On elements of degree g in the separating semigroup of a real curve of genus g.

G. B. Mikhalkin, S. Yu. Orevkov

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1 Introduction

By a real curve (C, conj) we mean a connected closed Riemann surface of genus g, equipped with an involution of complex conjugation conj, which sends multiplication of tangent vectors by i to multiplication by $-i$. The set of fixed points $\mathbb{R}C$ of such an involution is a collection of $r \le g+1$ smooth circles.

Real curves are divided into two types: those for which the set $C \setminus \mathbb{R}C$ is disconnected are of type I, and the rest are of type II. The Riemann surface C is canonically oriented by its complex orientation. For curves of type I (also called *separating*), the real locus $\mathbb{R}C$ inherits two opposite orientations from the two components of the complement $C \setminus \mathbb{R}C$. Thus, among the 2^r possible orientations of the multicomponent curve $\mathbb{R}C$, there are two preferred orientations. Each of these two opposite orientations is called a *complex orientation* of $\mathbb{R}C$.

A rational function $f : C \to \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is *real* if it is equivariant with respect to conj, i.e., if $f \circ \mathrm{conj} = \mathrm{conj} \circ f$. It is called *totally real* if $f^{-1}(\widehat{\mathbb{R}}) = \mathbb{R}C$, where $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. In this case the restriction $f|_{\mathbb{R}C} : \mathbb{R}C \to \widehat{\mathbb{R}}$ is an oriented covering for the standard orientation of $\widehat{\mathbb{R}}$ and one of the complex orientations of $\mathbb{R}C$. Such a covering has a *polydegree* $\delta = (d_1, \dots, d_r) \in \mathbb{N}^r$, consisting of the degrees d_j of the restrictions of the covering to the components of $\mathbb{R}C$ (which we assume to be numbered).

Clearly, if f is totally real, then C is separating. Moreover, L. Ahlfors [1] showed that on a separating curve one can always find a totally real rational function. Therefore, totally real functions, considered as morphisms $C \to \mathbb{P}^1$, are called in [2] *separating morphisms.* A divisor consisting only of real points is called *totally real*, and the divisor of zeros of a totally real rational function is called *separating.* It is easy to see that a separating divisor cannot contain multiple points, therefore, with some abuse of language, we will not always distinguish between separating divisors and their supports.

M. Kummer and K. Shaw [2] considered the set

$$\mathrm{Sep}(C) = \{d(f) \mid f : C \to \widehat{\mathbb{C}} \text{ is totally real}\} \subset \mathbb{Z}^r$$

consisting of all polydegrees of separating divisors on a real curve C, and showed that it is closed under addition. This subset is called the *separating semigroup* of the real curve C. The separating semigroups of M-curves, hyperelliptic curves, and curves of genus ≤ 4 are computed in [2, 3, 5].

Definition 1 *An element $\delta \in \mathrm{Sep}(C)$ is called* extendable *if*

$$\delta + \mathbb{N}_0^r \subset \mathrm{Sep}(C), \qquad \textit{where} \quad \mathbb{N}_0 = \{n \in \mathbb{Z} \mid n \geq 0\}.$$

Kummer and Shaw proved that if a separating divisor D is nonspecial, then its polydegree is a extendable element in $\mathrm{Sep}(C)$. The main result of our paper gives a sufficient condition for the extendability of elements corresponding to the special case, i.e., the case $h^1((f)_0) > 0$.

Theorem 1 *Let C be a separating real algebraic curve of genus g, and $f : C \to \widehat{\mathbb{C}}$ be a totally real rational function of degree g, such that its divisor of zeros $(f)_0$ has projective rank 1, i.e., $h^0((f)_0) = 2$. Then for all, except finitely many, points $y \in \widehat{\mathbb{R}}$, the preimage $f^{-1}(y)$ defines a divisor $D = (f)_y$ with the following property.*

For a sufficiently small arc $A \subset \mathbb{R}C$ centered at any point $p \in f^{-1}(y)$, and for any choice of points $p', p'' \in A$ lying in different components of $A \setminus \{p\}$, the divisor $D - p + p' + p''$ is a separating nonspecial divisor of degree $g + 1$.

Corollary 2 *Under the assumptions of Theorem 1, the polydegree of the divisor D is a extendable element of the semigroup* $\mathrm{Sep}(C)$.

2 Divisors of projective rank 1

The following statement is an adaptation of an observation made in [4] for curves embedded in surfaces, to the case of abstract real curves. Let C be a separating real curve, and $f : C \to \widehat{\mathbb{C}}$ be a real rational function on it. Let ω be a real holomorphic form on the curve C, and $(\omega)_0$ be its divisor of zeros.

Proposition 3 *Suppose that $p', p'' \in f^{-1}(0)$ are two distinct points such that the divisor $(\omega)_0 - (f)_0$ is not effective, but the divisor $(\omega)_0 - (f)_0 + p' + p''$ is effective, i.e., at one of the points p', p'' the order of zero of the function f is one greater than the order of zero of the form ω. Then the divisors $(\omega)_0 - (f)_0 + p'$ and $(\omega)_0 - (f)_0 + p''$ are not effective, i.e., at each of the points p', p'' the order of zero of the function f is one greater than the order of zero of the form ω. Moreover, the orientations on $\mathbb{R}C$ defined by the forms ω and df coincide in a neighborhood of one of the points p', p'', and are opposite in a neighborhood of the other point.*

Proof. The preimage $f^{-1}([-\varepsilon, \varepsilon])$, $\varepsilon > 0$, defines a system of paths on C. At the points of the preimage $f^{-1}(0)$ other than p' and p'', the order of zero of ω is not less than the order of zero of f, and hence the integral of the form ω over the corresponding components of the path system is $o(\varepsilon)$. At one of the points p', p'', or at both, the order of zero of ω is one less than the order of zero of f; consequently, the integral of ω over the corresponding components of the path system is significant, i.e., has order ε. For the same reason, at this point (at each of these points) the orders of zero of the forms ω and df are equal, and thus the orientations they define either coincide or are opposite.

By Abel's theorem, the integral of ω over the entire system of paths $f^{-1}([-\varepsilon, \varepsilon])$ is zero, and hence there must be two significant integrals and their signs in the total sum must be opposite. $\square$

This statement is especially convenient to apply to movable divisors of projective rank 1, as shown by the following observation (Proposition 4). Starting from this observation, in this section it will not be important for us whether C is separating or even a real curve. Recall that a *movable divisor* is an effective divisor D whose linear system $|D|$ has no base points.

Proposition 4 *Let D be an effective divisor of projective rank 1 (i.e., $h^0(D) = 2$) with no multiple points. Then for any two distinct points p', p'' of the support of D there exists a form ω vanishing on all of $D \setminus \{p', p''\}$, but not on all of D, and such that $\omega(p') \neq 0$ and $\omega(p'') \neq 0$, if the divisor D is movable.*

If D is a real divisor, then the form ω can be chosen real.

Proof. Set $D'' = D - p' - p''$, $d = \deg D$, $d'' = \deg D''$. Note that $\dim \Omega^1(D) = h^0(K - D)$, where $\Omega^1(D)$ is the space of holomorphic forms vanishing on

D. We need to prove that $\Omega^1(D'') \not\subset \Omega^1(D)$. This fact follows from the Riemann–Roch theorem. Indeed, since $h^0(D) = 2$ and $h^0(D'') > 0$, we have

$$\begin{aligned} h^0(K - D'') &= h^0(D'') + g - d'' - 1 = h^0(D'') + g - d + 1 \\ &> g - d + 1 = h^0(K - D) - h^0(D) + 2 = h^0(K - D). \end{aligned} \tag{1}$$

Hence $\dim \Omega^1(D'') > \dim \Omega^1(D)$, i.e., there exists a form $\omega \in \Omega^1(D'') \setminus \Omega^1(D)$.

Now let us prove that $\omega(p') = 0$ implies $\omega(p'') = 0$ if the divisor D is movable. This can be derived either from Abel's theorem (as in the proof of Proposition 3), or again from the Riemann–Roch theorem. Indeed, set $D' = D - p'$. From the movability of D it follows that $h^0(D') = h^0(D) - 1$. Therefore $h^0(K - D') = h^0(K - D)$ (cf. (1)), and since $\Omega^1(D) \subset \Omega^1(D')$, we conclude that $\Omega^1(D) = \Omega^1(D')$, i.e., $\omega(p'') = 0$ implies $\omega(p') = 0$ for $\omega \in \Omega^1(D'')$.

In the real case, the space $\Omega^1(D - p' - p'')$ and its subspace $\Omega^1(D)$ are real, and hence the form ω can also be chosen real. $\square$

The converse statement also holds.

Proposition 5 *Suppose that D is an effective movable divisor without multiple points, such that for any two distinct points $p', p'' \in \mathrm{Supp}(D)$ there exists a form ω vanishing on $D \setminus \{p', p''\}$, but not at p'. Then the projective rank of the divisor D is equal to one.*

Proof. The projective rank of a movable divisor is positive. If it is greater than one, then for any point $p' \in \mathrm{Supp}(D)$, there exists a deformation of the divisor D leaving the point p' fixed. If $p'' \in \mathrm{Supp}(D)$ is a point that moves under this deformation, then the existence of a form ω from the hypothesis of the theorem contradicts Abel's theorem (cf. the proof of Proposition 4). $\square$

3 Duplicating points on separating special divisors of projective rank 1

Now let the curve C be real and separating. If D is a totally real divisor on C (i.e., $\mathrm{Supp}(D) \subset \mathbb{R}C$), and D is movable, without multiple points, then, as noted in [5], Proposition 3 sometimes gives both necessary and sufficient conditions for the divisor D to be separating. It turns out that this observation is most conveniently applied when D has projective rank 1.

Note that the linear system of such a divisor D defines a real rational function f such that $D = (f)_0$, and 0 is a regular value of this function. This function is defined uniquely up to an automorphism of $\widehat{\mathbb{R}}$. The increase of the function f defines an orientation in a neighborhood of the preimage $f^{-1}(0)$.

Proposition 6 (cf. [2, Proposition 2.11]) *Let $f : C \to \widehat{\mathbb{C}}$ be a real rational function with a totally real divisor of zeros $(f)_0$ without multiple points. The orientation of a neighborhood of the preimage $f^{-1}(0)$, given by the form df, extends to an orientation on all of $\mathbb{R}C$ if and only if f is totally real.*

Proof. By definition, all $d = \deg f$ points of the totally real divisor are real. If the orientation extension condition is satisfied, then the degree of the map $f_{\mathbb{R}} : \mathbb{R}C \to \widehat{\mathbb{R}}$ in the obtained orientation of the curve $\mathbb{R}C$ is equal to $\pm d$. Thus, this is a totally real map. Conversely, if the function f is totally real, then df defines an orientation on $\mathbb{R}C$. $\square$

Corollary 7 *If the orientation of a neighborhood of the preimage $f^{-1}(0)$, defined by the form df, extends to an orientation on all of $\mathbb{R}C$, then it coincides with one of the complex orientations of the curve $\mathbb{R}C$.*

Proof. By Proposition 6, the function f is totally real. Therefore, the orientation on $\mathbb{R}C$ defined by the form df is a complex orientation, since it coincides with the boundary orientation induced from the preimage of the upper half-plane. $\square$

Putting the proven statements together, we obtain the following two facts.

Proposition 8 *Let D be a separating divisor, and $p', p'' \in \mathrm{Supp}(D)$ two points. Then any holomorphic form vanishing on all of $\mathrm{Supp}(D) \setminus \{p', p''\}$, but not at p', determines local orientations at the points p' and p'' that are not consistent with the complex orientation of $\mathbb{R}C$.*

Proof. Let $D = (f)_0$, where f is a totally real function. By Proposition 3, the orientations given by the forms ω and df are not consistent at the points p' and p''. Moreover, the orientation given by the form df for a totally real function f is a complex orientation. $\square$

Proposition 9 *Let D be a movable totally real divisor without multiple points. Suppose that for any distinct points $p', p'' \in \operatorname{Supp}(D)$ there exists a holomorphic form $\omega_{p',p''}$, vanishing on $\operatorname{Supp}(D) \setminus \{p', p''\}$, but not at p', which defines local orientations at the points p' and p'' not consistent with the complex orientation of $\mathbb{R}C$. Then the divisor D is separating.*

Proof. Since D is movable, $D = (f)_0$, where f is a real rational function. For any $p', p'' \in D$, the orientation given by the form $\omega_{p',p''}$ is not consistent in neighborhoods of these points with the complex orientation (by hypothesis), nor with the orientation given by the form df (by Proposition 3), Consequently, the latter two orientations are consistent, and hence f is totally real by Proposition 6. $\square$

It is convenient to introduce the following notation. Fix a complex orientation on C and for a real holomorphic form ω on C and a point $p \in \mathbb{R}C$, set $\operatorname{sign}_\omega(p) = \operatorname{sign} \omega(v)$, where v is a tangent vector to $\mathbb{R}C$ at the point p, positively oriented with respect to the chosen complex orientation.

If p is a simple zero of a real rational function $f : C \to \widehat{\mathbb{C}}$, then $\operatorname{sign}_{df}(p)$ is the local degree of $f|_{\mathbb{R}C} : \mathbb{R}C \to \widehat{\mathbb{R}}$ at p. In these notations, Proposition 6 and Corollary 7 take the following form.

Proposition 10 *Let $f : C \to \widehat{\mathbb{C}}$ be a real rational function $f : C \to \widehat{\mathbb{C}}$ with a totally real divisor of zeros $D = (f)_0$ without multiple points. The function f is totally real if and only if for any points $p', p'' \in \operatorname{Supp}(D)$ one has*

$$\operatorname{sign}_{df}(p') = \operatorname{sign}_{df}(p''). \tag{2}$$

To verify condition (2), it is convenient to use the following fact, which follows directly from Propositions 3 and 10.

Proposition 11 *Let f and D be as in Proposition 10. Let $p', p'' \in \operatorname{Supp}(D)$ and ω be a real holomorphic form on C, vanishing everywhere on $f^{-1}(0) \setminus \{p', p''\}$, but not at p'. Then condition (2) is equivalent to condition*

$$\operatorname{sign}_\omega(p') \neq \operatorname{sign}_\omega(p''). \tag{3}$$

4 Proof of the main theorem

Lemma 12 *Let $f : C \to \widehat{\mathbb{C}}$ be a rational function of degree $d \leq g$ such that the divisor $D = (f)_0$ has projective rank 1. Then for all but finitely many values $y \in \widehat{\mathbb{C}}$, there exists a form ω_y having first-order zeros at all points of $f^{-1}(y)$.*

Proof. By the Riemann–Roch theorem, $h^0(K - D) = g - d + 1 \geq 1$. Consequently, there exists a divisor E such that $K_y = E + (f)_y$ belongs to the canonical class for all y. Therefore, if y is a regular value of the function f and $y \notin f(\mathrm{Supp}(E))$, then the holomorphic form ω_y, corresponding to the divisor K_y, has the required property. Indeed, in this case the divisors $(f)_y$ and E have no common points, hence, at each point of $f^{-1}(y)$, the multiplicity of zero of the form ω_y is equal to the multiplicity of zero of the function f, and since y is a regular value, these multiplicities are equal to one. $\square$

Proof of Theorem 1. Let $f : C \to \mathbb{P}^1$ be a totally real rational function of degree g and projective rank 1, and $D = (f)_y$, $y \in \mathbb{R}$. Since then $h^1(D) = h^0(K - D) = 1$, there exists a unique (up to multiplication by a constant) holomorphic form ω_D vanishing on $\mathrm{Supp}(D)$. Note that, since the divisor D is movable, the point p is not a base point of it, hence $h^0(D - p) < h^0(D)$, and thus $h^1(D - p) = 1$, i.e., any form vanishing on $\mathrm{Supp}(D) \setminus \{p\}$ is also proportional to the form ω_D. If the arc A is sufficiently small, then this form does not vanish at p', and thus the divisor $D^+ = D - p + p' + p''$ of degree $g + 1$ is nonspecial and has projective rank 1, as does the divisor D.

We prove that the divisor D^+ is movable. Indeed, if q is a base point of it, then $h^0(D^+ - q) = h^0(D^+) = 2$, consequently, the divisor $D^+ - q$ is special, i.e., there exists a holomorphic form ω_q vanishing everywhere on $\mathrm{Supp}(D^+) \setminus \{q\}$. If $q = p''$, then ω_q vanishes on $\mathrm{Supp}(D) \setminus \{p\}$, but then, as shown above, ω_q cannot vanish at the point p'. The case $q = p'$ is analogous. Suppose now that $q \notin \{p', p''\}$. If such a form ω_q exists for any points p', p'' arbitrarily close to p, then, passing to the limit, we obtain a holomorphic form which vanishes everywhere on $\mathrm{Supp}(D) \setminus \{q\}$ and has a multiple zero at p. As shown above, such a form is proportional to the form ω_D, and by Lemma 12 we can assume that ω_D has a first-order zero at p. The obtained contradiction shows that the divisor D^+ is movable. Let f^+ be a real rational function such that $D^+ = (f^+)_0$.

Since the vector space of forms vanishing on $\mathrm{Supp}(D)$ is defined over $\mathbb{R}$, we can assume that the form ω_D is real. Let $q \in \operatorname{Supp}(D) \setminus \{p\}$. Consider the real holomorphic form ω for the pair of points q and p in the divisor D, provided by Proposition 4. Since D is separating, by Propositions 10 and 11 we have

$$\operatorname{sign}_\omega(p) \neq \operatorname{sign}_\omega(q). \tag{4}$$

The point p'' is a zero of the form $\widetilde{\omega} = \omega_D + \varepsilon\omega$ for some $\varepsilon \in \mathbb{R}$. If the arc A is sufficiently small, then ε is also small, and then

$$\operatorname{sign}_{\widetilde{\omega}}(p') = \operatorname{sign}_{\widetilde{\omega}}(p) = \operatorname{sign}_\omega(p) \tag{5}$$

(since p is a simple zero of the form ω_D) and

$$\operatorname{sign}_{\widetilde{\omega}}(q) = \operatorname{sign}_\omega(q). \tag{6}$$

From (4)–(6) it follows that

$$\operatorname{sign}_{\widetilde{\omega}}(p') = \operatorname{sign}_\omega(p) \neq \operatorname{sign}_\omega(q) = \operatorname{sign}_{\widetilde{\omega}}(q).$$

The form $\widetilde{\omega}$ vanishes on $\operatorname{Supp}(D^+) \setminus \{p', q\}$, but not at q. Consequently, $\operatorname{sign}_{df^+}(p') = \operatorname{sign}_{df^+}(q)$ by Proposition 11. Similarly, $\operatorname{sign}_{df^+}(p'') = \operatorname{sign}_{df^+}(q)$. Since q is an arbitrary point of $\operatorname{Supp}(D^+) \setminus \{p', p''\}$, we conclude that the function $\operatorname{sign}_{df^+}$ is constant on $\operatorname{Supp}(D^+)$, and the theorem follows from Proposition 10. □

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