PROPER ANALYTIC EMBEDDING OF $\mathbb{CP}^1$ MINUS A CANTOR SET INTO $\mathbb{C}^2$

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In this note we construct a proper embedding of $\mathbb{CP}^1 \setminus K \to \mathbb{C}^2$ where K is a Cantor set. This answers affirmatively to a question asked to me by Burglind Jöricke.

Such a curve is constructed as a limit of algebraic curves A_n obtained from each other by a birational transformation $F_n : \mathbb{C}^2 \to \mathbb{C}^2$. For some exhaustion of $\mathbb{C}^2$ by nested bidisks $B_1 \subset B_2 \subset \dots$ the topological type of $A_n \cap B_n$ does not change under further transformations.

Let us fix any complex numbers $a_1, a_2, \dots$ whose absolute values are strictly increasing and tend to infinity. Let us define inductively a sequence of birational mappings $F_n : \mathbb{C}^2 \to \mathbb{C}^2$ by setting F_0 to be the identity mapping and by setting $F_n = f_n \circ F_{n-1}$ where

$$f_n(x,y) = \begin{cases} \big(x,\, y + g_n(x)\big), & n \text{ odd}, \\ \big(x + g_n(y),\, y\big), & n \text{ even}, \end{cases} \qquad g_n(t) = \frac{\varepsilon_n}{t - a_n}, \quad 0 < \varepsilon_n \ll \varepsilon_{n-1}.$$

Let us denote the one-point compactifications of $\mathbb{C}$ and $\mathbb{C}^2$ by $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $\bar{\mathbb{C}}^2 = \mathbb{C}^2 \cup \{\infty\}$ respectively. Let $\gamma_n : \bar{\mathbb{C}} \to \bar{\mathbb{C}}^2$ be defined by $\gamma_n(z) = F_n(z, 0)$. Then, for a suitable choice of the small parameters ε_n, the limit of γ_n is a continuous mapping (let us denote it by $\gamma : \bar{\mathbb{C}} \to \bar{\mathbb{C}}^2$) such that $K = \gamma^{-1}(\infty)$ is a Cantor set and the restriction of γ to $\bar{\mathbb{C}} \setminus K$ is a proper embedding of of the open Riemann surface $\bar{\mathbb{C}} \setminus K$ into $\mathbb{C}^2$.

Let us describe more precisely the choice of the parameters ε_n, and this will explain why γ satisfies the required properties. Let us fix positive numbers R_n such that $|a_n| < R_n < |a_{n+1}|$. Let $A_n = F_n(\bar{\mathbb{C}})$, $D_n = \{z \in \mathbb{C} : |z| < R_n\}$. Let us denote the projection $(z_1, z_2) \mapsto z_i$ by $\mathrm{pr}_i : \mathbb{C}^2 \to \mathbb{C}$, $i = 1, 2$ and let us set $C_n^{(i)} = \mathrm{pr}_i^{-1}(D_n)$, $B_n = C_n^{(1)} \cap C_n^{(2)} = D_n \times D_n$, and $C_n = C_n^{(1)} \cup C_n^{(2)}$. Then $B_1 \subset B_2 \subset \dots$ and $\bigcup_n B_n = \mathbb{C}^2$. We define ε_n inductively so that they satisfy:

(1) $A_n \subset C_n$;
(2) $A_n \cap (C_n^{(i)} \setminus B_n)$, $i = 1, 2$, has a finite number of connected components each being mapped biholomorphically onto $\mathbb{C} \setminus D_n$ by the projection pr_i;
(3) For any fixed n, all the curves $A_p \cap B_n$ for $p \ge n$, are isotopic to each other in B_n and they $\mathcal{C}^\infty$-smoothly converge to an analytic curve which is also isotopic to all of them;
(4) $\lim_{n\to\infty} d_n = 0$ where d_n is the maximum of the diameters (with respect to some fixed metric on $\bar{\mathbb{C}}$) of the connected components of $F_n^{-1}(A_n \setminus B_n)$.

Let us call a boundary component of $A_n \cap B_n$ *horizontal* if it is contained in $(\partial D_n) \times D_n$ and *vertical* if it is contained in $D_n \times (\partial D_n)$ (it follows from the condition (1) that there are no other components). If the parameters ε_n are chosen as is described, then, up to a small perturbation, $A_{2n+1} \cap B_{2n+1}$ is obtained from $A_{2n} \cap B_{2n}$ by attaching an annulus to each vertical boundary component and by attaching a pair of pants (an annulus with a hole) to each horizontal one. So each vertical component at the $2n$-th step provides a single vertical components at the next step, but each horizontal component provides one horizontal and one vertical component at the next step. When passing from $A_{2n+1} \cap B_{2n+1}$ to $A_{2n+2} \cap B_{2n+2}$, the roles of vertical and horizontal boundary components are exchanged.

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