# CUBIC HECKE ALGEBRAS AND INVARIANTS OF TRANSVERSAL LINKS

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**Introduction.** Let $\alpha$ be a differential 1-form which defines the standard (tight) contact structure in $\mathbb{R}^3$, e. g., $\alpha = x\,dy - y\,dx + dz$. A link $L$ in $\mathbb{R}^3$ is called *transversal* if $\alpha|_L$ does not vanish on $L$. Transversal links are considered up to isotopies such that the link remains transversal at every moment. Transversal links and their invariants are being actively studied, see, e. g., [2, 6, 7, 11] and numerous references therein. In the present paper, we propose a purely algebraic approach to construct invariants of transversal links (similar to Jones' approach [5] to construct invariants of usual links). The only geometry used is the transversal analogue of Alexander's and Markov's theorems proved in [1] and [10] respectively.

Let $B_n$ be the group of $n$-braids. We denote its standard (Artin's) generators by $\sigma_1, \dots, \sigma_{n-1}$. Let $B_\infty = \lim B_n$ be the limit under the embeddings $B_n \to B_{n+1}$, $\sigma_i \mapsto \sigma_i$. Let $k$ be a commutative ring and $u, v$ indeterminates. We set $A = k[u]$, $A_v = k[u, v]$ and we denote the corresponding group algebras by $kB_\infty$, $AB_\infty$ and $A_vB_\infty$. Let $\pi : kB_\infty \to H_\infty$ be a surjective morphism of $k$-algebras. We extend it to the morphisms (also denotes by $\pi$) of $A$- and $A_v$-algebras $AB_\infty \to AH_\infty = H_\infty \otimes_k A$ and $A_vB_\infty \to A_vH_\infty = H_\infty \otimes_k A_v$.

Let $R$ be the $A$-submodule of $AB_\infty$ generated by all the elements of the form

$$XY - YX, \quad X\sigma_n - uX \quad \text{where } X, Y \in B_n,\ n \ge 1, \tag{1}$$

and let $R_v$ be the $A_v$-submodule of $A_vB_\infty$ generated by (1) and also by $X\sigma_n^{-1} - vX$ for $X \in B_n$, $n \ge 1$. Let $M = AH_\infty/\pi(R)$ and $M_v = A_vH_\infty/\pi(R_v)$. We say that the quotient map $t_v : A_vH_\infty \to M_v$ is the *universal Markov trace* on $H_\infty$. Due to Alexander's and Markov's theorems, it defines a link invariant $P_{t_v}(L) = u^{(-n-e)/2}v^{(-n+e)/2}t_v(X) \in M_v \otimes_{A_v} k[u^{\pm 1/2}, v^{\pm 1/2}]$ where $L$ is the closure of an $n$-braid $X$ and $e = e(X) = \sum_j e_j$ for $X = \prod_j \sigma_{i_j}^{e_j}$.

Similarly, by the transversal analogue of Alexander's and Markov's theorems, the quotient map[1] $t : AH_\infty \to M$ defines a transversal link invariant $P_t(L) = u^{-n}t(X) \in M \otimes_A k[u^{\pm 1}]$ where $L$ is the closure of an $n$-braid $X$.

Of course, these invariants do not make much sense unless there is a reasonable solution to the identity problem in $M$ or in $M_v$. For example, if $\ker \pi = 0$, then $P_t$ is not really better than the tautological invariant $I(L) = L$. However, if $k = \mathbb{Z}[\alpha]$ and $A_vH_\infty = A_vB_\infty/(\sigma_1^2 + \alpha\sigma_1 + 1)$, then $M_v = A_v/(u + \alpha + v) \cong A$ and $P_{t_v}$ is the HOMFLY-PT polynomial up to variable change.

[1] I propose to call it *universal semi-Markov trace* on $H_\infty$.

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In [8], a description of $M_v$ is given when $H_\infty$ is a quotient of $kB_\infty$ by cubic relations of the form

$$\sigma_1^3 - \alpha\sigma_1^2 + \beta\sigma_1 = 1, \quad \sigma_2^\delta \sigma_1^{-\delta} \sigma_2^\delta = \sum_{\varepsilon \in E} c_{\varepsilon,\delta} \sigma_1^{\varepsilon_1} \sigma_2^{\varepsilon_2} \sigma_1^{\varepsilon_3}, \quad \delta = \pm 1, \tag{2}$$

$E = \{\varepsilon \in \{-1,0,1\}^3 \mid \varepsilon_2 = 0 \Rightarrow \varepsilon_1\varepsilon_3 = 0\}$, $\alpha, \beta, c_{\varepsilon,\delta} \in k$. We have in this case $M_v = A_v/I_v$ and a Gröbner base of $I_v$ can be computed at least theoretically. Moreover, $I_v$ is computed in practice in a particular case when $H_\infty$ is the Funar algebra [4] and $\beta = 0$. The computations may be fastened using [9].

In this paper we adapt the construction from [8] for the computation of $M$ when $H_\infty$ is defined by (2). We show that in this case $M \cong \hat{A}/\hat{I}$ where $\hat{A} = A[v_1, v_2, \dots]$ and $\hat{I}$ is an ideal of $\hat{A}$. For any $d$, we give an algorithm to compute the ideal $\hat{I} + (v_{d+1}, v_{d+2}, \dots)$. Thus, we define an infinite sequence (indexed by $d$) of computable transversal link invariants which carries the same information as the universal semi-Markov trace on the cubic Hecke algebra given by (2).

§**1. Monoid of braids with marked points.** Let $\hat{B}_n$ be the monoid of $n$-braids with a finite number of points marked on the strings. Algebraically it can be described as the monoid generated by $\sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}, q_1, \dots, q_n$ (see Figure 1), subject to the standard braid group relations and the relations $q_i q_j = q_j q_i$, $i, j = 1, \dots, n-1$, and $q_i \sigma_j = \sigma_j q_{T_j(i)}$ where $T_j$ is the transposition $(j, j+1)$. Each element of $\hat{B}_n$ can be written in a unique way in the form $q_1^{a_1} \dots q_n^{a_n} X$, $X \in B_n$, $a_i \geq 0$, so, $\hat{B}_n = Q_n \rtimes B_n$ where $Q_n$ is the free abelian monoid generated by $q_1, \dots, q_n$.

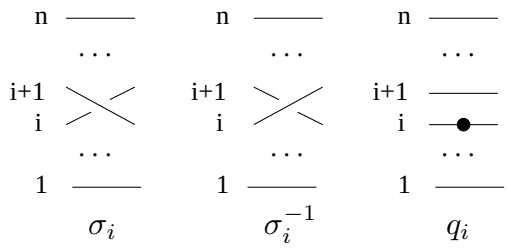


FIGURE 1. Generators of $\hat{B}_n$

Let $\hat{B}_\infty^{\sqcup}$ be the disjoint union $\bigsqcup_{n=1}^\infty \hat{B}_n$. If an ambiguity is possible, we use the notation $(X)_n$ to emphasize that a word $X$ represents an element of $\hat{B}_n$, for example, the braid closure of $(1)_n$ is the trivial $n$-component link.

**Theorem 1.** *Transversal links are in bijection with the quotient of $\hat{B}_\infty^{\sqcup}$ by the equivalence relation generated by*

$$\begin{aligned}
&(XY)_n \sim (YX)_n, && X, Y \in \hat{B}_n,\ n \geq 1 && (\textit{conjugations}),\\
&(X)_n \sim (X\sigma_n)_{n+1}, && X \in \hat{B}_n,\ n \geq 1 && (\textit{positive Markov moves}),\\
&(Xq_n)_n \sim (X\sigma_n^{-1})_{n+1}, && X \in \hat{B}_n,\ n \geq 1 && (\textit{negative Markov } q\textit{-moves})
\end{aligned}$$

*Proof.* Follows easily from Lemma 2. □

Let $\overset{s}{\sim}$ (strong equivalence) be the equivalence relation on $\hat{B}_\infty^{\sqcup}$ generated by conjugations and positive Markov moves only.

**Lemma 1.** (Key Lemma) *Let* $X \in \hat{B}_n$, $\varepsilon = \pm 1$, $X'_\varepsilon = X\sigma_n^{-1}\sigma_{n-1}^{2\varepsilon}$ *and* $X''_\varepsilon = X\sigma_{n-1}^{2\varepsilon}\sigma_n^{-1}$. *Then* $(X'_\varepsilon)_{n+1} \overset{s}{\sim} (X''_\varepsilon)_{n+1}$.

*Proof.* Let $a = \sigma_{n-1}$, $b = \sigma_n$, $c = \sigma_{n+1}$, $\bar{a} = a^{-1}$, $\bar{b} = b^{-1}$, $\bar{c} = c^{-1}$. Then

$$\begin{aligned}
X'_1 =& X\bar{b}a\,a \xrightarrow{\text{Mm}} X\,\underline{\bar{b}ab}\,\underline{bc\bar{b}\bar{b}}\,a = Xab\,\underline{\bar{a}\bar{c}\bar{c}}\,b\,\underline{ca} = Xab\bar{c}\bar{c}\bar{a}ba\,\underline{c}\\
&\xrightarrow{\text{cyc}} \underline{cXa}\,b\bar{c}\bar{c}\,\underline{\bar{a}ba} = Xa\,\underline{cb\bar{c}\bar{c}}\,ba\bar{b} = Xa\,\underline{\bar{b}\bar{b}cbb}\,a\bar{b} \xrightarrow{\text{Mm}} X''_1\\
X'_{-1} =& X\,\underline{\bar{b}\bar{a}\bar{b}}\,b\bar{a} = X\bar{a}\bar{b}\,\bar{a}b\bar{a} \xrightarrow{\text{Mm}} X\bar{a}\,\underline{\bar{b}\bar{b}cb}\,\bar{a}b\bar{a} = \underline{X\bar{a}c}\,b\bar{c}\bar{c}\bar{a}b\bar{a} = \underline{c}X\bar{a}b\bar{c}\bar{c}\bar{a}b\bar{a}\\
&\xrightarrow{\text{cyc}} X\bar{a}b\,\underline{\bar{c}\bar{c}\bar{a}}\,b\,\underline{\bar{a}c} = X\bar{a}b\bar{a}\,\underline{\bar{c}\bar{c}bc}\,\bar{a} = X\bar{a}b\bar{a}\,\underline{bc\bar{b}}\,\bar{b}\bar{a} \xrightarrow{\text{Mm}} X\bar{a}b\,\underline{\bar{a}\bar{b}\bar{a}} = X''_{-1}. \quad \square
\end{aligned}$$

Let $\deg_q : \hat{B}_n \to \mathbb{Z}$ be the monoid homomorphism such that $\deg_q(q_i) = 1$ and $\deg_q(\sigma_i) = 0$ for any $i$. We call $\deg_q(X)$ the *q-degree* of $X$.

**Lemma 2.** (Diamond Lemma) *If* $(Xq_n)_n \overset{s}{\sim} (X'q_m)_m$, *then either* $(X\sigma_n^{-1})_{n+1} \overset{s}{\sim} (X'\sigma_m^{-1})_{m+1}$ *or there exist* $Z, Z', Z'', Z''' \in \hat{B}^{\sqcup}_\infty$ *related to* $X\sigma_n^{-1}$ *and* $X'\sigma_m^{-1}$ *as follows* (*the arrows represent negative Markov q-moves which decrease the q-degree*)*:*

$$\begin{array}{ccccccc}
X\sigma_n^{-1} & \overset{s}{\sim} & Z & & Z''' & \overset{s}{\sim} & X'\sigma_m^{-1}\\
 & & \downarrow & & \downarrow & & \\
 & & Z' & \overset{s}{\sim} & Z'' & &
\end{array} \tag{3}$$

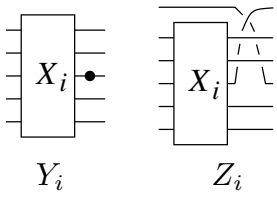


FIGURE 2

*Proof.* Since $Xq_n \overset{s}{\sim} X'q_m$, there exists a sequence of words $Xq_n = Y_0, Y_1, \ldots, Y_t$ of the form $Y_i = X_i q_{k_i} \in \hat{B}_{n_i}$ such that $Y_t$ is a cyclic permutation of $X'q_m$ and for any pair of consecutive indexes $i, j$ ($j = i \pm 1$) one of the following possibilities holds up to exchange of $i$ and $j$:

(*i*) $n_j = n_i$, $k_j = k_i$, $X_i$ and $X_j$ represent the same element of $\hat{B}^{\sqcup}_\infty$;
(*ii*) $n_j = n_i + 1$, $k_j = k_i$, $X_i = UV$, $X_j = U\sigma_{n_i}V$;
(*iii*) $n_j = n_i$, $X_i = U\sigma_\ell^\varepsilon$, $X_j = \sigma_\ell^\varepsilon U$, $k_j = T_\ell(k_i)$, $\varepsilon = \pm 1$;
(*iv*) $n_j = n_i$, $k_j = k_i \neq \ell$, $X_i = Uq_\ell$, $X_j = q_\ell U$.

For $i < j$, we denote $\sigma_i\sigma_{i+1}\ldots\sigma_{j-1}$ by $\pi_{i,j}$ and we set $\pi_{i,i} = 1$. Let $Z_i = X_i\pi_{k_i,n_i}\sigma_{n_i}^{-1}\pi_{k_i,n_i}^{-1} \in \hat{B}_{n_i+1}$ (see Figure 2). It is enough to prove that:

(a) $Z_i \overset{s}{\sim} Z_j$ in all cases (*i*)–(*iv*) (this implies $X\sigma_n^{-1} = Z_0 \overset{s}{\sim} Z_t$) and
(b) either $Z_t = X'\sigma_m^{-1}$ or we have $Z_t \overset{s}{\sim} Z \to Z' \overset{s}{\sim} Z'' \leftarrow Z''' \overset{s}{\sim} X'\sigma_m^{-1}$ where the arrows mean the same as in (3).

Assertion (a) either is evident or follows from Lemma 1. For example, in Case $(ii)$, we have $Z_i \overset{s}{\sim} Z_j$ because

$$Z_i = UV\pi_{k_i,n_i}\sigma_{n_i}^{-1}\pi_{k_i,n_i}^{-1} \overset{s}{\sim} U\sigma_{n_i}^{-1}\sigma_{n_i+1}\sigma_{n_i}V\pi_{k_i,n_i}\sigma_{n_i}^{-1}\pi_{k_i,n_i}^{-1} \overset{\text{def}}{=} Z_i',$$
$$Z_j = U\sigma_{n_i}V\pi_{k_i,n_i+1}\sigma_{n_i+1}^{-1}\pi_{k_i,n_i+1}^{-1} \qquad \text{and} \quad \sigma_{n_j}Z_j\sigma_{n_j}^{-1} = Z_i'.$$

In Case $(iii)$, $k_i = \ell+1$, $\varepsilon = -1$ we have $Z_i \overset{s}{\sim} Z_j$ by Lemma 1 because

$$Z_i = U\sigma_\ell^{-1}\pi_{\ell+1,n_i}\sigma_{n_i}^{-1}\pi_{\ell+1,n_i}^{-1} = V\sigma_{n_i-1}^{-2}\sigma_{n_i}^{-1}W,$$
$$Z_j = U\pi_{\ell,n_i}\sigma_{n_i}^{-1}\pi_{\ell,n_i}^{-1}\sigma_\ell^{-1} = V\sigma_{n_i}^{-1}\sigma_{n_i-1}^{-2}W \qquad \text{for}$$
$$V = U\pi_{\ell+1,n_i}\pi_{\ell,n_i-1}\sigma_{n_i-1}, \qquad W = \pi_{\ell,n_i-1}^{-1}\pi_{\ell+1,n_i}^{-1}.$$

In all the other cases Assertion (a) is either similar or easier.

It remains to prove Assertion (b). We know that $Y_t$ is a cyclic permutation of $X'q_m$. If $Y_t = X'q_m$, then $Z_t = X'\sigma_m^{-1}$ and we are done. Otherwise we have $X' = Uq_kV$ and $Y_t = Vq_mUq_k$ for some $k \le m$. Then we have:

$$Z_t = Vq_mU\pi_{k,m}\sigma_m^{-1}\pi_{k,m}^{-1} \overset{s}{\sim} \sigma_mU\pi_{k,m}\sigma_m^{-1}\pi_{k,m}^{-1}V\sigma_m^{-1}q_{m+1} \overset{\text{def}}{=} Z \to Z'$$
$$X'\sigma_m^{-1} = Uq_kV\sigma_m^{-1} \overset{s}{\sim} \pi_{k,m+1}^{-1}V\sigma_m^{-1}U\pi_{k,m+1}q_{m+1} \overset{\text{def}}{=} Z''' \to Z''.$$

It is easy to check that $Z'$ and $Z''$ are conjugate. $\square$

**Remark 1.** Theorem 1 admits also a geometric proof based on the interpretation of the marked points as local modifications introduced in [3] which increase the Thurston-Bennequin number (see the extended version of [10]).

§**2. From $A$ to $\hat{A}$.** Let the notation be as in the introduction and let $\hat{A}\hat{B}_\infty$ be the semigroup algebra of $\hat{B}_\infty$ with coefficients in $\hat{A}$. We have $kB_\infty \subset AB_\infty \subset \hat{A}\hat{B}_\infty$. Let $\hat{H}_\infty$ be the quotient of $\hat{A}\hat{B}_\infty$ by the bilateral ideal generated by $\ker\pi$ and let $\hat{\pi} : \hat{A}\hat{B}_\infty \to \hat{H}_\infty$ be the quotient map.

Let $\hat{R}$ be the submodule of $\hat{A}\hat{B}_\infty$ generated by all the elements of the form

$$XY - YX, \quad X\sigma_n - uX, \quad X\sigma_n^{-1} - Xq_n, \quad q_{n+1}^aX - v_aX, \quad q_1^a - v_a$$

with $X, Y \in \hat{B}_n$ and $n, a \ge 1$. Let $\hat{M} = \hat{H}_\infty/\hat{\pi}(\hat{R})$ and let $\hat{t} : \hat{H}_\infty \to \hat{M}$ be the quotient map.

**Theorem 2.** *(a). $M$ and $\hat{M}$ are isomorphic as $A$-modules. (b). If, moreover, $H_\infty$ is given by* (2)*, then $\hat{M}$ is generated by $\hat{t}(1)$ as an $\hat{A}$-module.*

*Proof.* (a). Follows from Theorem 1. (b). Follows from the fact that $\hat{H}_{n+1} = \langle q_{n+1}\rangle\hat{H}_n + \hat{H}_n\sigma_n\hat{H}_n + \hat{H}_n\sigma_n^{-1}\hat{H}_n$ where $\langle q_{n+1}\rangle = \{1, q_{n+1}, q_{n+1}^2, \dots\}$. $\square$

Thus $\hat{M} = \hat{A}/\hat{I}$ where $\hat{I}$ is the annihilator of $\hat{M}$.

§**3. Description of $\hat{I}$.** In this section we assume that $\hat{H}_\infty$ is defined by (2). Let $F_n^+$ (resp. $\hat{F}_n$) be the free monoid freely generated by $x_1^{\pm1}, \dots, x_{n-1}^{\pm1}$ (resp. by $x_1^{\pm1}, \dots, x_{n-1}^{\pm1}, q_1, \dots, q_n$) and let $\hat{A}\hat{F}_n$ be the semigroup algebra of $\hat{F}_n$ over $\hat{A}$. We define the *basic replacements* as in [8; §2.1, $(i)$–$(viii)$] and we add to them

$(ix)$ $x_iq_j \to q_{T_i(j)}x_i$

We define $\hat{A}\hat{F}_n^{\mathrm{red}}$ and $\mathbf{r} : \hat{A}\hat{F}_n \to \hat{A}\hat{F}_n^{\mathrm{red}}$ similarly to [8; §2.2] using the replacements $(i)$–$(ix)$. Then $\hat{A}\hat{F}_n^{\mathrm{red}}$ is the free $\hat{A}$-module freely generated by the elements of the form $qX_1X_2\dots X_{n-1}$, $q \in Q_n$, $X_i \in S_i$ where $S_i$ are as in [8; (5)]. We define $\hat{\tau}_n : \hat{A}\hat{F}_n^{\mathrm{red}} \to \hat{A}\hat{F}_{n-1}^{\mathrm{red}}$ by setting $\hat{\tau}_n(qq_n^aXx_{n-1}Y) = \mathbf{r}(qXq_{n-1}^aY)$, $\hat{\tau}_n(qq_n^aXx_{n-1}^{-1}Y) = \mathbf{r}(qXq_{n-1}^{a+1}Y)$, $\hat{\tau}_n(qq_n^aX) = v_aqX$ for $q \in Q_{n-1}$, $X, Y \in F_{n-1}^+$. We extend $\hat{\tau}_n$ to $\hat{A}\hat{F}_n$ by setting $\hat{\tau}_n(X) = \hat{\tau}_n(\mathbf{r}(X))$ and we define $\hat{\tau} : \hat{A}\hat{F}_\infty \to \hat{A}\hat{F}_0 = \hat{A}$ by $\hat{\tau}(X) = \hat{\tau}_1\hat{\tau}_2\dots\hat{\tau}_n(X)$ for $X \in \hat{A}\hat{F}_n$.

Let $\mathrm{sh}^n$ be the $\hat{A}$-algebra endomorphism of $\hat{A}\hat{F}_\infty$ defined by $\mathrm{sh}\,\sigma_i = \sigma_{i+n}$, $\mathrm{sh}\,q_i = q_{i+n}$. We set $\mathrm{sh} = \mathrm{sh}^1$. For $X \in F_{n+1}^+$, we define $\rho_{n,X} \in \mathrm{End}_{\hat{A}}(\hat{A}\hat{F}_n^{\mathrm{red}})$ by setting $\rho_{n,X}(Y) = \hat{\tau}_{n+1}(X\,\mathrm{sh}\,Y)$.

Let $\hat{J}_4$ be the minimal $\hat{A}$-submodule of $\hat{A}\hat{F}_4^{\mathrm{red}}$ which satisfies the conditions

(J1) $\mathbf{r}(\mathbf{r}(X_3X_2)X_1) - \mathbf{r}(X_3\mathbf{r}(X_2X_1)) \in \hat{J}_4$ for any $X_j \in \mathrm{sh}^{3-j}\,S_j \setminus \{1\}$, $j = 1, 2, 3$;

(J2) $\rho_{4,X}(\hat{J}_4) \subset \hat{J}_4$ for any $X \in S_4$.

Similarly, let $\hat{J}_3$ be the minimal $\hat{A}$-submodule of $\hat{A}\hat{F}_3^{\mathrm{red}}$ which satisfies

(J1′) $q_i\mathbf{r}(X) - \mathbf{r}(\mathbf{r}(X)q_j) \in \hat{J}_3$ for any $X = x_2^{\varepsilon_1}x_1^{\varepsilon_2}x_2^{\varepsilon_3}$, $\varepsilon_1, \varepsilon_3 \in \{-1, 1\}$, $\varepsilon_2 \in \{-1, 0, 1\}$, $i = 1, 2, 3$, $j = T_2T_1^{\varepsilon_2}T_2(i)$.

(J2′) $\rho_{3,X}(\hat{J}_3) \subset \hat{J}_3$ for any $X \in S_3$.

Let $\hat{N} = \hat{A}\hat{F}_2^{\mathrm{red}} \otimes_{\hat{A}} \hat{A}\hat{F}_2^{\mathrm{red}}$. We define $\hat{A}$-linear mappings $\hat{\tau}_N : \hat{N} \to \hat{A}$ and $\rho_\delta : \hat{N} \to \hat{N}$, $\delta = (\delta_1, \delta_2) \in \{-1, 0, 1\}^2$, by setting $\hat{\tau}_N(Y_1 \otimes Y_2) = \hat{\tau}(Y_1Y_2)$, $\rho_\delta(Y_1 \otimes Y_2) = x_1^{\delta_1} \otimes \hat{\tau}_3((\mathrm{sh}\,Y_1)x_1^{\delta_2}\,\mathrm{sh}\,Y_2)$. Let $\hat{L}$ be the minimal $\hat{A}$-submodule of $\hat{N}$ satisfying

(L1) $\hat{\tau}_3(x_2^{\varepsilon_1}x_1^{\varepsilon_2}x_2^{\varepsilon_3}) \otimes x_1^{\varepsilon_4} - x_1^{\varepsilon_2} \otimes \hat{\tau}_3(x_2^{\varepsilon_3}x_1^{\varepsilon_4}x_2^{\varepsilon_1}) \in \hat{L}$ for any $\varepsilon_1, \varepsilon_3 \in \{-1, 1\}$ and for any $\varepsilon_2, \varepsilon_4 \in \{-1, 0, 1\}$;

(L2) $\rho_\delta(\hat{L}) \subset \hat{L}$ for any $\delta \in \{-1, 0, 1\}^2$.

**Theorem 3.** $\hat{I} = \hat{\tau}(\hat{J}_4) + \hat{\tau}(\hat{J}_3) + \hat{\tau}_N(\hat{L})$.

A proof repeats almost word by word the proof of Main Theorem in [8] (we ignore the variables $q_i$ when we define the weight function on $\hat{F}_\infty$).

Each of the modules $\hat{J}_4$, $\hat{J}_3$, $\hat{L}$ is defined as the limit of an increasing sequence of submodules of a finite rank $\hat{A}$-module. Since $\hat{A}$ is not Noetherian, this does not give yet a way to compute them. However, we can approximate $\hat{A}$ by Noetherian rings $\hat{A}_d = A[v_1, \dots, v_d]$ and the projections $\mathrm{pr}_d(\hat{I})$ can be effectively computed where $\mathrm{pr}_d : \hat{A} \to \hat{A}_d$ is the quotient by the ideal $(v_{d+1}, v_{d+2}, \dots)$. Namely, let $(\hat{A}\hat{F}_n^{\mathrm{red}})_d$, $(\hat{J}_4)_d$, $(\hat{J}_3)_d$, $(\hat{N})_d$, $(\hat{L})_d$ be the $\hat{A}_d$-modules obtained by the above procedure but with the additional relations $q_i^{d+1} = 0$ for any $i$. Then we have $\mathrm{pr}_d(\hat{I}) = \hat{\tau}(\hat{J}_4)_d + \hat{\tau}(\hat{J}_3)_d + \hat{\tau}_N(\hat{L})_d$ and these modules (at least theoretically) can be computed as limits of increasing sequences of Noetherian modules. The rank of $(\hat{A}\hat{F}_4^{\mathrm{red}})_d$ (the module where $(\hat{J}_4)_d$ sits) is equal to $315(d+1)^4$. We hope that, at least for $d = 1$ or 2, the computations can be performed in practice.

**Remark 2.** If $\beta = 0$ (the case when the Groebner base of $I_v$ was computed in [8]), then the obtained transversal link invariants a priori cannot detect transversally non-simple links. Indeed, in this case we have $1 = \alpha\sigma_1^{-1} + \sigma_1^{-3}$, hence $q_1 = q_1(\alpha\sigma_1^{-1} + \sigma_1^{-3}) = (\alpha\sigma_1^{-1} + \sigma_1^{-3})q_2 = q_2$. Thus $q_1 = q_2 = q_3 = \dots$ whence $v_1 = v_2 = \dots$ and we obtain $M = M_v$, $t = t_v$ and $P_t(L) = (v/u)^{(n-e)/2}P_{t_v}(L)$, i. e., the invariant $P_t$ reduces to a usual link invariant $P_{t_v}$ and Thurston-Bennequin number $n - e$.

**Remark 3.** By [9], all the computations in the huge module $(\hat{A}\hat{F}_4^{\mathrm{red}})_d$ can be done with the coefficients in $\mathbb{Q}$ or in $\mathbb{Z}/m\mathbb{Z}$ for $m$ not very big.

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