

SEIFERT FORMS AND SLICE EULER CHARACTERISTIC OF LINKS

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ABSTRACT. We define the Witt coindex β of a rational bilinear form (not necessarily symmetric) as the codimension minus dimension of a maximal isotropic subspace. Levine gave a computable necessary and sufficient condition for $\beta = 0$, but no algorithm of computation of β had been known when $\beta > 0$. Our main result is a computable necessary and sufficient condition for $\beta = 1$ in the case when the symmetrized form is non-degenerate. As shown by Taylor, all upper bounds of the slice genera of knots based on their Seifert form can be derived from the Witt coindex of the Seifert form. The case of an odd β does not apply to knots but we show that it provides an estimate of the slice Euler characteristic for the links with an even number of components. We illustrate our results with examples where the Witt coindex is used to show that a two-component link does not bound a locally flat cylinder in the four-ball, whereas any other known restriction does not show it.

1. INTRODUCTION

A knot in S^3 is (topologically) *slice* if it bounds a locally flat disk in B^4 . If K is a slice knot then any Seifert form of K is *metabolic* (it vanishes on a half-dimensional subspace). In this case K is said *algebraically slice*. In [Lev69a, Lev69b] Levine gave necessary and sufficient conditions for a Seifert form of a knot to be metabolic. These are:

- (1) the Alexander polynomial has the form $f(t)f(t^{-1})$ (the Fox-Milnor condition);
- (2) the Tristram-Levine signature function vanishes;
- (3) certain invariants depending on a prime number p vanish for all p .

Livingston [Liv10] has shown that it is enough to check (3) for a finite number of primes only. It follows that there is an algorithm to check whether a given knot is algebraically slice or not.

Our aim is to extend the work of Levine to links with non-trivial Alexander polynomial. It is convenient to formulate our results in terms of the *coindex* $\beta(L)$ of an oriented link L in S^3 . We define $\beta(L)$ as the *Witt coindex* of any Seifert form of L (on the homology with rational coefficients), which is the codimension minus dimension of a maximal isotropic subspace (see §2.4). It does not depend on the choice of the form (see §2.5). In the case of knots, the coindex coincides with an invariant introduced by Taylor [Tay79].

The paper consists of an algebraic part (§§2–4) and a topological part (§5). The main algebraic result is Theorem 4.3 which completely describes rational forms with coindex 1 up to concordance under the condition that the symmetrized form is non-degenerate. Namely, we prove that each such a form is concordant to a very special form determined

by the Alexander polynomial (see §4.2). Combined with the classical result by Levine [Lev69a] (extended by Livingston [Liv10]), Theorem 4.3 provides an algorithm to decide whether a given form has coindex 1. The topological part of the paper is devoted to an extension of the known properties of $\beta(L)$ from knots to links (see §1.1 for a more detailed discussion).

In our opinion, new estimates of the slice Euler characteristic (see §1.1) via Seifert-form-based invariants are still of interest because they can be effectively computed for links with more than hundred crossings, for example for those that appear in the topological classification of plane real algebraic curves (see [Ore99], [Ore02]).

1.1. Concordance and slice Euler characteristic. Two oriented links are *concordant* if there exists a properly embedded collection of locally flat $S^1 \times I$ in $S^3 \times I$, intersecting $S^3 \times 0$ and $S^3 \times 1$ at the given links. We first observe that for links whose Alexander polynomial is not zero, the coindex is invariant by concordance:

Theorem 1.1. *Let L be a link in S^3 with $\Delta_L \neq 0$. If L' is concordant to L then*

$$\beta(L) = \beta(L').$$

Note that the property $\Delta_L \neq 0$ is invariant by concordance [Kaw78, NP17]; see Proposition 5.4.

The *slice Euler characteristic* is defined as $\chi_s(L) = \max_F \chi(F)$, where the maximum is taken over all locally flat oriented surfaces with boundary L , without closed components in B^4 and transverse to S^3 . We show that the coindex provides an upper bound for χ_s .

Theorem 1.2. *Let L be a link in S^3 . Then the following inequality holds*

$$\chi_s(L) \leq 1 - \beta(L).$$

Closely related to the slice Euler characteristic is the *slice genus* of L , defined as $g_s(L) = \min_F \text{genus}(F)$. Since $2g_s(L) \geq 2 - n - \chi_s(L)$ where n is the number of components of L , Theorem 1.2 provides also a lower bound for g_s (note that this inequality between g_s and χ_s is not always sharp; see the end of §6.2). In the case of a knot K , Theorem 1.2 reduces to the estimation $2g_s(K) \leq \beta(K)$ proved by Taylor [Tay79].

We observe that Theorem 1.2 implies the Murasugi-Tristram inequality [Mur65b, Tri69], see Corollary 5.8. In the case $\chi_s(L) = 1$, the result implies also the Fox-Milnor theorem for the Alexander polynomial of links [FM66, Flo04], see Corollary 5.7 (more precisely, the *one-variable* version). In particular, the Alexander polynomial of a slice knot has the form $f(t)f(t^{-1})$. We illustrate in Section 6 that Theorem 1.2 may offer better estimates of $\chi_s(L)$ than the Murasugi-Tristram inequality and the Fox-Milnor Theorem. We present in particular (see Example 6.6) two-component links with vanishing Arf-Robertello invariant [Rob65] where the coindex is used to show that the links do not bound a cylinder in B^4 .

Let us mention that Taylor [Tay79] observed that a Seifert form is realizable by a knot of slice genus g if and only if its coindex is less or equal to $2g$. This motivates naturally the definition of the *algebraic slice genus* of a knot K as $g_s^{\text{alg}} = \frac{1}{2}\beta(K)$. In particular, K algebraically slice corresponds to $g_s^{\text{alg}}(K) = 0$. In view of Theorem 1.2, one might naturally

define the *algebraic slice Euler characteristic* as $1 - \beta(L)$. However, the condition that the coindex of a Seifert form of a link is $1 - \chi$ does not imply that the form is realizable by a link L with $\chi_s(L) = \chi$. For this reason, we do not know for the moment what should be the appropriate notion of algebraic slice Euler characteristic. See §6.3.

1.2. Algebraic concordance and isometric structures. In order to apply Theorem 1.2 one needs to estimate $\beta(L)$. We do not know how to compute $\beta(L)$ in general. However, by Levine [Lev69a] (and [Liv10]), there is an algorithm to decide whether $\beta(L) = 0$ (see above). We extend algebraic concordance techniques of Levine on isometric structures and decide algorithmically whether $\beta(L) = 1$ (in the case $\Delta_L(-1) \neq 0$).

In [Lev69a, Lev69b], Levine considers integral matrices of even size with $\det(A - A^\top) = \pm 1$ and defines the group $\mathcal{G}^{\mathbb{Z}}$ of their concordance classes. The construction of $\mathcal{G}^{\mathbb{Z}}$ works as well for rational matrices and $\mathcal{G}^{\mathbb{Z}}$ is in fact a subgroup of the rational algebraic concordance group, denoted $\mathcal{G}^{\mathbb{Q}}$. The strategy to study $\mathcal{G}^{\mathbb{Q}}$ is then to consider *isometric structures*. These are triple (V, b, T) where b is a nonsingular symmetric bilinear form on a finite dimensional space V and T is an isometry with respect to b , with some hypothesis on the characteristic polynomial of T . The Witt group $\mathcal{G}_{\mathbb{Q}}$ of (concordance) classes of isometric structures is then shown to be isomorphic to $\mathcal{G}^{\mathbb{Q}}$, see [Lev69b].

We extend the notion of concordance to matrices of arbitrary size and non-trivial Alexander polynomial and show that the coindex is an algebraic concordance invariant. In order to characterize matrices with coindex 1, we extend also the notion of isometric structures and study structures corresponding to forms of coindex 1. Then we show that any Seifert form of a link L with $\Delta_L(-1) \neq 0$ has coindex 1 if and only if it is concordant to a very special one determined by Δ_L . See Definition 4.1 and Theorem 4.3 for a precise statement.

Finally, one may recall that there is a group homomorphism from the concordance group of knots to the algebraic concordance group $\mathcal{G}^{\mathbb{Z}}$, sending (the class of) a knot to the concordance class of a Seifert matrix. This homomorphism is not injective in dimension 3 (see [CG86, CG78]) but it is an isomorphism in higher odd dimensions (see [Ker71]). We do not know if the coindex may characterize completely the existence of bounding manifolds with prescribed Euler characteristic.

Moreover Theorem 1.2 should be extended to Seifert matrices whose Alexander polynomial is zero. This is the object of a future work, as well as the generalization to Seifert forms of colored links [CF08].

In §2, we define the concordance of admissible matrices and give basic properties of the coindex. In §3 we introduce isometric structures. §4 is devoted to the main result concerning the forms of coindex 1. In §5 we relate the topological and algebraic invariants discussed in the paper. We conclude with §6 where we illustrate our results by several examples and discuss some related topics.

2. ALGEBRAIC CONCORDANCE AND COINDEX

Let \mathbb{F} be a field of characteristic zero (in fact, we need only the case when it is the field of rational, real, or p -adic rational numbers: \mathbb{Q} , \mathbb{R} , or \mathbb{Q}_p). In this section, we consider square matrices with entries in \mathbb{F} .

2.1. Normalized Alexander polynomial. Given a square matrix A , we define its normalized Alexander polynomial (which we shall call just *Alexander polynomial*) as

$$\Delta_A(t) = t^{-k} \det(A - tA^\top), \quad k \text{ is such that } \Delta_A \in \mathbb{F}[t] \text{ and } \Delta_A(0) \neq 0$$

(in particular, $\Delta_A(t) = \det(A - tA^\top)$ when A is non-singular). It is clear that

$$\Delta_A(t) = (-t)^n \Delta_A(t^{-1}), \quad n = \deg \Delta_A. \quad (1)$$

One readily checks that $\Delta_A(t)$ is invariant under the S-equivalence, and then the Alexander polynomial $\Delta_L(t)$ of a link L is a well-defined invariant (not up to multiplication by ct^n for $c \in \mathbb{F}^*$). It can be equivalently defined as $\Delta_L(t) = t^{n/2} \nabla_L(t^{1/2} - t^{-1/2})$ with $n = \text{span } \nabla_L$, where $\nabla_L(z)$ is the Conway polynomial defined by $\nabla_{L_+}(z) - \nabla_{L_-}(z) = z \nabla_{L_0}(z)$ and $\nabla_O(z) = 1$.

Our choice of the sign of ∇_L corresponds to that in [Kau87] and [Kaw96, §8.2] but it is opposite to the choice in [Kaw96, §5.4] and [Lic97] in the case of even-component links.

2.2. The group $\bar{\mathcal{G}}^\mathbb{F}$.

Definition 2.1. A matrix A is *admissible* if its Alexander polynomial is non-zero.

In particular, any nonsingular matrix is admissible. Two matrices A and A' are *congruent* if there exists a matrix P with $\det P \neq 0$ such that $P^\top A P = A'$.

With any square matrix A we associate (in the standard way) a bilinear form on an \mathbb{F} -vector space V whose dimension equals the size of A .

Definition 2.2. A matrix A of size $2k$ is *metabolic* if there exists a k -dimensional subspace U of \mathbb{F}^{2k} , totally isotropic for A , i.e., $x^\top A y = 0$ for all $x, y \in U$. Equivalently A is congruent to a matrix of the form

$$\begin{pmatrix} 0 & B \\ C & D \end{pmatrix},$$

where B, C and D are $k \times k$ matrices and 0 is the zero matrix. Two matrices A_1 and A_2 are *concordant* if the orthogonal sum $A_1 \oplus -A_2$ is metabolic.

The *Witt index* of A is defined as the maximal dimension of a totally isotropic subspace for A .¹ We denote this index by $\text{ind}(A)$ or $\text{ind}_\mathbb{F}(A)$. Note that if $\det A \neq 0$ and A has size n , then $\dim U \leq n/2$ for any totally isotropic subspace U .

Proposition 2.3. *The concordance of admissible matrices is an equivalence relation. Congruent matrices are concordant.*

¹This is already a standard term for symmetric non-singular forms. However, it is used very rarely for degenerate or non-symmetric forms, and the definitions vary in these cases.

The symmetry is obvious. The concordance of congruent matrices (in particular, the reflexivity): $P^\top \begin{pmatrix} Q^{\top A Q} & 0 \\ 0 & -A \end{pmatrix} P = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$ for $P = \begin{pmatrix} I & I \\ Q & 0 \end{pmatrix}$. The transitivity follows from Lemma 2.5, which is a generalization of the Witt cancellation lemma (cf. [Lev69b, §3]).

Remark 2.4. The concordance is no longer transitive if we omit the requirement that the matrices are admissible. Indeed, according to our definition, any matrix is concordant to the zero matrix of the same size. It would be interesting to find an adequate notion of concordance, relevant to Seifert matrices of arbitrary links.

Equivalence classes of admissible matrices form an abelian group under orthogonal sum, denoted by $\mathcal{G}^{\mathbb{F}}$, with subgroup $\mathcal{G}^{\mathbb{F}}$ the set of concordance classes of matrices of even size introduced by Levine. The metabolic class constitutes the identity element, and the inverse of the class of A is the class of $-A$.

Lemma 2.5. *Let A and B be square matrices. Suppose that B is admissible and metabolic of size $2k$. Then*

$$\text{ind}(A \oplus B) = \text{ind}(A) + k.$$

In particular, if both B and $A \oplus B$ are metabolic, then A is metabolic.

Proof. In essential, the proof follows that of [Lev69b, Lemma 1] or the lemma on p. 87 in [Ker71]. We first show that $\text{ind}(A \oplus B) \geq \text{ind} A + k$. Let X and Y be respectively the maximal isotropic subspaces for A and B . Denote θ and η the forms on vector spaces V and W associated with A and B respectively. Then, for all $x, x' \in X$ and $y, y' \in Y$, one has

$$(\theta \oplus \eta)(x \oplus y, x' \oplus y') = \theta(x, x') + \eta(y, y') = 0.$$

Hence $X \oplus Y$ is isotropic for $\theta \oplus \eta$ and

$$\text{ind}(A \oplus B) \geq \dim(X \oplus Y) = \dim X + \dim Y = \text{ind}(A) + k.$$

Let Z be a supplementary subspace of Y in W . Let $U \subset V \oplus W$ be a totally isotropic subspace for $A \oplus B$, with basis $(\alpha_1, \dots, \alpha_n)$. Suppose that U is maximal, i.e.

$$n = \text{ind}(A \oplus B).$$

For all $i \in \{1, \dots, n\}$, there exist $x_i \in V$, $y_i \in Y$ and $z_i \in Z$ such that $\alpha_i = x_i + y_i + z_i$. Let r be the rank of the family $\{z_i\}_i$. After re-ordering, we can suppose that z_1, \dots, z_r are linearly independent. Then after the replacement of α_i with $\alpha_i + \sum_{j=1}^r \lambda_{ij} \alpha_j$, $i = r+1, \dots, n$, for suitable λ_{ij} , we may achieve that $z_{r+1} = \dots = z_n = 0$. Similarly, if s is the rank of the family $\{x_i\}_{r < i \leq n}$, we may achieve that x_{r+1}, \dots, x_{r+s} are linearly independent and $x_{r+s+1} = \dots = x_n = 0$. We obtain the following family of rank n :

$$\begin{aligned} \alpha_i &= x_i + y_i + z_i, & i &= 1, \dots, r, \\ \alpha_i &= x_i + y_i, & i &= r+1, \dots, r+s, \\ \alpha_i &= y_i, & i &= r+s+1, \dots, n. \end{aligned}$$

For all $i, j \in \{r+1, \dots, r+s\}$, we have

$$\theta(x_i, x_j) = \theta(x_i, x_j) + \eta(y_i, y_j) = (\theta \oplus \eta)(\alpha_i, \alpha_j) = 0.$$

Hence there exists a subspace of V of dimension s on which θ vanishes. It remains to show that

$$s \geq n - k. \quad (2)$$

Since the polynomial Δ_B is non zero, there exists $\lambda \in \mathbb{F}$ such that $B^\lambda := B + \lambda B^\top$ is non-singular. The family $\{y_i\}_{r+s < i \leq n}$ is linearly independent and the subspace Y' of Y spanned by this family has dimension $n - r - s$. Moreover, for all $i > r + s$ and all j , we have

$$\eta(y_i, z_j) = \eta(y_i, y_j) + \eta(y_i, z_j) = \theta(0, x_j) + \eta(y_i, y_j + z_j) = (\theta \oplus \eta)(\alpha_i, \alpha_j) = 0.$$

and similarly $\eta(z_j, y_i) = 0$. Hence, if η^λ is the form associated with B^λ then

$$\eta^\lambda(y, z_j) = 0 \text{ for } y \in Y' \text{ and } 1 \leq j \leq r,$$

and Y' is orthogonal to the subspace $\langle z_1, \dots, z_r \rangle$ of Z with respect to η^λ . If we extend the basis of Y' to a basis $(\tilde{y}_1, \dots, \tilde{y}_k)$ of Y , and (z_1, \dots, z_r) to a basis of Z , then B^λ has an $(n - r - s) \times (k + r)$ -block of zeros whose rows correspond to (y_{r+s+1}, \dots, y_n) and columns correspond to $(\tilde{y}_1, \dots, \tilde{y}_k, z_1, \dots, z_r)$. Since B^λ is non-singular, we conclude that $(n - r - s) + (k + r) \leq 2k$, thus (2) holds. \square

2.3. A concordance condition for proportional matrices. For a polynomial $F(t) \in \mathbb{F}[t]$, we denote $\bar{F}(t) = t^n F(t^{-1})$ where $n = \deg F$.

Let $F(t) \in \mathbb{F}[t]$, $F(0) \neq 0$. We say that F is *symmetric* if $\bar{F} = cF$ with $c \in \mathbb{F}^\times$. Since $\bar{\bar{F}} = F$, we have $c = \pm 1$ in this case. Thus any symmetric polynomial F is either *palindromic*, i.e. $F = \bar{F}$, or *antipalindromic*, i.e. $F = -\bar{F}$. Evaluating F at $t = \pm 1$, we see that $F(t)$ is a multiple of $t - 1$ when F is antipalindromic, and a multiple of $t + 1$ when F is palindromic of odd degree. The equation (1) means that the Alexander polynomial of an admissible matrix of even (resp. odd) size is palindromic (resp. antipalindromic).

Definition 2.6. Two polynomials $P, Q \in \mathbb{F}[t]$ are *Fox-Milnor equivalent* and if there exist $F, G \in \mathbb{F}[t]$ with $F(0)G(0) \neq 0$, such that $F\bar{F}P = G\bar{G}Q$. We denote $P \stackrel{\text{FM}}{\sim} Q$

Proposition 2.7. *If A is a metabolic nonsingular matrix, then there exists $F \in \mathbb{F}[t]$ with $F(0) \neq 0$ such that $\Delta_A(t) = F(t)\bar{F}(t)$, in particular, $\Delta_A \stackrel{\text{FM}}{\sim} 1$.*

Proof. Suppose that $A = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$, where the blocks have size $k \times k$. Then $\Delta_A(t) = (-1)^k \det(C^\top - tB) \det(B - tC^\top) = (-1)^k (-t)^k \det(B - t^{-1}C^\top) \det(B - tC^\top)$. Hence $\Delta_A(t) = t^k F(t^{-1})F(t) = \bar{F}(t)F(t)$ (here $k = \deg F$ since C is nonsingular). \square

Lemma 2.8. *If two nonsingular matrices A and B are concordant, then $\Delta_A \stackrel{\text{FM}}{\sim} \Delta_B$.*

Proof. Since $A \oplus -B$ is metabolic, by Proposition 2.7 there exists $F \in \mathbb{F}[t]$ with $F(0) \neq 0$ such that $\Delta_A(t)\Delta_{-B}(t) = F(t)\bar{F}(t)$. Let k be the dimension of B . Then $\Delta_{-B} = (-1)^k \Delta_B$ and (1) reads $\Delta_B(t) = (-1)^k \bar{\Delta}_B(t)$. Thus

$$\Delta_A(t) = \frac{F(t)\bar{F}(t)}{(-1)^k \Delta_B(t)} = \Delta_B(t) \frac{F(t)\bar{F}(t)}{(-1)^k \Delta_B^2(t)} = \Delta_B(t) \frac{F(t)\bar{F}(t)}{\Delta_B(t)\bar{\Delta}_B(t)}.$$

□

Lemma 2.9. *Let A and B be two nonsingular matrices of odd size such that $\Delta_B \stackrel{\text{FM}}{\sim} \mu \Delta_A$ with $\mu \in \mathbb{F}^\times$. Suppose that λA and B are concordant for some $\lambda \in \mathbb{F}^\times$. Then $\lambda = a^2 \mu$ for some $a \in \mathbb{F}^\times$. In particular, μA and B are concordant.*

Proof. Let $2n + 1$ be the dimension of A . Then by Lemma 2.8

$$\lambda^{2n+1} \Delta_A = \Delta_{\lambda A} \stackrel{\text{FM}}{\sim} \Delta_B \stackrel{\text{FM}}{\sim} \mu \Delta_A,$$

hence $\lambda = a^2 \mu$, $a \in \mathbb{F}^\times$. Note that $a^2 \mu A$ is congruent (and hence concordant) to μA . □

2.4. The coindex of a concordance class. Remind that the Witt index $\text{ind}_{\mathbb{F}}(A)$ of a matrix A is the maximal dimension of a totally isotropic subspace for A , see Section 2.2.

Definition 2.10. The *Witt coindex* of a matrix A of dimension n is the integer

$$\beta(A) = \beta_{\mathbb{F}}(A) = n - 2 \text{ind}_{\mathbb{F}}(A).$$

By definition, A is metabolic if and only if $\beta(A) = 0$. Note also that if $\det A \neq 0$, then $\beta(A)$ is a non-negative integer.

Proposition 2.11. *If admissible matrices A and A' are concordant, then $\beta(A) = \beta(A')$.*

Proof. Suppose that A and A' are concordant, which means that there exists a metabolic M with $A \oplus -A' \equiv M$. Hence $A \oplus -A' \oplus A' \equiv M \oplus A'$. By reflexivity, there exists M' metabolic such that $-A' \oplus A' \equiv M'$ and hence $A \oplus M' \equiv M \oplus A'$. By Lemma 2.5, it follows that $\beta(A) = \beta(A')$. □

Example 2.12. Let $A = \begin{pmatrix} 0 & B_{n+1} \\ B_n & 0 \end{pmatrix}^{-1}$ where B_n and B_{n+1} are nonsingular square matrices of size n and $n + 1$ respectively. Using the cofactor matrix we obtain $\beta(A) = 1$.

2.5. The coindex of an S-equivalence class. We say that two square matrices are S-equivalent (over \mathbb{F}) if they are obtained from each other by a sequence of congruences and the following replacements:

$$A \leftrightarrow B = \begin{pmatrix} A & 0 & u \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A \leftrightarrow C = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 1 \\ v & 0 & 0 \end{pmatrix} \quad (3)$$

where u (resp. v) is a column (resp. row) with entries in \mathbb{F} .

Proposition 2.13. [Lev69a, Claim in §8] *S-equivalent matrices are concordant.*

This fact combined with Proposition 2.11 implies the invariance of β under S-equivalence of admissible matrices. However, the following proposition shows that the invariance of β under S-equivalence holds without the admissibility assumption and thus it is a well-defined link invariant (see §5.1).

Proposition 2.14. *If two matrices A and A' are S-equivalent, then $\beta(A) = \beta(A')$.*

Proof. The proof is almost the same as for Proposition 2.5. It is enough to show that $\text{ind } B = \text{ind } C = 1 + \text{ind } A$ for A , B , and C in (3). We do it for B (for C the arguments are the same). It is clear that $\text{ind } B \geq 1 + \text{ind } A$. Let us show that $\text{ind } B \leq 1 + \text{ind } A$.

Let θ and Θ denote the forms on vector spaces V and $W = V \oplus Y \oplus Z$ associated with A and B respectively where Y and Z are the one-dimensional subspaces of W corresponding to the last two rows and columns of B . Let X and U be the maximal isotropic subspaces for A and B . Let $(\alpha_1, \dots, \alpha_n)$ be a basis of U , thus $n = \text{ind } B$. Write $\alpha_i = x_i + y_i + z_i$ with $x_i \in V$, $y_i \in Y$, $z_i \in Z$. We define r and s as in the proof of Proposition 2.5, and we impose the same assumptions on the vectors y_i 's and z_i 's (here $r \leq 1$ and $n - r - s \leq 1$). For all $i, j \in \{r + 1, \dots, r + s\}$, we have $\Theta(x_i, y_j) = \Theta(y_i, x_j) = \Theta(y_i, y_j) = 0$, whence

$$\theta(x_i, x_j) = \Theta(x_i + y_i, x_j + y_j) = \Theta(\alpha_i, \alpha_j) = 0.$$

Thus there is a subspace of V of dimension s on which θ vanishes, and it remains to show that $s \geq n - 1$ (that is (2) with $k = 1$). Suppose that $s < n - 1$. In our context this means that $r = 1$ and $s = n - 2$, i.e., $\alpha_1 = x_1 + y_1 + z_1$ and $\alpha_n = y_n$ with $z_1 \neq 0$ and $y_n \neq 0$. We have $\Theta(x_1, y_n) = \Theta(y_1, y_n) = 0 \neq \Theta(z_1, y_n)$ (see the penultimate column of B), which contradicts the fact that $\Theta(x_1 + y_1 + z_1, y_n) = \Theta(\alpha_1, \alpha_n) = 0$. \square

2.6. Computation of the coindex of a symmetric form over \mathbb{Q} . Let Q be a non-degenerate symmetric $n \times n$ matrix over \mathbb{Q} . The computation of $\beta(Q)$ is known for more than a century and it is exposed (sometimes more sometimes less explicitly) in all textbooks on quadratic forms. For the reader's convenience we present it here in the form given in Serre's book [Ser70]. In [MH73] (see also [Liv10, Appendix B]) a computation of $\beta(Q)$ is presented in another form, without explicit usage of the Hasse symbol.

Let $[a_1, \dots, a_n]$ be a diagonalization of Q such that all the a_i 's are square-free integers. Let P be the set of prime divisors of $\prod a_i$ which do not divide $\text{gcd}(a_1, \dots, a_n)$. Then

$$\beta(Q) = \max \left\{ \beta_{\mathbb{R}}(Q), \max_{p \in P \cup \{2\}} \beta_{\mathbb{Q}_p}(Q) \right\} \quad \text{and} \quad \beta_{\mathbb{R}}(Q) = |\sigma|$$

where σ is the signature of Q .

The $\beta_{\mathbb{Q}_p}(Q)$ are computed as follows. Let $\varepsilon = \varepsilon_p(Q) = \prod_{i < j} (a_i, a_j)_p$ be the *Hasse symbol* (the “invariant ε ” in [Ser70, Ch. 4, §2]); here $(,)_p$ is the Hilbert symbol. Note that a different definition of the Hasse symbol (with $\prod_{i \leq j}$ instead of $\prod_{i < j}$) is used in [Lev69a]. Let $d = d_p(Q)$ be the *discriminant* of Q , that is the class of $\det Q$ in $\mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2$. We define

the *normalized discriminant* \tilde{d} and *normalized Hasse invariant* $\tilde{\varepsilon}$ as follows²

$$\tilde{d} = (-1)^l d \in \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2, \quad \tilde{\varepsilon} = (-1, d)_p^l (-1, -1)_p^{l(l+1)/2} \varepsilon, \quad \text{where } l = \lfloor n/2 \rfloor.$$

Note that $(-1, -1)_p = (-1)^{p-1}$.

If $n = 2l$, then

$$\beta_{\mathbb{Q}_p}(Q) = \begin{cases} 0 & \text{if } \tilde{d} = 1 \text{ and } \tilde{\varepsilon} = 1, \\ 2 & \text{if } \tilde{d} \neq 1, \\ 4 & \text{if } \tilde{d} = 1 \text{ and } \tilde{\varepsilon} = -1. \end{cases}$$

If $n = 2l + 1$, then

$$\beta_{\mathbb{Q}_p}(Q) = \begin{cases} 1 & \text{if } \tilde{\varepsilon} = 1, \\ 3 & \text{if } \tilde{\varepsilon} = -1. \end{cases}$$

3. ISOMETRIC STRUCTURES

Let \mathbb{F} be a field of characteristic zero (as above, the most interesting cases for us are \mathbb{F} is \mathbb{Q} , \mathbb{R} , or \mathbb{Q}_p).

Definition 3.1. An *isometric structure over* \mathbb{F} is a triple (V, b, T) where b is a non-singular symmetric bilinear form on a finite dimensional \mathbb{F} -vector space V and T is an isometry of (V, b) , i.e.

$$b(x, y) = b(Tx, Ty) \text{ for all } x, y \in V. \quad (4)$$

Let $\Delta_T(t) = \det(T - tI)$ be the characteristic polynomial of T .

If Q and T are $n \times n$ matrices, T is nonsingular, $Q^\top = Q$, and $T^\top QT = Q$, then the corresponding isometric structure on the coordinate space \mathbb{F}^n is denoted by (Q, T) .

3.1. The concordance group $\bar{\mathcal{G}}_{\mathbb{F}}^*$.

Definition 3.2. An isometric structure (V, b, T) is *admissible* if $\Delta_T(-1) \neq 0$.

The *sum* of two isometric structures is defined as

$$(V, b, T) \oplus (V', b', T') = (V \oplus V', b \oplus b', T \oplus T').$$

Definition 3.3. An isometric structure (V, b, T) is *metabolic* if V contains a subspace of dimension $\frac{1}{2} \dim V$, totally isotropic for b and invariant under T . Two isometric structures (V_1, b_1, T_1) and (V_2, b_2, T_2) are *concordant* if $(V_1, b_1, T_1) \oplus (V_2, -b_2, T_2)$ is metabolic.

The concordance of admissible isometric structures is an equivalence relation. Equivalence classes form an abelian group under orthogonal sum denoted $\bar{\mathcal{G}}_{\mathbb{F}}^*$, and the neutral element is the class of admissible metabolic structures [Lev69a, §6].

Let A be a non-singular admissible matrix with $\det A \neq 0$. Then the congruence class of the matrix $A + A^\top$ and the similarity class of $A^{-1}A^\top$ are determined by the congruence

²Some authors define the discriminant of Q to be \tilde{d} .

class of A . This implies that the congruence class of A determines a well-defined isometric structure $(Q, T) = (A + A^\top, A^{-1}A^\top)$. Note that

$$\Delta_A(t) = \det(A)\Delta_T(t). \quad (5)$$

Let $\bar{\mathcal{G}}_*^{\mathbb{F}}$ be the subgroup of $\bar{\mathcal{G}}^{\mathbb{F}}$ of concordance classes of admissible matrices A such that $\Delta_A(-1) \neq 0$.

Proposition 3.4. [Lev69a, §§8–9] *The morphism*

$$\bar{\mathcal{G}}_*^{\mathbb{F}} \longrightarrow \bar{\mathcal{G}}_{\mathbb{F}}^*, \quad A \longmapsto (A + A^\top, A^{-1}A^\top), \quad \det A \neq 0,$$

is well-defined and it is an isomorphism.

The inverse is given by $(Q, T) \mapsto Q(I + T)^{-1}$ (see [LN, Thm. 3.5.16] for a proof that this is indeed the inverse). The proof of Proposition 3.4 follows mainly from the following lemma.

Lemma 3.5. [Lev69a, Lemma 8] *Every matrix A with $\Delta_A(-1) \neq 0$ is S -equivalent (and then concordant by Proposition 2.13) to an admissible non-singular matrix.*

The isomorphism in Proposition 3.4 can be defined in invariant terms by $(V, \theta) \mapsto (V, b, T)$ where θ is a non-degenerate bilinear form on a vector space V , $b = \theta + \theta^\top$, and $T : V \rightarrow V$ is such that $\theta(x, y) = \theta(y, Tx)$ for any $x, y \in V$.

3.2. Decomposition of $\bar{\mathcal{G}}_{\mathbb{F}}$. Given an admissible isometric structure (V, b, T) , the \mathbb{F} -vector space V can be viewed as an $\mathbb{F}[t^{\pm 1}]$ -module through the action of t , as follows:

$$tv = T(v) \text{ for all } v \in V.$$

Note that V is a torsion module, annihilated by $\Delta_T(t)$. The δ -primary component of V , where δ is an irreducible factor of Δ_T , is defined as

$$V_\delta = \ker \delta(T)^m \text{ for a large } m.$$

Recall that $t + 1$ cannot be a factor of Δ_T since (V, b, T) is admissible.

By [Lev69a, Lemma 10], if δ_i and δ_j are irreducible factors of Δ_T such that $\delta_i(t)$ and $\delta_j(t^{-1})$ are relatively prime, then V_{δ_i} and V_{δ_j} are orthogonal. In particular, V_δ and $V_{\bar{\delta}}$ are orthogonal if δ is not symmetric (see the definition in Section 2.3). Note that an irreducible symmetric polynomial different from $t \pm 1$ is palindromic of even degree. Thus there is an orthogonal decomposition

$$V = \left(\bigoplus_{\delta} V_\delta \right) \oplus \left(\bigoplus_{\lambda} (V_\lambda \oplus V_{\bar{\lambda}}) \right)$$

where the δ are symmetric and the λ are non-symmetric factors of Δ_T . The restriction of (b, T) to each summand $V_\lambda \oplus V_{\bar{\lambda}}$ is metabolic (because λ is orthogonal to itself and hence b vanishes on $V_\lambda \oplus 0$). As a direct consequence of these observations, we obtain that the isometric structure (b, T) is metabolic if and only if its restriction to each of the factors V_δ is metabolic. This fact can be equivalently formulated as follows.

Proposition 3.6. [Lev69a, Lemma 11] *The group $\bar{\mathcal{G}}_{\mathbb{F}}$ decomposes as a sum $\bar{\mathcal{G}}_{\mathbb{F}} \simeq \bigoplus_{\delta} \bar{\mathcal{G}}_{\mathbb{F}}^{\delta}$, where δ runs over all irreducible symmetric polynomials and $\bar{\mathcal{G}}_{\mathbb{F}}^{\delta}$ is the subgroup of concordance classes of isometric structures whose characteristic polynomial is a power of δ .*

We also need the following Levine's result.

Proposition 3.7. [Lev69a, Lemma 12] *If δ is an irreducible symmetric polynomial and (V, b, T) has characteristic polynomial δ^k for some positive k , then (V, b, T) is metabolic or concordant to an isometric structure with minimal polynomial δ .*

3.3. Minimal isometric structures. An isometric structure (V, b, T) is called *minimal* if $\dim V$ is minimal in its concordance class.

Lemma 3.8. *Let (V, b, T) be a minimal isometric structure. Then V does not have nontrivial isotropic T -invariant subspaces.*

Proof. Suppose that b vanishes on a T -invariant subspace $W \neq \{0\}$. Without loss of generality we suppose that W is generated by some vector w , as a $\mathbb{F}[t^{\pm 1}]$ -submodule of V , and that the minimal polynomial which annihilates w is an irreducible factor δ of Δ_T . Let us choose a vector $w' \in V_{\delta} \setminus W$ such that $b(w, w') \neq 0$. Such a vector exists because otherwise b would vanish on V_{δ} , which is impossible since b is non-singular and V_{δ} is an orthogonal summand of V . Let W' be the submodule generated by w' . By Proposition 3.6, the minimal polynomial of w' is δ . Hence $W \cap W' = \{0\}$.

Let us show that $b|_{W \oplus W'}$ is non-singular. Suppose that $u + u'$ is orthogonal to $W \oplus W'$ for some $u \in W$ and $u' \in W'$. Then $(u + u') \perp W$ and $u \perp W$ (because $b|_W = 0$), hence $u' \perp W$. Since W' is cyclic and δ irreducible, W' is generated by any its non-zero element. Therefore, if u' were non-zero, we would have $W' \perp W$, which contradicts the condition $b(w, w') \neq 0$. Thus $u' = 0$ and hence $u = u + u'$ is orthogonal to W' . Since W is also cyclic, the same arguments show that $u = 0$.

Thus $b|_{W \oplus W'}$ is non-singular, hence V splits into the orthogonal sum of two T -invariant subspaces $(W \oplus W') \oplus (W \oplus W')^{\perp}$. Moreover, the restriction of b to $W \oplus W'$ is metabolic, which contradicts the minimality of $\dim V$ in the concordance class. \square

We do not use the following fact, but it seems to be interesting by itself and it is an almost immediate consequence from the previous lemma.

Proposition 3.9. *If two minimal isometric structures are concordant, then they are isomorphic.*

In other words, each class in $\mathcal{G}_{\mathbb{F}}^*$ contains a unique minimal isometric structure up to isomorphism.

Proof. Let (V_1, b_1, T_1) and (V_2, b_2, T_2) be two concordant minimal isometric structures and let $n = \dim V_1 = \dim V_2$. Then $(V, b, T) := (V_1 \oplus V_2, b_1 \oplus -b_2, T_1 \oplus T_2)$ has a T -invariant isotropic subspace U of dimension n . Let $\pi_k : U \rightarrow V_k$, $k = 1, 2$, be the restriction of the canonical projection $V \rightarrow V_k$. Then $\ker \pi_1$ is a T -invariant isotropic subspace of V_2 , hence $\ker \pi_1 = 0$ by Lemma 3.8, i.e., π_1 is an isomorphism of \mathbb{F} -vector spaces. The same is true

for π_2 . Then it is straightforward to check that $-\pi_2 \circ \pi_1^{-1}$ is an isomorphism between the considered isomorphic structures. \square

4. FORMS OF COINDEX 1

4.1. The main idea. Levine's theory [Lev69a] starts from the observation that if a Seifert matrix A is metabolic then any totally isotropic subspace U is T -invariant (following the notations of §3). Indeed U and $T(U)$ are contained in the right orthogonal complement U_A^\top of U , and then $T(U) = U_A^\top = U$ since the condition $\beta(A) = 0$ (i.e. A is metabolic) implies that the dimensions of all the three subspaces coincide.

If $\beta(A) = 1$, then the dimensions are no longer equal, but U and $T(U)$ are two hyperplanes in U_A^\top , and $U_1 = U \cap T(U)$ is a hyperplane in U . Then we show that $U_2 = T^2(U) \cap U$ is a hyperplane in U_1 etc. In this way we construct a basis whose first elements generate U , and the matrix of T is the companion matrix of Δ_A . Then the T -invariance of $Q = A + A^\top$ means that the entries of Q verify a system of linear equations which admits a unique solution. We obtain that the isometric structure is concordant to a standard one, determined uniquely by Δ_A . Unfortunately these arguments do not work in the case $\beta(A) > 1$.

4.2. Standard coindex-one matrices. In this section \mathbb{F} is a field of characteristic zero (the most important case being $\mathbb{F} = \mathbb{Q}$). Let $n \geq 0$ and let $P(t)$ be an antipalindromic polynomial (see Section 2.3) of degree $2n + 1$ in $\mathbb{F}[t]$ such that $P(0) \neq 0$. Then P can be written

$$P(t) = -c_0 t^{2n+1} - c_1 t^{2n} - \dots - c_n t^{n+1} + c_n t^n + \dots + c_1 t + c_0, \quad c_0 \neq 0.$$

Consider the matrices

$$A_P = \begin{pmatrix} 0 & M_{n+1}^\top \\ M_n & 0 \end{pmatrix}^{-1} \quad \text{where} \quad M_i = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{i-1} \\ & c_0 & c_1 & \cdots & c_{i-2} \\ & & \ddots & \ddots & \vdots \\ & & & c_0 & c_1 \\ & & & & c_0 \end{pmatrix}$$

(here the M_i are upper triangular and then the matrix A_P^{-1} is “butterfly-shaped”). It is easy to see that $\beta(A_P) = 1$ (see Example 2.12).

Definition 4.1. The *standard coindex-one matrix associated with P* is A_P . We denote it by $\text{Std}(P)$.

Let T_P be the companion matrix of P , i.e., the matrix of the multiplication by t in the $\mathbb{F}[t]$ -module $\mathbb{F}[t]/(P)$ in the basis $(1, t, \dots, t^{2n})$. A straightforward computation shows that $T_P(A_P^{-1})^\top = A_P^{-1}$, hence $A_P^{-1}A_P^\top = T_P$ and $(A_P + A_P^\top, T_P)$ is the corresponding isometric structure.

Since T_P is the companion matrix of P , we have $\Delta_{T_P}(t) = P(t)/c_0$. It is also clear that $\det A_P^{-1} = \det M_n \det M_{n+1} = c_0^{2n+1}$. Hence, by (5), we obtain

$$\Delta_{\text{Std}(P)}(t) = \Delta_{A_P}(t) = c_0^{-2(n+1)} P(t). \quad (6)$$

Proposition 4.2. *Let P, A_P, T_P be as above. Let (Q, T_P) be an isometric structure such that the left upper $(n \times n)$ -submatrix of Q is zero. Then $Q = \lambda(A_P + A_P^\top)$, $\lambda \in \mathbb{F}$.*

Proof. Without loss of generality we may assume that $c_0 = 1$. Then the last column of T_P is $(1, c_1, \dots, c_n, -c_n, \dots, -c_1)^\top$. We have $T_P^\top Q T_P = Q$. Hence Q is a Toeplitz matrix (i.e., $Q_{i,j} = Q_{i+1,j+1}$ for each $i, j < 2n + 1$) with the first line starting with n zeros: $(0, \dots, 0, x_0, x_1, \dots, x_n)$. The first n entries of the last column of $Q - T_P^\top Q T_P$ provide the following equations, which allow us to express x_1, \dots, x_n via x_0 :

$$\begin{aligned} c_n x_0 + c_{n-1} x_1 + \dots + c_2 x_{n-2} + c_1 x_{n-1} + x_n &= 0, \\ c_{n-1} x_0 + \dots + c_2 x_{n-3} + c_1 x_{n-2} + x_{n-1} &= 0, \\ &\vdots \\ c_2 x_0 + c_1 x_1 + x_2 &= 0, \\ -x_0 + c_1 x_0 + x_1 &= 0. \end{aligned}$$

This means that Q is determined by P up to a scalar factor whence the result. \square

4.3. The main result. If A is a matrix of odd dimension, then (1) implies that Δ_A is antipalindromic of odd degree and hence the matrix $\text{Std}(\Delta_A)$ is well defined. Moreover, the parity of degree and the property to be (anti)palindromic are evidently invariant under the Fox-Milnor equivalence, hence $\text{Std}(P)$ is well defined if $P \stackrel{\text{FM}}{\sim} \Delta_A$.

Theorem 4.3. *Let A be an admissible matrix of odd dimension such that $\Delta_A(-1) \neq 0$ and $\beta(A) = 1$. Then for any polynomial $P \in \mathbb{F}[t]$ such that $\Delta_A(t) \stackrel{\text{FM}}{\sim} P(t)$, the matrix A is concordant to the standard coindex-one matrix $\text{Std}(P)$ associated with P .*

The proof of Theorem 4.3 can be mainly brought back to a particular case stated in the following lemma which will be proven below.

Lemma 4.4. *Let A be a matrix of odd dimension such that $\Delta_A(-1) \neq 0$. Let $P = \prod_{i=1}^s \delta_i$ be a product of polynomials, where the δ_i 's are pairwise distinct symmetric irreducible. Suppose that $\Delta_A(t) \stackrel{\text{FM}}{\sim} P(t)$. If $\beta(A) = 1$ then A is concordant to $\text{Std}(P)$.*

Proof of Theorem 4.3. Let A and P be as in Theorem 4.3. It is clear that $\Delta_A \stackrel{\text{FM}}{\sim} P_0$ where P_0 is a product of pairwise distinct symmetric polynomials (because any symmetric polynomial F with $F(0) \neq 0$ is Fox-Milnor equivalent to such a product). Let $B = \text{Std}(P)$. By (6) we have $\Delta_B \stackrel{\text{FM}}{\sim} P$, hence $\Delta_B \stackrel{\text{FM}}{\sim} P \stackrel{\text{FM}}{\sim} \Delta_A \stackrel{\text{FM}}{\sim} P_0$. Thus $\Delta_A \stackrel{\text{FM}}{\sim} P_0$ and $\Delta_B \stackrel{\text{FM}}{\sim} P_0$. Moreover, $\beta(A) = \beta(B) = 1$, hence both matrices A and B are concordant to $\text{Std}(P_0)$ by Lemma 4.4 and thus A is concordant to $B = \text{Std}(P)$. \square

In order to prove Lemma 4.4, we use the following Lemma 4.5, which states that if Lemma 4.4 holds for a nonsingular matrix A then it holds for any nonsingular matrix concordant to A .

Lemma 4.5. *Let A and B be nonsingular matrices such that $\Delta_A \stackrel{\text{FM}}{\sim} P$ and $\Delta_B \stackrel{\text{FM}}{\sim} Q$ where P and Q are products of pairwise distinct symmetric irreducible polynomials. If B is concordant to A and A is concordant to $\text{Std}(P)$, then B is concordant to $\text{Std}(Q)$.*

Proof. Since A and B are concordant and nonsingular, one has $\Delta_A \stackrel{\text{FM}}{\sim} \Delta_B$ by Lemma 2.8. Hence $P \stackrel{\text{FM}}{\sim} \Delta_A \stackrel{\text{FM}}{\sim} \Delta_B \stackrel{\text{FM}}{\sim} Q$. Since P and Q are products of pairwise distinct symmetric polynomials, it follows that there exists $c \in \mathbb{F}^\times$ such that $Q(t) = c^2 P(t)$, whence $\text{Std}(Q) = c^2 \text{Std}(P)$. In particular, $\text{Std}(P)$ and $\text{Std}(Q)$ are concordant and the result follows. \square

Proof of Lemma 4.4. Since A is admissible, A is S-equivalent to a non-singular matrix \bar{A} by Lemma 3.5. Then $\Delta_A = \Delta_{\bar{A}}$ (see §2.1) and A is concordant to \bar{A} , hence the concordance between \bar{A} and $\text{Std}(\Delta_{\bar{A}})$ implies the concordance between A and $\text{Std}(\Delta_A)$. Due to this observation, we assume from now on that A is nonsingular. By Lemma 4.5, we may also assume that the size of A is minimal in its concordance class.

Let $2n + 1$ be the size of A and (V, b, T) be the isometric structure related to A , i.e., $V = \mathbb{F}^{2n+1}$, $T = A^{-1}A^\top$, and $b(x, y) = x^\top(A + A^\top)y$. Since the size of A is minimal, the isometric structure is minimal. Then it decomposes according to the irreducible factors of Δ_A as $V = \bigoplus_{i=1}^s V_{\delta_i}$, where the δ_i are symmetric and irreducible and, moreover, $V_{\delta_i} = \ker \delta_i(T)$ for all i (see Propositions 3.6 and 3.7). Let U be an n -dimensional totally isotropic subspace of V . For all x and y in U , we have

$$y^\top A(Tx) = y^\top A(A^{-1}A^\top)x = y^\top A^\top x = (x^\top A y)^\top = 0.$$

Hence $T(U)$ is a subspace of the right orthogonal supplementary space of U with respect to A :

$$T(U) \subset U_A^\perp = \{z \in V \mid y^\top A z = 0 \text{ for any } y \in U\}.$$

Let us set inductively:

$$U_0 = U, \quad U_{i+1} = T(U_i) \cap U. \quad (7)$$

We have $U_1 \subset U_0$ and, for any $i \geq 1$, the condition $U_i \subset U_{i-1}$ implies

$$U_{i+1} = T(U_i) \cap U \subset T(U_{i-1}) \cap U = U_i.$$

Hence, by induction, we have $U = U_0 \supset U_1 \supset U_2 \supset \dots$. Let k be such that

$$n = \dim U_0 > \dim U_1 > \dots > \dim U_k = \dim U_{k+1}.$$

The fact that A is non-singular implies that $\dim U_A^\perp = \dim V - \dim U = n + 1$. This fact combined with $U \subset U_A^\perp$, $T(U) \subset U_A^\perp$, and $\dim U = \dim T(U) = n$ yields

$$\dim U_1 = \dim(U \cap T(U)) \geq n - 1. \quad (8)$$

Since T induces an isomorphism $U_{i-1}/U_i \rightarrow T(U_{i-1})/T(U_i)$, we have also

$$\dim U_{i-1}/U_i \geq \dim (T(U_{i-1}) \cap U) / (T(U_i) \cap U) = \dim U_i/U_{i+1}. \quad (9)$$

By combining (8) and (9), we obtain

$$\dim U_i = n - i \quad \text{for } 0 \leq i \leq k. \quad (10)$$

By definition, $U_{k+1} \subset T(U_k)$ and $\dim U_{k+1} = \dim U_k = \dim T(U_k)$, hence $U_{k+1} = T(U_k)$. We also have $U_{k+1} = U_k$. Thus U_k is T -invariant whence $U_k = \{0\}$ by Lemma 3.8. Hence $k = n$ by (10). Choose $u_n \in U_{n-1} \setminus \{0\}$ and set

$$u_i = T^{-(n-i)}(u_n) \quad \text{for } i = 1, \dots, n-1.$$

By (7) we have $T^{-1}(U_{i+1}) = U_i \cap T^{-1}(U) \subset U_i$ whence, by induction, $u_i \in U_{i-1} \setminus U_i$ for $i = 1, \dots, n$. Hence (u_1, \dots, u_n) is basis of U .

Let W be the $\mathbb{F}[t^{\pm 1}]$ -submodule of V generated by u_1 . Let us show that $\dim W = 2n+1$.

Case 1. Suppose that $\dim W = m < 2n$. Since $b|_U = 0$ we have $b(u_n, u_i) = 0$ for $i = 1, \dots, n$. If $n < i \leq m$, then $1 < i - n + 1 \leq n$, hence

$$b(u_n, u_i) = b(T^{n-1}u_1, T^{n-1}u_{i-n+1}) = b(u_1, u_{i-n+1}) = 0.$$

Thus $b(u_n, w) = 0$ for each $w \in W$. Since W is generated by u_n as an $\mathbb{F}[t^{\pm 1}]$ -module, it follows that $b|_W = 0$, which contradicts Lemma 3.8.

Case 2. Suppose that $\dim W = 2n$. There is an orthogonal sum decomposition $W = \bigoplus_i W_{\delta_i}$ where $W_{\delta_i} = W \cap V_{\delta_i}$. The module W is cyclic, generated by the single element u_1 . Hence for all $i = 1, \dots, s$, $W_{\delta_i} = \ker \delta_i(T|_W)$ is cyclic as well and $\dim W_{\delta_i} = \deg \delta_i$. Since δ_i is symmetric for all $i \neq i_0$, it follows that $\deg \delta_i$ is even for all $i \neq i_0$. Hence $t-1$ is the only odd degree factor of Δ_A and then of Q . From the hypothesis $\dim W = 2n$ we obtain $W_{t-1} = 0$, and then the orthogonal decomposition $V = W \oplus V_{t-1}$. The restriction $A|_W$ is metabolic, hence $b|_W$ is metabolic, which contradicts the minimality of $\dim V$ in the concordance class of (V, b, T) .

We have then shown that $\dim W = 2n+1$. Hence $V = W$ and (u_1, \dots, u_{2n+1}) is a basis of V . In this basis, the matrix of T is the companion matrix of Δ_T and the matrix of b has a zero $n \times n$ submatrix at the left upper corner. Since the $\mathbb{F}[t]$ -module V is cyclic, the characteristic polynomial of T coincides with its minimal polynomial, proportional to P . Then by Proposition 4.2 the matrix $A + A^\top$ is proportional to $B + B^\top$, where $B := \text{Std } P$. Hence $B = \mu A$ with $\mu \in \mathbb{F}^\times$. We also have $\Delta_B \stackrel{\text{FM}}{\sim} P \stackrel{\text{FM}}{\sim} \Delta_A$ (see (6) for the first equivalence), thus A is concordant to $B = \text{Std}(P)$ by Lemma 2.9. \square

4.4. The $(t-1)$ -primary component. By Theorem 4.3, determining whether $\beta(A) = 1$ or not is reduced to decide if A is concordant to a given standard matrix (associated with Δ_A). This can be done by comparing their isometric structures. By Proposition 3.6, two isometric structures are concordant if and only if all their primary components are concordant. The study of the $(t-1)$ -primary component could *a priori* be sufficient in some cases to prove that $\beta(A) > 1$, see Example 6.4. However (see the discussion after Example 6.4), the most interesting case would be when A is a Seifert matrix of a

two-component link. The following proposition shows in fact that the $(t - 1)$ -primary component does not give any restriction in this case.

Proposition 4.6. *Let $L = k_1 \cup k_2$ be a two-component link such that $\Delta_L(-1) \neq 1$. Let A_L be a Seifert matrix of L and $B_L = \text{Std}(\Delta_L)$ be the standard coindex-one matrix associated with $\Delta_L(t)$. Then the $(t - 1)$ -primary components of the isometric structures corresponding to A_L and B_L are concordant.*

The rest of the section is devoted to the proof of Proposition 4.6. We need two preliminary results.

Proposition 4.7. *Let A be a matrix such that $\Delta_A(-1) \neq 0$. If 1 is a simple root of $\Delta_A(t)$. Then the $(t - 1)$ -primary component of the isometric structure of A is concordant to (V, b, id_V) , where $\dim V = 1$ and the value of b on a generator of V is $-2\Delta'_A(1)$.*

Notice that if $k_1 \cup k_2$ is a two-component link with Seifert matrix A , then $\Delta'_A(1) = -\text{lk}(k_1, k_2)$ (cf. [Hos58]).

Proof. A is congruent to $(a) \oplus B$ for some matrix B . Hence $\Delta_A(t) = a(1 - t)\Delta_B(t)$ and $\Delta'_A(1) = -a\Delta_B(1) = -a\det(B - B^\top)$. It remains to note that the determinant of any skew-symmetric matrix is a square. \square

Lemma 4.8. *Let A be a matrix with $\Delta_A(-1) \neq 0$ and $\dim \ker(A - A^\top) = 1$. Then A is concordant to a matrix such that the $(t - 1)$ -primary component of its isometric structure is one-dimensional.*

Proof. By Lemma 3.5 any admissible matrix is S-equivalent (over \mathbb{F}) to a non-singular matrix. One easily shows that $\dim \ker(A - A^\top)$ is invariant under S-equivalence (see, however, Remark 4.9). Thus we may assume that A is non-singular.

Let (V, b, T) be the $(t - 1)$ -primary component of the corresponding isometric structure. According to Proposition 3.7, (V, b, T) is either metabolic or concordant to an isometric structure with minimal polynomial $t - 1$. This fact is proven in [Lev69a] by induction: if the minimal polynomial of T is δ^a with $a > 1$, then it is shown that (V, b, T) is concordant to (V', b', T') and $\dim V' < \dim V$, where $V' = W^\perp/W$, $W = \delta(T)^{a-1}V$, and b' and T' are induced by b and T .

In order to prove the lemma, it is enough to show that the condition $\dim \ker(T - I) = 1$ is preserved during the inductive process. Indeed, we have $A(T - I) = A^\top - A$, hence $\dim \ker(T - I) = \dim \ker(A - A^\top) = 1$. Therefore, the matrix of T in a suitable base v_1, \dots, v_m of V is a Jordan block, i.e., $T_{i,i} = T_{i,i+1} = 1$ for all i . The T -invariance of b yields a system of linear equations for the entries of the Gram matrix Q of b . By solving some of them, we obtain that Q is lower triangular with respect to the secondary diagonal (i.e. $Q_{ij} = 0$ when $i + j < m$) and the entries on that diagonal are $Q_{i,m-i} = (-1)^i c$ with $c \in \mathbb{F}$; we may set $c = 1$ because Q is non-singular. Since Q is symmetric, this fact implies, in particular, that m , and hence the dimension of A , is odd. Thus the subspaces W and W^\perp are spanned by v_1 and by (v_1, \dots, v_{m-1}) respectively. If $m > 1$, then $\dim V' = \dim V - 2$ and T' is again a Jordan block, thus $\dim \ker(T' - I) = 1$.

By induction we reduce the lemma to the case where the minimal polynomial of T is $t - 1$ and the dimension of $\ker(T - I)$ is still equal to 1. In this case the $(t - 1)$ -component coincides with $\ker(T - I)$ and hence it is one-dimensional. \square

Remark 4.9. In general the dimension of $\ker(A - A^\top)$ is not invariant under concordance. For example, for $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have $A - A^\top = 0$ whereas A is metabolic. Note that this matrix is realizable as a Seifert matrix of the 3-component Hopf link L where the orientation of one component is reversed, and we have $\chi_s(L) = 1$.

Proof of Proposition 4.6. In virtue of Proposition 4.7 and Lemma 4.8, it is enough to show that the kernels of $A_L - A_L^\top$ and $B_L - B_L^\top$ are one-dimensional.

Indeed, the module $\mathbb{F}[t]/\Delta_L(t)$ is cyclic and every primary component is cyclic. Then $\dim \ker(T - I) = 1$, where $T = B_L^{-1}B_L^\top$. Since $\ker(B_L - B_L^\top) = \ker(T - I)$, it follows that $\dim \ker(B_L - B_L^\top) = 1$.

Furthermore, $A_L - A_L^\top$ is the matrix of the intersection form on a Seifert surface of L . Its corank is 1 because L has two components. Thus $\dim \ker(A_L - A_L^\top) = 1$. \square

4.5. Livingston's theorem. Recall that the *discriminant* of a polynomial $P(t) = \sum a_k t^k \in \mathbb{F}[t]$ of degree n is $\text{Disc } P = a_n^{2n-2} \prod_{i < j} (\alpha_i - \alpha_j)^2$, where $\alpha_1, \dots, \alpha_n$ are the roots of P in an extension of \mathbb{F} . This is a polynomial in a_0, \dots, a_n with integral coefficients.

Theorem 4.10. [Liv10, Theorem 5.1], [LN, Theorem 6.4.2]³ *Let A be a non-singular matrix with entries in \mathbb{Z} , such that $\Delta_A(1)\Delta_A(-1) \neq 0$. Let $(Q, T) = (A + A^\top, A^{-1}A^\top)$ be the isometric structure of A . Suppose that all irreducible symmetric factors of $\Delta_T(t)$ have even exponent. Then for any prime p that does not divide $2 \det(A) \text{Disc } \tilde{\Delta}_A$, the image of (Q, T) in $\bar{\mathcal{G}}_{\mathbb{Q}_p}^*$ is zero, where $\tilde{\Delta}_A(t)$ denotes the product of all the distinct irreducible factors of $\Delta_A(t)$ in $\mathbb{Z}[t]$.*

This theorem is formulated in [Liv10, Theorem 5.1] for Seifert matrices of knots, however its proof goes in our more general setting. The corresponding analog of [Liv10, Lemma 5.2] (which is used in the proof of [Liv10, Theorem 5.1]) can be also derived from the following fact.

Lemma 4.11. *Let $P(t)$ be a palindromic polynomial of even degree with coefficients in a ring Λ . Then there exists $\lambda \in \Lambda$ such that $\text{Disc } P = \lambda P(1)P(-1)$.*

Proof. It is enough to assume that $\Lambda = \mathbb{Z}[x_0, \dots, x_n]$ and $P = \sum_{k=0}^{2n} x_k t^k$ with $x_k = x_{2n-k}$ for $k > n$. Then $\text{Disc } P$ and $P(\pm 1)$ are polynomials in $x = (x_0, \dots, x_n)$ that we denote by $D(x)$ and $P_{\pm 1}(x)$. It is clear that $P_{\pm 1}(x)$ is irreducible in Λ and in $\bar{\Lambda} = \mathbb{C}[x_0, \dots, x_n]$. Furthermore, $P_{\pm 1}(z) = 0$ implies $D(z) = 0$ for any $z \in \mathbb{C}^{n+1}$. Then by Hilbert's Nulstellensatz D is divisible by $P_{\pm 1}$ in $\bar{\Lambda}$ and hence in Λ . \square

Remark 4.12. By computations similar to those in the proof of [Liv10, Lemma 5.2], one can show that the coefficient λ in Lemma 4.11 is equal to $(-1)^n \text{Disc } S$, where S is the polynomial such that $P(t) = t^n S(t + t^{-1})$.

³We corrected misprints in [Liv10] in the definitions of Disc and $\tilde{\Delta}_A$ (denoted in [Liv10] by $\bar{\Delta}_A$).

Remark 4.13. When completing details in the proof of Lemma 4.11, one should take into account that setting $a_n = 0$ in the discriminant of $\sum_{k=0}^n a_k t^k$ (considered as a polynomial of degree n) does not give the discriminant of $\sum_{k=0}^{n-1} a_k t^k$ (considered as a polynomial of degree $n - 1$).

Remark 4.14. In the proof of [Liv10, Theorem 5.1] (p.1701, l.1; here Q instead of Q^δ is a misprint), the assertion that $\det Q^\delta$ is a square follows from [Liv10, Theorem 2.6(3)], which is [Lev69a, Lemma 7(c)]. A proof of [Liv10, Thm. C.1] can be found in [DF04, §14.6, Exercise 30].

4.6. An algorithm to decide if $\beta(A) = 1$. Let A be a square matrix of odd dimension, with entries in \mathbb{Q} , such that $\Delta_A(-1) \neq 0$. In particular A is admissible. We use Levine's theory combined with the characterization of Theorem 4.3 to provide an algorithm deciding whether $\beta(A) = 1$. Indeed, one has $\beta(A) = 1$ if and only if A is concordant (over \mathbb{Q}) to the standard coindex-one matrix $\text{Std}(\Delta_A)$ associated with Δ_A (or in fact to any standard coindex-one matrix associated with a polynomial FM-equivalent to Δ_A).

Let $M = A \oplus -\text{Std}(\Delta_A)$. The matrix of even dimension M is admissible and $\beta(A) = 1$ if and only if M is metabolic.

Following the proof of Lemma 3.5 due to Levine, we can transform M into a non-singular matrix, S-equivalent to M . After these transformations, we assume now that M is a non-singular matrix with $\Delta_M(-1) \neq 0$.

Let $(Q, T) = (M + M^\top, M^{-1}M^\top)$ be the isometric structure of M . Since $\Delta_M(-1) \neq 0$, the isometric structure (Q, T) is admissible and M is metabolic if and only if the class of (Q, T) in $\bar{\mathcal{G}}_{\mathbb{Q}}^*$ is zero (see Proposition 3.4).

By [Lev69b, Proposition 17], the class of (Q, T) is trivial in $\bar{\mathcal{G}}_{\mathbb{Q}}^*$ if and only if its images in $\bar{\mathcal{G}}_{\mathbb{R}}^*$ and $\bar{\mathcal{G}}_{\mathbb{Q}_p}^*$ for all primes p are trivial. By [Lev69b, Section 20], the signature function σ_M vanishes if and only if the image of the class of M in $\bar{\mathcal{G}}^{\mathbb{R}}$ is zero. See also [Section 2][Liv10]. Write

$$\Delta_T(t) = (t - 1)^{2\alpha_0} \delta_1^{2\alpha_1} \dots \delta_n^{2\alpha_n} \bar{F}(t)F(t),$$

where $\delta_1, \dots, \delta_n$ are distinct irreducible symmetric factors of Δ_A in $\mathbb{Q}[t]$, the exponents $\alpha_0, \dots, \alpha_n$ are integers and F is a polynomial in $\mathbb{Q}[t]$. Note that if some irreducible symmetric factor has odd exponent, then the structure (Q, T) is not metabolic. According to Section 3.2, the class of (Q, T) in $\bar{\mathcal{G}}_{\mathbb{Q}}^*$ decomposes as a sum of classes of (Q_{t-1}, T_{t-1}) and (Q', T') , with respective characteristic polynomials $(t - 1)^{2\alpha_0}$ and the product $\delta_1^{2\alpha_1} \dots \delta_n^{2\alpha_n}$ (the restriction to the $\bar{F}F$ -primary component being metabolic).

Let $M' = Q'(I + T')^{-1}$ be a matrix with entries in \mathbb{Q} . After multiplication by a factor in \mathbb{Q}^* , we may assume now that M' has entries in \mathbb{Z} . Denote $\tilde{\Delta}_{M'} = \delta_1 \dots \delta_n$. By Livingston's Theorem 4.10, the image of (Q', T') in $\bar{\mathcal{G}}_{\mathbb{Q}_p}^*$ is zero for all prime p that does not divide $2 \det(M') \text{Disc } \tilde{\Delta}_{M'}$. Then, in order to determine if M is metabolic, we have to determine the classes:

- (Q_{t-1}, T_{t-1}) in $\bar{\mathcal{G}}_{\mathbb{Q}}^{t-1}$. By Proposition 3.7, (Q_{t-1}, T_{t-1}) is metabolic or concordant to an isometric structure whose minimal polynomial is $t - 1$ (ie. $T' = I$).
- the image of (Q', T') in $\bar{\mathcal{G}}_{\mathbb{Q}_p}^*$ for all prime p dividing $2 \det(M') \text{Disc } \tilde{\Delta}_{M'}$. Let δ be an irreducible factor of $\Delta_{M'}$ in \mathbb{Q}_p . By [Lev69a, Proposition 16], the restriction of (Q', T') to the δ -primary component is trivial in $\bar{\mathcal{G}}_{\mathbb{Q}}^{\delta}$ if the restriction of Q' is metabolic. In order to determine this, one refers to Section 2.6 (over \mathbb{Q}_p).

5. LINK INVARIANTS AND SURFACES IN THE FOUR-BALL

5.1. The coindex of a link. A Seifert surface is a compact, connected, oriented surface S embedded in S^3 whose boundary is a given oriented link. Let $H_1(S)$ be the first homology group, with rational coefficients.

Definition 5.1. Let L be an oriented link and S be a Seifert surface of L , with Seifert matrix A in some basis of $H_1(S)$. The coindex

$$\beta(L) := \beta(A)$$

depends only on the isotopy class of L and does not depend on the choice of S .

The fact that $\beta(L)$ is well-defined follows from the invariance by S-equivalence (Proposition 2.14). The rest of this subsection is devoted to a proof of Theorem 1.1 about the invariance of $\beta(L)$ by (topological) concordance under the assumption $\Delta_L \neq 0$. We start with the following two elementary lemmas.

Lemma 5.2. *Let E, F , and G be \mathbb{Q} -vector spaces of finite dimension. Consider linear maps $f : E \rightarrow F$ and $g : F \rightarrow G$. Then, the following holds*

$$\dim \ker(g \circ f) \geq \dim E - \dim \text{im } g.$$

Proof. Indeed, $\dim \ker(g \circ f) = \dim E - \dim \text{im}(g \circ f) \geq \dim E - \dim \text{im } g$. \square

Lemma 5.3. *Let Σ be a compact connected oriented surface and M be a compact connected three-manifold with $\partial M = \Sigma$. Let $i_* : H_1(\Sigma) \rightarrow H_1(M)$ be the map induced by the inclusion in homology with rational coefficients. Then $\dim \ker i_* = \dim \text{im } i_* = \frac{1}{2} \dim H_1(\Sigma)$.*

Proof. Consider the following commutative diagram, whose rows come from the long exact sequence in rational (co)homology of the pair (M, Σ) and the vertical arrows follow from Poincaré duality.

$$\begin{array}{ccccc} H_2(M, \Sigma) & \xrightarrow{\partial} & H_1(\Sigma) & \xrightarrow{i_*} & H_1(M) \\ \downarrow & & \downarrow & & \\ H_1(M)^* & \xrightarrow{i^*} & H_1(\Sigma)^* & & \end{array}$$

Hence $\dim \ker i_* = \text{rank } \partial = \text{rank } i^* = \text{rank } i_*$ and the result follows from the rank theorem applied to i_* . \square

Proposition 5.4. [Kaw78] *If a link L is concordant to L' and $\Delta_L \neq 0$, then $\Delta_{L'} \neq 0$.*

In [Kaw78] this result is proven for PL-concordance, but the proof easily extends to the locally flat case. Otherwise, this fact follows from [NP17, Theorem 4.2].

Proof of Theorem 1.1. Let $C \subset S^3 \times I$ be the (locally flat) concordance with boundary $L \subset S^3 \times \{0\}$ and $L' \subset S^3 \times \{1\}$. Consider Seifert surfaces S and S' for L and L' , and let θ and θ' be the corresponding Seifert forms. One can show that there exists a compact connected oriented 3-submanifold M in $S^3 \times I$, with boundary

$$\partial M = S \cup_L C \cup_{L'} S'.$$

The proof of this fact uses the same arguments as Lemma 5.6. Let $h = \text{genus}(\partial M)$ and let K be the kernel of the map $i_* \circ j_*$ induced by the inclusions, with rational coefficients:

$$H_1(S \sqcup S') \simeq H_1(S) \oplus H_1(S') \xrightarrow{j_*} H_1(\partial M) \xrightarrow{i_*} H_1(M).$$

By computing the Euler characteristics, we obtain $\dim H_1(S \sqcup S') = \dim H_1(\partial M) = 2h$. By Lemma 5.3 we also have $\dim \text{im } i_* = \frac{1}{2} \dim H_1(\partial M) = h$. Hence, by Lemma 5.2,

$$\dim K \geq \dim H_1(S \sqcup S') - \dim \text{im } i_* = 2h - h = h.$$

Consider the bilinear form $\Theta = \theta \oplus (-\theta')$ on $H_1(S) \oplus H_1(S')$ and let $x \oplus x'$ and $y \oplus y'$ be two elements of K . Let α, β be 2-chains in M such that $\partial\alpha = x \oplus x'$ and $\partial\beta = y \oplus y'$, and β_+ denote the push of β from M in the positive normal direction in $S^3 \times I$. Then Θ is computed by the intersections in $S^3 \times I$:

$$\Theta(x \oplus x', y \oplus y') = \theta(x, y) - \theta'(x', y') = \alpha \cdot \beta_+.$$

Since $\alpha \subset M$ and $\beta_+ \subset (S^3 \times I) \setminus M$, these chains are disjoint and $\Theta(x \oplus x', y \oplus y') = 0$. Hence K is totally isotropic and $\dim K \geq h$, which means that Θ is metabolic. Thus θ' is concordant to θ . Since Δ_L and $\Delta_{L'}$ are non-zero (see Proposition 5.4), the Seifert matrices are admissible and the result follows from Proposition 2.11. \square

5.2. Bounding the slice Euler characteristic via the coindex. The following theorem is equivalent to Theorem 1.2.

Theorem 5.5. *If a link L bounds a locally flat embedded surface F in B^4 without closed components, then the following inequality holds:*

$$\chi(F) \leq 1 - \beta(L).$$

We need the next lemma.

Lemma 5.6. *If L bounds a locally flat surface F in B^4 and a Seifert surface S in S^3 , then there is a compact oriented 3-manifold M in B^4 with $\partial M = F \cup_L S$.*

The proof of Lemma 5.6 is folklore; the detailed proof (for knots) can be found in [FL18, Section 1.5]. However the same arguments work for links and follow essentially from topological transversality [FQ90, Chapter 9].

Proof of Theorem 5.5. By Lemma 5.6, the closed surface $F \cup_L S$ bounds a 3-manifold M in B^4 . Let $i : F \cup_L S \hookrightarrow M$ and $j : S \hookrightarrow F \cup_L S$ be the inclusions. Consider the composition

$$H_1(S) \xrightarrow{j_*} H_1(F \cup S) \xrightarrow{i_*} H_1(M).$$

The Seifert form θ on $H_1(S)$ vanishes on $\ker(i_* \circ j_*)$. Indeed, suppose that x and y are two elements of $\ker(i_* \circ j_*)$. There exist 2-chains α and β in M with $\partial\alpha = x$ and $\partial\beta = y$. Then, as in the proof of Theorem 1.1, we push β off from M and we obtain $\theta(x, y) = 0$. Thus $\text{ind } A \geq \dim \ker(i_* \circ j_*)$ whence, by Lemma 5.2,

$$\text{ind } A \geq \dim H_1(S) - \dim \text{im } i_*.$$

By Lemma 5.3 we also have $2 \dim \text{im } i_* = \dim H_1(F \cup S) = 2 - \chi(F \cup S)$, thus

$$\begin{aligned} \beta(A) &= \dim H_1(S) - 2 \text{ind } A \\ &\leq 2 \dim \text{im } i_* - \dim H_1(S) = (2 - \chi(F \cup S)) - (1 - \chi(S)) = 1 - \chi(F). \end{aligned}$$

□

5.3. Alexander polynomial and signatures. Let L be an oriented link. The next result was due to Fox and Milnor in the case of knots, and extended to (colored) links by the first author. Note that since we work with rational Seifert forms, the statement is slightly weaker than the original ones.

Corollary 5.7. [FM66, Flo04] *If L bounds a locally flat surface in B^4 with Euler characteristic 1, then there exists $F \in \mathbb{Z}[t]$ such that $\Delta_L(t) = F(t)\bar{F}(t)$ (here Δ_L and \bar{F} are as defined in Sections 2.1 and 2.3 respectively).*

Proof. Combine Theorem 5.5 and Proposition 2.7. □

Let A be a Seifert matrix of L . Given $\omega \in S^1 \subset \mathbb{C}$, the *Levine-Tristram signature and nullity* of L are defined as the signature and nullity of the hermitian matrix $(1 - \omega)A + (1 - \bar{\omega})A^\top$. They give rise to functions $\sigma_L, \eta_L : S^1 \rightarrow \mathbb{Z}$.

Corollary 5.8. [Mur65b, Tri69] *If L bounds a locally flat surface F in B^4 without closed components, then for all $\omega \in S^1 \setminus 1$, the following inequality holds*

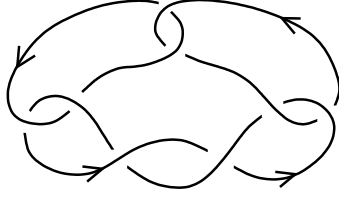
$$|\sigma_L(\omega)| - \eta_L(\omega) \leq 1 - \chi(F).$$

Note that Corollary 5.8 was first proven for smooth surfaces. A proof for locally flat surfaces can be found in [Pow17].

Proof. Let A be a Seifert matrix of L . Let i_- and i_+ be the number of positive and negative eigenvalues of $A(\omega) = (1 - \omega)A + (1 - \bar{\omega})A^\top$. Without loss of generality, suppose that $i_- \leq i_+$. By definition, $\sigma_L(\omega) = i_+ - i_-$ and $i_- + i_+ + \eta(\omega) = \dim V$, and $\beta(L) = \dim V - 2 \text{ind}(V, \theta)$. Since $\text{ind}(V, \theta) \leq i_- + \eta(\omega)$, one gets

$$\beta(L) = i_+ + i_- + \eta(\omega) - 2 \text{ind}(V, \theta) \geq i_+ + i_- + \eta(\omega) - 2i_- - 2\eta(\omega) = |\sigma_L(\omega)| - \eta(\omega).$$

By Theorem 1.2, $|\sigma_L(\omega)| - \eta(\omega) \leq \beta(L) \leq 1 - \chi(F)$. □

FIGURE 1. The link $L_{15}^{8a}(0,0)$.

6. SOME EXAMPLES

6.1. **Levels of lower bounds for χ_s .** Let L be a link with $\Delta_L \neq 0$. Note that L can be a knot. Let A be a Seifert matrix of L . The coindex of L is denoted $\beta(L) = \beta(A)$, see Definition 5.1. By Theorem 1.2, we have

$$\chi_s(L) \geq 1 - \beta(L).$$

It is natural to distinguish the following levels of lower bounds of $\beta(A)$ (on each level we discuss only the bounds which are stronger than those on lower levels).

- (Level 1) The Murasugi-Tristram inequality and the Fox-Milnor Theorem, see Corollaries 5.7 and 5.8.
- (Level 2) $\beta(A) \geq \beta(A + A^\top)$ (this can be used to prove that $\beta(A) \geq c$ for some constant c , only in the case $c \leq 4$; see 2.6).

Let α_L be the concordance class of the isometric structure of A . The next two levels are applicable only for proving the estimate $\beta(A) \geq c$ for $c = 2$ or $c = 3$.

- (Level 3) Application of Levine's theorem [Lev69a] (for $c = 2$), or Theorem 1.2 (for $c = 3$) using the splitting of α_L according to the factorization of Δ_L over \mathbb{Q} .
- (Level 4) The same as Level 3 but for a splitting of α_L according to the factorization of Δ_L over \mathbb{Q}_p for some prime p .

Example 6.1. A computation of $\chi_s(L)$ using a Level 2 lower bound for $\beta(A)$. Let L be the link $L_{15}^{8a}(0,0)$ according to the notation of LinkInfo [LM]. See Figure 1.

The Seifert surface constructed by the standard algorithm via the Seifert circles of the link diagram in Figure 1 has Euler characteristic -3 , thus $\chi_s(L) \geq -3$. The corresponding Seifert matrix (in some basis) is

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The Alexander polynomial is $-(t-1)^2(t^2-5t+1)$. All its roots are real and the signature of $A + A^\top$ is 2. Hence there is no level 1 proof that $\chi_s(L) \leq -3$.

Let us show that $\beta(A + A^\top) = 4$. The diagonalization of $A + A^\top$ is $[2, 2, 2, -7/2]$. The Hilbert symbol over \mathbb{Q}_2 of any two diagonal entries is 1, hence the corresponding Hasse

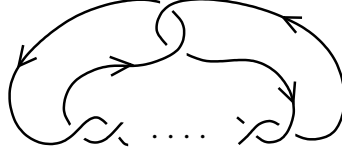
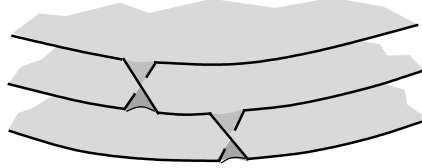
FIGURE 2. The 21/2 rational knot 12_{803}^a .

FIGURE 3. Seifert surface of the (3, 1)-cable.

invariant ε and the discriminant d (see §2.6) are 1 and -28 respectively. We also have $(-1, -1) = -1 \neq \varepsilon$, and -28 is a square in \mathbb{Q}_2^\times (see [Ser70, Ch. II, Thm. 4]). Hence the form is anisotropic (see [Ser70, Ch. IV, Thm. 6]). We obtain $\beta(A + A^\top) = 4$ and then $\chi_s(L) \leq -3$. This implies $\chi_s(L) = -3$. We also deduce that $g_s(L) = 1$.

Example 6.2. *A bound $\beta(L) > 0$ of Level 3 for a knot L .* The idea of this example was suggested by Charles Livingston. Let $K = 12_{803}^a$ be the rational knot 21/2 in Figure 2. Let J be the (3, 1)-cable along K .

The standard construction via Seifert circles according to the knot diagram yields a genus 1 Seifert surface S_K of K . By taking three parallel copies of it and connecting them by two half-twisted bands (see Figure 3), we obtain a Seifert surface S_J of J .

The corresponding Seifert forms are

$$\theta_K = \theta = \begin{pmatrix} 1 & 1 \\ 0 & -5 \end{pmatrix}, \quad \theta_J = \begin{pmatrix} \theta & \theta & \theta \\ \theta^\top & \theta & \theta \\ \theta^\top & \theta^\top & \theta \end{pmatrix}.$$

(for the base of $H_1(S_J)$ we choose three parallel copies of the base of $H_1(S_K)$). The Alexander polynomials are

$$\Delta_K = 5t^2 - 11t + 5, \quad \Delta_J = 5t^6 - 11t^3 + 5.$$

As shown in [Lev69a, Corollary 23(c)], θ_K represents an element of order 4 in $\mathcal{G}_\mathbb{Q}$. Let $L = K \# K \# J \# J$. The symmetrized forms $\theta_K + \theta_K^\top$ and $\theta_J + \theta_J^\top$ are concordant over \mathbb{Q} . Indeed,

$$P^\top(\theta_J + \theta_J^\top)P = (\theta_K + \theta_K^\top) \oplus \begin{pmatrix} 0 & \theta - \theta^\top \\ \theta^\top - \theta & 0 \end{pmatrix}, \quad P = \begin{pmatrix} I & -I & -I \\ -I & I & 0 \\ I & 0 & I \end{pmatrix}.$$

Hence the symmetrized Seifert form of L is metabolic, i.e., there is no level-2 proof that L is not slice. However, the (Δ_K) -primary component of θ_L is non-trivial in $\mathcal{G}_{\mathbb{Q}}$ (because the order of θ_K in $\mathcal{G}_{\mathbb{Q}}$ is 4), hence L is not algebraically slice by [Lev69a].

Example 6.3. *A bound $\beta(L) > 1$ of Level 3.* Let $L_1 = L \# l$ where L is as in Example 6.2 and l is any link such that $\chi_s(l) = 0$ and Δ_l is coprime with Δ_L (for example, the two-component Hopf link). Then $\chi_s(L_1) < 0$ by Theorem 4.3 because the (Δ_K) - and (Δ_J) -primary components are non-trivial.

Example 6.4. *A Level 3 bound $\beta(L) > 1$ based on the $(t-1)$ -primary component.* Let $L = K \# l_{-1} \# l_3 \# l_5$ where K is the figure-eight knot and l_k is the $(2, 2k)$ -torus link oriented so that its Seifert matrix is the (1×1) -matrix (k) (it does not matter which pairs of components are joint by the connected sums). A Seifert matrix of L is $A = \text{diag}(-1, 3, 5) \oplus \begin{pmatrix} 1 & \\ 0 & -1 \end{pmatrix}$. All the Levine-Tristram signatures are ± 1 and the form $A + A^{\top}$ vanishes on the subspace spanned by $(1, 0, 0, 1, 0)^{\top}$ and $(1, 0, 1, 0, 2)^{\top}$. So, there is no level-2 proof that $\chi_s(L) < 0$. However, the $(t-1)$ -primary component of A is $\text{diag}(-1, 3, 5)$, and this is an anisotropic form. Thus $\beta(L) > 1$ by Theorem 4.3, and hence $\chi_s(L) < 0$.

Note that the bound $\chi_s(L) < 0$ in Example 6.4 can be obtained in a much easier way. Indeed, L has four components, hence any surface F in B^4 with $\partial F = L$ and $\chi(F) = 0$ has at least two components, whereas L is not a disjoint union of two sublinks with zero linking number between them.

In general, for links with more than two components, Theorem 4.3 hardly can give bounds better than those coming from the consideration of the sublinks bounding components of F , in particular, using the techniques from [Flo04], [CF08].

Example 6.5. *A bound $\beta(L) > 1$ of Level 4.* Let L be the Montesinos two-component link $(\frac{1}{18}; \frac{11}{4}; -\frac{1}{2})$ oriented as in Figure 4. On the right hand side in Figure 4 we show a ribbon surface for L with Euler characteristic -2 , thus $\chi_s(L) \geq -2$. We are going to show that $\chi_s(L) < 0$ and thus $\chi_s(L) = -2$ (note that the bound $\chi_s(L) < 0$ also follows from Theorem 6.7 below because $\text{Arf}(L) = 1$).

Consider the Seifert surface S shown in Figure 4. Choose the basis of $H_1(S)$ formed by the loops projected to the boundaries of the regions marked by $1, \dots, 5$ in Figure 4. The Seifert matrix A and the corresponding monodromy $T = A^{-1}A^{\top}$ in this basis are

$$A = \begin{pmatrix} 8 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -8 & 0 & 0 \\ 0 & 1 & -8 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

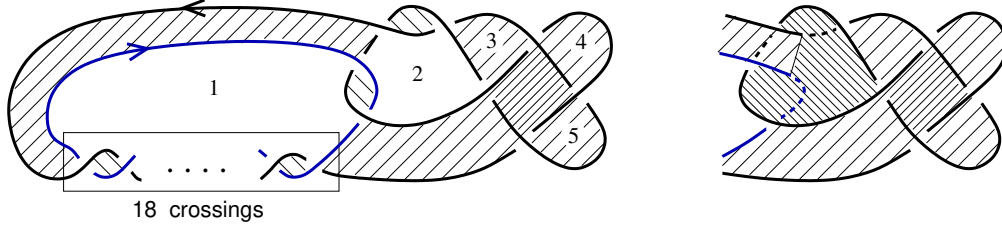


FIGURE 4. Montesinos link $(\frac{1}{18}; \frac{11}{4}; -\frac{1}{2})$; its Seifert and ribbon surfaces.

The form $Q = A + A^\top$ vanishes on the subspace spanned by $(0, 1, 1, 0, 0)^\top$ and $(1, 0, 2, 0, 2)^\top$. Hence $\beta(Q) = 1$ and there is no level-2 proof that $\chi_s(L) < 0$.

We have the following factorization over \mathbb{Q} :

$$\Delta_A = \Delta_T = 1 + 5t - 12t^2 + 12t^3 - 5t^4 - t^5 = (1 - t)(1 + 6t - 6t^2 + 6t^3 + t^4).$$

Denote the irreducible factor of degree 4 by δ . Let A_0 be the standard coindex-one matrix for Δ_A , and let $Q_0 = A_0 + A_0^\top$ and $T_0 = A_0^{-1}A_0^\top$ (the companion matrix of Δ_A).

The $(T - I)$ -invariant vector is $(1, 0, 0, 0, 0)^\top$, hence the $\delta(T)$ -invariant subspace is generated by the last four columns of the matrix $T - I$. Similarly we find the primary components for T_0 (see §4.4 for the $(t - 1)$ -component). The δ -components of Q and Q_0 are:

$$Q_\delta = \begin{pmatrix} -2 & 9 & -1 & -1 \\ 9 & -18 & 8 & 2 \\ -1 & 8 & -2 & 0 \\ -1 & 2 & 0 & -2 \end{pmatrix}, \quad Q_{0,\delta} = \begin{pmatrix} 0 & -1 & 6 & -41 \\ -1 & 0 & -1 & 6 \\ 6 & -1 & 0 & -1 \\ -41 & 6 & -1 & 0 \end{pmatrix}.$$

We have $\det Q_\delta = \det Q_{0,\delta} = -128$ and the Hasse invariant of each matrix is -1 in \mathbb{Q}_2 and 1 in \mathbb{Q}_p for $p \neq 2$. Thus there is no Level 3 bound for $\chi_s(L)$.

The factorization of δ over \mathbb{Q}_2 is $\delta = \delta_1 \delta_2$ where

$$\delta_1 = t^2 - 20t + 1 \pmod{2^8}, \quad \delta_2 = t^2 + 26t + 1 \pmod{2^8}.$$

The δ_1 -primary components of Q and Q_0 are congruent to the diagonal matrices

$$Q_{\delta_1} \cong \text{diag}(26, 18) \pmod{32}, \quad Q_{0,\delta_1} \cong \text{diag}(36, 20) \pmod{64}.$$

Their Hasse invariants are respectively -1 and 1 , whence $\beta(A) = 1$ and $\chi_s(L) = -2$. The following is the `pari-gp` code used for these computations:

```

A0=[0,0,1,0,0; 0,0,5,1,0; 0,0,-12,5,1; 1,5,0,0,0; 0,1,0,0,0]^(-1);
A=[8,1,0,0,0; 1,0,0,0,0; 0,1,-1,1,0; 0,0,0,-1,1; 0,0,0,0,-1];
Q=A+A~; T=1/A*A~; Q0=A0+A0~; T0=1/A0*A0~; /* define A,A0,T,T0,Q,Q0 */
de = t^4 + 6*t^3 - 6*t^2 + 6*t + 1 ; /* define delta(t) */
fp=factorpadic(de,2,8); /* factorize delta in Q_2[t] up to 0(2^8) */
de1=fp[1,1]; de2=fp[2,1]; /* define delta1, delta2 */
V=matker(subst(de1,t,T)); /* compute a 5x2-matrix whose */
/* columns span ker(delta1(T)) */
Q1=V~*Q*V; /* compute the restriction of Q to ker(delta1(T)) */
D=qfgaussred(Q1); /* diagonalize it */
hilbert(D[1,1],D[2,2],2) /* compute the Hasse invariant */
V=matker(subst(de1,t,T0)); /* do all these for Q0, ker(delta1(T0)) */
D=qfgaussred(V~*Q0*V); hilbert(D[1,1],D[2,2],2)

```

Example 6.6. *A Level 4 lower bound $\beta(L) > 1$ for links with zero Arf invariant. Let*

$$A_1 = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 10 & -1 \\ -2 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & -3 & 1 & -1 \\ -3 & 0 & 0 & -3 & 1 \\ 1 & -3 & 0 & 0 & -3 \\ -1 & 1 & -3 & 0 & 0 \\ -1 & -1 & 1 & -3 & 0 \end{pmatrix}.$$

One easily checks that $U_j^\top(A_j - A_j^\top)U_j = (0) \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for some integral matrices U_1, U_2 . Hence, by [Kaw96, Theorem 5.1.4], A_j is a Seifert matrix of some 2-component link L_j . These links are rather big, for example, the linking number between the components is 38 for A_1 and -110 for A_2 . We have

$$\Delta_{L_1} = 2(1 + 7t - 7t^2 + 7t^3 - 7t^4 - t^5), \quad \Delta_{L_2} = 10(1 - 5t - t^2 + t^3 + 5t^4 - t^5).$$

The pairs of vectors $(e_1, -2e_3 + e_4 + 2e_5)$ and (e_2, e_5) generate totally isotropic subspaces for $A_1 + A_1^\top$ and $A_2 + A_2^\top$, thus there is no level-2 bound for $\chi_s(L_j)$. There is no level-3 bound neither. However, there are level-4 bounds $\chi_s(L_j) < 0$ coming from the factorizations of Δ_{L_j} over \mathbb{Q}_2 for L_1 and over \mathbb{Q}_5 for L_2 . The computations are the same as in Example 6.5.

In contrary to Example 6.5, the Arf-Robertello invariant (see §6.2) vanishes for L_1 and L_2 , and we do not know any proof of the bound $\chi_s(L_j) < 0$ without Theorem 4.3.

6.2. Arf-Robertello invariant: $g_s(L)$ vs. $\chi_s(L)$. The *Arf invariant* $\text{Arf}(K)$ of a knot K can be defined as the Arf invariant (see, e.g. [MH73]) of the quadratic form $q : H_1(S; \mathbb{F}_2) \rightarrow \mathbb{F}_2$, $q(x) = \theta(x, x)$, where θ is the reduction mod 2 of a Seifert form on a Seifert surface S of K (see [Rob65]). By [Mur65a], $\text{Arf}(K) = 0$ iff $\Delta_K(-1) \equiv \pm 1 \pmod{8}$.

Following Robertello [Rob65], we say that a knot K is *related* to an oriented link L if there exists a locally flat surface F of genus 0 in $S^3 \times I$ such that $\partial F \cap (S^3 \times 0) = L$ and $\partial F \cap (S^3 \times 1) = K$. A link $L = k_1 \cup \dots \cup k_\mu$ (k_i are knots) is called in [Rob65] *proper* if all the linking numbers $\text{lk}(k_i, L \setminus k_i)$ are even.

Let L be a proper link. By [Rob65, Theorem 2], the Arf invariant of any knot related to a proper link L is an invariant of L . We call it the *Arf-Robertello invariant* of L and denote by $\text{Arf}(L)$. Note that some interesting interpretations of the Arf-Robertello invariant are found in [Mou] for algebraically split links. If $g_s(L) = 0$, then the unknot is related to L , which gives the following lower bound for the slice genus.

Theorem 6.7. (Essentially [Rob65]) *If L is a proper link and $g_s(L) = 0$, then $\text{Arf}(L) = 0$.*

Given a diagram of a proper link L with μ components, one can compute $\text{Arf}(L)$ using any knot obtained from L by resolving $\mu - 1$ crossings. $\text{Arf}(L)$ can be also computed from a Seifert form of L .

Proposition 6.8. *Let S be a Seifert surface of a proper link L , and θ be the Seifert form on $V = H_1(S; \mathbb{F}_2)$. Let $q(x) = \theta(x, x)$ be the corresponding quadratic form on V . Let W be any complementary subspace for $\ker \theta$, i.e., $V = W \oplus \ker \theta$. Then $\text{Arf}(L)$ is the Arf invariant of $q|_W$.*

As we pointed out in the introduction, for an n -component link L , the bound $2g_s(L) \geq 2 - n - \chi_s(L)$ is not always sharp. A strict inequality takes place when the maximal Euler characteristic cannot be attained on a connected surface in B^4 .

The simplest example is the Whitehead link $L = L_1^{5a}$ (the braid closure of $\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1} \sigma_1$). By switching the central crossing σ_1 , one obtains a trivial 2-component link. Hence L bounds a disjoint union of a disk and a genus 1 surface. Thus $\chi_s(L) = 0$ and $g_s(L) \leq 1$. On the other hand, resolving one of the crossings σ_2^{-1} , one obtains a trefoil. Hence $\text{Arf}(L) = 1$ and $g_s(L) \geq 1$. So, we obtain $2g_s(L) = 2 > 0 = 2 - n - \chi_s(L)$.

6.3. What is a good definition of algebraically slice Euler characteristic. As shown in [Tay79, Theorem 3], if A is a Seifert matrix of a knot K , then there exists a knot K' whose ribbon genus (and hence the slice genus as well) is $\frac{1}{2}\beta(A)$. So, it is natural to call $\frac{1}{2}\beta(A)$ the *algebraic slice genus* of A and of K .

The corresponding statement for links is false as one can see in the following example.

Example 6.9. Let L be the connected sum of the trefoil and the torus link $T(2, -8)$ oriented so that the Seifert matrix of L is $A = (-4) \oplus \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We have $x^\top A x = 0$ for $x = (1, 2, 0)^\top$, hence $\beta(A) = 1$. Let us show that there does not exist any link L' such that A is a Seifert matrix of L' and $\chi_s(L') = 0$. Indeed, suppose that such a link L' does exist. The kernel of $A - A^\top$ is generated by $x_0 = (1, 0, 0)^\top$, hence $L' = k_1 \cup k_2$ is a 2-component link, $\pm x_0$ is the homology class of k_1 , and $\text{lk}(k_1, k_2) = -4$, in particular, L' is a proper link (see §6.2). By Proposition 6.8 we have $\text{Arf}(L') = \text{Arf}(K) = 1$, hence $g_s(L') > 0$ by Theorem 6.7. Since $\text{lk}(k_1, k_2) \neq 0$, any surface in B^4 bounding L' is connected, whence $\chi_s(L') < 0$.

A candidate for a good notion of algebraic slice Euler characteristic of a link with Seifert matrix A is $1 - \beta_{\text{sat}}(A)$ where $\beta_{\text{sat}}(A) = n - 2k$, n is the size of A , and k is the maximal rank of a totally isotropic subgroup U of \mathbb{Z}^n such that $U + \ker(A - A^\top)$ is a saturated subgroup of \mathbb{Z}^n (a subgroup L of \mathbb{Z}^n is called *saturated* if \mathbb{Z}^n/L is torsion-free).

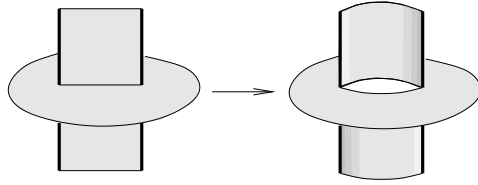


FIGURE 5. Kauffman's trick; cf. [Kau87, pp. 212–213].

We do not know whether β_{sat} is invariant even under S-equivalence. We neither know any upper bound for $\beta_{\text{sat}}(A)$ better than $\beta(A)$. However, the following two observations (Propositions 6.10 and 6.11) give a hope that β_{sat} could play some rôle. We define the *ribbon Euler characteristic* $\chi_r(L)$ of a link L as the maximum of $\chi(F)$ over all ribbon immersions of surfaces without closed components $r : F \rightarrow S^3$ such that $L = r(\partial F)$. It is well-known that $\chi_s(L) \geq \chi_r(L)$

Proposition 6.10. *Any link L has a Seifert matrix A such that $\chi_r(L) \leq 1 - \beta_{\text{sat}}(A)$.*

Proof. Let $r : F \rightarrow S^3$, $r(\partial F) = L$, be a ribbon immersion with $\chi(F) = \chi_r(L)$. Without loss of generality we may assume that $r(F)$ is connected. Using Kauffman's trick (see Figure 5), we transform F into a Seifert surface S . Denote the Seifert form on $H_1(S)$ by θ . Let x_1, \dots, x_k be the simple closed loops corresponding to the boundaries the disks depicted in Figure 5. They generate a totally isotropic subgroup U of $H_1(S)$ of rank k . It is clear that $\chi(S) = \chi(F) - 2k$. The kernel of $\theta - \theta^\top$ is generated by the components of ∂L . Thus $U + \ker(\theta - \theta^\top)$ is generated by pairwise disjoint simple closed curves on S , hence it is a saturated subgroup of $H_1(S)$ whence the result. \square

Proposition 6.11. *Let A be a Seifert matrix of a link L . Then there exists a link L' such that $\chi_r(L') \geq 1 - \beta_{\text{sat}}(A)$ and A is a Seifert matrix of L' .*

Proof. The construction of a ribbon surface is more or less the same as in the proof of [Tay79, Theorem 3]. Let (V, θ) , $V = \mathbb{Z}^n$, be the bilinear form defined by A and let $\iota = \theta - \theta^\top$ (the intersection form on the Seifert surface of L). Let $K = \ker \iota$ and let U be a totally isotropic subgroup of V of the appropriate rank such that $U + K$ is saturated. Since K is generated by the boundary components of a Seifert surface of L , it is saturated. It is clear that U is saturated as well. Then we have $V = V_1 \oplus \dots \oplus V_4$ where $V_1 \oplus V_2 = K$ and $V_2 \oplus V_3 = U$. Since the restriction of ι to $V_3 \oplus V_4$ is unimodular, V_4 splits into $V'_4 \oplus V''_4 \oplus V'''_4$ so that, for some bases (x_1, \dots, x_p) , (x'_1, \dots, x'_p) , (x''_1, \dots, x''_q) , (x'''_1, \dots, x'''_q) of, respectively, V_3 , V'_4 , V''_4 , V'''_4 , we have $\iota(x_i, x'_j) = \delta_{ij}$, $\iota(x''_i, x'''_j) = \delta_{ij}$, and $V_3 \oplus V'_4$ is ι -orthogonal to $V''_4 \oplus V'''_4$. Let (u_1, \dots, u_r) and (v_1, \dots, v_s) be some bases of V_1 and V_2 respectively.

By the classical Seifert's method [Sei36] (see also [Kaw96, Theorem 5.1.4]), we construct L' as the boundary of an embedded surface S shown in Figure 6 where the dashed parts of the bands are supposed to be twisted and linked with each other and with the solid

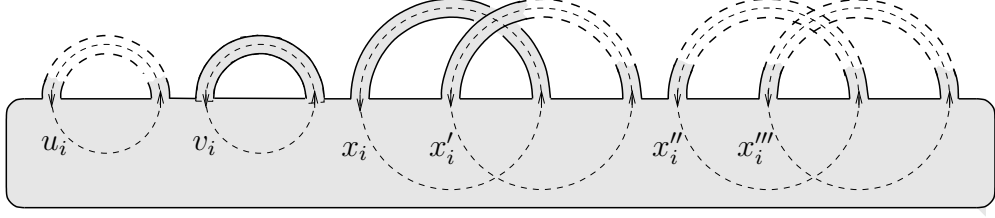


FIGURE 6. Seifert surface in the proof of Proposition 6.11.

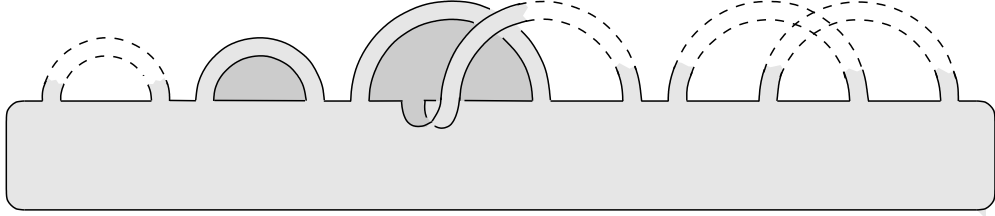


FIGURE 7. Ribbon surface in the proof of Proposition 6.11.

bands so that they realize the given Seifert form. Then, near the bands corresponding to the cycles v_i and x_i , we modify S as shown in Figure 7. After this modification, each band that was linked with v_i and/or x_i gets ribbon intersections with the modified piece of the surface. \square

6.4. On Milnor's remark. The results of Levine [Lev69a] are based, in particular, on Milnor's theorem [Mil69, Thm. 2.1] which states that an isometric structure (V, b, T) over a field \mathbb{F} is determined up to isomorphism by (V, b) and the minimal polynomial for T under the condition that \mathbb{F} is \mathbb{R} or \mathbb{Q}_p , and the minimal polynomial is irreducible.

It is said in [Mil69, p. 87]: “*Note that the corresponding statement for an arbitrary field would definitely be false (e.g. for the field of rational numbers)*”. The following example (proposed by Andrey Levin) illustrates that this statement is indeed false for $\mathbb{F} = \mathbb{Q}$.

Example 6.12. Let $\rho = \exp(2\pi i/5)$, $V = \mathbb{Q}(\rho)$, and let $T : V \rightarrow V$ be given by $x \mapsto \rho x$. For $k \in \{1, 2\}$, we set $a_k = \rho^k + \rho^{-k} = 2 \cos(2\pi k/5)$ and we define the symmetric bilinear form $b_k : V \times V \rightarrow \mathbb{Q}$ as $b_k(x, y) = \text{trace}(a_k x \bar{y})$. Then T preserves b_k , the bilinear forms b_1 and b_2 are congruent, and the minimal polynomial of T is $\Phi_5(t) = t^4 + t^3 + t^2 + t + 1$ which is irreducible over \mathbb{Q} .

Let us show that the isometric structures (V, b_1, T) and (V, b_2, T) are not isomorphic. We have $\Phi_5(t) = \delta_1(t)\delta_2(t)$ where $\delta_k(t) = t^2 - a_k t + 1$. Let $V_k = \ker \delta_k(T_{\mathbb{R}})$ where $T_{\mathbb{R}}$ is the \mathbb{R} -linear extension of T to $V \otimes \mathbb{R}$. Then $V \otimes \mathbb{R} = V_1 \oplus V_2$ and this decomposition is orthogonal for each of the forms b_1 and b_2 (we keep the notation b_k for the extension of b_k to $V \otimes \mathbb{R}$). A straightforward computation shows that the forms $b_1|_{V_1}$ and $b_2|_{V_2}$ are positive definite whereas the forms $b_1|_{V_2}$ and $b_2|_{V_1}$ are negative definite. Hence the discussed isometric structures are not isomorphic.

The matrices of T , b_1 , and b_2 in the basis $(1, \rho, \rho^2, \rho^3)$ are, respectively,

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} -2 & 3 & -2 & -2 \\ 3 & -2 & 3 & -2 \\ -2 & 3 & -2 & 3 \\ -2 & -2 & 3 & -2 \end{pmatrix}, \quad \begin{pmatrix} -2 & -2 & 3 & 3 \\ -2 & -2 & -2 & 3 \\ 3 & -2 & -2 & -2 \\ 3 & 3 & -2 & -2 \end{pmatrix}.$$

The subspace V_k , $k = 1, 2$, is spanned by the vectors $(1, 1, 0, -a_k)$ and $(1, a_k^{-1}, 1, 0)$. The matrices of $b_1|_{V_k}$ and $b_2|_{V_k}$ in these bases are, respectively,

$$5a_k \begin{pmatrix} 2 & 1 \\ 1 & 2a_k^2 \end{pmatrix} \quad \text{and} \quad -5a_k^{-1} \begin{pmatrix} 2 & 1 \\ 1 & 2a_k^2 \end{pmatrix}.$$

REFERENCES

- [CF08] David Cimasoni and Vincent Florens, *Generalized Seifert surfaces and signatures of colored links*, Trans. Amer. Math. Soc. **360** (2008), no. 3, 1223–1264. MR 2357695
- [CG78] A. J. Casson and C. McA. Gordon, *On slice knots in dimension three*, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, R.I., 1978, pp. 39–53. MR 520521
- [CG86] ———, *Cobordism of classical knots*, À la recherche de la topologie perdue, Progr. Math., vol. 62, Birkhäuser Boston, Boston, MA, 1986, With an appendix by P. M. Gilmer, pp. 181–199. MR 900252
- [DF04] David S. Dummit and Richard M. Foote, *Abstract algebra*, third ed., John Wiley & Sons, Inc., Hoboken, NJ, 2004. MR 2286236
- [FL18] Peter Feller and Lukas Lewark, *On classical upper bounds for slice genera*, Selecta Math. (N.S.) **24** (2018), no. 5, 4885–4916. MR 3874707
- [Flo04] V. Florens, *On the Fox-Milnor theorem for the Alexander polynomial of links*, Int. Math. Res. Not. (2004), no. 2, 55–67. MR 2040323
- [FM66] Ralph H. Fox and John W. Milnor, *Singularities of 2-spheres in 4-space and cobordism of knots*, Osaka Math. J. **3** (1966), 257–267. MR 211392
- [FQ90] Michael H. Freedman and Frank Quinn, *Topology of 4-manifolds*, Princeton Mathematical Series, vol. 39, Princeton University Press, Princeton, NJ, 1990. MR 1201584
- [Hos58] Fujitsugu Hosokawa, *On ∇ -polynomials of links*, Osaka Math. J. **10** (1958), 273–282. MR 102820
- [Kau87] L. H. Kauffman, *On knots*, Ann. Math. Studies, vol. 115, Princeton University Press, Princeton, NJ, 1987.
- [Kaw78] Akio Kawauchi, *On the Alexander polynomials of cobordant links*, Osaka J. Math. **15** (1978), no. 1, 151–159.
- [Kaw96] ———, *A survey of knot theory*, Birkhäuser, Basel, 1996.
- [Ker71] Michel A. Kervaire, *Knot cobordism in codimension two*, Manifolds–Amsterdam 1970 (Proc. Nuffic Summer School), Lecture Notes in Mathematics, Vol. 197, Springer, Berlin, 1971, pp. 83–105. MR 0283786
- [Lev69a] J. Levine, *Invariants of knot cobordism*, Invent. Math. **8** (1969), 98–110; addendum, *ibid.* **8** (1969), 355. MR 253348
- [Lev69b] ———, *Knot cobordism groups in codimension two*, Comment. Math. Helv. **44** (1969), 229–244. MR 246314
- [Lic97] W. B. R. Lickorish, *An introduction to knot theory*, Graduate Texts in Math., vol. 175, Springer, New York, 1997.

- [Liv10] Charles Livingston, *The algebraic concordance order of a knot*, J. Knot Theory Ramifications **19** (2010), no. 12, 1693–1711. MR 2755496
- [LM] Charles Livingston and Allison H. Moore, *Linkinfo: Table of link invariants*, URL: linkinfo.math.indiana.edu.
- [LN] Charles Livingston and Swatee Naik, *Introduction to knot concordance (work in progress)*, URL: citeseerx.ist.psu.edu/document?doi=07e3225de8e1084a9d6d2d53c54b548cb1a93ac8.
- [MH73] John Milnor and Dale Husemoller, *Symmetric bilinear forms*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 73, Springer-Verlag, Berlin-Heidelberg, 1973.
- [Mil69] John Milnor, *On isometries of inner product spaces*, Invent. Math. **8** (1969), 83–97. MR 249519
- [Mou] Delphine Moussard, *Slice genus, t-genus and 4-dimensional clasp number*, Comm. Anal. Geom., arXiv: 2101.01553.
- [Mur65a] Kunio Murasugi, *The Arf invariant for knot types*, Trans. Amer. Math. Soc. **117** (1965), 387–422.
- [Mur65b] ———, *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc. **117** (1965), 387–422. MR 171275
- [NP17] Matthias Nagel and Mark Powell, *Concordance invariance of Levine-Tristram signatures of links*, Documenta Math. **22** (2017), 25–43.
- [Ore99] S. Yu. Orevkov, *Link theory and oval arrangements of real algebraic curves*, Topology **38** (1999), 779–810.
- [Ore02] ———, *Classification of flexible m -curves of degree 8 up to isotopy*, Geom. Funct. Anal. **12** (2002), 723–855.
- [Pow17] Mark Powell, *The four-genus of a link, Levine-Tristram signatures and satellites*, J. Knot Theory Ramifications **26** (2017), 1740008, 28.
- [Rob65] Raymond A. Robertello, *An invariant of knot cobordism*, Comm. Pure Appl. Math. **18** (1965), 543–555.
- [Sei36] H. Seifert, *Über das Geschlecht von Knoten*, Math. Ann. **110** (1936), 571–592.
- [Ser70] Jean-Pierre Serre, *Cours d'arithmétique*, Collection SUP: “Le Mathématicien”, vol. 2, Presses Universitaires de France, Paris, 1970. MR 0255476
- [Tay79] Laurence R. Taylor, *On the genera of knots*, Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977), Lecture Notes in Math., vol. 722, Springer, Berlin, 1979, pp. 144–154. MR 547461
- [Tri69] A. G. Tristram, *Some cobordism invariants for links*, Proc. Cambridge Philos. Soc. **66** (1969), 251–264. MR 248854

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