PLANE ALGEBRAIC CURVES IN FANCY BALLS

N. G. Kruzhilin and S. Yu. Orevkov

Abstract. Boileau and Rudolph [2] called an oriented link L in the 3-sphere a $\mathbb{C}$-*boundary* if it can be realized as the intersection of an algebraic curve A in $\mathbb{C}^2$ with the boundary of a smooth embedded closed 4-ball B. They showed that some links are not $\mathbb{C}$-boundaries. We say that L is a *strong* $\mathbb{C}$-*boundary* if $A \setminus B$ is connected. In particular, all quasipositive links are strong $\mathbb{C}$-boundaries.

In this paper we give examples of non-quasipositive strong $\mathbb{C}$-boundaries and non-strong $\mathbb{C}$-boundaries. We give a complete classification of (strong) $\mathbb{C}$-boundaries with at most 5 crossings.

§1. Introduction

Let $B \subset \mathbb{C}^2$ be diffeomorphic to a closed 4-ball and A be a complex analytic curve in a neighbourhood of B which is transverse to ∂B (since we consider only topological properties, we may assume that A is a piece of an algebraic curve). Let $L = A \cap \partial B$ be a link in the 3-sphere ∂B, endowed with the boundary orientation from $A \cap B$. Which links can be obtained in this way? (All links in this paper are assumed to be oriented.)

If we impose no additional restrictions then, as Lee Rudolph showed in [17],[1] the answer is "any link". Moreover, any embedded oriented surface without closed components can be realized as $A \cap B$.

If B is strictly pseudoconvex (for example, a usual round ball), then it was shown in [1] that a link is realizable in this way if and only if it is *quasipositive* (the "if" part being earlier proven in [16]), i.e., it is the braid closure of a quasipositive braid (an n-braid is called *quasipositive* if it is a product of conjugates of the standard generators $\sigma_1, \dots, \sigma_{n-1}$ of the braid group B_n). This is a rather strong restriction on the class of possible links (see [2, 7]).

In [2], a link is called a $\mathbb{C}$-*boundary* if it is realizable as $A \cap \partial B$ where B is diffeomorphic to a closed 4-ball (without any pseudoconvexity assumptions) and A is a whole algebraic curve in $\mathbb{C}^2$, not just a piece of an algebraic curve as in [17]. It is also natural to distinguish the case when L is realizable as $A \cap \partial B$ as above, and moreover $A \setminus B$ is connected. We call such links *strong* $\mathbb{C}$-*boundaries.* It was observed in [2] that Kronheimer and Mrowka's result [8] implies some restrictions on this class of links, in particular, there exist knots and links that are not concordant to any $\mathbb{C}$-boundary. In fact, some results stated in [2] for arbitrary $\mathbb{C}$-boundaries are true only for strong $\mathbb{C}$-boundaries; see more details in §3.

Key words and phrases. Quasipositive link, $\mathbb{C}$-boundary, Thom Conjecture.

[1] The title of our paper is a rephrasing of the title of [17].

Michel Boileau (private communication) asked if there exist non-quasipositive $\mathbb{C}$-boundaries. Here we give an affirmative answer to this question. Moreover, we show that all the following inclusions are strict:

$$\mathcal{Q} := \{\text{quasipositive links}\} \subset \mathcal{SB} := \{\text{strong } \mathbb{C}\text{-boundaries}\} \subset \mathcal{B} := \{\mathbb{C}\text{-boundaries}\} \subset \{\text{all links}\}.$$

To prove that some $\mathbb{C}$-boundaries are not quasipositive, we use the following facts.

Theorem 1.1. ([7, Cor. 1.5]). *If a link and its mirror are both quasipositive, then the link is trivial (cf. Remark 1.3).*

Theorem 1.2. ([15, Thms. 1.1 and 1.2]). *If the split sum or a connected sum of links L_1 and L_2 is quasipositive, then L_1 and L_2 are quasipositive links.*

Another necessary condition for the quasipositivity of links follows from the Franks–Williams–Morton inequality (see Theorem 6.1 below).

Remark 1.3. Let $-L$ be the link L with the opposite orientation. Let $\mathcal{C}$ be one of the classes $\mathcal{Q}$, $\mathcal{SB}$, $\mathcal{B}$. Then $L \in \mathcal{C}$ if and only if $-L \in \mathcal{C}$. Indeed, let $L = \partial(A \cap B)$ with A and B as above, and let $\bar{A}$, $\bar{B}$, and $\bar{L}$ be the images of A, B, and L under the complex conjugation $\mathbf{c} : (z, w) \mapsto (\bar{z}, \bar{w})$. Then $\bar{A}$ is an algebraic curve and we endow $\bar{L}$ with the boundary orientation induced by the complex orientation of $\bar{A} \cap \bar{B}$. Since $\mathbf{c}|_A$ is antiholomorphic, we deduce that $(\partial\bar{B}, \bar{L})$ has the oriented link type of $(\partial B, -L)$. Notice also that the equivalence $L \in \mathcal{Q} \Leftrightarrow -L \in \mathcal{Q}$ can also be seen algebraically: if L is represented by a braid $\sigma_{i_1}^{\pm1}\sigma_{i_2}^{\pm1}\dots\sigma_{i_n}^{\pm1}$, then $-L$ is represented by the braid $\sigma_{i_n}^{\pm1}\dots\sigma_{i_2}^{\pm1}\sigma_{i_1}^{\pm1}$.

The definition of strong $\mathbb{C}$-boundary can be reformulated equivalently by replacing the condition that $A \setminus B$ is connected by the condition that $A \setminus B$ does not have bounded components. If B is strictly pseudoconvex, bounded components may appear: see Wermer's example [14, p. 34] (but no component of $A \setminus B$ cannot be a disk by Nemirovski's result in [13]). Nonetheless, when B is strictly pseudoconvex, $A \cap B$ is a strong $\mathbb{C}$-boundary, because it is a quasipositive link by [1], thus it can be realized on the standard round sphere by [16], and the absence of bounded components in that case follows from the maximum principle. So, we have $\mathcal{Q} \subset \mathcal{SB}$. Notice also that Wermer's example also provides a non-quasipositive $\mathbb{C}$-boundary; see Remark 4.1.

Plan of the paper. In §2 we give simplest examples of non-quasipositive $\mathbb{C}$-boundaries. They are obtained by choosing a "fancy 4-ball" which is a small thickening of a 3-ball embedded in the standard 3-sphere.

In §3 we present some tools to prove that some links are not (strong) $\mathbb{C}$-boundaries. All of them are based on the Kronheimer–Mrowka Theorem.

In §§4–5 we discuss some links which are cut by a complex line on an embedded 3-sphere. If L is such a link, then both L and its mirror L^* are $\mathbb{C}$-boundaries, thus one of them is a non-quasipositive $\mathbb{C}$-boundary by Theorem 1.1. In §5 we show that these links are iterated torus links and we describe their Eisenbud–Neumann splice diagrams.

In §6 we give a complete classification of $\mathbb{C}$-boundaries and strong $\mathbb{C}$-boundaries with up to five crossings, which shows, in particular, that all the inclusions $\mathcal{Q} \subset \mathcal{SB} \subset \mathcal{B} \subset \{\text{all links}\}$ are strict. This classification easily follows from the general facts established in the previous sections, except for the $\mathbb{C}$-boundary realization of the link 5_1^2 which is a little bit tricky.

§2. The simplest examples of non-quasipositive $\mathbb{C}$-boundaries

For a link L, let L^* denote its mirror image and let $-L$ denote L with the opposite orientation.

Theorem 2.1. *Let B and B_0 be closed 4-balls smoothly embedded in $\mathbb{C}^2$ such that the B_0 is contained in the interior of B. Let A be an algebraic curve in $\mathbb{C}^2$ which is transverse to ∂B and ∂B_0. Let L and L_0 be the links cut by A on ∂B and ∂B_0, respectively. Then the split sum $L \sqcup (-L_0^*)$ and a connected sum $L\#(-L_0^*)$ (see Remark 2.2) are $\mathbb{C}$-boundaries.*

If B_0 is, moreover, strictly pseudoconvex and L_0 is non-trivial, then $L \sqcup (-L_0^*)$ and $L\#(-L_0^*)$ are non-quasipositive $\mathbb{C}$-boundaries.

Remark 2.2. A connected sum of two oriented links $L = L_1\#L_2$ usually depends on the choice of the components which are joined into a single component of L. In Theorem 2.1, the components of $A\cap\partial B$ and $A\cap\partial B_0$ that we choose should adjoin the same connected component of $A \cap (B \setminus B_0)$.

Proof. Let us show that the link under discussion is a $\mathbb{C}$-boundary. Indeed, let I be an embedded line segment in $B \setminus B_0$ which connects $A\cap\partial B$ with $A\cap\partial B_0$. Let U be a small tubular neighbourhood of I. Then $B \setminus (B_0 \cup U)$ is a 4-ball, and the link cut on it by A is $L \sqcup (-L_0^*)$ (if I is disjoint from A) or $L\#(-L_0^*)$ (if $I \subset A$).

If B_0 is strictly pseudoconvex, then L_0 is a quasipositive link. If it is non-trivial, Theorem 1.1 (see also Remark 1.3) implies that $-L_0^*$ is not quasipositive and the result follows from Theorem 1.2. □

Corollary 2.3. *Let L be a non-trivial quasipositive link. Then $L \sqcup (-L^*)$ and $L\#(-L^*)$ are non-quasipositive $\mathbb{C}$-boundaries. If, moreover, L is a knot, then $L\#(-L^*)$ is a non-quasipositive strong $\mathbb{C}$-boundary.*

This construction admits the following generalization.

Theorem 2.4. *Let L be a $\mathbb{C}$-boundary link in S^3 transverse to an equatorial 2-sphere $S^2 \subset S^3$. Let H be one of the halves of $S^3 \setminus S^2$ and $\xi : S^3 \to S^3$ be the symmetry with respect to S^2. Then the link $(L \cap H) \cup \xi(-L \cap H)$ is a non-quasipositive $\mathbb{C}$-boundary unless it is a trivial link.*

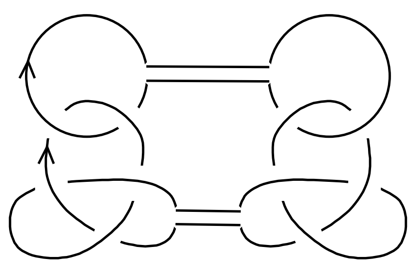

Figure 1. L10n36(1) in [10].

Proof. Let (A, B) be a $\mathbb{C}$-boundary realization of L. Let $f : S^3 \to \partial B$ be a diffeomorphism that maps L to $A \cap \partial B$. Let B' be a small thickening of $f(H)$. Then (A, B') is a $\mathbb{C}$-boundary realization of the required link. Being amphicheiral, it is either trivial or non-quasipositive by Theorem 1.1. □

This theorem allows us to construct a lot of non-quasipositive $\mathbb{C}$-boundaries. In Figure 1 we give an example of such a link. Starting with any quasipositive braid, one can construct many others.

§3. Restrictions on (strong) $\mathbb{C}$-boundaries

All the known restrictions on (strong) $\mathbb{C}$-boundaries are more or less immediate consequences of the Kronheimer–Mrowka theorem [8] (also known as the Thom Conjecture or the Adjunction Inequality) and its version for immersed 2-surfaces in $\mathbb{CP}^2$ with negative double points (see [4], [11, §2]), which was actually proven in [8] but was not explicitly formulated there.

Theorem 3.1. (the Immersed Thom Conjecture). *Let Σ be a connected oriented closed surface of genus g and $j : \Sigma \to \mathbb{CP}^2$ be an immersion which has only negative ordinary double points as self-crossings. Let $j_*([\Sigma]) = d[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2)$ with $d > 0$. Then g is bounded below by the genus of a smooth algebraic curve of degree d, that is, $g \geq (d-1)(d-2)/2$.*

Given a link L in $S^3 = \partial B^4$, we define the *slice Euler characteristic* of L by $\chi_s(L) = \max_\Sigma \chi(\Sigma)$ where the maximum is taken over all embedded smooth oriented surfaces Σ without closed components, such that $\partial\Sigma = L$.

Similarly, we define the *slice negatively immersed Euler characteristic* of L by $\chi_s^-(L) = \max_{(\Sigma,j)} \chi(\Sigma)$ where the maximum is taken over all immersions $j : (\Sigma, \partial\Sigma) \to (B^4, S^3)$ of oriented surfaces Σ without closed components such that $j(\Sigma)$ has only negative double points and $j(\partial\Sigma) = L$.

Theorem 3.1 immediately implies the following.

Proposition 3.2. (cf. [2, Thm. 1.3]) *Let A be a smooth algebraic curve in $\mathbb{C}^2$ which is transverse to the boundary of a 4-ball B smoothly embedded in $\mathbb{C}^2$ and such that $A \setminus B$* **is connected***, and let $L = A \cap \partial B$. Then $\chi_s^-(L) = \chi_s(L) = \chi(A \cap B)$.*

The connectedness condition in Proposition 3.2 can be replaced by the condition that $A \setminus B$ does not have bounded components. Indeed, in this case $A \setminus B$ becomes connected after a perturbation of the union of A and a generic line, far from B.

Proof. We replace $A \setminus B$ by a negatively immersed surface $j(\Sigma)$ of maximum $\chi(\Sigma)$ and apply Theorem 3.1. □

Remark 3.3. The connectedness condition is missing in [2, Thm. 1.3]. Without this condition, Proposition 3.2 is wrong. Indeed, let $A = \{w = 0\}$ and let B be the unit ball "drilled" along the line segment $[(0,0),(0,1)]$. Then $\chi(A \cap B) = 0$ whereas $\chi_s(L) = 2$. The proof fails because, when we replace $A \cap B$ with Σ, the Euler characteristic increases due to splitting out a 2-sphere, while the Euler characteristic of the unbounded component does not change. Note that [2, Prop. 1.4] is wrong even for strong $\mathbb{C}$-boundaries if both links are multi-component. A correct version is Proposition 3.6 below.

Remark 3.4. A similar inaccuracy appears in [11]: it has not been checked if the auxiliary surface (analog of $(A \setminus B) \cup \Sigma$ in the proof of Theorem 3.1) is connected. So the conclusion of [11, Thm. 1] is wrong e.g. in the case when both curves are real conics with non-empty but mutually disjoint real loci. However, this inaccuracy can be easily corrected and does not affect the most interesting case when the curves have common real points.

Definition 3.5. A component of a $\mathbb{C}$-boundary link L is said to be *outer* if it is adjacent to an unbounded component of $A \setminus B$ where A and B are as in the definition of $\mathbb{C}$-boundaries. In particular, all components of a strong $\mathbb{C}$-boundary are outer.

Proposition 3.6. (cf. [2, Prop. 1.4]) *Let K_1 and K_2 be outer components of $\mathbb{C}$-boundary links L_1 and L_2, respectively. Then $L_1 \sqcup L_2$ and $L_1 \# L_2 = L_1 \#_{(K_1,K_2)} L_2$ are $\mathbb{C}$-boundaries. If, moreover, L_1 and L_2 are strong $\mathbb{C}$-boundaries, then so are $L_1 \sqcup L_2$ and $L_1 \# L_2$, and $\chi_s(L_1 \# L_2) + 1 = \chi_s(L_1 \sqcup L_2) = \chi_s(L_1) + \chi_s(L_2)$.*

Proof. For $j = 1, 2$ let (A_j, B_j) be a realization of L_j as a (strong) $\mathbb{C}$-boundary. By translating A_1 and B_1 sufficiently far away, we can achieve that $A_1 \cap B_2 = A_2 \cap B_1 = B_1 \cap B_2 = \varnothing$. Perturbing $A_1 \cup A_2 \cup L$ for a suitable line L we can achieve that $L_j = A \cap \partial B_j$, $j = 1, 2$, for a smooth projective algebraic curve A. Set $B = B_1 \cup B_2 \cup T$ where T is a small tubular neighbourhood of an embedded arc connecting a point in K_1 with a point in K_2. Then (A, B) realizes $L_1 \sqcup L_2$ (resp. $L_1 \# L_2$) if this arc does not (resp. does) lie on A. Then the required relation on the $\chi_s(\dots)$ easily follows from Proposition 3.2. $\square$

Proposition 3.7. *If L is a strong $\mathbb{C}$-boundary and $-L^*$ is a (not necessarily strong) $\mathbb{C}$-boundary, then $\chi_s(L) = \chi_s^-(L) \geq 1$.*

Proof. Let $\hat{L} = L \#_{(K,-K^*)} (-L^*)$ where $-K^*$ is an outer component of $-L^*$, and K is the corresponding component of L. Let $A \cap B$ and $A^* \cap B^*$ be (strong) $\mathbb{C}$-boundary realizations of L and $-L^*$, respectively. The construction in the proof of Proposition 3.6 provides a curve $\hat{A}$ and a smooth ball $\hat{B}$ such that $\hat{A} \cap \hat{B}$ is $\hat{L}$ and all the components of $\hat{L}$ inherited from L (including $K \# (-K^*)$) lie on the boundary of a single unbounded component of $\hat{A} \setminus \hat{B}$. It is well known (and easy to see) that such an $\hat{L}$ bounds a surface $\Sigma = \Sigma_1 \cup \dots \cup \Sigma_r$ in $\hat{B}$, where Σ_1 is a disk bounded by $K \# (-K^*)$ and, for $i \geq 2$, Σ_i is an annulus bounded by $K_i \sqcup (-K_i^*)$ where $K, K_2, \dots, K_r$ are the components of L. Thus $\chi(\Sigma) = 1$. Let $A' = (\hat{A} \setminus \hat{B}) \cup \Sigma$. By construction, A' is connected. Hence $\chi(A') \leq \chi(\hat{A})$ by Theorem 3.1. Again by construction, we have $\chi(\hat{A} \cap \hat{B}) = \chi(A \cap B) + \chi(A^* \cap B^*) - 1 \leq 2\chi_s(L) - 1$. Thus

$$0 \leq \chi(\hat{A}) - \chi(A') = \chi(\hat{A} \cap \hat{B}) - \chi(\Sigma) \leq (2\chi_s(L) - 1) - 1.$$

Finally, $\chi_s(L) = \chi_s^-(L)$ by Proposition 3.2. $\square$

Lemma 3.8. *Let L be a $\mathbb{C}$-boundary which is not a strong $\mathbb{C}$-boundary. Then there exists a proper sublink of L which has zero linking number with its complement in L.*

Proof. Let $L = A \cap B$ be as in the definition of a $\mathbb{C}$-boundary. Without loss of generality we may assume A to be smooth. Then $A \setminus B$ has a bounded connected component A_0, because otherwise, for some line C, a perturbation of $A \cup C$ would realize L as a strong $\mathbb{C}$-boundary. Let $A_1 = (A \setminus B) \setminus A_0$, and let B' be the complement of B in the one-point compactification of $\mathbb{C}^2$. Then B' is a ball and A_0 is disjoint from the closure of A_1 in B'. Hence the linking number of ∂A_0 and ∂A_1 is zero. $\square$

§4. A non-quasipositive $\mathbb{C}$-boundary coming from Wermer's example

Let $S^3 = \{|z|^2 + |w|^2 = 1\} \subset \mathbb{C}^2$ and let $0 < \varepsilon \ll 1$. Let G_f be the graph of the function

$$f(z) = \begin{cases} 2\varepsilon/\bar{z}, & |z| \geq \varepsilon, \\ 2z/\varepsilon, & |z| \leq \varepsilon. \end{cases}$$

which is endowed with the orientation induced by the projection onto the z-axis. It is easy to check that $L_f := G_f \cap S^3$ is the link in Figure 2 where the horizontal circle represents the component of L_f close to the z-axis.

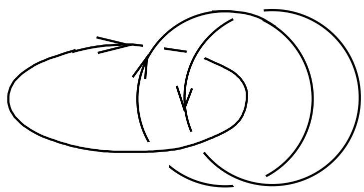

Figure 2. Example of a non-quasipositive $\mathbb{C}$-boundary

This link is evidently non-trivial, hence either L_f or its mirror image $L_{\bar{f}}$ is not quasipositive by Theorem 1.1. The mapping $T_f : (z,w) \mapsto (z, w - f(z))$ transforms G_f into the line $\{w = 0\}$. Similarly, $T_{\bar{f}}(G_{\bar{f}}) = \{w = 0\}$. Thus either the pair $\big(T_f(B^4), \{w = 0\}\big)$ or its image under $(z,w) \mapsto (z,\bar{w})$ is a desired example of a "fancy ball" in $\mathbb{C}^2$ and an algebraic curve which cuts a non-quasipositive link on its boundary sphere, i.e., either L_f or $L_{\bar{f}}$ is a non-quasipositive $\mathbb{C}$-boundary.

So far it is a non-constructive proof of existence, because we do not know yet which link is not quasipositive (we will establish this later). However, if we replace $f(z)$ by $f(z+\frac{1}{2}) + \overline{f(z-\frac{1}{2})}$, the resulting link will be amphicheiral because it is invariant under $(z,w) \mapsto (-z,-\bar{w})$. Thus it is not quasipositive (again by Theorem 1.1) and it is a $\mathbb{C}$-boundary by the same reasons as above.

Let us show that the link $L_{\bar{f}}$ is quasipositive (and hence, by Theorem 1.1, L_f is not). In fact, $L_{\bar{f}}$ is isotopic to the braid closure of the 3-braid $(\sigma_1\sigma_2\sigma_1^{-1})(\sigma_1^{-1}\sigma_2\sigma_1)$. We do not know whether L_f is a strong $\mathbb{C}$-boundary or not.

The quasipositivity of $L_{\bar{f}}$ can also be seen geometrically as follows. The components of L_f are parametrized, obeying the orientation, as follows

$$t \mapsto (e^{it},\, 2\varepsilon e^{it}), \qquad t \mapsto (2\varepsilon e^{-it},\, e^{-it}), \qquad t \mapsto (\tfrac{1}{2}\varepsilon e^{it},\, e^{it})$$

(here we have approximated the coefficients up to $O(\varepsilon^2)$). Therefore,

$$L_f = S^3 \cap \Big(\{w = 2\varepsilon z\} \cup \{z = 2\varepsilon w\}^{\mathrm{op}} \cup \{2z = \varepsilon w\}\Big)$$

where $\{\dots\}^{\mathrm{op}}$ means the opposite orientation is introduced on a complex line. Any two triples of distinct complex lines through the origin are isotopic to each other. Hence

$$L_f \sim S^3 \cap \Big(\{w = 0\} \cup \{w = \varepsilon z\} \cup \{z = \varepsilon w\}^{\mathrm{op}}\Big). \tag{1}$$

Thus $L_{\bar{f}}$ is isotopic to the image of the right hand side of (1) under $(z,w) \to (z,\bar{w})$. It can be parametrized by $t \mapsto (e^{it}, 0)$, $t \mapsto (e^{it},\, \varepsilon e^{-it})$, $t \mapsto (\varepsilon e^{-it},\, e^{it})$ but this is (again up to $O(\varepsilon^2)$) a parametrization of $S^3 \cap \{w(zw - \varepsilon) = 0\}$. Thus $L_{\bar f} \sim S^3 \cap \{w(zw - \varepsilon) = 0\}$ is quasipositive.

Remark 4.1. A third way to see that $L_{\bar f}$ is quasipositve is to observe that it can be constructed starting from Wermer's example (see [14, p. 34]), which consists in exhibiting the function $F(z) = (1+i)\bar z - iz\bar z^2 - z^2\bar z^3$ with the following properties: $F'_{\bar z} \neq 0$ on the unit disk Δ and $F|_{\partial\Delta} = 0$. Then the graph of F is totally real, hence it has a small neighbourhood which is a smooth pseudoconvex ball B. One can check that the link cut by the z-axis on ∂B is $L_{\bar f}$; thus $L_{\bar f}$ is quasipositive by [1].

§5. Further examples of $\mathbb{C}$-boundaries cut out by a complex line and their properties

It is clear that in §4 we could take any function $f : \mathbb{C} \to \mathbb{C}$ whose graph G_f is transverse to S^3 and cuts a non-trivial link L_f on it. In this case Hayden's theorem (Theorem 1.1) guarantees that either L_f or its mirror image $L_{\bar f}$ is non-quasipositive. This is, however, a very small family of links, which we are going to describe in this section. All they are iterated torus links, so an appropriate language to describe them are *EN-diagrams* (called also *splice diagrams*), which are certain graphs introduced by Eisenbud and Neumann in [3]. More precisely, considering diagrams obtained one from another by certain simple operations as equivalent (see [3, Thm. 8.1]), each iterated torus link corresponds to a unique equivalence class of diagrams.

The computation of the iterated torus link structure of L_f from the initial data is very similar to that in [6] (in both cases, the initial data is an arrangement of disjoint circles on the plane equipped with some extra information).

So, let f, G_f, and L_f be as above and let $\mathrm{pr}_1 : \mathbb{C}^2 \to \mathbb{C}$ be the projection $(z,w) \mapsto z$. Without loss of generality we assume that L_f is disjoint from the z-axis. Then $\mathrm{pr}_1(L_f)$ is a disjoint union of smooth embedded circles $C_1 \cup \dots \cup C_n$. Let $D_1, \dots, D_n$ be the bounded connected components of $\mathbb{C} \setminus \mathrm{pr}_1(L_f)$ numbered so that C_j is the exterior component of ∂D_j. We say that D_j is *positive* or *negative* according to the sign of $|f(z)|^2 + |z|^2 - 1$ for $z \in D_j$ (thus $\mathrm{pr}_1(G_f \cap B^4)$ is the union of all negative D_j's). We endow each C_j with the boundary orientation coming from the adjoining negative component of $\mathbb{C} \setminus \mathrm{pr}_1(L_f)$ (which is the orientation induced by the projection of L_f). Let a_j be the increment of $(\operatorname{Arg} f)/(2\pi)$ along C_j. Then the link L_f is determined by the following combinatorial data: the partial order $\prec$ on $\{C_1, \dots, C_n\}$ defined by $C_i \prec C_j$ if C_i lies inside C_j, and the numbers a_j corresponding to the non-maximal C_j with respect to this order. Such data are realizable if and only if $\sum_{k \in I(j)} a_k = 0$ each time when D_j is positive; here $I(j) = \{k \mid C_k \subset \partial D_j\}$.

Definition 5.1. Let K be a component of an oriented link L. Let p, q, and d be integers such that $\gcd(p,q) = 1$ and $d \geq 1$. We say that $L \cup L'$ (resp. $(L \setminus K) \cup L'$) is the *(pd, qd)-cable of L along K with the core retained* (resp. *with the core removed*) if, for some tubular neighbourhood T of K disjoint from $L \setminus K$, we have

- $L' \subset \partial T$, and L' is a disjoint union of knots: $L' = K_1 \cup \dots \cup K_d$;
- $[K_j] = p[K]$ in $H_1(T)$ and $\mathrm{lk}(K, K_j) = q$ for each $j = 1, \dots, d$.

An *iterated torus link* is a link obtained from the unknot by repeated cabling of either kind.

Remark 5.2. Reversing the orientations of some components of an iterated torus link leads to another iterated torus link. Indeed, reversing the orientation of a component K is the $(-1,0)$-cable along K with the core removed.

Proposition 5.3. *L_f is an iterated torus link.*

Proof. This follows from Lemma 5.4 below. □

Lemma 5.4. *Let $L = K_1 \cup \dots \cup K_n$ be a link in $S^3 = \partial B^4$ such that $\mathrm{pr}_1|_L$ is injective. Then L is an iterated torus link.*

Proof. We can assume that L is disjoint from the z-axis. Let $K_1, \dots, K_n$ be the components of L and let $C_j = \mathrm{pr}_1(K_j)$. We shall call the C_j ovals. Due to Remark 5.2, on each component we may chose any orientation. So we fix on K_j the orientation induced by the counter-clockwise orientation of C_j. Let a_j be the linking number of K_j with the z-axis (the increment of $\operatorname{Arg} F_j/(2\pi)$ where K_j is considered as the graph of a function $F_j : C_j \to \mathbb{C}$). The link L is determined by $\mathrm{pr}_1(L)$ and the numbers $a_1, \dots, a_n$ (if C_j is an outermost oval, then L does not depend on this a_j up to isotopy).

We shall prove the lemma for a larger class of links, namely we shall allow that some components of L are fibers of pr_1 positively linked with the z-axis (in fact the link does not change if we replace such a component by a small oval with $a_j = 1$).

Without loss of generality we may assume that $\mathrm{pr}_1(L)$ has a single outermost oval. Otherwise L is a split sum of sublinks each of which corresponds to an outermost oval and all the ovals surrounded by it. If $\mathrm{pr}_1(L)$ consists of a single oval and a point inside it, then L is the Hopf link which is the $(1,1)$-cable over the unknot. So it is enough to check that the following operations (i)–(iii) are cablings (see the first row in Figure 3). Let K be a component of L of the form $\mathrm{pr}_1^{-1}(p)$ for a point p, and D be a disk such that $D \cap \mathrm{pr}_1(L) = \{p\}$. The operations are:

(i) Adding a new component whose projection is ∂D and whose linking number with the z-axis is any given integer a.
(ii) The operation (i) followed by the removal of K.
(iii) Replacing K by $\mathrm{pr}_1^{-1}(P)$ where $P = \{p_1, \dots, p_k\} \subset D$.

Then (i) (resp. (ii)) is the $(a,1)$-cabling along K with the core retained (resp. removed), and (iii) is the $(k,0)$-cabling along K with the core removed. □

The second row in Figure 3 represents the evolution of the EN-diagram under the iterated cablings considered in the proof of Lemma 5.4.

In Figure 4 we give two EN-diagrams of a link L_f for the arrangement of ovals and the linking numbers as on the left-hand side of this figure. The grey area is $\mathrm{pr}_1(G_f \cap B^4)$ (recall that the sum of the linking numbers should be zero along the boundary of each bounded white (=positive) domain). The left-hand EN-diagram corresponds to the proof of Lemma 5.4, and the right-hand one is obtained from the former using admissible operations with EN-diagrams described in [3, Thm. 8.1 (3) and (6)]. In general, such operations can be applied to each piece of an EN-diagram which corresponds to an annular component of $\mathrm{pr}_1(G_f \setminus B^4)$ (a white annular component for the coloring as in Figure 4).

Remark 5.5. We do not know wheither any of the non-quasipositive links considered in this section is a strong $\mathbb{C}$-boundary.

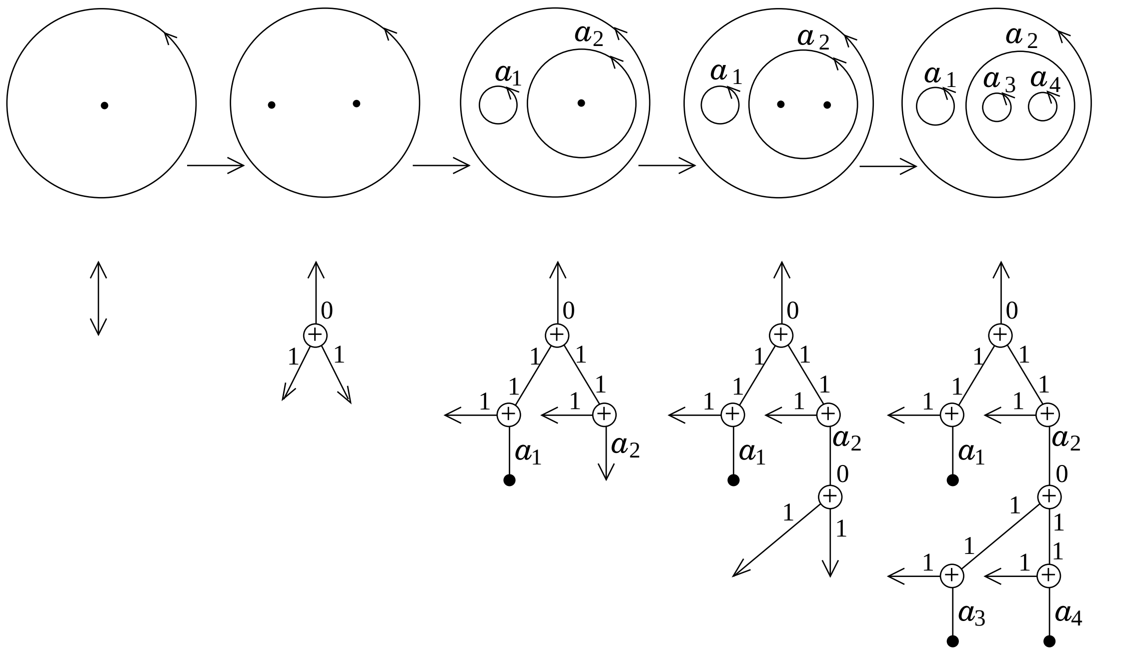

FIGURE 3. Evolution of $\mathrm{pr}_1(L)$ and of the EN-diagram.

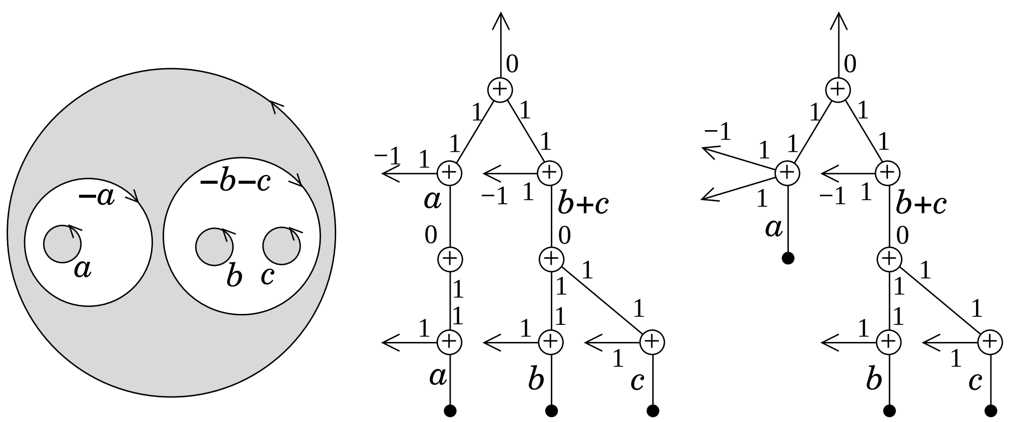

FIGURE 4. $\mathrm{pr}_1(L_f)$ and two EN-diagrams for a $\mathbb{C}$-boundary L_f

§6. LINKS WITH AT MOST 5 CROSSINGS

In this section, for each link admitting a plane projection with at most five crossings, we determine if it belongs to the classes $\mathcal{Q}$, $\mathcal{SB}$ or $\mathcal{B}$. In Table 1 we give the answers for all such links which do not have an unknot split component (do not have the form $L \sqcup O$ where O is the unknot) but the answers for links of the form $L \sqcup O \sqcup \cdots \sqcup O$ with ≤ 5 crossings easily follow. The implication $L \in \mathcal{C} \Rightarrow L \sqcup O \in \mathcal{C}$ ($\mathcal{C}$ is $\mathcal{Q}$, $\mathcal{SB}$, or $\mathcal{B}$) is evident. The reverse implication is true for $\mathcal{Q}$ (see [15]) but we do not know if it takes place in general for the classes $\mathcal{SB}$ or $\mathcal{B}$. However, the nature of our proofs is such that each time when we prove that $L \notin \mathcal{C}$ (where $\mathcal{C}$ is the class $\mathcal{SB}$ or $\mathcal{B}$), the same arguments can be easily adapted to prove that $L \sqcup O \sqcup \cdots \sqcup O \notin \mathcal{C}$.

The list of prime links up to 5 crossings is taken from [9, 10] (but we abbreviate 2_1^2 to 2_1). We denote with 4_{1-}^2 the link 4_1^2 (as it is given in [10]) with the reversed orientation of one of its components. Notice that any choice of orientation of the components of 5_1^2 provides the same orientated link type. In the second column we give the braid notation. It serves to identify the link as well as to make evidence of its quasipositivity (when applicable). The braid words also help to estimate $\chi_s^-(L)$ from below by using the observation that if a braid β' is obtained from β by replacing some σ_i^{-1} with σ_i, then $\chi_s^-(\beta) \ge \chi_s^-(\beta')$. (In fact, we only need lower bounds for χ_s^- and upper bounds for χ_s for our proofs, and the reader can assume that Table 1 contains just these bounds for the Euler characteristics.) For example, $\chi_s^-(4_{1-}^2) = \chi_s^-(\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}) \ge \chi_s^-(\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_1\sigma_2) = \chi^-(\sigma_1) = 2$ (the "σ_1" in "$\chi_s^-(\sigma_1)$" is viewed as a 3-braid here).

We also use Murasugi's inequality in estimates for the links 4_1^2, 4_{1-}^2, and 5_1^2.

The comments at the end of this section are referred to as "(a)", "(b)", etc. in Table 1. Almost all proofs are based on general results in §§2-3. Here we present some specific results used in the table.

Table 1

L	braid	$L \in \mathcal{Q}$	$L \in \mathcal{SB}$	$L \in \mathcal{B}$	$\chi_s(L)$	$\chi_s^-(L)$
2_1	σ_1^2	yes	yes	yes		
2_1^*		no	no (a)	no (b)	0	2
3_1	σ_1^3	yes	yes	yes		
3_1^*		no	no (a)	no (b)	-1	1
4_1	$(\sigma_1^{-1}\sigma_2)^2$	no	no (a)	no (b)	-1	1
4_1^2	σ_1^4	yes	yes	yes		
4_1^{2*}		no	no (a)	no (b)	-2	2
4_{1-}^2		no	no (a)	no (b)	0	2
4_{1-}^{2*}	$\sigma_1^{-1}\sigma_2\sigma_1^2\sigma_2$	yes	yes	yes		
$2_1\#2_1$		yes	yes	yes		
$2_1\#2_1^*$		no (c)	yes (d)	yes		
$2_1^*\#2_1^*$		no	no	no (f,e)	-1	3
$2_1 \sqcup 2_1$		yes	yes	yes		
$2_1 \sqcup 2_1^*$		no	no (a)	yes (d)	0	2
$2_1^* \sqcup 2_1^*$		no	no	no (f,e)	0	4
5_1	σ_1^5	yes	yes	yes		
5_1^*		no	no (a)	no (b)	-3	1
5_2	$\sigma_1^2\sigma_2^2\sigma_1\sigma_2^{-1}$	yes	yes	yes		
5_2^*		no	no (a)	no (b)	-1	1
5_1^2	$(\sigma_1\sigma_2^{-1})^2\sigma_1$	no (i)	yes (j)	yes		
5_1^{2*}		no	no	no (f)	0	2
$3_1\#2_1$		yes	yes	yes		
$3_1\#2_1^*$		no (c)	yes (d)	yes		
$3_1^*\#2_1$		no	no	no (f,g)	0	
$3_1^*\#2_1^*$		no	no	no (f,e)	-2	2
$3_1 \sqcup 2_1$		yes	yes	yes		
$3_1 \sqcup 2_1^*$		no (c)	no (a,g)	yes (d)	-1	1
$3_1^* \sqcup 2_1$		no	no (f,g)	no (h)	-1	1
$3_1^* \sqcup 2_1^*$		no	no	no (f,e)	-1	3

The following theorem is immediate from the Franks–Williams–Morton inequality [5, 12] in combination with Proposition 3.2 (in [9] this result is used in most cases to prove the non-quasipositivity of knots).

Theorem 6.1. ([2, Theorem 3.2]). *If L is a quasipositive link, then* $\operatorname{ord}_v P_L \geq 1 - \chi_s(L)$*, where $P_L(v,z)$ is the HOMFLY polynomial normalized by $P_{\text{unknot}} = 1$, $P_{L_+} = vzP_{L_0} + v^2 P_{L_-}$.*

Proposition 6.2. *The link $3_1^* \sqcup 2_1$ is not a $\mathbb{C}$-boundary.*

Proof. Let $L = L_1 \sqcup L_2$ where L_1 is 3_1^* and L_2 is 2_1. Suppose that L is a $\mathbb{C}$-boundary $\partial(A \cap B)$ where A is a smooth algebraic curve in $\mathbb{CP}^2$ and B is a 4-ball embedded in $\mathbb{C}^2$ which is considered as an affine chart in $\mathbb{CP}^2$. Without loss of generality we may assume that $A \setminus B$ has only one unbounded (in $\mathbb{C}^2$) component. Let Σ be a disjoint union of two surfaces $\Sigma_1 \cup \Sigma_2$, and let $j : (\Sigma, \partial\Sigma) \to (B, \partial B)$ be an immersion with negative crossings such that $j(\partial\Sigma_i) = L_i$, $i = 1, 2$. We may assume that Σ_1 is a disk and Σ_2 is an annulus. Let A' be $A \setminus B$ glued with Σ along the boundary, and let us extend j to A' so that $j(A') = (A \setminus B) \cup j(\Sigma)$.

We have $\chi_s(L) = -1$ and $\chi_s^-(L) = 1$ (see Table 1), so $\chi_s^-(L) > \chi_s(L)$, and hence $\chi(A') > \chi(A)$. Thus A' cannot be connected by Theorem 3.1. Recall that $A \setminus B$ has only one unbounded component, hence there exists a component A'_0 of A' such that $j(A'_0)$ is a bounded subset of $\mathbb{C}^2$. Since $j(A' \setminus A'_0)$ also intersects B, for some $k \in \{1, 2\}$ we have $j(A'_0) \cap B = j(\Sigma_k)$ and hence $j(A'_0) \cap \partial B = L_k$. Since $[j(A' \setminus A'_0)] = [j(A)]$ in $H_2(\mathbb{CP}^2)$, by Theorem 3.1 we have $\chi(A) \geq \chi(A' \setminus A'_0)$, hence

$$\chi(A) + \chi(A'_0) \geq \chi(A') = \chi(A \setminus B) + \chi(\Sigma) \geq \chi(A) - \chi_s(L) + \chi(\Sigma),$$

thus $\chi(A'_0) \geq \chi(\Sigma) - \chi_s(L) = 2$. This is impossible. Indeed, let $\Sigma_0 = A'_0 \setminus \Sigma_k$. Then $j(\Sigma_0)$ can be regarded as a smooth surface with boundary $-L_k$ which is embedded in the one-point compactification of $\mathbb{C}^2$, thus $\chi(\Sigma_0) \leq \chi_s(L_k) \leq 0$ (because $\chi_s(3_1) = -1$ and $\chi_s(2_1) = 0$) and we have assumed that $\chi(\Sigma_1) = 1$ and $\chi(\Sigma_2) = 0$. $\square$

Proposition 6.3. *The link 5_1^2 (see Figure 5) is a strong $\mathbb{C}$-boundary.*

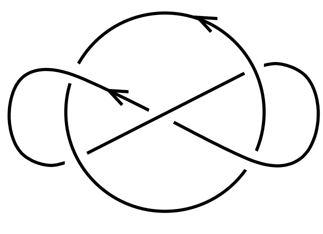

FIGURE 5. The link 5_1^2

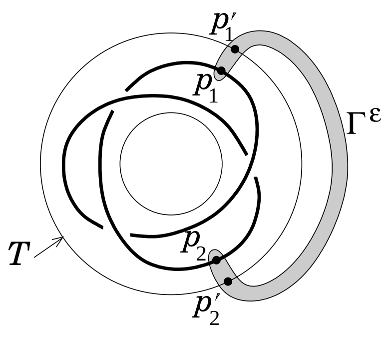

FIGURE 6

Proof. Let $A = \{(z, w) \mid w^2 = z^2 + z^3\}$. Let $1 \ll r \ll R$ and let $\Delta_r, \Delta_R \subset \mathbb{C}$ be the disks of the respective radii. Let $U_t = ([t, r] \times \Delta_R) \cup \partial(\Delta_r \times \Delta_R)$. Let $z_1 = r\,\exp(\pi i/3)$, $z_2 = \bar{z}_1$, and let w_j, $j = 1, 2$, be the solution of $w^2 = z_j^2 + z_j^3$ such that $\operatorname{Im} w_j > 0$, so that $w_1 \approx w_2 \approx r^{3/2} i$. Let $p_j = (z_j, w_j)$ and $p'_j = (z_j, Ri)$. Let γ be an embedded arc in Δ_r which connects z_1 with z_2 avoiding the interval $[0, r]$, and let $\Gamma = [p_1, p_1'] \cup (\gamma \times \{Ri\}) \cup [p_2', p_2]$. For a set $X \subset \mathbb{C}^2$ and $\varepsilon > 0$, we denote the ε-neighbourhood of X in $\mathbb{C}^2$ by X^ε. Finally, for $0 \ll \delta \ll \varepsilon$, let B_t be a small smoothing of $(U_t \setminus \Gamma^\varepsilon)^\delta$.

Then $A \cap \partial B_0$ is a strong $\mathbb{C}$-boundary link isotopic to 5_1^2 in the embedded 3-sphere ∂B_0. Indeed, we have $U_r = \partial(\Delta_r \times \Delta_R)$, and $A \cap U_r$ is the trefoil knot sitting in the "vertical" solid torus $T = (\partial\Delta_r) \times \Delta_R$; see Figure 6 where we represent the piecewise smooth 3-sphere U_r via the central projection onto the unit sphere followed by a suitable stereographic projection onto the 3-space.

Hence the link $A \cap B_r$ is as on the left-hand side of Figure 7 (cf. Theorem 2.4 and its proof). Consider the family of links $(B_t, A \cap B_t)$ where t varies from r to 0. In this deformation, the link changes only in a small area in its "inner" part, namely, the portion of the link in the sector $-\eta < \operatorname{Arg} z < \eta$ $(0 < \eta \ll 1)$ of the solid torus $(\partial\Delta_{r-\delta}) \times \Delta_R$ is deformed as t varies. When the parameter t crosses the value $t = \delta$, the sphere ∂B_t crosses the double point of A (at the origin), and the link bifurcates as shown in the middle of Figure 7. The resulting link is exactly 5_1^2 (see the right-hand side of Figure 7). $\square$

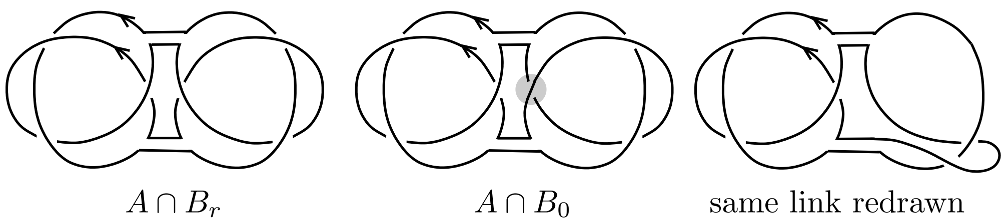

FIGURE 7

Comments to Table 1.

(a). Because $\chi_s(L) \neq \chi_s^-(L)$ (see Proposition 3.2).

(b). By Lemma 3.8 combined with the fact that $L \notin \mathcal{SB}$.

(c). By Theorem 1.2.

(d). By Theorem 2.1 applied to the nodal or cuspidal cubic $y^2 = ax^2 + x^3$ $(a = 0$ or $1)$ where B_0 is a small ball with center at the origin and B a small (for $2_1 \# 2_1^*$ and $2_1 \sqcup 2_1^*$) or a large (in the other cases) ball containing B_0. One easily checks that the resulting link is a strong $\mathbb{C}$-boundary in the corresponding cases.

(e). Use that $\chi_s(L) = \chi_s(L^*)$ and apply Proposition 3.6 to compute $\chi_s(L^*)$.

(f). By Proposition 3.7.

(g). An embedded surface in a 4-ball that is bounded by L cannot contain a disk as a component; hence $\chi_s(L) \leq 0$. (In the case of $3_1^* \# 2_1$ one can also compute $\chi_s(L) = \chi_s(L^*)$ using Proposition 3.2.)

(h). See Proposition 6.2.

(i). By Theorem 6.1.

(j). See Proposition 6.3.

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Steklov Mathematical Institute of Russian Academy of Sciences (N.K. and S.O.)

IMT, l'université Paul Sabatier, 118 route de Narbonne, Toulouse, France (S.O.)