to Proposition 4.1(h), this fact implies the following nice property of the normal form of $P(L)$. Let us define the degree of an element of $\bar{A}/\bar{I}$ represented by $f \in \bar{A}$ as $\min_{g \in f + \bar{I}} \max_T \deg T$ where T runs over all monomials of g and $\deg u^a v^b \bar{u}^c \bar{v}^d \overset{\text{def}}{=} a + b - c - d$.

Proposition 4.6. *If $f(u, v, \bar{u}, \bar{v})$ is the normal form of an element of $\bar{A}/\bar{I}$ of degree ≤ 2, then $f(v, u, \bar{v}, \bar{u})$ is the normal form of the corresponding element. In particular, if $\bar{L}$ is the mirror image of a link L, then the normal form of $P_{0,0;\mathbb{Z}}(\bar{L})$ is obtained from the normal form of $P_{0,0;\mathbb{Z}}(L)$ by swapping u with v and $\bar{u}$ with $\bar{v}$.*

Note that if the degree of an element of $\bar{A}/\bar{I}$ is greater than two, then the swapping of u and v can drastically change the normal form. For example, u^3 is already in its normal form whereas the normal form of v^3 is $3uv + 3u^3 + 3$.

In Table 2 we give the normal forms of the invariant $P_{0,0;\mathbb{Z}}(K)$ for all knots K up to 10 crossings. We see in this table that there are many repetitions. In Table 3, for each $n = 0, 1, \dots, 12$, we give the number of different values (up to exchange of u and v) that $P_{0,0,;\mathbb{Z}}$ takes on knots with $\leq n$ crossings.

Table 1. $P_{\alpha,0;\mathbb{Q}}(K)$ for knots K up to 9 crossings ($\bar{K}$ is the mirror of K)

K	coefficients of $P_{\alpha,0;\mathbb{Q}}(K)$								coefficients of $P_{\alpha,0;\mathbb{Q}}(\bar{K})$ if differ							
	1	α^3	α^6	$\alpha^2 u$	$\alpha^5 u$	αu^2	$\alpha^4 u^2$	u^3	1	α^3	α^6	$\alpha^2 u$	$\alpha^5 u$	αu^2	$\alpha^4 u^2$	u^3
0_1	1	0	0	0	0	0	0	0								
3_1	-20/9	-61/36	-1/24	-44/3	-5/2	26/1	27/8	-35/3	-145/81	-899/324	-317/864	605/108	275/288	-20/3	-23/24	56/27
4_1	-10/9	25/9	43/96	-127/12	-49/32	10/1	5/4	-7/3								
5_1	22/3	965/12	365/32	-1221/4	-1449/32	367/1	375/8	-119/1	-796/729	-3229/1458	-1565/7776	4769/972	659/2592	-55/9	-1/72	791/243
5_2	34/9	311/9	491/96	-1823/12	-737/32	194/1	99/4	-203/3	146/243	-446/243	-923/2592	2399/324	1313/864	-82/9	-59/36	203/81
6_1	19/9	161/36	47/96	49/12	31/32	-16/1	-17/8	28/3	136/81	1799/324	22/27	-437/27	-179/72	50/3	53/24	-119/27
6_2	-55/27	-481/54	-95/72	365/9	19/3	-55/1	-57/8	182/9	-445/243	-5/972	7/648	-340/81	-29/108	13/3	1/24	-154/81
6_3	-157/81	-1265/324	-23/54	83/27	7/36	0	1/4	-28/27								
7_1	178/3	2753/12	785/32	-1257/4	-1053/32	131/1	171/8	77/1	-2936/2187	-20393/4374	-26419/23328	52159/2916	39997/7776	-1819/81	-3925/648	2569/729
7_2	169/9	1265/9	449/24	-1289/3	-491/8	468/1	60/1	-392/3	-665/243	-970/243	-59/81	832/81	145/54	-344/27	-167/54	196/81
7_3	1351/2187	2123/8748	2471/5832	-2084/729	-2461/972	89/27	737/216	2338/729	43/1	521/2	33/1	-678/1	-747/8	673/1	699/8	-154/1
7_4	-28/81	-989/324	-155/216	326/27	13/4	-410/27	-815/216	77/27	223/9	6365/36	2291/96	-6803/12	-2621/32	636/1	651/8	-560/3
7_5	28/1	309/2	621/32	-1601/4	-1761/32	399/1	417/8	-91/1	8650/2187	20987/8748	18569/23328	-16697/2916	-28751/7776	187/27	1051/216	2401/729
7_6	20/27	1385/54	1093/288	-4009/36	-1603/96	139/1	141/8	-427/9	-542/243	-1789/972	-895/2592	1039/324	1081/864	-43/9	-115/72	49/81
7_7	-34/27	89/54	7/18	-118/9	-55/24	50/3	59/24	-49/9	-67/81	46/81	55/864	773/108	335/288	-16/1	-15/8	224/27
8_1	-35/9	-286/9	-14/3	424/3	43/2	-184/1	-47/2	196/3	-173/243	1121/243	521/648	-1457/81	-659/216	172/9	26/9	-392/81
8_2	-94/9	-2165/36	-701/96	1493/12	515/32	-101/1	-105/8	35/3	1928/729	2591/1458	4099/7776	-5215/972	-6301/2592	179/27	665/216	203/243
8_3	226/81	812/81	1127/864	-1307/108	-437/288	2/3	1/12	133/27								
8_4	-98/81	-721/324	-385/864	2905/108	1327/288	-45/1	-49/8	511/27	-244/243	3245/486	2293/2592	-5785/324	-1723/864	43/3	25/24	-259/81
8_5	185/729	2443/2916	4015/7776	-6967/972	-7765/2592	245/27	203/54	98/243	-148/9	-1735/18	-299/24	785/3	293/8	-269/1	-69/2	203/3
8_6	-151/27	-5915/108	-137/18	1745/9	343/12	-228/1	-117/4	644/9	-101/729	1091/2916	-35/243	-413/243	323/324	4/3	-19/12	-644/243
8_7	-29/243	-4489/972	-257/324	1117/81	601/216	-133/9	-205/72	238/81	77/27	1235/54	241/72	-907/9	-91/6	129/1	131/8	-406/9
8_8	217/243	-1895/486	-283/648	589/81	25/54	-13/3	5/24	70/81	89/81	1621/324	193/216	-1012/27	-221/36	55/1	59/8	-574/27
8_9	53/27	361/108	31/72	14/9	5/24	-28/3	-11/12	56/9								
8_{10}	-707/243	-3593/486	-3007/2592	6283/324	3229/864	-193/9	-137/36	406/81	-10/27	2287/108	119/36	-1039/9	-53/3	155/1	79/4	-511/9
8_{11}	-238/27	-3049/54	-551/72	1613/9	313/12	-202/1	-207/8	539/9	-2135/729	-1750/729	-3973/7776	3793/972	5059/2592	-16/3	-61/24	-140/243
8_{12}	-4/27	661/108	253/288	-325/36	-127/96	2/3	1/3	35/9								
8_{13}	-296/243	-545/486	29/2592	-1073/324	-923/864	17/3	35/24	-119/81	-82/81	2521/324	1159/864	-5191/108	-2209/288	65/1	69/8	-637/27
8_{14}	-76/27	-2183/108	-719/288	1511/36	533/96	-34/1	-9/2	35/9	-392/729	-4261/2916	-3889/7776	5545/972	6523/2592	-70/9	-29/9	-35/243
8_{15}	53/3	1331/6	1057/32	-3605/4	-4353/32	1109/1	279/2	-378/1	18166/2187	72113/8748	13781/5832	-17984/729	-21011/1944	2449/81	1106/81	2485/729
8_{16}	-5/27	377/27	73/36	-542/9	-53/6	74/1	37/4	-224/9	-239/81	-749/162	-169/216	259/27	181/72	-94/9	-101/36	28/27
8_{17}	-79/81	-91/162	1/216	125/27	29/72	-28/3	-2/3	140/27								
8_{18}	-146/81	-2347/324	-751/864	1975/108	613/288	-58/3	-5/3	175/27								
8_{19}	-1486/729	-10007/2916	-547/972	2327/243	643/324	-329/27	-58/27	875/243	85/3	668/3	965/32	-2881/4	-3333/32	809/1	207/2	-238/1
8_{20}	22/27	779/108	23/18	-497/9	-53/6	81/1	21/2	-287/9	253/243	-673/243	-979/2592	3175/324	1057/864	-11/1	-1/1	322/81
8_{21}	-52/9	-428/9	-19/3	416/3	79/4	-147/1	-75/4	119/3	-71/729	-6985/2916	-4057/7776	7873/972	5755/2592	-29/3	-31/12	322/243

Table 1 (continued-1)

	1	α^3	α^6	α^2u	α^5u	αu^2	α^4u^2	u^3	1	α^3	α^6	α^2u	α^5u	αu^2	α^4u^2	u^3
9_1	-265/3	-10993/6	-9047/32	28819/4	34923/32	-8614/1	-8433/8	2940/1	230278/19683	510649/39366	23542/6561	-260498/6561	-285377/17496	11800/243	39745/1944	22519/6561
9_2	448/9	5303/36	157/12	-71/3	79/8	-210/1	-177/8	553/3	-4463/2187	-45685/8748	-30283/23328	54187/2916	45445/7776	-1864/81	-4537/648	1708/729
$\bar{9}_3$	134/3	-7367/12	-3599/32	14167/4	18003/32	-4741/1	-4617/8	1841/1	-76616/6561	-211865/13122	-281941/69984	448729/8748	415435/23328	-15509/243	-42239/1944	7315/2187
9_4	232/3	689/12	-35/4	827/1	1203/8	-1450/1	-1377/8	707/1	32555/6561	161341/26244	140143/69984	-194719/8748	-226657/23328	736/27	2653/216	6692/2187
$\bar{9}_5$	637/9	2606/9	191/6	-1316/3	-49/1	232/1	69/2	196/3	-6533/2187	-14083/2187	-9679/5832	16951/729	14725/1944	-2356/81	-740/81	1960/729
9_6	31/3	-2465/6	-2263/32	8147/4	10203/32	-2618/1	-2541/8	980/1	-105118/6561	-289351/13122	-24527/4374	154610/2187	145757/5832	-21332/243	-59483/1944	7637/2187
9_7	94/3	-166/3	-535/32	2795/4	3759/32	-1046/1	-501/4	455/1	49886/6561	62557/6561	206071/69984	-292507/8748	-327541/23328	3334/81	5717/324	7049/2187
9_8	172/243	1003/486	209/2592	43/324	913/864	-43/9	-133/72	133/81	298/81	9575/324	3647/864	-12359/108	-4865/288	139/1	139/8	-1253/27
9_9	124/3	-4849/12	-2443/32	9771/4	12483/32	-3296/1	-1599/4	1295/1	-100552/6561	-610997/26244	-432305/69984	691145/8748	656723/23328	-23828/243	-33541/972	7469/2187
$\bar{9}_{10}$	98/1	263/2	-3/4	732/1	1125/8	-1418/1	-1335/8	735/1	65669/6561	70606/6561	209935/69984	-287731/8748	-318841/23328	1088/27	3707/216	6776/2187
$\bar{9}_{11}$	313/9	3637/18	2471/96	-6467/12	-2393/32	546/1	567/8	-392/3	11050/2187	20971/4374	7685/5832	-10079/729	-1438/243	448/27	1609/216	1435/729
9_{12}	329/27	9265/108	799/72	-2122/9	-787/24	240/1	123/4	-532/9	-2825/729	-10549/2916	-212/243	2083/243	1193/324	-308/27	-505/108	-56/243
$\bar{9}_{13}$	67/1	499/4	39/8	326/1	555/8	-740/1	-339/4	420/1	61103/6561	314719/26244	15613/4374	-90109/2187	-49067/2916	4096/81	6827/324	6944/2187
9_{14}	-199/243	347/486	119/324	-868/81	-559/216	133/9	223/72	-322/81	-307/81	-8675/324	-815/216	2804/27	553/36	-129/1	-129/8	1190/27
9_{15}	-1082/729	-3905/1458	-1675/1944	2521/243	344/81	-374/27	-1157/216	49/243	491/27	3295/27	4681/288	-13429/36	-5119/96	408/1	417/8	-1036/9
$\bar{9}_{16}$	7/1	-201/1	-1107/32	3751/4	4683/32	-1173/1	-561/4	434/1	-43018/2187	-255323/8748	-45233/5832	71738/729	68693/1944	-29651/243	-42163/972	2597/729
9_{17}	313/81	3619/162	2543/864	-6395/108	-2321/288	58/1	55/8	-392/27	2/3	29/6	11/24	-29/3	-1/6	56/9	-61/72	-7/3
9_{18}	52/1	75/4	-279/32	2415/4	3447/32	-1014/1	-120/1	483/1	83000/6561	371311/26244	275863/69984	-385519/8748	-419725/23328	4390/81	3649/162	7133/2187
9_{19}	26/9	541/36	211/96	-595/12	-245/32	167/3	179/24	-49/3	652/243	2977/486	2249/2592	-1565/324	-911/864	-11/3	7/24	469/81
9_{20}	124/9	2153/36	671/96	-1487/12	-509/32	104/1	57/4	-35/3	13120/2187	52589/8748	39173/23328	-53933/2916	-59471/7776	68/3	115/12	1183/729
$\bar{9}_{21}$	404/27	12997/108	4669/288	-13957/36	-5359/96	434/1	111/2	-1141/9	-3116/729	-15901/2916	-9553/7776	15529/972	13483/2592	-554/27	-341/54	553/243
9_{22}	-172/81	667/324	79/864	-439/108	227/288	-4/9	-65/36	-7/27	52/81	6689/324	2507/864	-7979/108	-3041/288	84/1	41/4	-707/27
9_{23}	21/1	12/1	-99/32	791/4	1167/32	-336/1	-303/8	168/1	78434/6561	201803/13122	39467/8748	-114556/2187	-123355/5832	5222/81	17129/648	7301/2187
9_{24}	-94/81	2155/324	277/216	-1366/27	-607/72	215/3	59/6	-721/27	-227/243	-1619/486	-967/2592	4675/324	1405/864	-61/3	-5/3	742/81
9_{25}	233/27	1078/27	173/36	-742/9	-253/24	67/1	69/8	-70/9	-1591/729	-9443/2916	-1997/1944	2690/243	1603/324	-389/27	-1361/216	-238/243
9_{26}	-917/729	-3094/729	-911/972	2938/243	2449/648	-121/9	-319/72	70/243	-101/27	-3445/108	-307/72	838/9	161/12	-99/1	-103/8	238/9
9_{27}	167/81	676/81	143/108	-970/27	-427/72	137/3	155/24	-406/27	451/243	-541/972	-1/162	715/81	145/216	-41/3	-17/24	574/81
9_{28}	-188/27	-907/27	-155/36	706/9	131/12	-73/1	-19/2	133/9	-2951/729	-20467/2916	-10141/7776	17197/972	12271/2592	-181/9	-97/18	574/243
9_{29}	-38/81	1310/81	521/216	-2105/27	-415/36	100/1	99/8	-959/27	-227/81	-283/81	-625/864	1309/108	943/288	-154/9	-289/72	112/27
9_{30}	-4/81	901/81	1531/864	-5023/108	-2149/288	167/3	185/24	-469/27	-62/243	2159/972	1145/2592	-569/324	-743/864	-11/3	13/24	385/81
9_{31}	-26/27	287/108	245/288	-2117/36	-923/96	95/1	95/8	-371/9	-1208/729	-4432/729	-10057/7776	18949/972	13735/2592	-203/9	-437/72	679/243
9_{32}	-61/9	-1469/36	-67/12	401/3	79/4	-154/1	-20/1	140/3	-2981/729	-12391/2916	-1801/1944	1918/243	2275/648	-82/9	-79/18	-392/243
$\bar{9}_{34}$	-98/243	1061/972	31/81	-347/81	-175/108	3/1	7/4	133/81	19/81	722/81	1199/864	-3107/108	-1373/288	89/3	55/12	-182/27
9_{35}	772/9	3560/9	4361/96	-8597/12	-2795/32	506/1	279/4	7/3	-13832/2187	-18799/2187	-47401/23328	76165/2916	67963/7776	-2650/81	-3431/324	1897/729
9_{36}	259/9	5969/36	247/12	-1205/3	-217/4	378/1	99/2	-224/3	5821/2187	33725/8748	3811/2916	-11393/729	-6301/972	514/27	439/54	1120/729
9_{37}	7/9	641/36	127/48	-361/6	-147/16	197/3	209/24	-56/3	139/243	4327/486	1705/1296	-2497/162	-1117/432	19/3	37/24	280/81

Table 1 (continued-2)

	1	α^3	α^6	α^2u	α^5u	αu^2	α^4u^2	u^3	1	α^3	α^6	α^2u	α^5u	αu^2	α^4u^2	u^3
9_{38}	$\frac{125}{3}$	$\frac{1033}{12}$	$\frac{157}{32}$	$\frac{411}{4}$	$\frac{855}{32}$	$\frac{-304}{1}$	$\frac{-261}{8}$	$\frac{196}{1}$	$\frac{111548}{6561}$	$\frac{524689}{26244}$	$\frac{48191}{8748}$	$\frac{-137809}{2187}$	$\frac{-146401}{5832}$	$\frac{6278}{81}$	$\frac{20291}{648}$	$\frac{7385}{2187}$
$\bar{9}_{39}$	$\frac{308}{27}$	$\frac{2011}{27}$	$\frac{2857}{288}$	$\frac{-8437}{36}$	$\frac{-3223}{96}$	$\frac{261}{1}$	$\frac{267}{8}$	$\frac{-679}{9}$	$\frac{-1882}{729}$	$\frac{-14795}{2916}$	$\frac{-10757}{7776}$	$\frac{17957}{972}$	$\frac{16763}{2592}$	$\frac{-635}{27}$	$\frac{-1715}{216}$	$\frac{371}{243}$
9_{40}	$\frac{-103}{27}$	$\frac{-727}{54}$	$\frac{-503}{288}$	$\frac{1331}{36}$	$\frac{533}{96}$	$\frac{-41}{1}$	$\frac{-23}{4}$	$\frac{98}{9}$	$\frac{-3302}{729}$	$\frac{-9667}{2916}$	$\frac{-1759}{1944}$	$\frac{1336}{243}$	$\frac{2467}{648}$	$\frac{-65}{9}$	$\frac{-181}{36}$	$\frac{-749}{243}$
$\bar{9}_{41}$	$\frac{163}{81}$	$\frac{233}{324}$	$\frac{77}{216}$	$\frac{-176}{27}$	$\frac{-167}{72}$	$\frac{94}{9}$	$\frac{55}{18}$	$\frac{-56}{27}$	$\frac{-61}{81}$	$\frac{-5789}{324}$	$\frac{-265}{108}$	$\frac{1709}{27}$	$\frac{325}{36}$	$\frac{-74}{1}$	$\frac{-9}{1}$	$\frac{644}{27}$
9_{42}	$\frac{148}{81}$	$\frac{2165}{324}$	$\frac{755}{864}$	$\frac{-1475}{108}$	$\frac{-497}{288}$	$\frac{10}{1}$	$\frac{1}{1}$	$\frac{-35}{27}$								
9_{43}	$\frac{-505}{729}$	$\frac{-553}{1458}$	$\frac{301}{1944}$	$\frac{-607}{243}$	$\frac{-205}{162}$	$\frac{3}{1}$	$\frac{13}{8}$	$\frac{182}{243}$	$\frac{41}{9}$	$\frac{1651}{36}$	$\frac{151}{24}$	$\frac{-460}{3}$	$\frac{-89}{4}$	$\frac{173}{1}$	$\frac{177}{8}$	$\frac{-154}{3}$
9_{44}	$\frac{52}{27}$	$\frac{631}{54}$	$\frac{509}{288}$	$\frac{-1841}{36}$	$\frac{-755}{96}$	$\frac{65}{1}$	$\frac{67}{8}$	$\frac{-203}{9}$	$\frac{418}{243}$	$\frac{2705}{972}$	$\frac{1133}{2592}$	$\frac{-2069}{324}$	$\frac{-1091}{864}$	$\frac{17}{3}$	$\frac{29}{24}$	$\frac{-35}{81}$
9_{45}	$\frac{137}{9}$	$\frac{3409}{36}$	$\frac{149}{12}$	$\frac{-829}{3}$	$\frac{-313}{8}$	$\frac{295}{1}$	$\frac{303}{8}$	$\frac{-238}{3}$	$\frac{-761}{729}$	$\frac{-5267}{1458}$	$\frac{-1717}{1944}$	$\frac{3103}{243}$	$\frac{320}{81}$	$\frac{-425}{27}$	$\frac{-1019}{216}$	$\frac{406}{243}$
9_{46}	$\frac{-16}{9}$	$\frac{-311}{9}$	$\frac{-491}{96}$	$\frac{1823}{12}$	$\frac{737}{32}$	$\frac{-194}{1}$	$\frac{-99}{4}$	$\frac{203}{3}$	$\frac{340}{243}$	$\frac{446}{243}$	$\frac{923}{2592}$	$\frac{-2399}{324}$	$\frac{-1313}{864}$	$\frac{82}{9}$	$\frac{59}{36}$	$\frac{-203}{81}$
$\bar{9}_{47}$	$\frac{-80}{27}$	$\frac{-556}{27}$	$\frac{-889}{288}$	$\frac{3301}{36}$	$\frac{1363}{96}$	$\frac{-120}{1}$	$\frac{-31}{2}$	$\frac{385}{9}$	$\frac{-620}{243}$	$\frac{-1355}{486}$	$\frac{-1105}{2592}$	$\frac{709}{324}$	$\frac{859}{864}$	$\frac{-4}{3}$	$\frac{-7}{6}$	$\frac{-119}{81}$
9_{48}	$\frac{-71}{27}$	$\frac{-397}{108}$	$\frac{-101}{144}$	$\frac{191}{18}$	$\frac{133}{48}$	$\frac{-125}{9}$	$\frac{-233}{72}$	$\frac{28}{9}$	$\frac{95}{27}$	$\frac{3251}{54}$	$\frac{1283}{144}$	$\frac{-4739}{18}$	$\frac{-1907}{48}$	$\frac{333}{1}$	$\frac{339}{8}$	$\frac{-1036}{9}$
$\bar{9}_{49}$	$\frac{191}{3}$	$\frac{4015}{12}$	$\frac{41}{1}$	$\frac{-773}{1}$	$\frac{-825}{8}$	$\frac{705}{1}$	$\frac{741}{8}$	$\frac{-126}{1}$	$\frac{12389}{2187}$	$\frac{10621}{2187}$	$\frac{8287}{5832}$	$\frac{-9835}{729}$	$\frac{-3151}{486}$	$\frac{49}{3}$	$\frac{199}{24}$	$\frac{2366}{729}$

Example 4.7. Let us compute $P_{0,0,\mathbb{Z}}(K)$ where K is the figure-eight knot 4_1. We represent K by the 3-braid $X = \bar{x}y\bar{x}y$ where $x, \bar{x}, y, \bar{y}$ are as in Introduction. Then

$$\begin{aligned}
t(X) &= t\big(\bar{x}(-x - y - \bar{x}\bar{y} - \bar{y}\bar{x} - \bar{x}yx - x\bar{y}x - xy\bar{x})\big) && \text{by (2)}\\
&= t(-1 - \bar{x}y - \bar{x}^2\bar{y} - \bar{x}\bar{y}\bar{x} - \bar{x}^2yx - \bar{y}x - y\bar{x})\\
&= t(-1 - u\bar{x} - v\bar{x}^2 - v\bar{x}^2 - u\bar{x} \quad - vx - u\bar{x}) && \text{by the Markov relation}\\
&= t(-1 - 3u\bar{x} - 3vx) && \text{since } \bar{x}^2 = x \text{ by (1)}\\
&= \quad -1 - 3uv - 3uv = -1 - 6uv && \text{by the Markov relation}
\end{aligned}$$

whence

$$\begin{aligned}
P(K) &= u^{(1-3-0)/2}v^{(1-3+0)/2}t(X) && \text{by the definition of } P\\
&= -\bar{u}\bar{v} - 6 && \text{see } t(X) \text{ computed above}\\
&= 7 + u^2\bar{v} + \bar{u}v^2 && \text{Gröbner reduction } -\bar{u}\bar{v} \to u^2\bar{v} + \bar{u}v^2 - 3
\end{aligned}$$

Table 3. Here $\mathcal{K}_n$ is the set of knots with $\le n$ crossings; $f(u,v) \sim f(v,u)$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
Card $\mathcal{K}_n$	1	1	1	2	3	5	8	15	36	85	250	802	2978
Card $P_{0,0;\mathbb{Z}}(\mathcal{K}_n)/\!\sim$	1	1	1	2	3	4	5	7	11	19	29	47	86

4.5. Relation between $P_{0,0;\mathbb{Z}}$ and the HOMFLY polynomial. It was discovered by Cabanes and Marin [2] that $P_{0,0;\mathbb{Z}/4\mathbb{Z}}$ is a specialization of the HOMFLY polynomial. In this subsection we reproduce their arguments in another form, and then we show that if links L_1 and L_2 have equal HOMFLY polynomials, than

Table 2. $P_{0,0,\mathbb{Z}}(K)$ for knots K up to 10 crossings

K	$P_{0,0,\mathbb{Z}}(K)$
$0_1 6_2 6_3 8_6 8_7 8_8 8_9 8_{16} 8_{17} 9_{26} 9_{27} 9_{32} 9_{33} 10_{12} 10_{16} 10_{20}$ $10_{22} 10_{23} 10_{26} 10_{27} 10_{34} 10_{41} 10_{43} 10_{48} 10_{52} 10_{54} 10_{79}$ $10_{81} 10_{83} 10_{86} 10_{91} 10_{94} 10_{102} 10_{109} 10_{110} 10_{116}$ $10_{118} 10_{123} 10_{125} 10_{129} 10_{135} 10_{153} 10_{155} 10_{156} 10_{162}$	1
$\overline{3}_1 8_5 8_{10} \overline{8}_{11} \overline{8}_{20} \overline{8}_{21} \overline{9}_{24} \overline{9}_{28} \overline{9}_{29} 10_5 10_9 \overline{10}_{32} 10_{40} 10_{59}$ $10_{62} 10_{64} 10_{76} 10_{77} \overline{10}_{82} 10_{84} \overline{10}_{85} 10_{87} \overline{10}_{103} 10_{106}$ $\overline{10}_{112} 10_{113} \overline{10}_{114} 10_{122} 10_{136} \overline{10}_{141} \overline{10}_{143} 10_{147} \overline{10}_{159}$	$\bar{u}^2 v$
$4_1 8_4 8_{13} 9_{22} 9_{30} 9_{42} 9_{44} 10_{11} 10_{15} 10_{17}$ $10_{28} 10_{37} 10_{70} 10_{71} 10_{90} 10_{93} 10_{104} 10_{119}$	$7 + \bar{u}v^2 + u^2\bar{v}$
$\overline{5}_1 \overline{5}_2 \overline{7}_1 \overline{7}_5 \overline{7}_6 \overline{8}_2 \overline{8}_{14} 9_3 \overline{9}_7 \overline{9}_8 \overline{9}_9 \overline{9}_{18} \overline{9}_{20} \overline{9}_{31} \overline{10}_7 \overline{10}_8 \overline{10}_{18}$ $\overline{10}_{24} \overline{10}_{25} \overline{10}_{44} 10_{46} 10_{47} 10_{50} 10_{51} 10_{56} 10_{57} 10_{92} 10_{95}$ $\overline{10}_{100} 10_{105} 10_{111} 10_{117} \overline{10}_{121} \overline{10}_{126} \overline{10}_{127} \overline{10}_{130} \overline{10}_{131}$ $\overline{10}_{132} \overline{10}_{133} \overline{10}_{148} \overline{10}_{149} 10_{150} 10_{151} \overline{10}_{161}$	$3 + 3\bar{u}^3 + 3\bar{u}^2 v$
$\overline{6}_1 7_7 \overline{9}_{17} \overline{9}_{34} 10_4 10_{10} \overline{10}_{19} \overline{10}_{29} \overline{10}_{31} \overline{10}_{42}$ $\overline{10}_{68} \overline{10}_{107} 10_{108} \overline{10}_{146} 10_{158} \overline{10}_{164}$	$4 + 3\bar{u}^2 v + \bar{u}v^2 + u^2\bar{v}$
$\overline{7}_2 7_3 \overline{9}_{12} 9_{13} \overline{9}_{25} 9_{36} 9_{43} \overline{9}_{45} 9_{49} \overline{10}_2$ $\overline{10}_6 \overline{10}_{38} \overline{10}_{39} 10_{72} \overline{10}_{73} 10_{157}$	$2 + 2\bar{u}^3 + 2\bar{u}^2 v + 3\bar{u}^4 v^2$
$7_4 9_{11} 9_{15} \overline{10}_{14} \overline{10}_{21} \overline{10}_{36} \overline{10}_{67} 10_{69} \overline{10}_{89} 10_{160} 10_{165}$	$5 + \bar{u}^3 + 3\bar{u}^4 v^2$
$\overline{8}_1 9_{14} \overline{9}_{41} \overline{10}_{60} \overline{10}_{137} 10_{138}$	$13 + \bar{u}^3 + \bar{u}^2 v + \bar{u}v^2 + u^2\bar{v}$
$8_3 8_{12} 9_{19} 10_{33} 10_{45} 10_{88}$	$15 + 3\bar{u}^2 v + 2\bar{u}v^2 + 2u^2\bar{v} + 3u\bar{v}^2$
$\overline{8}_{15} 8_{19} \overline{9}_1 \overline{9}_6 9_{16} \overline{9}_{23} \overline{9}_{38} \overline{10}_{66} \overline{10}_{78} 10_{139}$	$3\bar{u}^2 v + 3\bar{u}^5 v + 3\bar{u}^4 v^2$
$8_{18} 10_{99}$	$11 + 3\bar{u}v^2 + 3u^2\bar{v}$
$\overline{9}_2 \overline{9}_4 9_{10} \overline{10}_{145}$	$5 + \bar{u}^3 + 2\bar{u}^2 v + \bar{u}^5 v$
9_5	$7 + 3\bar{u}^3 + 3\bar{u}^2 v + \bar{u}^5 v + 3\bar{u}^4 v^2$
$9_{21} 9_{39} \overline{10}_{30}$	$12 + \bar{u}^3 + \bar{u}^2 v + 3\bar{u}^4 v^2$
9_{35}	$2 + 2u\bar{v}^2 + 2\bar{v}^3 + 2u^2\bar{v}^4 + u\bar{v}^5$
9_{37}	$13 + 3\bar{u}^2 v + 3\bar{u}v^2 + 3u^2\bar{v} + 3u\bar{v}^2$
$\overline{9}_{40} 10_{61} 10_{65} \overline{10}_{140} \overline{10}_{144} 10_{163}$	$12 + 3\bar{u}^3 + 2\bar{u}^2 v$
$\overline{9}_{46} 9_{47} 10_{75}$	$15 + \bar{u}^3 + \bar{u}^2 v$
$9_{48} \overline{10}_{74}$	$5 + 2\bar{u}^3 + 2\bar{u}^2 v$
10_1	$10 + \bar{u}v^2 + u^2\bar{v} + 2u\bar{v}^2 + 2\bar{v}^3 + u^2\bar{v}^4$
$\overline{10}_3 10_{35}$	$1 + \bar{u}^3 + 2\bar{u}v^2 + 2u^2\bar{v} + 3u\bar{v}^2$
$10_{13} 10_{58}$	$7 + 3\bar{u}^2 v + 3\bar{u}v^2 + 3u^2\bar{v} + \bar{v}^3$
$\overline{10}_{49} \overline{10}_{53} \overline{10}_{55} \overline{10}_{80} 10_{101} 10_{124} 10_{128} 10_{134}$ $\overline{10}_{152} 10_{154}$	$9 + 2\bar{u}^3 + \bar{u}^6 + 2\bar{u}^2 v + 2\bar{u}^5 v + \bar{u}^4 v^2$
$\overline{10}_{63} 10_{142}$	$8 + \bar{u}^3 + \bar{u}^6 + 2\bar{u}^2 v + 3\bar{u}^5 v + 2\bar{u}^4 v^2$
10_{96}	$12 + \bar{u}^3 + 2\bar{u}v^2 + 2u^2\bar{v}$
10_{97}	$10 + 3\bar{u}^3 + 2\bar{u}^2 v + 2\bar{u}^4 v^2$
10_{98}	$u^2\bar{v}^4$
10_{115}	$9 + 3\bar{u}^2 v + \bar{u}v^2 + u^2\bar{v} + 3u\bar{v}^2$
10_{120}	$11 + 3u\bar{v}^2 + u^2\bar{v}^4 + u\bar{v}^5 + \bar{v}^6$

$P_{0,0;\mathbb{Z}}(L_1) - P_{0,0;\mathbb{Z}}(L_2) \in \{0, 8\}$. So, $P_{0,0;\mathbb{Z}}$ can be thought of as a sort of spin of a certain specialization of the HOMFLY polynomial.

Let $k = \mathbb{Z}/4\mathbb{Z}$, $K_\infty = K_\infty(0,0;k)$, $A = k[u,v]$, and $I = I(0,0;k)$. Let $\hat{k} = k[j]/(j^3-1)$ and let $\hat{A} = \hat{k}[u,v] = k[j,u,v]/(j^3-1)$. Let H_∞ be the Hecke algebra $\hat{k}B_\infty/(\sigma_1^2 + j\sigma_1 + j^2)$ (we denote the image of σ_i in H_∞ by g_i). A straightforward computation using (1) and (2) shows that the correspondence $s_i \mapsto g_i$ defines a ring homomorphism $p : K_\infty \to H_\infty$. Let $t_H : H_\infty \otimes_{\hat{k}} \hat{A} \to M_H \overset{\text{def}}{=} \hat{A}/(v+ju+j^2) \cong \hat{k}[u]$ be the Markov trace on H_∞. The ring M_H is the quotient of $k[j,u,v]$ by the ideal $(j^3-1, v+ju+j^2)$. For the lexicographic monomial order with $j > w > v' > u'$ ($u = wu'$, $v = wv'$; see Remark 4.5), the reduced Gröbner base $\mathcal{G}_H$ of this ideal is

$$\{v^3+u^3+uv+1,\ j(u^2-v)+uv-1,\ j(uv-1)+v^2-u,\ j(v^2-u)-u^2+v,\ j^2+ju+v\}.$$

Since $I = (v^3+u^3+uv+1)$, it follows that we can identify A/I with the subring of M_H generated by u and v. Then we have $t = t_H \circ p$, i. e., the Markov trace on $K_\infty(0,0;\mathbb{Z}/4\mathbb{Z})$ is determined by the Markov trace on H_∞, hence $P_{0,0;\mathbb{Z}/4\mathbb{Z}}$ is determined by the HOMFLY polynomial. For example, if we normalize the latter (we denote it $h(a,z)$) by $h(\text{unknot}) = 1$, $ah(\times) - zh(\asymp) - a^{-1}h(\times) = 0$, then we have:

Proposition 4.8. *$P_{0,0;\mathbb{Z}/4\mathbb{Z}}(u,v) = h(ij^2u^{1/2}v^{-1/2}, -i)$ where $i = \sqrt{-1}$.*

If we plug $a = ij^2u^{1/2}v^{-1/2}$, $z = -i$ into $h(a,z)$, we obtain a Laurent polynomial in $i, j, u^{1/2}, v^{1/2}$. To get rid of the square roots of u, v, and -1, we just multiply the result by $(-uv)^{-1/2}$ in the case of a link L with an even number of components (this corresponds to the normalization (14) of $P(L)$). To eliminate j and to put the result to the canonical form, we reduce it using the Gröbner base $\bar{\mathcal{G}}_H$ of the ideal $(j^3-1,\ v+ju+j^2,\ u\bar{u}-1, v\bar{v}-1)$ of the ring $k[j,u,v,\bar{u},\bar{v}]$ for the lexicographic monomial order with $j > \bar{v} > \bar{u} > w > v' > u'$ ($u = wu'$, $v = wv'$; see Remark 4.5). We have $\bar{\mathcal{G}}_H = \mathcal{G}_H \cup \{\bar{u}u - 1, \bar{v}v - 1, u^3\bar{v} + \bar{v} + v^2 + u, \bar{u}\bar{v} + u^2\bar{v} + v^2\bar{u} + 1, j(\bar{u} - v) - v^2\bar{u} + 1, j(\bar{v} - u) + u\bar{v} - v\}$.

Example 4.9. Let K be the trefoil knot given by the 2-braid σ_1^{-3}. Then $h(K) = -a^4 + a^2z^2 + 2a^2$, hence $P_{0,0;\mathbb{Z}/4\mathbb{Z}}(K) = -j^2u^2\bar{v}^2 - ju\bar{v}$ by Proposition 4.8. The following code for `Singular` reduces $P(K)$ to $u\bar{v}^2$.

```
> ring r=(integer,4),(j,V,U,v,u),(lp(3),dp(2));
> ideal I=j3-1,v+ju+j2,uU-1,vV-1; reduce(-j2u2V2-juV,std(I));
V2u
```

Proposition 4.10. *If $h(L_1) = h(L_2)$, then $P_{0,0;\mathbb{Z}}(L_1) - P_{0,0;\mathbb{Z}}(L_2) \in \{0, 8\}$.*

Proof. We have shown in Section 4.4 that the coefficients of all monomials in the normal form of $P(L)$ except the constant term are in $\{0,1,2,3\}$. Thus they are determined by $h(L)$ due to Proposition 4.8. The constant term is determined mod 8 by the other coefficients due to Proposition 4.1(g). □

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