CUBIC HECKE ALGEBRAS AND INVARIANTS OF TRANSVERSAL LINKS

S. Yu. Orevkov

Introduction. Let α be a differential 1-form which defines the standard (tight) contact structure in $\mathbb{R}^3$, e. g., $\alpha = x\,dy - y\,dx + dz$. A link L in $\mathbb{R}^3$ is called *transversal* if $\alpha|_L$ does not vanish on L. Transversal links are considered up to isotopies such that the link remains transversal at every moment. Transversal links and their invariants are being actively studied, see, e. g., [2, 6, 7, 11] and numerous references therein. In the present paper, we propose a purely algebraic approach to construct invariants of transversal links (similar to Jones' approach [5] to construct invariants of usual links). The only geometry used is the transversal analogue of Alexander's and Markov's theorems proved in [1] and [10] respectively.

Let B_n be the group of n-braids. We denote its standard (Artin's) generators by $\sigma_1,\dots,\sigma_{n-1}$. Let $B_\infty = \lim B_n$ be the limit under the embeddings $B_n \to B_{n+1}$, $\sigma_i \mapsto \sigma_i$. Let k be a commutative ring and u, v indeterminates. We set $A = k[u]$, $A_v = k[u,v]$ and we denote the corresponding group algebras by kB_∞, AB_∞ and A_vB_∞. Let $\pi : kB_\infty \to H_\infty$ be a surjective morphism of k-algebras. We extend it to the morphisms (also denotes by π) of A- and A_v-algebras $AB_\infty \to AH_\infty = H_\infty \otimes_k A$ and $A_vB_\infty \to A_vH_\infty = H_\infty \otimes_k A_v$.

Let R be the A-submodule of AB_∞ generated by all the elements of the form

$$XY - YX, \;\; X\sigma_n - uX \quad \text{where} \;\; X, Y \in B_n, \; n \ge 1, \tag{1}$$

and let R_v be the A_v-submodule of A_vB_∞ generated by (1) and also by $X\sigma_n^{-1} - vX$ for $X \in B_n$, $n \ge 1$. Let $M = AH_\infty/\pi(R)$ and $M_v = A_vH_\infty/\pi(R_v)$. We say that the quotient map $t_v : A_vH_\infty \to M_v$ is the *universal Markov trace* on H_∞. Due to Alexander's and Markov's theorems, it defines a link invariant $P_{t_v}(L) = u^{(-n-e)/2}v^{(-n+e)/2}t_v(X) \in M_v \otimes_{A_v} k[u^{\pm1/2}, v^{\pm1/2}]$ where L is the closure of an n-braid X and $e = e(X) = \sum_j e_j$ for $X = \prod_j \sigma_{i_j}^{e_j}$.

Similarly, by the transversal analogue of Alexander's and Markov's theorems, the quotient map[1] $t : AH_\infty \to M$ defines a transversal link invariant $P_t(L) = u^{-n}t(X) \in M \otimes_A k[u^{\pm1}]$ where L is the closure of an n-braid X.

Of course, these invariants do not make much sense unless there is a reasonable solution to the identity problem in M or in M_v. For example, if $\ker\pi = 0$, then P_t is not really better than the tautological invariant $I(L) = L$. However, if $k = \mathbb{Z}[\alpha]$ and $A_vH_\infty = A_vB_\infty/(\sigma_1^2 + \alpha\sigma_1 + 1)$, then $M_v = A_v/(u + \alpha + v) \cong A$ and P_{t_v} is the HOMFLY-PT polynomial up to variable change.

[1] I propose to call it *universal semi-Markov trace* on H_∞.

In [8], a description of M_v is given when H_∞ is a quotient of kB_∞ by cubic relations of the form

$$\sigma_1^3 - \alpha\sigma_1^2 + \beta\sigma_1 = 1, \quad \sigma_2^\delta\sigma_1^{-\delta}\sigma_2^\delta = \sum_{\varepsilon\in E} c_{\varepsilon,\delta}\sigma_1^{\varepsilon_1}\sigma_2^{\varepsilon_2}\sigma_1^{\varepsilon_3}, \quad \delta = \pm 1, \tag{2}$$

$E = \{\varepsilon \in \{-1,0,1\}^3 \mid \varepsilon_2 = 0 \Rightarrow \varepsilon_1\varepsilon_3 = 0\}$, $\alpha, \beta, c_{\varepsilon,\delta} \in k$. We have in this case $M_v = A_v/I_v$ and a Gröbner base of I_v can be computed at least theoretically. Moreover, I_v is computed in practice in a particular case when H_∞ is the Funar algebra [4] and $\beta = 0$. The computations may be fastened using [9].

In this paper we adapt the construction from [8] for the computation of M when H_∞ is defined by (2). We show that in this case $M \cong \hat{A}/\hat{I}$ where $\hat{A} = A[v_1, v_2, \dots]$ and $\hat{I}$ is an ideal of $\hat{A}$. For any d, we give an algorithm to compute the ideal $\hat{I} + (v_{d+1}, v_{d+2}, \dots)$. Thus, we define an infinite sequence (indexed by d) of computable transversal link invariants which carries the same information as the universal semi-Markov trace on the cubic Hecke algebra given by (2).

§**1. Monoid of braids with marked points.** Let $\hat{B}_n$ be the monoid of n-braids with a finite number of points marked on the strings. Algebraically it can be described as the monoid generated by $\sigma_1^{\pm1}, \dots, \sigma_{n-1}^{\pm1}, q_1, \dots, q_n$ (see Figure 1), subject to the standard braid group relations and the relations $q_iq_j = q_jq_i$, $i,j = 1,\dots,n-1$, and $q_i\sigma_j = \sigma_j q_{T_j(i)}$ where T_j is the transposition $(j, j+1)$. Each element of $\hat{B}_n$ can be written in a unique way in the form $q_1^{a_1}\dots q_n^{a_n}X$, $X \in B_n$, $a_i \geq 0$, so, $\hat{B}_n = Q_n \rtimes B_n$ where Q_n is the free abelian monoid generated by $q_1, \dots, q_n$.

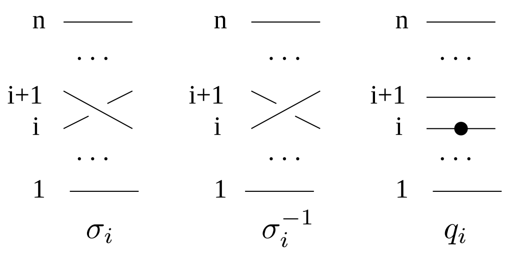


FIGURE 1. Generators of $\hat{B}_n$

Let $\hat{B}_\infty^\sqcup$ be the disjoint union $\bigsqcup_{n=1}^\infty \hat{B}_n$. If an ambiguity is possible, we use the notation $(X)_n$ to emphasize that a word X represents an element of $\hat{B}_n$, for example, the braid closure of $(1)_n$ is the trivial n-component link.

Theorem 1. *Transversal links are in bijection with the quotient of $\hat{B}_\infty^\sqcup$ by the equivalence relation generated by*

$$(XY)_n \sim (YX)_n, \qquad X, Y \in \hat{B}_n,\ n \geq 1 \qquad (\textit{conjugations}),$$

$$(X)_n \sim (X\sigma_n)_{n+1}, \qquad X \in \hat{B}_n,\ n \geq 1 \qquad (\textit{positive Markov moves}),$$

$$(Xq_n)_n \sim (X\sigma_n^{-1})_{n+1}, \qquad X \in \hat{B}_n,\ n \geq 1 \qquad (\textit{negative Markov q-moves})$$

Proof. Follows easily from Lemma 2. □

Let $\overset{s}{\sim}$ (strong equivalence) be the equivalence relation on $\hat{B}_\infty^\sqcup$ generated by conjugations and positive Markov moves only.

Lemma 1. (Key Lemma) *Let $X \in \hat{B}_n$, $\varepsilon = \pm 1$, $X'_\varepsilon = X\sigma_n^{-1}\sigma_{n-1}^{2\varepsilon}$ and $X''_\varepsilon = X\sigma_{n-1}^{2\varepsilon}\sigma_n^{-1}$. Then $(X'_\varepsilon)_{n+1} \overset{s}{\sim} (X''_\varepsilon)_{n+1}$.*

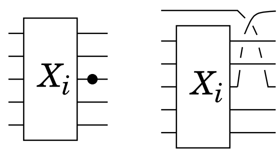
Proof. Let $a = \sigma_{n-1}$, $b = \sigma_n$, $c = \sigma_{n+1}$, $\bar{a} = a^{-1}$, $\bar{b} = b^{-1}$, $\bar{c} = c^{-1}$. Then

$$\begin{aligned}
X'_1 &= X\bar{b}a\,a \xrightarrow{\text{Mm}} X\,\underline{\bar{b}ab}\,\underline{bc\bar{b}\bar{b}}\,a = Xab\,\underline{\bar{a}\bar{c}\bar{c}}\,b\,\underline{ca} = Xab\bar{c}\bar{c}\bar{a}ba\,\underline{c}\\
&\xrightarrow{\text{cyc}} \underline{cXa}\,b\bar{c}\bar{c}\,\underline{\bar{a}ba} = Xa\,\underline{cb\bar{c}\bar{c}}\,ba\bar{b} = Xa\,\underline{\bar{b}\bar{b}cbb}\,a\bar{b} \xrightarrow{\text{Mm}} X''_1\\
X'_{-1} &= X\,\underline{\bar{b}\bar{a}\bar{b}}\,b\bar{a} = X\bar{a}\bar{b}\,\bar{a}b\bar{a} \xrightarrow{\text{Mm}} X\bar{a}\,\underline{\bar{b}\bar{b}cb}\,\bar{a}b\bar{a} = \underline{X\bar{a}c}\,b\bar{c}\bar{c}\bar{a}b\bar{a} = \underline{c}X\bar{a}b\bar{c}\bar{c}\bar{a}b\bar{a}\\
&\xrightarrow{\text{cyc}} X\bar{a}b\,\underline{\bar{c}\bar{c}\bar{a}}\,b\,\underline{\bar{a}c} = X\bar{a}b\bar{a}\,\underline{\bar{c}\bar{c}bc}\,\bar{a} = X\bar{a}b\bar{a}\,\underline{bc\bar{b}}\,\bar{b}\bar{a} \xrightarrow{\text{Mm}} X\bar{a}b\,\underline{\bar{a}\bar{b}\bar{a}} = X''_{-1}. \qquad \square
\end{aligned}$$

Let $\deg_q : \hat{B}_n \to \mathbb{Z}$ be the monoid homomorphism such that $\deg_q(q_i) = 1$ and $\deg_q(\sigma_i) = 0$ for any i. We call $\deg_q(X)$ the *q-degree* of X.

Lemma 2. (Diamond Lemma) *If $(Xq_n)_n \overset{s}{\sim} (X'q_m)_m$, then either $(X\sigma_n^{-1})_{n+1} \overset{s}{\sim} (X'\sigma_m^{-1})_{m+1}$ or there exist $Z, Z', Z'', Z''' \in \hat{B}_\infty^\sqcup$ related to $X\sigma_n^{-1}$ and $X'\sigma_m^{-1}$ as follows (the arrows represent negative Markov q-moves which decrease the q-degree):*

$$\begin{array}{ccccccc}
X\sigma_n^{-1} & \overset{s}{\sim} & Z & & Z''' & \overset{s}{\sim} & X'\sigma_m^{-1}\\
& & \downarrow & & \downarrow & & \\
& & Z' & \overset{s}{\sim} & Z'' & &
\end{array} \qquad (3)$$



Y_i Z_i

FIGURE 2

Proof. Since $Xq_n \overset{s}{\sim} X'q_m$, there exists a sequence of words $Xq_n = Y_0, Y_1, \ldots, Y_t$ of the form $Y_i = X_i q_{k_i} \in \hat{B}_{n_i}$ such that Y_t is a cyclic permutation of $X'q_m$ and for any pair of consecutive indexes i, j ($j = i \pm 1$) one of the following possibilities holds up to exchange of i and j:

- (*i*) $n_j = n_i$, $k_j = k_i$, X_i and X_j represent the same element of $\hat{B}_\infty^\sqcup$;
- (*ii*) $n_j = n_i + 1$, $k_j = k_i$, $X_i = UV$, $X_j = U\sigma_{n_i}V$;
- (*iii*) $n_j = n_i$, $X_i = U\sigma_\ell^\varepsilon$, $X_j = \sigma_\ell^\varepsilon U$, $k_j = T_\ell(k_i)$, $\varepsilon = \pm 1$;
- (*iv*) $n_j = n_i$, $k_j = k_i \neq \ell$, $X_i = Uq_\ell$, $X_j = q_\ell U$.

For $i < j$, we denote $\sigma_i\sigma_{i+1}\ldots\sigma_{j-1}$ by $\pi_{i,j}$ and we set $\pi_{i,i} = 1$. Let $Z_i = X_i\pi_{k_i,n_i}\sigma_{n_i}^{-1}\pi_{k_i,n_i}^{-1} \in \hat{B}_{n_i+1}$ (see Figure 2). It is enough to prove that:

- (a) $Z_i \overset{s}{\sim} Z_j$ in all cases (*i*)–(*iv*) (this implies $X\sigma_n^{-1} = Z_0 \overset{s}{\sim} Z_t$) and
- (b) either $Z_t = X'\sigma_m^{-1}$ or we have $Z_t \overset{s}{\sim} Z \to Z' \overset{s}{\sim} Z'' \leftarrow Z''' \overset{s}{\sim} X'\sigma_m^{-1}$ where the arrows mean the same as in (3).

Assertion (a) either is evident or follows from Lemma 1. For example, in Case (ii), we have $Z_i \overset{s}{\sim} Z_j$ because

$$Z_i = UV\pi_{k_i,n_i}\sigma_{n_i}^{-1}\pi_{k_i,n_i}^{-1} \overset{s}{\sim} U\sigma_{n_i}^{-1}\sigma_{n_i+1}\sigma_{n_i}V\pi_{k_i,n_i}\sigma_{n_i}^{-1}\pi_{k_i,n_i}^{-1} \overset{\text{def}}{=} Z_i',$$
$$Z_j = U\sigma_{n_i}V\pi_{k_i,n_i+1}\sigma_{n_i+1}^{-1}\pi_{k_i,n_i+1}^{-1} \qquad \text{and} \quad \sigma_{n_j}Z_j\sigma_{n_j}^{-1} = Z_i'.$$

In Case (iii), $k_i = \ell+1$, $\varepsilon = -1$ we have $Z_i \overset{s}{\sim} Z_j$ by Lemma 1 because

$$Z_i = U\sigma_\ell^{-1}\pi_{\ell+1,n_i}\sigma_{n_i}^{-1}\pi_{\ell+1,n_i}^{-1} = V\sigma_{n_i-1}^{-2}\sigma_{n_i}^{-1}W,$$
$$Z_j = U\pi_{\ell,n_i}\sigma_{n_i}^{-1}\pi_{\ell,n_i}^{-1}\sigma_\ell^{-1} = V\sigma_{n_i}^{-1}\sigma_{n_i-1}^{-2}W \qquad \text{for}$$
$$V = U\pi_{\ell+1,n_i}\pi_{\ell,n_i-1}\sigma_{n_i-1}, \qquad W = \pi_{\ell,n_i-1}^{-1}\pi_{\ell+1,n_i}^{-1}.$$

In all the other cases Assertion (a) is either similar or easier.

It remains to prove Assertion (b). We know that Y_t is a cyclic permutation of $X'q_m$. If $Y_t = X'q_m$, then $Z_t = X'\sigma_m^{-1}$ and we are done. Otherwise we have $X' = Uq_kV$ and $Y_t = Vq_mUq_k$ for some $k \le m$. Then we have:

$$Z_t = Vq_mU\pi_{k,m}\sigma_m^{-1}\pi_{k,m}^{-1} \overset{s}{\sim} \sigma_mU\pi_{k,m}\sigma_m^{-1}\pi_{k,m}^{-1}V\sigma_m^{-1}q_{m+1} \overset{\text{def}}{=} Z \to Z'$$
$$X'\sigma_m^{-1} = Uq_kV\sigma_m^{-1} \overset{s}{\sim} \pi_{k,m+1}^{-1}V\sigma_m^{-1}U\pi_{k,m+1}q_{m+1} \overset{\text{def}}{=} Z''' \to Z''.$$

It is easy to check that Z' and Z'' are conjugate. □

Remark 1. Theorem 1 admits also a geometric proof based on the interpretation of the marked points as local modifications introduced in [3] which increase the Thurston-Bennequin number (see the extended version of [10]).

§**2. From A to $\hat{A}$.** Let the notation be as in the introduction and let $\hat{A}\hat{B}_\infty$ be the semigroup algebra of $\hat{B}_\infty$ with coefficients in $\hat{A}$. We have $kB_\infty \subset AB_\infty \subset \hat{A}\hat{B}_\infty$. Let $\hat{H}_\infty$ be the quotient of $\hat{A}\hat{B}_\infty$ by the bilateral ideal generated by $\ker\pi$ and let $\hat\pi : \hat{A}\hat{B}_\infty \to \hat{H}_\infty$ be the quotient map.

Let $\hat{R}$ be the submodule of $\hat{A}\hat{B}_\infty$ generated by all the elements of the form

$$XY - YX, \quad X\sigma_n - uX, \quad X\sigma_n^{-1} - Xq_n, \quad q_{n+1}^aX - v_aX, \quad q_1^a - v_a$$

with $X, Y \in \hat{B}_n$ and $n, a \ge 1$. Let $\hat{M} = \hat{H}_\infty/\hat\pi(\hat{R})$ and let $\hat{t} : \hat{H}_\infty \to \hat{M}$ be the quotient map.

Theorem 2. *(a). M and $\hat{M}$ are isomorphic as A-modules. (b). If, moreover, H_∞ is given by* (2)*, then $\hat{M}$ is generated by $\hat{t}(1)$ as an $\hat{A}$-module.*

Proof. (a). Follows from Theorem 1. (b). Follows from the fact that $\hat{H}_{n+1} = \langle q_{n+1}\rangle\hat{H}_n + \hat{H}_n\sigma_n\hat{H}_n + \hat{H}_n\sigma_n^{-1}\hat{H}_n$ where $\langle q_{n+1}\rangle = \{1, q_{n+1}, q_{n+1}^2, \dots\}$. □

Thus $\hat{M} = \hat{A}/\hat{I}$ where $\hat{I}$ is the annihilator of $\hat{M}$.

§**3. Description of $\hat{I}$.** In this section we assume that $\hat{H}_\infty$ is defined by (2). Let F_n^+ (resp. $\hat{F}_n$) be the free monoid freely generated by $x_1^{\pm1}, \dots, x_{n-1}^{\pm1}$ (resp. by $x_1^{\pm1}, \dots, x_{n-1}^{\pm1}, q_1, \dots, q_n$) and let $\hat{A}\hat{F}_n$ be the semigroup algebra of $\hat{F}_n$ over $\hat{A}$. We define the *basic replacements* as in [8; §2.1, (i)–$(viii)$] and we add to them

(ix) $x_iq_j \to q_{T_i(j)}x_i$

We define $\hat A\hat F_n^{\rm red}$ and $\mathbf r:\hat A\hat F_n\to\hat A\hat F_n^{\rm red}$ similarly to [8; §2.2] using the replacements (i)–(ix). Then $\hat A\hat F_n^{\rm red}$ is the free $\hat A$-module freely generated by the elements of the form $qX_1X_2\dots X_{n-1}$, $q\in Q_n$, $X_i\in S_i$ where S_i are as in [8; (5)]. We define $\hat\tau_n:\hat A\hat F_n^{\rm red}\to\hat A\hat F_{n-1}^{\rm red}$ by setting $\hat\tau_n(qq_n^aXx_{n-1}Y)=\mathbf r(qXq_{n-1}^aY)$, $\hat\tau_n(qq_n^aXx_{n-1}^{-1}Y)=\mathbf r(qXq_{n-1}^{a+1}Y)$, $\hat\tau_n(qq_n^aX)=v_aqX$ for $q\in Q_{n-1}$, $X,Y\in F_{n-1}^+$. We extend $\hat\tau_n$ to $\hat A\hat F_n$ by setting $\hat\tau_n(X)=\hat\tau_n(\mathbf r(X))$ and we define $\hat\tau:\hat A\hat F_\infty\to\hat A\hat F_0=\hat A$ by $\hat\tau(X)=\hat\tau_1\hat\tau_2\dots\hat\tau_n(X)$ for $X\in\hat A\hat F_n$.

Let sh^n be the $\hat A$-algebra endomorphism of $\hat A\hat F_\infty$ defined by $\mathrm{sh}\,\sigma_i=\sigma_{i+n}$, $\mathrm{sh}\,q_i=q_{i+n}$. We set $\mathrm{sh}=\mathrm{sh}^1$. For $X\in F_{n+1}^+$, we define $\rho_{n,X}\in\mathrm{End}_{\hat A}(\hat A\hat F_n^{\rm red})$ by setting $\rho_{n,X}(Y)=\hat\tau_{n+1}(X\,\mathrm{sh}\,Y)$.

Let $\hat J_4$ be the minimal $\hat A$-submodule of $\hat A\hat F_4^{\rm red}$ which satisfies the conditions

(J1) $\mathbf r(\mathbf r(X_3X_2)X_1)-\mathbf r(X_3\mathbf r(X_2X_1))\in\hat J_4$ for any $X_j\in\mathrm{sh}^{3-j}\,S_j\setminus\{1\}$, $j=1,2,3$;
(J2) $\rho_{4,X}(\hat J_4)\subset\hat J_4$ for any $X\in S_4$.

Similarly, let $\hat J_3$ be the minimal $\hat A$-submodule of $\hat A\hat F_3^{\rm red}$ which satisfies

(J1′) $q_i\mathbf r(X)-\mathbf r(\mathbf r(X)q_j)\in\hat J_3$ for any $X=x_2^{\varepsilon_1}x_1^{\varepsilon_2}x_2^{\varepsilon_3}$, $\varepsilon_1,\varepsilon_3\in\{-1,1\}$, $\varepsilon_2\in\{-1,0,1\}$, $i=1,2,3$, $j=T_2T_1^{\varepsilon_2}T_2(i)$.
(J2′) $\rho_{3,X}(\hat J_3)\subset\hat J_3$ for any $X\in S_3$.

Let $\hat N=\hat A\hat F_2^{\rm red}\otimes_{\hat A}\hat A\hat F_2^{\rm red}$. We define $\hat A$-linear mappings $\hat\tau_N:\hat N\to\hat A$ and $\rho_\delta:\hat N\to\hat N$, $\delta=(\delta_1,\delta_2)\in\{-1,0,1\}^2$, by setting $\hat\tau_N(Y_1\otimes Y_2)=\hat\tau(Y_1Y_2)$, $\rho_\delta(Y_1\otimes Y_2)=x_1^{\delta_1}\otimes\hat\tau_3((\mathrm{sh}\,Y_1)x_1^{\delta_2}\,\mathrm{sh}\,Y_2)$. Let $\hat L$ be the minimal $\hat A$-submodule of $\hat N$ satisfying

(L1) $\hat\tau_3(x_2^{\varepsilon_1}x_1^{\varepsilon_2}x_2^{\varepsilon_3})\otimes x_1^{\varepsilon_4}-x_1^{\varepsilon_2}\otimes\hat\tau_3(x_2^{\varepsilon_3}x_1^{\varepsilon_4}x_2^{\varepsilon_1})\in\hat L$ for any $\varepsilon_1,\varepsilon_3\in\{-1,1\}$ and for any $\varepsilon_2,\varepsilon_4\in\{-1,0,1\}$;
(L2) $\rho_\delta(\hat L)\subset\hat L$ for any $\delta\in\{-1,0,1\}^2$.

Theorem 3. $\hat I=\hat\tau(\hat J_4)+\hat\tau(\hat J_3)+\hat\tau_N(\hat L)$.

A proof repeats almost word by word the proof of Main Theorem in [8] (we ignore the variables q_i when we define the weight function on $\hat F_\infty$).

Each of the modules $\hat J_4$, $\hat J_3$, $\hat L$ is defined as the limit of an increasing sequence of submodules of a finite rank $\hat A$-module. Since $\hat A$ is not Noetherian, this does not give yet a way to compute them. However, we can approximate $\hat A$ by Noetherian rings $\hat A_d=A[v_1,\dots,v_d]$ and the projections $\mathrm{pr}_d(\hat I)$ can be effectively computed where $\mathrm{pr}_d:\hat A\to\hat A_d$ is the quotient by the ideal $(v_{d+1},v_{d+2},\dots)$. Namely, let $(\hat A\hat F_n^{\rm red})_d$, $(\hat J_4)_d$, $(\hat J_3)_d$, $(\hat N)_d$, $(\hat L)_d$ be the $\hat A_d$-modules obtained by the above procedure but with the additional relations $q_i^{d+1}=0$ for any i. Then we have $\mathrm{pr}_d(\hat I)=\hat\tau(\hat J_4)_d+\hat\tau(\hat J_3)_d+\hat\tau_N(\hat L)_d$ and these modules (at least theoretically) can be computed as limits of increasing sequences of Noetherian modules. The rank of $(\hat A\hat F_4^{\rm red})_d$ (the module where $(\hat J_4)_d$ sits) is equal to $315(d+1)^4$. We hope that, at least for $d=1$ or 2, the computations can be performed in practice.

Remark 2. If $\beta=0$ (the case when the Groebner base of I_v was computed in [8]), then the obtained transversal link invariants a priori cannot detect transversally non-simple links. Indeed, in this case we have $1=\alpha\sigma_1^{-1}+\sigma_1^{-3}$, hence $q_1=q_1(\alpha\sigma_1^{-1}+\sigma_1^{-3})=(\alpha\sigma_1^{-1}+\sigma_1^{-3})q_2=q_2$. Thus $q_1=q_2=q_3=\dots$ whence $v_1=v_2=\dots$ and we obtain $M=M_v$, $t=t_v$ and $P_t(L)=(v/u)^{(n-e)/2}P_{t_v}(L)$, i. e., the invariant P_t reduces to a usual link invariant P_{t_v} and Thurston-Bennequin number $n-e$.

Remark 3. By [9], all the computations in the huge module $(\hat{A}\hat{F}_4^{\rm red})_d$ can be done with the coefficients in $\mathbb{Q}$ or in $\mathbb{Z}/m\mathbb{Z}$ for m not very big.

References

1. D. Bennequin, *Entrelacements et équation de Pfaff*, Astérisque **107–108** (1983), 87–161.
2. J. S. Birman, W. M. Menasco, *Stabilization in the braid groups II: Transversal simplicity of knots*, Geom. and Topol. **10** (2006), 1425–1452.
3. D. Fuchs, S. Tabachnikov, *Invariants of Legendrian and transverse knots in the standard contact space*, Topology **36** (1997), 1025–1053.
4. L. Funar, *On cubic Hecke algebras*, Commun. Math. Phys. **173** (1995), 513–558.
5. V. F. R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. **126** (1987), 335–388.
6. R. Lipshitz, L. Ng, S. Sarkar, *On transverse invariants from Khovanov homology*, arXiv:1303.6371.
7. L. Ng, P. Ozsváth, D. Thurston, *Transverse knots distinguished by knot Floer homology*, J. Symplectic Geom. **6** (2008), 461–490.
8. S. Yu. Orevkov, *Markov trace on the Funar algebra*, arXiv:1206.0765.
9. S. Yu. Orevkov, *On modular computation of Groebner bases with integer coefficients*, Zapiski Nauchn. Semin. POMI (to appear) (Russian); English transl., J. of Math. Sciences (to appear).
10. S. Yu. Orevkov, V. V. Shevchishin, *Markov theorem for transversal links*, J. of Knot Theory and its Ramifications **12** (2003), 905–913; Extended version: arXiv:math/0112207.
11. P. S. Ozsváth, Z. Szabó, D. P. Thurston, *Legendrian knots, transverse knots and combinatorial Floer homology*, Geom. and Topol. **12** (2008), 941–980.

Steklov Math. Institute, Moscow

IMT, Université Paul Sabatier (Toulouse-3)
E-mail address: orevkov@math.ups-tlse.fr