A FLEXIBLE AFFINE M-SEXTIC WHICH IS ALGEBRAICALLY UNREALIZABLE

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Abstract. We prove that the union of a real algebraic curve of degree six and a real line on $\mathbf{RP}^2$ cannot be isotopic to the arrangement in Figure 1. Previously, the second author [5] realized this arrangement with flexible curves. Here we show that these flexible curves are pseudo-holomorphic in a suitable tame almost complex structure on $\mathbf{CP}^2$.

For the proof of the algebraic non-realizability we consider all possible positions of the curve with respect to certain pencils of lines. Using the Murasugi-Tristram inequality for certain links in S^3, we show that all the positions but one are unrealizable. Then, we prohibit the last position (the one which is realizable by a flexible curve) by studying its behaviour with respect to an auxiliary pencil of cubics.

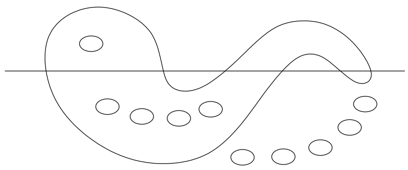

Fig. 1

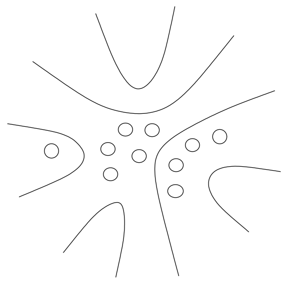

Fig. 2

Introduction

The main result of the paper is:

Theorem. *Let C_6 be a real algebraic curve of degree 6 and L a line, where $C_6, L \subset \mathbf{RP}^2$. Then there does not exist an ambient isotopy of $\mathbf{RP}^2$ which deforms C_6 and L into the curve and the line in Figure 1.*

Since the line L can be considered as the line at infinity, Theorem 1 is equivalent to the fact that a real affine sextic can not be arranged on $\mathbf{R}^2$ as in Figure 2. This result continues the classification of affine M-sextics started in [7,4,5,6]. In our opinion, the main interest of this theorem is that the curve in Figure 1 is realizable by a real pseudo-holomorphic curve (see the definition in Section 1). This means that the methods commonly used to obtain restrictions for real algebraic curves

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cannot be sufficient for the proof of the Theorem because almost all of them prove non-existance of real pseudo-holomorphic curves (see Section 1 for details).

We prove the Theorem in three steps. In Section 2, using link-theoretical methods from [5], we show that a curve $C_6 \cup L$ isotopic to Fig. 1 can be arranged in a unique way with respect to certain pencils of lines. In Section 3, using Bezout's theorem for auxiliary lines and conics, quadratic transformations etc., we find the unique position of $C_6 \cup L$ with respect to a certain auxiliary nodal cubic N. In Section 5, we prohibit this arrangement of C_6, L, and N (which is realizable pseudo-holomorphically!). The results of Sections 2 and 3 are valid in pseudo-holomorphic case (hence, something in Section 5 cannot work for pseudo-holomorphic curves).

In Section 6, we give two pseudo-holomorphic realizations of Figure 1 as the union of a line and a sextic. The construction in Section 6.1 is more general (in essential, it was given already in [5; Sect. 7.2, curve $B_2(1, 4, 5)$] not mentioning that the constructed curve is pseudo-holomorphic). In the construction in Section 6.2, the pseudo-holomorphic curve $C_6 \cup L$ is obtained as a deformation of an algebraic curve.

As we already mentioned, the most of known topological restrictions for real algebraic curves are also valid for real pseudo-holomorphic curves. Thus, to prove the Theorem, we need a tool which might distinguish these objects. As such a tool we use an auxiliary pencil of cubics. Choosing 8 real base points in a suitable way, we consider the pencil $\{C^t\}$ of all real cubics passing through them. The arrangement of one of the cubics of this pencil (let it be C^0) with respect to C_6, L, and N can be determined very precisely. Then we determine the position of the 9th base point of the pencil (note that its existence is crucial for our proof because the continuous family of pseudo-holomorphic cubics through 8 fixed points, in general, does not have any other base points). Now we change the parameter t and see how C^t transforms. If the base points are chosen in general position, the pencil of cubics must have at least 8 nodal cubics and when we pass through them, we perform a Morse bifurcation. Bezout's theorem bounding the number of real intersections of C^t with C_6, L, N, and some auxiliary lines imposes very strong restrictions which allow us to trace the pencil of cubics (i.e. to find the arrangement of the nodal cubics we pass successively) up to a certain moment and to show that a further bifurcation is impossible. This method was developed in [1].

We mention in Section 1 some known examples of algebraically unrealizable real pseudo-holomorphic curves. However, the methods used to prove the non-realizability of these examples work in very special cases whereas the method used in this paper (Bezout's theorem for an auxiliary pencil of cubics) seems to be more general. Note that auxiliary pencils of cubics were used (in another way) by Rokhlin [9; Section 3.6].

The results of Sections 3 – 5 are obtained by the first author; The results of Sections 2 and 6 are obtained by the second author.

1. Real pseudo-holomorphic curves

We say that a Riemann surface C, embedded (or immersed) in $\mathbf{CP}^2$, is a *real pseudo-holomorphic curve* if C is a J-holomorphic curve in some tame almost complex structure J (see [3]) such that $\mathrm{Conj}(C) = C$ and $\mathrm{Conj}_* \circ J = J^{-1} \circ \mathrm{Conj}_* : T_x \to T_{\bar{x}}$ for all $x \in \mathbf{CP}^2$ (here $\mathrm{Conj} : \mathbf{CP}^2 \to \mathbf{CP}^2$ denotes complex conjugation $x \mapsto \bar{x}$).

It is easy to show that any flexible curve compatible with a pencil of lines in the sense of [5; Section 3.1] is J-holomorphic in a suitable Conj-anti-invariant almost complex structure J (see Sect. 6.1). Hence, all flexible curves constructed in [5] are realizable by real pseudo-holomorphic curves. In fact, a complete classification up to isotopy of real affine pseudo-holomorphic M-sextics is obtained in [5].[1]

Viro [11] defined a *flexible curve of degree m* as a smooth embedded Conj-invariant surface C in $\mathbf{CP}^2$ such that $[C] = m[\mathbf{CP}^1] \in H_2(\mathbf{CP}^2)$, the genus of C is $(m-1)(m-2)/2$, and the planes tangent to C at real points are the complexifications of real lines. A lot of restrictions for the topology of real algebraic curves are valid for flexible curves (see [11] where such restrictions are called topological), e.g., Harnack inequality, Petrovski inequalities, Gudkov-Arnold-Rokhlin congruence, Arnold inequalities, formulas for complex orientations, etc. Real pseudo-holomorphic curves form a subclass (in our opinion, the most important one) of flexible curves. Moreover, they satisfy Bezout's theorem because all intersections are positive. Due to results of Gromov [3], there exists a real pseudo-holomorphic line (resp. conic) through any 2 (resp. 5) real points. This means that the restrictions coming from Bezout's theorem for auxiliary lines and conics also are valid for real pseudo-holomorphic curves (for instance, if the set of a real points of a real pseudo-holomorphic quartic has two ovals one inside another then it has no other ovals). The behaviour of pseudo-holomorphic curves in pencils of pseudo-holomorphic lines is the same as in holomorphic case, thus, almost all of the known restrictions in the literature for the topology of real algebraic curves are valid in the pseudo-holomorphic case.

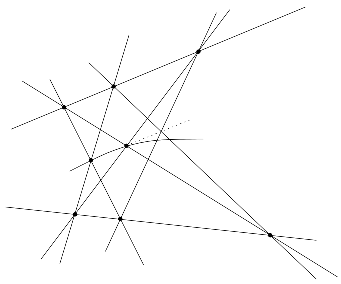


FIG. 3

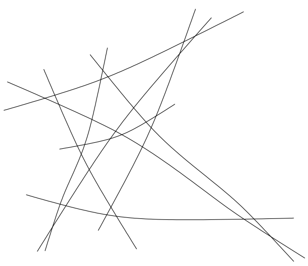


FIG. 4

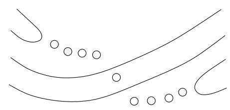


FIG. 5

However, reducible real pseudo-holomorphic curves in $\mathbf{RP}^2$ algebraically unrealizable were known long ago. To get the simplest example, consider a line arrangement where triple points are dependent, and perturb one of them into 3 double points (for instance, the Pappus arrangement of 9 real lines with 9 triple points perturbed as in Fig. 3). A less trivial example is provided by the Pappus-Ringel arrangement of pseudo-lines [8] (see Fig. 4) which is also a perturbation of the Pappus arrangement. Another example is provided by a smooth M-perturbation of four tangent real branches (Fig. 5) which was prohibited by Shustin in the 80's in his thesis (unpublished) using the Hilbert-Rohn-Gudkov approach (see Sect. 6.3 for a pseudo-holomorphic realization).

[1]There is a misprint in [5]. The line " 1 4 5 " in the table corresponding to $A_3(\alpha_1, \alpha_2, \beta)$ in Figure 1 (page 781) should be replaced by " 4 1 5 ".

Question. *Does there exist a smooth (hence, irreducible) real pseudo-holomorphic curve in* $\mathbf{CP}^2$ *such that the isotopy class of the set of its real points (with or without complex orientations) is not realizable by a real algebraic curve?*

2. Arrangement of the curve with respect to pencils of lines

The following fact easily follows from Bezout's theorem for a conic through 5 points in different empty ovals.

Lemma 2.1. *Let* O *be the non-empty oval of an* M*-sextic* C_6*.*

a). If one chooses a point in each interior oval of C_6 *then there exists a convex polygon contained inside* O *whose vertices are the chosen points.*

b). A line ℓ *through two exterior ovals of* C_6 *cannot separate interior ovals, i.e. all interior ovals lie in the same connected component of* $\operatorname{Int} O \setminus \ell$*.*

c). Let points p*,* p_1*,* p_2 *lie in 3 different interior ovals of* C_6*. Then all the exterior ovals are in the same connected component of* $\mathbf{RP}^2 \setminus ((pp_1) \cup (pp_2))$*.* $\square$

Suppose that there exists a sextic C_6 arranged with respect to a line L (up to an isotopy) as in Figure 1.

Lemma 2.2. *The curve* C_6 *is arranged with respect to the dashed lines as in Fig. 6, where 4 exterior ovals are in the shaded zone. In particular, if we choose points* $B, C, \dots, H$ *on empty ovals according to Fig. 6 (where* D *is in some of the 4 exterior ovals in the shaded zone) then the pencils of lines sweep out the points in the following orders:*

a). B, E, F, G, H, C, D *for the pencil of lines through* A*;*

b). A, F, G, H, C, D, E *for the pencil of lines through* B*;*

c). B, F, G, H, D, A, E *for the pencil of lines through* C*.*

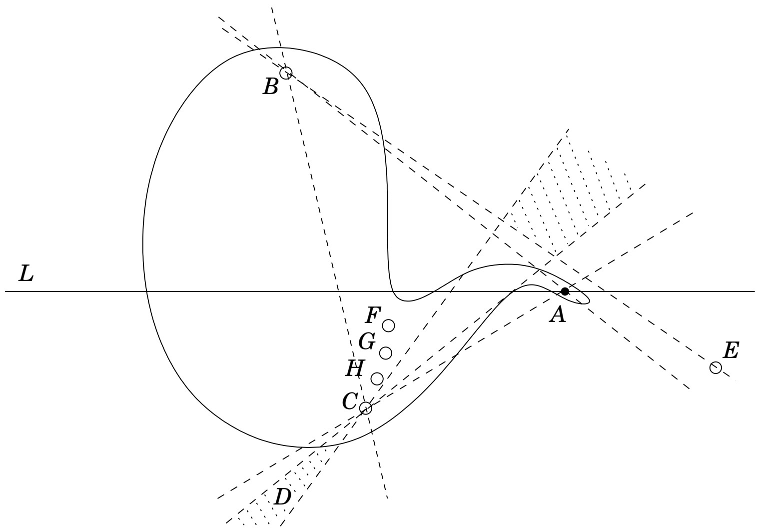


Fig. 6

Proof. Choose the points A, B, C, F, G, H according to Fig. 6 obeying the order in the pencil through A.

a). The order of the empty ovals in the pencil through C is the required one. This is proved in [5; Sect. 5.2] using link-theoretical methods (the Murasugi-Tristram inequality). For the reader's convenience, let us outline the proof omitting everything which does not concern the arrangement from Fig. 1. The arrangement of the curve with respect to the pencil of lines through C is depicted in Fig. 7, where C is the infinite point of vertical lines, A is the infinite point of L, and 9 empty ovals are somehow distributed in the areas indicated by digits $2, 3, 4$ (this picture should be considered up to transformations described in [5; Proposition 3.6]).

Let $O_1, \dots, O_9$ be the empty ovals absent in Fig. 7, numbered from left to right. Let $[i_1 \dots i_d][i_{d+1} \dots i_9]$, $0 \le d \le 9$, $2 \le i_j \le 4$, be the sequence of their heights, i.e. i_j is the digit in Fig.7 indicating the area containing O_j, and d is the number of ovals to the left of the middle vertical line. We may always assume that either $d = 0$ or $i_d = 3$ (otherwise we may push O_d to the right by [5; Proposition 3.6]).

By Bezout's theorem for auxiliary conics and lines and the formula of complex orientations for C_6 and $C_6 \cup L$ (see [5; Lemma 5.4 and Corollary 5.6]), we exclude all the possibilities for $[i_1 \dots i_d][i_{d+1} \dots i_9]$ except [][432224444] (corresponds to Fig. 6) and

[][222344444]	[3333][22234]	[][444322244]	[][444443222]
[33][2223444]	[33][4422234]	[][442223444]	[][444422234]

In each of the last 8 cases, we compute the braid $b = \prod_{j=1}^{n} \sigma_{k_j}^{e_j}$ according to [5; Proposition 3.8]. Let $e(b) = \sum e_j$, m be the number of strings, and $\det b$ be the determinant of the symmetrized Seifert matrix of the closure of b in S^3. In our case, $e(b) = 5$, $m = 7$, hence $e(b) < m - 1$. Thus, by [5; Corollary 2.2], $\det b$ must vanish if an arrangement is realizable by real algebraic (in fact, real pseudo-holomorphic!) curves. The computation shows that $\det b \ne 0$ in all 8 cases (to simplify the computation, one can use the fact that $\det b$ is equal to the determinant of a Goeritz matrix; see [2]). The correct order of F, G, H is provided by Lemma 2.1(a).

b,c). Applying Lemma 2.1(a), we also obtain the required order of all points except E in the other two pencils.

The arrangement of the curve with respect to the pencil of lines through B is depicted in Fig. 8, where B is the infinite point of vertical lines, A is the infinite point of L, and the oval containing E is not shown (this picture should also be considered up to transformations described in [5; Proposition 3.6]). Our goal is to prove that E is contained in the shaded zone.

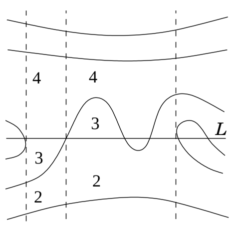


FIG. 7

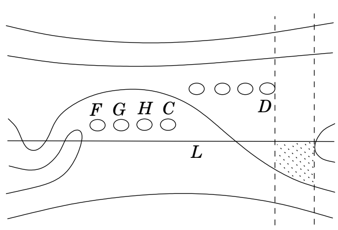


FIG. 8

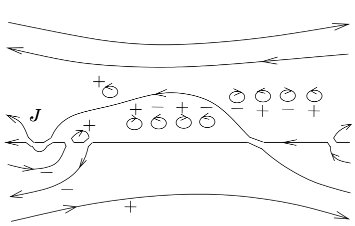


FIG. 9

If E is below L in Fig. 8 then it must be in the shaded zone because otherwise ED separates B from C, which contradicts Lemma 2.1(b). If E is above L in Fig. 8 then it must be either to the right of C or to the left of F because otherwise BD and BE separate C from F, which contradicts Lemma 2.1(c). Both latter cases will be prohibited by complex orientations.

Let C_7 be the 7th degree curve obtained by a smoothing of $C_6 \cup L$ coherent with complex orientations. C_7 is depicted in Fig. 9 in the case when E is to the left of F. The orientation of the oval containing B (the hyperbola) must be as in Fig. 9 because of the formula of complex orientations (see [9]) for C_6. But Fig. 9 contradicts the formula of complex orientations for C_7. The case where E is to the right of C but above L is analogous. $\square$

3. Auxiliary nodal cubic N

Assume that there exist C_6 and L contradicting the Theorem. Denote the nonempty oval of C_6 by O and let $A, \dots, H$ be as in Lemma 2.2. Let N be the nodal cubic passing through the points A, B, C, D, E, F, G, with a double point in C.

Lemma 3.1. *N is arranged with respect to $A, \dots, H, L, O$ up to isotopy as in Fig. 10.*

Let T_1 and T_2 be two of the 4 triangles defined by A, B, C and let $X \in T_1$ and $Y \in T_2$. Denote the segment of the line XY cutting the common edge of T_1 and T_2 by $[XY]$, and the other segment by $[XY]'$.

Lemma 3.2. *Let ℓ be the line which supports the common edge of T_1 and T_2, and let ℓ_1, ℓ_2 be the other two lines amongst AB, BC, CA. Let C_2 be a conic passing through X and Y, and let α be one of the 2 arcs of C_2 joining X and Y. If $[XY] \subset \operatorname{Int} C_2$ then α cuts ℓ once and cuts each of ℓ_1, ℓ_2 in 0 or 2 points. If $[XY]' \subset \operatorname{Int} C_2$ then α cuts ℓ in 0 or 2 points and cuts each of ℓ_1,ℓ_2 once.* $\square$

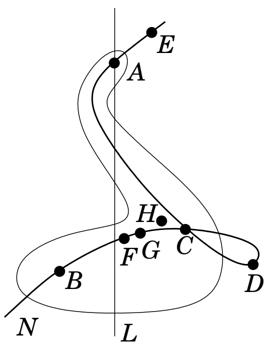


Fig. 10

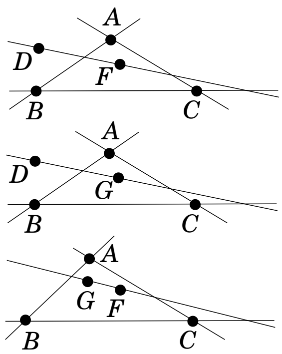


Fig. 11

Proof of Lemma 3.1.
Step 1. Let us prove that N is arranged with respect to $O \cup L$ and the points $A, \dots, G$ (not H yet) as in Fig. 10.

Let us perform a Cremona transformation $\mathrm{cr} : (x_0 : x_1 : x_2) \to (x_1x_2 : x_0x_2 : x_0x_1)$, with base points A, B, C. We shall denote the respective images of the lines

BC, AC and AB by A, B and C. For the other points we shall use the same notation as for their preimages under cr.

By Lemma 2.2, the 3 conics $ABFCD$, $ABGCD$ and $ABFGC$ are transformed into 3 lines as in Fig. 11. After the transformation, let us study the conic C_2 passing through D, F, G, E, and C.

From now on, when speaking of convexity, we refer to an affine plane where the points F, G, H are inside the triangle ABC and the points D and E are arranged with respect to the lines AB, BC, CA as in the left-hand parts of Figures 12–13 (such an affine plane exists by Lemma 2.2). Denote the triangle ABC containing F, G, H by T. We distinguish the 2 cases depending on whether G is separated from A in the triangle ABC by DF, or not.

Case 1. G is separated from A by the line DF in T. Clearly, the quadrangle $DFCE$ is convex. The diagonals divide it into four triangles. By Lemma 2.2(a,c), G lies in the one adjacent to DF. Hence, C_2 meets D, G, F, E, C in this order. Moreover, by Lemma 3.2, each of the arcs DG, GF, FE has 0 or 2 intersections with the line BC, hence, the second point of $C_2 \cap BC$ is on the arc ECD of C_2. Thus, there are two possibilities for C_2, depending on the order in which the arc FE intersects the lines AB and AC (see Fig. 12 where the tangency at C may be perturbed in the two ways).

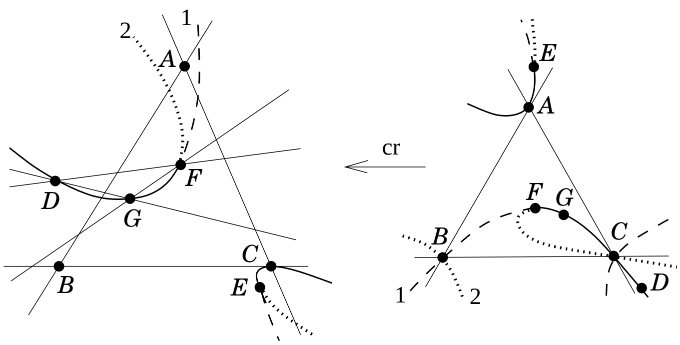


FIG. 12

Case 2. G is not separated from A by the line DF in T. Clearly, the points D, G, F, C, E lie in a convex position, hence they are arranged in this order on C_2 (see Fig. 13). By Lemma 3.2, each of the arcs ED, DG, GF has 0 or 2 intersections with the line AC, hence, the second point of $C_2 \cap AC$ is on the arc FCE of C_2. Thus, there are two possibilities for C_2, depending on the order in which the arc ED intersects the lines AB and BC (see Fig. 13 where, as above, the tangency at C may be perturbed in two ways).

Thus, we found that there are only four possibilities for the conic C_2 (two possibilities in each case). They are marked by digits $1, \dots, 4$ in the left-hand parts of Figures 12 and 13.

Now, we perform the inverse Cremona transformation. The conic C_2 is transformed into N. The corresponding possibilities for N are marked by the same digits in the right-hand parts of Figures 12 and 13 (again, the tangencies at A or B may be perturbed in the two ways). Cases 1, 2, and 4 are impossible because N would cut C_6 at ≥ 20 points. Hence, N follows the arc marked by 3 in Fig. 13.

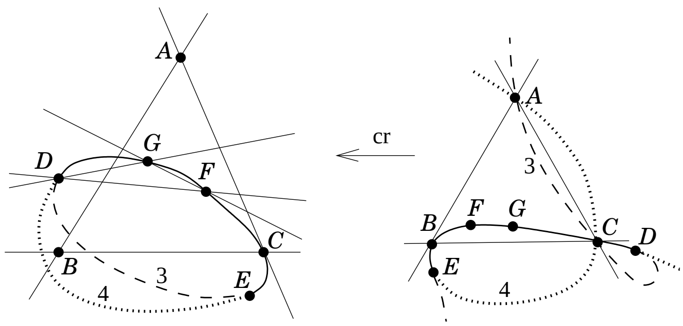


FIG. 13

Clearly, N cuts L in A, and 2 other points X, Y, situated respectively on the arcs BF, and CAE of N (see Fig. 13). Y cannot be on the arc AE (if it were, N would cut C_6 in 20 points), hence, Y is on the arc CA.

Step 2. Let us prove that H is arranged as in Fig. 10.

By Lemma 2.2 and the result of Step 1, this is equivalent to the fact that $H \in \operatorname{Int} C_2$ after the transformation cr. Replacing F, G by G, H in the arguments in Step 1, one gets that H is not separated from A by the line DG in T. Hence, by Lemma 2.2(c), H lies in the shaded zone in Fig. 14.

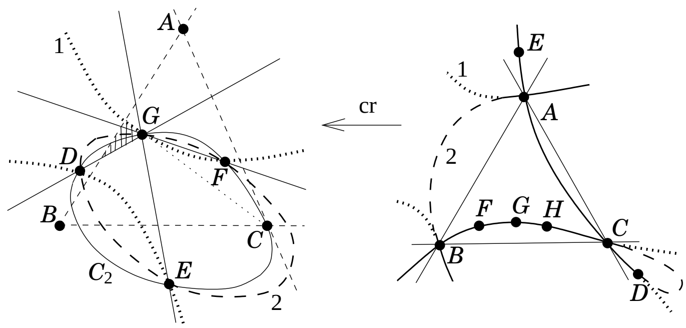


FIG. 14

Suppose that $H \notin \operatorname{Int} C_2$, and consider the conic C_2' passing through D, E, F, G, H. It may be arranged only in the two ways (see Fig. 14). Indeed, if we trace it starting from the shaded zone, each time we have no other choice because C_2' can cross C_2 and the solid lines in Fig. 14 only at D, E, F, G. Then $\mathrm{cr}^{-1}(C_2')$ is a quartic depicted in Fig. 14 (on the right). In both cases it cuts C_6 at ≥ 26 points. $\square$

4. Pencils of real cubics

Let $\mathbf{CP}^9$ be the space of all plane cubics. A *pencil of cubics* is a line in this $\mathbf{CP}^9$. A pencil of cubics $\mathcal{C}_3$ is *real* if this line is real; in this case we denote the set of real cubics from $\mathcal{C}_3$ by $\mathcal{C}_3(\mathbf{R})$. All of the cubics of a generic pencil $\mathcal{C}_3$ intersect in 9 distinct points (called the *base points* of the pencil), and are disjoint elsewhere. For any 8 generic points on $\mathbf{CP}^2$ there exists a unique pencil of cubics which has these points as base points. If the 8 points are real then the pencil is real and hence, the 9th base point is also.

We say that simple double point (node) of a real curve is *solitary* if the tangents are not real and *non-solitary* otherwise. If C_3 is a real nodal cubic with a non-solitary double point P then $C_3 \setminus \{P\} = \mathcal{J} \cup \mathcal{O}$ where $[\mathcal{J} \cup P] \neq 0$ and $[\mathcal{O} \cup P] = 0$ in $H_1(\mathbf{RP}^2)$. We say that $\mathcal{O}$ is the *loop* and $\mathcal{J}$ the *odd branch* of C_3.

In the space $\mathbf{CP}^9$ of all the cubics, the degree of the discriminant hypersurface is 12. Therefore, if a pencil $\mathcal{C}_3$ is generic, it contains exactly 12 singular cubics, all nodal. We denote the number of real cubics among them by n.

If $C_3 \in \mathcal{C}_3(\mathbf{R})$ is nodal, it may belong to 3 different types:

(1) C_3 has an solitary double point;
(2) C_3 has a non-solitary double point, and the loop of C_3 contains no base points;
(3) C_3 has a non-solitary double point, and the loop of C_3 contains some base points.

We denote the number of cubics of each type by n_1, n_2, n_3. The cubics of the third type will be called *distinguished cubics*. (It is easily seen that if a cubic of $\mathcal{C}_3(\mathbf{R})$ has an oval or a loop $\mathcal{O}_3$, then $\mathcal{O}_3$ contains an even number of base points.)

The following observation was communicated to us by V. Kharlamov. Let us compute the Euler characteristic of $\mathbf{RP}^2$ by fibering $\mathbf{RP}^2$ by the cubics of $\mathcal{C}_3(\mathbf{R})$. Each of the 9 base points and the n_1 solitary double points contributes 1; each of the $n - n_1$ non-solitary double points contributes -1. Hence, $1 = \chi(\mathbf{RP}^2) = 9 + n_1 - (n - n_1)$, i.e. $n - 2n_1 = 8$. Thus, $n = 8, 10$ or 12 which implies $n_1 = 0, 1$ or 2.

Consider a motion in $\mathcal{C}_3(\mathbf{R})$ from a nodal cubic of type 1 (with an solitary node) to the next nodal cubic. If we choose the direction of the motion properly then an oval appears, grows, and attaches itself to the odd component forming a loop which has no base points. Conversely, starting with any nodal cubic of type 2, we can perform this process in the opposite direction. Thus, $n_2 = n_1$ and $n_3 = 8$ independently of n.

5. Auxiliary pencil of cubics (proof of the Theorem)

5.1. Choice of base points. We shall prohibit C_6 using a pencil of cubics $\mathcal{C}_3$ with base points A, B, C, D, E, F, G, H, and an unknown ninth point P. We shall construct the sequence of successive distinguished cubics of this pencil until we are no longer able to continue because of Bezout's theorem. This will give us the contradiction proving the Theorem.

Definition. 1) We call A, B, C, D, E, F, G, H the *principal* base points of $\mathcal{C}_3$.
2) Let C^t be a cubic of $\mathcal{C}_3$. If X and Y are 2 base points (resp. principal base points) of $\mathcal{C}_3$, a *base arc* (resp. *principal arc*) XY of C^t is an arc of C^t connecting X and Y which does not contain any other base point (resp. principal base points).

5.2. Starting cubic C^0. Let C^0 be a generic cubic of $\mathcal{C}_3$ passing through a supplementary exterior oval of C_6. Let us denote by the odd component of C^0 by $\mathcal{J}^0$ and the oval of C^0 (if it exists) by $\mathcal{O}^0$.

Lemma 5.1. *(a). C^0 is arranged as in Fig. 15.*

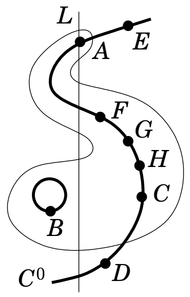


FIG. 15

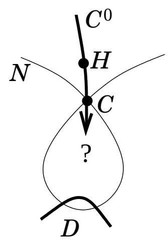


FIG. 16

(b). the 9th base point P lies on $\mathcal{O}^0$

Proof. Note, that by Bezout's theorem, C^0 cuts O in at most 2 points (recall that O is the non-empty oval of C_6). Indeed, C^0 cuts 8 empty ovals in ≥ 2 points each, but the total number of real points where C^0 cuts C_6 is ≤ 18.

Step 1. $A \in \mathcal{J}^0$.

Suppose $A \in \mathcal{O}^0$. All of B, C, F, G, H cannot lie on $\mathcal{O}^0$ because $\mathcal{O}^0$ is convex and $\#(\mathcal{O}^0 \cap O) \leq 2$. Hence, one of these points is on $\mathcal{J}^0$, hence, $\mathcal{J}^0$ meets O, and $\mathcal{O}^0 \subset \operatorname{Int} O$. Since $\mathcal{O}^0$ is convex, $B, C \in \mathcal{J}^0$. By Lemma 2.2(a), $\mathcal{J}^0$ intercepts C, D, B, E in this order, hence, it cuts O in 4 points. Contradiction.

Step 2. F, G, H, and C lie on $\mathcal{J}^0$.

Suppose $X = F, G, H$ or C is on $\mathcal{O}^0$. Then $\mathcal{O}^0 \subset \operatorname{Int} O$. The line AE cuts C^0 in 3 points of $\mathcal{J}^0$, and separates X from B in O. Hence, B is on $\mathcal{J}^0$. The points A, E, B, D are arranged in this order on $\mathcal{J}^0$, given by the pencil of lines through X (see Lemma 2.2). Hence, $\mathcal{J}^0$ cuts O in 4 points. Contradiction.

Step 3. $B \in \mathcal{O}^0$.

Suppose $B \in \mathcal{J}^0$. Bezout's theorem for C^0 and L yields that A and B are the extremal points amongst A, B, C, F, G, H on $\mathcal{J}^0 \cap \operatorname{Int} O$, hence, $\{X, Y\} \subset \{A, B, F, G, H\}$ where CX and CY are the principal arcs of $\mathcal{J}^0$. Hence, X or Y coincides with B or H (otherwise the arc XCY would meet the line BH in 2 points) and the last point of $C^0 \cap N$ is on the arc BDE, hence, CH cannot be a principal arc (see Fig. 16). Thus, CB and CZ ($Z = F$ or G) are principal arcs of $\mathcal{J}^0$ (we already know that one of CB, CH must be principal). Since C is separated from B and Z by the line AH in O, this line cuts both principal arcs CB and CZ, thus, $C^0 \cdot AH \geq 4$. Contradiction.

Step 4. It remains to apply Lemma 2.2(b) and Bezout's theorem. □

5.3. Construction of the pencil. Let us parametrize our pencil of cubics $\mathcal{C}_3 = \{C^t\}$, $t \in \mathbf{R} \cup \{\infty\}$ so that C^0 is as in Section 5.2 and $C^1, C^2, \dots$ are the successive distinguished cubics.

In Fig. 17 we show $C^0, \dots, C^4$ by solid lines. A cubic $C^{k+\varepsilon}$, $0 < \varepsilon \ll 1$, $k = 0, \dots, 4$ is depicted near C^k by the dashed line. We denote the base points B and P by B_1 and B_2 when we cannot distinguish them. To pass from C^k to C^{k+1}, we have to choose a pair of base arcs of $C^{k+\varepsilon}$ (they are connected by the dotted line) which join each other when $\varepsilon \to 1$. Note that during the passage from $C^{k+\varepsilon}$ to C^{k+1}, the mutual positions of all base arcs do not change even if we pass through a pair of non-distinguished nodal cubics.

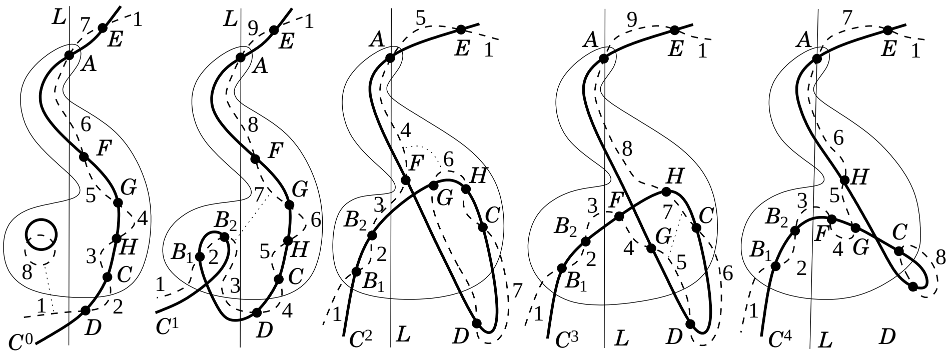


FIG. 17

Remark 5.2. If $C^{k+\varepsilon}$ is an M-cubic then one of the two chosen base arcs must belong to its oval and the other to the odd component.

The choice of the arcs can actually be made each time in a unique way (except the last time when this is impossible at all). The reasons why the other pairs of arcs cannot be joined are explained in Section 5.4.

5.4. Comments to Figure 17. P_j, $j = 1, 2, \dots$ always means "a point of the arc number j". When we pass from C^k to C^{k+1}, we suppose $k < t < k+1$, and we use the abbreviation: "$(x_1, x_2 \dots // y_1, y_2, \dots)$ by Z" means that each arc x_i is separated from each arc y_j by $C^k \cup Z$.

From C^0 to C^1. (See Remark 5.2).

$(8//2,7)$ by CH because A is separated from F in O by the line CH, hence, $CH \cap C^t = \{C, H, P_6\}$; see Fig. 18.

$(8//3,5)$ by L ($L \cap C^t = \{A, P_1, P_k\}$ where $k = 6$ or 7);

$(8//4,6)$ by O (to avoid C^0, we need to cross O twice!)

From C^1 to C^2.

$(1,3,5,7//6,8)$, $(4//5,7)$, by O.

$(1,3//4,5)$ by CH because F is separated from A in O by the line CH, hence $CH \cap C^t = \{C, H, P_8\}$; see Fig. 19.

$(1//3)$, $(4//6,8)$, $(5//7)$ by B_1H; $(6//8)$ by B_1G (for $X = H, G$, we have $B_1X \cap C^t = \{B_1, X, P_k\}$ where $k = 1, 2$, or 3). All the other pairs of arcs except $(3,7)$ are separated by L because $L \cap C^t = \{A, P_3, P_8\}$.

From C^2 to C^3. (See Remark 5.2).

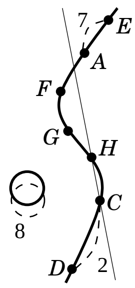
$(6//1,2,3,5)$ by O. $(7//1,\dots,5)$ by N. Indeed, let $\mathcal{J}^t$ be the odd component of C^t. Since the intersection of $\mathcal{J}_t$ and N is odd, the single non-base point of $N \cap C^t$ is on $\mathcal{J}^t$. Hence, the oval of C^t is arranged as in Fig. 20.

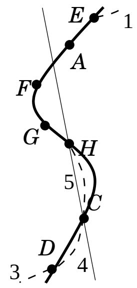
From C^3 to C^4.

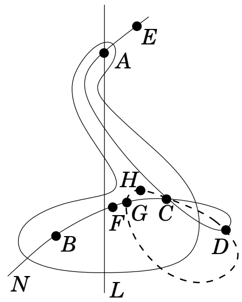
$(1//3)$, $(6//1,8)$ by CB_2 because $CB_2 \cap C^t = \{C, B_2, P_5\}$. All the other pairs of arcs except $(5,7)$ are separated by $L \cup O$ because $L \cap C^t = \{A, P_3, P_8\}$.

From C^4 to the contradiction. (See Remark 5.2).

$(8//2,3,4,5)$ by O;







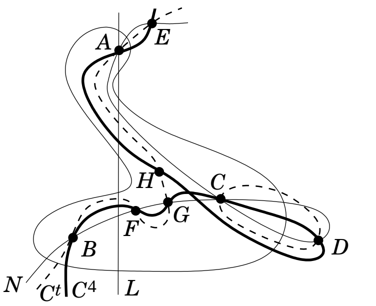
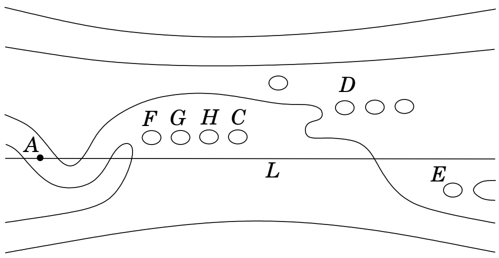


FIG. 18 FIG. 19 FIG. 20 FIG. 21

$(8//1,6,7)$ by N because either 8 is inside the loop of N or the non-base point of $N \cap C^t$ is on the loop of N; see Fig. 21 where the tangency of C^4 and C^t at B may be perturbed in the two ways.

6. Pseudo-holomorphic realizations



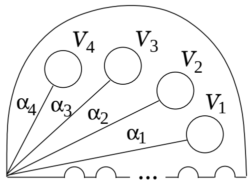


FIG. 22 FIG. 23

6.1. The first realization of the curve in Fig. 1. Let us fix a smooth curve C_6 and a line L arranged on the real affine plane (x, y) as in Figure 22 (compare with Fig. 8). Let (x_j, y_j), $j = 1, \dots, 28$ be the points where C_6 either meets L or has a vertical tangent. Denote the discs $\{|x - x_j| \le \varepsilon^2\}$ in $\mathbf{C}$ by U_j and let $U_0 = \{|x| \ge R\} \subset \mathbf{C}$. Choose R and ε such that $U_0, \dots, U_{28}$ are pairwise disjoint, and set $U = \bigcup_{j=0}^{28} U_j$, $D = \{\operatorname{Im} x \ge 0\} \setminus U$.

We may suppose that the part of C_6 over $U \cap \mathbf{R}$ is the union of segments of lines and arcs of parabolas of the form $(x - x_j) = \pm(y - y_j)^2$. In other words, this part of $C_6 \cup L$ is $F \cap \mathbf{R}^2$, where F is the graph of a 7-valued function f defined on U, all of whose branches are either linear or of the form $y = y_j \pm \sqrt{x - x_j}$. Along C_6 we can extend f to a half S_1 of $\mathbf{R} \setminus U$, but only 5 branches of F can be extended to the other half S_2. On each $[x_j + \varepsilon^2, x_k - \varepsilon^2] \subset S_2$, define the 2 missing branches $f_\pm$ linearly, setting $f_\pm(x_j + \varepsilon^2) = y_j \pm i\varepsilon$ and $f_\pm(x_k - \varepsilon^2) = y_k \pm i\varepsilon$. It is easy to check (see [5; Sect. 3]) that $f|_{\partial D}$ is the braid

$$b = \sigma_3^{-1}\sigma_4^{-2}\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-5}\tau_{3,4}\,\sigma_4^{-1}\tau_{4,3}\,\sigma_4^{-4}\sigma_2^{-1}\tau_{4,3}\,\sigma_3^{-1}\Delta \qquad (1)$$

where $\tau_{3,4} = \sigma_4^{-1}\sigma_3$, $\tau_{4,3} = \sigma_3^{-1}\sigma_4$, and Δ (it corresponds to ∂U_0) is the Garside element $\Delta = (\sigma_1 \dots \sigma_6)(\sigma_1 \dots \sigma_5)(\sigma_1 \dots \sigma_4)(\sigma_1\sigma_2\sigma_3)(\sigma_1\sigma_2)\sigma_1$. The braid b is quasipositive: one can check that

$$b = (a_1^{-1}\sigma_5 a_1)(a_2^{-1}\sigma_6 a_2)(a_3^{-1}\sigma_1 a_3)(a_4^{-1}\sigma_4 a_4) \tag{2}$$

where $a_1 = \sigma_4 a$, $a_2 = \sigma_5 a$, $a_3 = \sigma_2\sigma_3$, $a_4 = \sigma_5\sigma_6$, and $a = \sigma_4^2\sigma_3\sigma_2\sigma_4\sigma_3^2$.

Let us choose pairwise disjoint discs V_j inside D centred on points z_j and segments α_j as in Fig. 23 ($j = 1,\dots,4$). Define f in each V_j: two branches $\pm\sqrt{x - z_j}$ and five constant branches such that the braids over $\partial V_1,\dots,\partial V_4$ are respectively σ_5, σ_6, σ_1, σ_4. Extend f to α_j to get the braid a_j over it, and fill $D \setminus (V \cup \bigcup_j \alpha_j)$, $V = \bigcup_j V_j$ by an isotopy between the right-hand sides of (1) and (2). Extend f to the lower half-plane by $f(\bar x) = \overline{f(x)}$, and smooth it preserving this symmetry and the complex analycity near $x_1,\dots,x_{28}$ and $z_1,\dots,z_4$.

Let $\omega = (dx \wedge d\bar x + dy \wedge d\bar y)/2i$ be the standard symplectic form on $\mathbf{C}^2$. Set $M = 2\max_{D\setminus V} |\partial f/\partial \bar x|$, and let F_1 be the graph of $f_1 = f/M$ with the orientation induced by the projection $(x, y) \mapsto x$. Then $|\partial f_1/\partial\bar x| \le 1/2$ everywhere, hence $\omega|_{F_1} > 0$. If we embed $\mathbf{C}^2 \to \mathbf{CP}^2$ by $(x, y) \mapsto (x : y : M_1)$ with a sufficiently large constant M_1 then the Fubini-Studi symplectic form will also be positive on F_1, and since the double points of $F_1 \cup L$ are positive, we can find a tame Conj-anti-invariant almost complex structure where F_1 and L are pseudo-holomorphic.

6.2. The second realization of the curve in Fig. 1. The zig-zag on O between C and D in Fig. 22 is not occasional. In fact, the construction in Sect. 6.1 gives a pseudo-holomorphic realization of a rational singular sextic and a line depicted in Fig. 24 (see also [5; Remark 7.2]; recall that E_8 and A_n are singularities defined by $y^3 = x^5$ and $y^2 = x^{n+1}$ respectively in suitable local coordinates). Smoothing E_8 in two ways, one obtains our curve $B_2(1, 4, 5)$ (the curve in Fig. 1) and the curve $B_2(1, 8, 1)$ which was realized algebraically in [6; Part I].

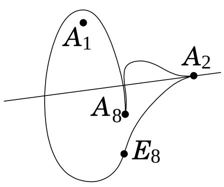


FIG. 24

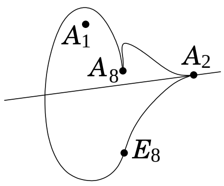


FIG. 25

The theorem proved in Sections 2–5 implies that Fig. 24 is not realizable by a real algebraic sextic (if it were, it could be algebraically smoothed into Fig. 1 due to [10; Lemma]), however, Fig. 25 is (see Proposition below). Given an algebraic curve as in Fig. 25, we can "pull" A_8 through the line keeping the curve symplectic (hence, J-holomorphic in a suitable tame J).

This pseudo-holomorphic realization of Fig. 1 is better than that in Sect. 6.1 because it shows immediately that the constructed arrangement $C_6 \cup L$ is isotopic (via pseudo-holomorphic curves) to a complex algebraic arrangement. Indeed, if we forget the Conj-invariance while we pull A_8, then we can avoid its meeting with L.

Proposition. *There exist a real algebraic rational curve of degree* 6 *and a real line arranged in* $\mathbf{RP}^2$ *as in Fig. 25.*

Proof. Let C_6 be a rational curve parametrized by $t \mapsto (x:y:z)$,

$$x = t^2(1+t)^2(1+bt+ct^2), \quad y = t^2(1+(b+2)t), \quad z = (1+t)^2(1+at),$$
$$a = \alpha, \qquad b = (10\alpha^2+19\alpha+10)/3, \quad c = -(130\alpha^2+232\alpha+160)/15$$

where $\alpha = -0.61351...$ is the unique real root of $5\alpha^3+12\alpha^2+12\alpha+4=0$. Let L be the line $u=0$ where $u=(a-1)x+(c-b+1)z$. One can check that $\deg_t(ay-(b+2)z)=1$, $\operatorname{ord}_{t=-1} u(t) = 3$, and

$$\operatorname{ord}_{t=0}(yz^3 - xz^3 - g_2x^2z^2 - g_3x^3z - g_4x^4) = 9, \quad g_2 = (10\alpha^2+14\alpha+15)/5,$$
$$g_3 = (1060\alpha^2+1804\alpha+1363)/75, \quad g_4 = (1490\alpha^2+2626\alpha+1923)/25.$$

Hence, C_6 has singularities of the types E_8, A_2, A_8 at $t=\infty,-1,0$ respectively. One can verify (for instance, as in [6]) that $C_6 \cup L$ is arranged as in Fig. 25. □

6.3. Realization of the curve in Fig. 4. As in Sect. 6.1, it suffices to check

$$\sigma_3^{-5}\tau_{3,2}\,\sigma_2^{-1}\tau_{2,1}\,\sigma_1^{-4}\Delta^2 = (a_1^{-1}\sigma_3 a_1)(a_2^{-1}\sigma_1 a_2)$$

where $\tau_{3,2} = \sigma_2^{-1}\sigma_3$, $\tau_{2,1} = \sigma_1^{-1}\sigma_2$, $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$, $a_2 = \sigma_3\sigma_2\sigma_3^4$, and $a_1 = \sigma_2\sigma_1^{-2}\sigma_3^{-2}\sigma_2 \cdot a_2$ (since all the 4 branches tends to parabolas far from the singular point, we have Δ^2 instead of Δ).

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