RIEMANN EXISTENCE THEOREM AND CONSTRUCTION OF REAL ALGEBRAIC CURVES

Stepan Yu. Orevkov

Abstract. We propose a method of construction of plane real algebraic curves given by $y^3+p(x)y+q(x)=0$ which has a required arrangement on the affine plane. The construction is based on a consideration of the arrangement of $f^{-1}(\mathbf{RP}^1)$ on $\mathbf{CP}^1$ where $f:\mathbf{CP}^1\to\mathbf{CP}^1$ is the homogenized discriminant, i.e. the rational function defined by $f(x)=D/q^2$, $D=4p^3+27q^2$.

As examples of applications, we construct some M-curves of degrees 7 and 9 on $\mathbf{RP}^2$ whose realizability was unknown.

Résumé. On propose une méthode de construction des courbes algébriques réelles planes données par $y^3+p(x)y+q(x)=0$, qui ont un arrangement préscrit sur le plan affine. La construction est basée sur la considération de l'arrangement de $f^{-1}(\mathbf{RP}^1)$ sur $\mathbf{CP}^1$ où $f:\mathbf{CP}^1\to\mathbf{CP}^1$ est le discriminant homogénisé, i.e. la fonction rationnelle définie par $f(x)=D/q^2$, $D=4p^3+27q^2$.

Comme exemple d'applications, on construit certaines M-courbes de degrés 7 et 9 dont la réalisabilité n'était pas connue.

1. Introduction. In this paper we propose a method of construction of plane real algebraic curves given by $f(x,y)=0$, $\deg_y f=3$ which has a required arrangement on the affine plane. This method allows one to obtain a complete classification of such curves (singular or not) up to semi-rigid isotopy of the plane. We call an isotopy *semi-rigid* if the image of any vertical line at any moment is a vertical line. In particular, this means that the same method allows to obtain some restrictions for trigonal curves.

This result will be published in the next paper. Here we just illustrate the method of construction by a realization of a complex M-scheme[1] of degree 7 and some real M-schemes of degree 9 on $\mathbf{RP}^2$ whose realisability was previousely unknown.

The proposed method was inspired by a construction of extremal polynomials for the Davenport's bound $\deg(p^3-q^2)\ge 1+(\deg p)/2$ in terms of so-called "dessins d'enfant" (see Sect. 3).

Proposition 1. *There exists an M-curve of degree 9 on $\mathbf{RP}^2$ whose real scheme is $\langle J\sqcup 1\langle 8\rangle\sqcup 1\langle 14\rangle\sqcup 4\rangle$.*

Following [3], we say that a curve of degree 7 on $\mathbf{RP}^2$ has a *jump* if it contains 5 ovals arranged with respect to some line as it is shown in Figure 1.

[1] See [12] for the definition and notation of real and complex schemes.

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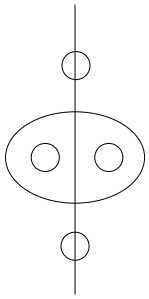


FIG. 1

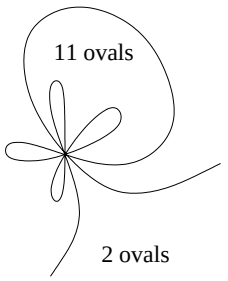


FIG. 2

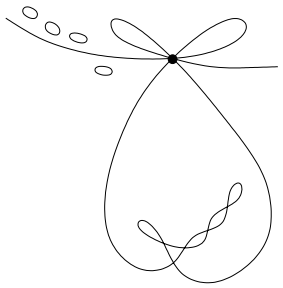


FIG. 3

Proposition 2. *There exists an M-curve of degree 7 on $\mathbf{RP}^2$ without a jump whose complex scheme is $\langle J \sqcup 5_+ \sqcup 4_- \sqcup 1_+\langle 2_+ \sqcup 3_-\rangle\rangle$.*

Combined with the results of [13, 3, 9, 10, 4], Proposition 2 provides the classification of complex schemes of M-curves of degree 7 without a jump (for curves with jump, this classification is already complete in the papers cited above).

In Sect. 5 (Proposition 3), we prove the realisability of some other M-schemes of degree 9. All the curves in Proposition 3 are constructed by glueing affine sextics into a 6-fold singular point of a curve of degree 9. We choosed this example because the same glueing was used in [6] but our method allows us to obtain more curves of degree 9.

2. Preparation. To construct these curves, we construct singular curves depicted in Figures 2 and 3 and then perturb the singularities glueing (see [13]) the affine sextic depicted in Figure 4 (resp. quartic in Figure 5) into the 6-fold (resp. quadruple) point.

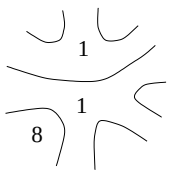


FIG. 4 (SEE [6])

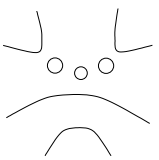


FIG. 5

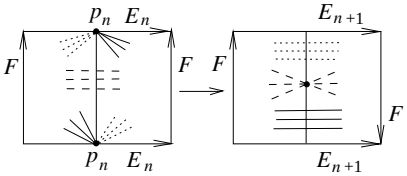


FIG. 6

Denote by F_n the Hirzebruch surface of degree n and let E_n be the exceptional section (whose self-intersection is $-n$). In particular, $F_0 = \mathbf{P}^1 \times \mathbf{P}^1$, F_1 is the blown-up $\mathbf{P}^2$. Let $\pi_n : F_n \to \mathbf{P}^1$ be the fibration with fiber $\mathbf{P}^1$. The surfaces $F_1, F_2, \dots$ can be obtained from F_0 by successive birational transformations $F_0 \overset{\beta_0(p_0)}{--\to} F_1 \overset{\beta_1(p_1)}{--\to} F_2 \overset{\beta_2(p_2)}{--\to} \dots$ where β_n is the blowup of a point $p_n \in E_n$ followed by the blowdown of the strict preimage of the fiber $\pi_n^{-1}(\pi_n(p_n))$. If the points $p_0, p_1, \dots$ are real then all F_n are also real.

We present F_n in pictures as a rectangle obtained by cutting F_n along E_n (horizontal edges) and a fiber (vertical edges). The interior of such a rectangle corresponds to an affine coordinate system on F_n where a generic smooth curve C is defined by a polynomial whose Newton polygon is $(0,0)$-$(l+kn,0)$-(l,k)-$(0,k)$

where k (resp. l) is the intersection of C with a fiber (resp. with E_n). We call (k,l) the *bidegree* of C. The action of β_n (for an even n) on the real part $\mathbf{R}F_n$ of F_n is shown in Fig. 6. We see in this picture that $\mathbf{R}F_n$ is a torus for an even n and a Klein bottle for an odd n.

Since $\mathbf{R}F_1$ is the blown-up $\mathbf{RP}^2$, the existence of the curve in Fig. 2 (resp. Fig. 3) follows from the existence of a curve of bidegree $(3,6)$ (resp. $(3,4)$) arranged on $\mathbf{R}F_1$ as it is shown in Fig. 7 (resp. Fig. 8).

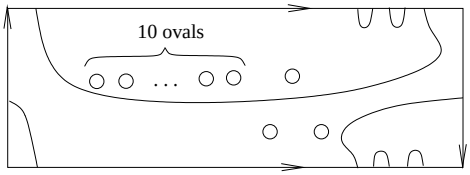


FIG. 7

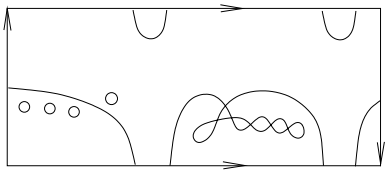


FIG. 8

If a curve C of bidegree (k,l) on F_n has multiplicity m at the point p_n then the strict transform of C on F_{n+1} under $\beta_n(p_n)$ has bidegree $(k, l-m)$. Hence, applying $\beta_1(p_1)\circ\cdots\circ\beta_h(p_h)$ to the curve C in Fig. 7 for $h=6$ and $\{p_1\dots p_h\} = C\cap E_1$, we obtain the curve of bidegree $(3,0)$ on F_7 whose real part is depicted in the upper part of Fig. 9. Analogously, for $h=4$, we obtain Fig. 10 from Fig. 8. The isolated points in Fig. 9 and Fig. 10 are simple double points with imaginary tangents (like $x^2+y^2=0$). The curves in Fig. 9 and Fig. 10 will be constructed in Sect. 4.

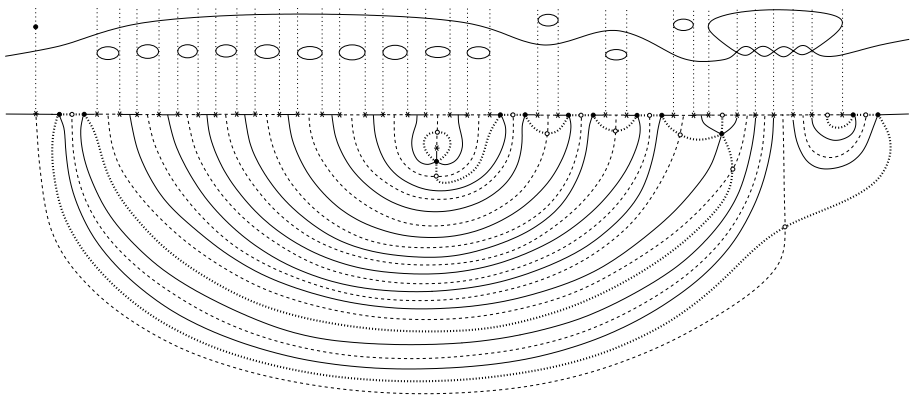


FIG. 9

3. Degression. Let $p(t), q(t) \in \mathbf{C}[t]$, $\deg p = 2k$, $\deg q = 3k$ and put

$$r(t) = p^3 - q^2. \tag{1}$$

Suppose r is not identically zero. How small can be $\deg r$? This question was posed in [1] in 1965. The same year Davenport [2] had shown that $\deg r \geq k+1$ but it was unknown if the estimate is sharp. This estimate is very natural. Indeed, if we write $p = t^{2k} + a_1 t^{2k-2} + \cdots + a_{2k-1}$, $q = t^{3k} + b_1 t^{3k-2} + \cdots + b_{3k-1}$ with indeterminate coefficients then the condition $\deg r \leq k+1$ imposes $5k-2$ equations

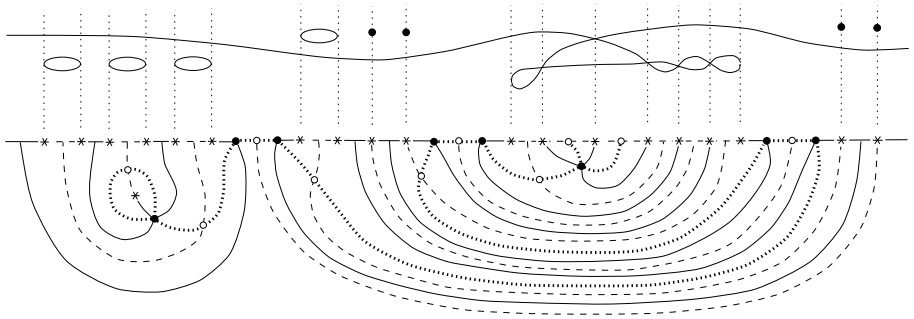


FIG. 10

that is equal exactly to the number of the unknowns. However, it is very hard to show algebraically that this system has solutions other than $p = s^2$, $q = s^3$, $s = t^k + c_1 t^{k-2} + \cdots + c_{k-1}$.

Stothers [11] proved the sharpness of Davenport's estimate for any k. His result was rediscovered by Zannier [14]. A. Zvonkin gave another (?) elegant proof but he did not publish it because he claims that his proof coincides with the Zannier's one. However, he kindly permitted us to publish his proof and we do it in this section.

The main idea is to divide the both sides of (1) by q^2. Denote the obtained rational function by f. Then $f(t) = r/q^2 = p^3/q^2 + 1$. This means that

(i) f has $3k$ poles of multiplicity 2 at the roots of q,

(ii) the equation $f = 1$ has $2k$ triple roots at the roots of p,

and if $\deg r = k + 1$ then

(iii) f has a zero of multiplicity $5k - 1$ at $t = \infty$.

Conversely, any rational function f of degree $6k$ satisfying (i)–(iii), defines the required p and q.

From the topological point of view f is a branched covering $\mathbf{CP}^1 \to \mathbf{CP}^1$. Denote the preimage of the real segment $[1, +\infty]$ by Γ. If (i)–(iii) hold then Γ is a graph on $\mathbf{CP}^1$ whose vertices $f^{-1}(\infty)$ have valence 2 (white vertices) and the vertices $f^{-1}(1)$ have valence 3 (black vertices). The graph Γ cuts $\mathbf{CP}^1$ into polygons homeomorphic to a disc, one of which should have $5k - 1$ white vertices and $5k - 1$ black ones.

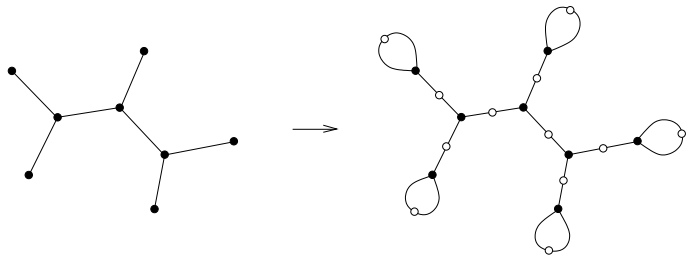


FIG. 11

Now we are ready to construct f. Let us start with any binary tree in S^2 with $k - 1$ triple vertices and $k + 1$ ends and transform it to the graph Γ as it is shown in Figure 11. Define the mapping $\Gamma \to [1, \infty]$ which takes white vertices to ∞

and black ones to 1, and extend it continuously to a mapping $f : S^2 \to \mathbf{CP}^1$ which maps each bigon homeomorphically onto $\mathbf{CP}^1 \setminus [1, \infty]$ and whose restriction onto the polygonal component of $S^2 \setminus \Gamma$ is a branched cyclic covering ramified in a single point t_0 such that $f(t_0) = 0$. Pull back the complex structure from $\mathbf{CP}^1$ to S^2. By Riemann's theorem, the obtained surface is isomorphic to $\mathbf{CP}^1$. Choose the isomorphism so that it takes t_0 into ∞. Than f becomes a rational function satisfying (i)–(iii).

Remark 1. Given any k and coprime a, b, similar arguments allow to construct polynomials p, q, and $r = p^b - q^a$ such that $\deg p = ak$, $\deg q = bk$, $\deg r = (ab - a - b)k + 1$. Like is the case $a = 3$, $b = 2$, this is the minimal possible value for $\deg r$.

4. Construction. Let us construct a curve C of bidegree $(3, 0)$ on F_n ($n = 7$ or 5) whose real part is depicted in Figures 9 or 10. It is defined by a polynomial whose Newton polygon is the triangle $(0, 0)$-$(3n, 0)$-$(0, 3)$. Killing the coefficient of y^2, we put C into the form

$$y^3 + p(x)y + q(x) = 0, \qquad \deg p = 2n, \quad \deg q = 3n. \tag{2}$$

The discriminant of (2) with respect to y is

$$D(x) = 4p^3 + 27q^2. \tag{3}$$

Let x_0 be a root of D ("$*$" in Figures 9 and 10). This means that either x_0 is the x-coordinate of a double point of C (then x_0 is a double root of D) or the vertical line $x = x_0$ is tangent to C (a simple root of D). Let $F(y) = y^3 + p(x_0)y + q(x_0) = (y - y_1)(y - y_2)^2$. Since the coefficient of y^2 vanishes, y_1 and y_2 have opposite signs. Hence, $q(x_0) = F(0) > 0$ when $y_1 < y_2$, and $q(x_0) = F(0) < 0$ when $y_2 < y_1$. This means that the real roots of q ("$\circ$" in Figures 9 and 10) must separate the root of D where $y_1 < y_2$ from those where $y_2 < y_1$. Thus, to construct C, we need to find polynomials $p(x)$, $q(x)$, and $D(x)$, satisfying (2), (3) such that the real roots of D and q are arranged as in Figures 9 and 10.

FIG. 12

Now we apply the main idea of Sect. 3: let us divide (3) by q^2. The result is a rational function $f(x) = D/q^2 = 4p^3/q^2 + 27$ whose poles are the roots of q taken with multiplicity 2, zeros are the roots of D, and the solutions of $f = 27$ are roots of p taken with multiplicity 3. To construct f, consider the graph $\Gamma \subset S^2$ depicted in the lower parts of Figures 9 and 10 (since Γ is symmetric, we show only half of it). Let us map Γ onto $\mathbf{RP}^1$ according to the coloring in Figure 12 and then continue this mapping to a branched covering $f : S^2 \to \mathbf{CP}^1$ sending homeomorphically each component of $S^2 \setminus \Gamma$ onto one of the half-planes of $\mathbf{CP}^1 \setminus \mathbf{RP}^1$ in an alternating order. The additional vertices (which are not mapped onto 0, 27, or ∞) are mapped to arbitrarily choosen points on the corresponding segment of $\mathbf{RP}^2$. Then the pull-back of the complex structure makes f to be a rational function which has the

needed properties. Due to the symmetry principle, f becomes real in suitable coordinates.

In conclusion of this section, let us give a list of condition on a colored embedded graph $\Gamma \subset S^2$ which are sufficient to apply our construction (these conditions are satisfied in the constructions of Section 5).

(1) The graph Γ is symmetric with respect to an equator of S^3 (which is included to Γ) and the coloring of the equator is imposed by the desired arrangement of the real algebraic curve as it is explained above;
(2) The valence of each "$\bullet$" is divisible by 6, and the incident edges are colored alternatively by the colors of the segments $[0, 27]$ and $[27, \infty]$;
(3) The valence of each "$\circ$" is divisible by 4, and the incident edges are colored alternatively by the colors of the segments $[27, \infty]$ and $[\infty, 0]$;
(4) The valence of each "$*$" is even, and the incident edges are colored alternatively by the colors of the segments $[\infty, 0]$ and $[0, 27]$;
(5) The valence of each non-colored vertex is even, and the incident edges are of the same color;
(6) Each connected component of $S^3 \setminus \Gamma$ is homeomorphic to an open disk whose boundary (considered as the set of Carathéodory boundary elements) is colored as a covering of $\mathbf{RP}^1$. Moreover, the orientations of the boundaries (induced by the coverings) of neighbouring disks are opposite.

5. Other M-curves of degree 9.

Proposition 3. *(a). There exist M-curves of degree 9 whose real schemes are*

2) $\langle J \sqcup \alpha \sqcup 1\langle\beta\rangle\rangle$, $\alpha = 27 - \beta$, $\beta = 17^*, 20, 21$
3) $\langle J \sqcup \alpha \sqcup 1\langle\beta\rangle \sqcup 1\langle\gamma\rangle\rangle$, $\alpha = 26 - \beta - \gamma$ *where*
$\beta = 1$, $\gamma = 20, 21$
$\beta = 2$, $\gamma = 11^*, 12^*, 13, 15, 16^*, 17, 19, 20$
$\beta = 3$, $\gamma = 14, 17$
$\beta = 4$, $\gamma = 9, 11, 13, 14, 17, 18$
$\beta = 5$, $\gamma = 12, 13^*, 14, 15, 16, 17$
$\beta = 6$, $\gamma = 11, 12, 14$
$\beta = 7$, $\gamma = 14, 15$
$\beta = 8$, $\gamma = 9, 11, 12, 13, 14$
$\beta = 9$, $\gamma = 9, 10, 11$

4) $\langle J \sqcup \alpha \sqcup 1\langle\beta\rangle \sqcup 1\langle\gamma\rangle \sqcup 1\langle\delta\rangle\rangle$, $\alpha = 25 - \beta - \gamma - \delta$ *where*
$(\beta, \gamma) = (1, 1)$, $\delta = 8, 12, 15, 16, 17, 22$
$(\beta, \gamma) = (1, 3)$, $\delta = 13, 14, 15, 16$
$(\beta, \gamma) = (1, 5)$, $\delta = 8, 12$
$(\beta, \gamma) = (1, 6)$, $\delta = 13$
$(\beta, \gamma) = (1, 8)$, $\delta = 9$
$(\beta, \gamma) = (1, 9)$, $\delta = 11, 12, 14$
$(\beta, \gamma) = (1, 10)$, $\delta = 13$
$(\beta, \gamma) = (2, 3)$, $\delta = 9$
$(\beta, \gamma) = (3, 3)$, $\delta = 8$
$(\beta, \gamma) = (3, 5)$, $\delta = 9$
$(\beta, \gamma) = (3, 7)$, $\delta = 8$
$(\beta, \gamma) = (4, 5)$, $\delta = 5$

$(\beta,\gamma)=(5,5)$, $\delta=8,10$
$(\beta,\gamma)=(7,7)$, $\delta=10$

6) $\langle J\sqcup\alpha\sqcup 1\langle\beta\sqcup 1\langle\gamma\rangle\rangle\rangle$, $\alpha=26-\beta-\gamma$ *where*
$\gamma=1$, $\beta=1^*,17,22$
$\gamma=3$, $\beta=1^*,4,5^*,8$

(b). There exist flexible[2] *M-curves of degree 9 whose real schemes are*

3) $\langle J\sqcup 3\sqcup 1\langle 5\rangle\sqcup 1\langle 18\rangle\rangle$, *i.e.* $\beta=5$, $\gamma=18$
4) $\langle J\sqcup\alpha\sqcup 1\langle\beta\rangle\sqcup 1\langle\gamma\rangle\sqcup 1\langle\delta\rangle\rangle$, $\alpha=25-\beta-\gamma-\delta$ *where*
$(\beta,\gamma,\delta)=(1,1,19),(1,7,13),(5,7,9)$

6) $\langle J\sqcup\alpha\sqcup 1\langle\beta\sqcup 1\langle\gamma\rangle\rangle\rangle$, $\alpha=26-\beta-\gamma$ *where*
$\gamma=5$, $\beta=2,3,5,8,9$
$\gamma=7$, $\beta=1,3,4$

Remark 2. The list of the real schemes in Proposition 3 is given is the same format as the list in [5; Theorem 6] (which explains, in particular, such a strange numbering of the series). We did not include in our list the M-schemes which are in [5] but we included those constructed in [6] (marked by the asterisk) and in Proposition 1.

Remark 3. We found the following misprints in the list in [5].

In [5; Theorem 6, Series 4], there should be "$(\beta,\gamma)=(5,7)$, $\delta=8,10$" instead of "$(\beta,\gamma)=(5,8)$, $\delta=8,10$".

In [5; Theorem 6, Series 5], there should be $(\alpha,\beta,\gamma,\delta)=(1,1,3,15)$ instead of $(1,1,3,13)$.

In [5; Theorem 7], "α,β,γ – even" in Series 4 means "each of α,β,γ is even" whearas "$\alpha,\beta,\gamma,\delta$ – even" in Series 6 means "one of α,β,γ is even".

We shall call *central block* the rectangles depicted in the upper parts of Figures 13.1–13.5 or their images under the reflection with respect to a vertical or a horizontal line.

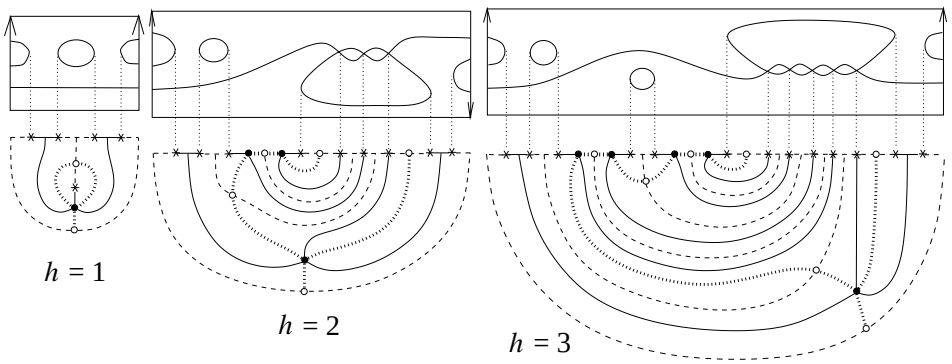


FIG. 13.1 FIG. 13.2 FIG. 13.3

[2]See [12] for the definition of a flexible curve.

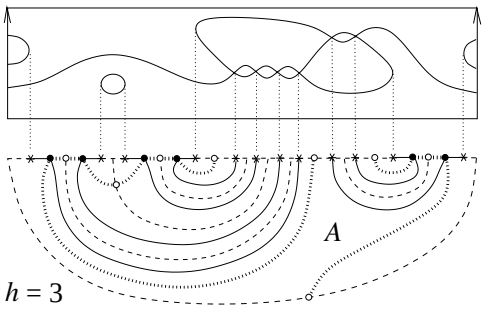


FIG. 13.4

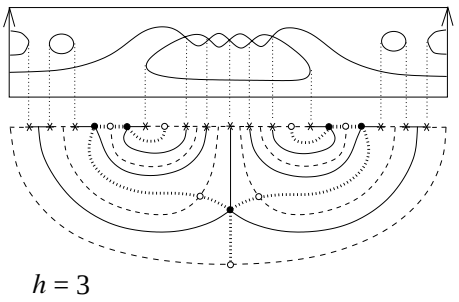


FIG. 13.5

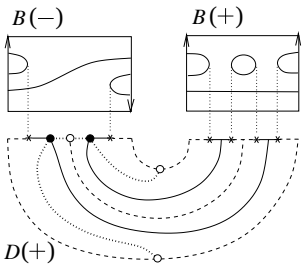


FIG. 14

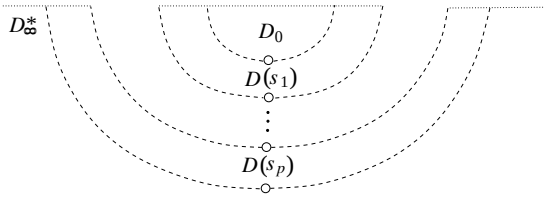


FIG. 15

Lemma 1. *Let B_0, B_∞ be two central blocks and h_0, h_∞ the corresponding values of the parameter h (indicated in 13.1–13.5). Then for any sequence of signs $s_1, \dots, s_p$, $p \geq 0$, there exists a real algebraic curve C of bidegree $(3,0)$ on F_n, $n = p + h_0 + h_\infty$ such that the pair $(F_n \setminus E_n, C)$ (recall that $E_n \subset F_n$, $E_n^2 = -n$) is obtained (up to a semi-rigid isotopy, see the Introduction) by the successive cyclic glueing (according to the arrows) of the blocks B_0, $B(s_1), \dots, B(s_p)$, B_∞, $B(-s_p), \dots, B(-s_1)$, where $B(+)$, $B(-)$ are shown in the upper part of Fig. 14.*

Pooof. Let D_0 and D_∞ be the half-discs shown below the blocks B_0 and B_∞ in Figures 13.1–13.5, and let D_∞^* be the image of D_∞ under the inversion. Let $D(+)$ be the half-annulus shown in the lower part of Fig. 14 and $D(-)$ be its mirror image. Let us fill the lower half-plane by the domains $D_0, D(s_1), \dots, D(s_p), D_\infty^*$ according to Fig. 15, and do symmetrically the upper half-plane. Let us deal with the obtained graph Γ in the same way as in Sect. 4 (in the case 13.4, to construct the mapping of the region A, one should take a double covering branched at a single point). $\square$

Example. We show in Fig.16 how the curve in Fig. 9 can be obtained from the central blocks 13.1 and 13.3 by applying Lemma 1 followed by a contracting of an oval (see Lemma 2 below).

Corollary 1. *Let $(s_1, \dots, s_p)$, $p \geq 1$, $s_i = \pm 1$, be an arbitrary sequence of signs such that $s_1 = 1$. Let $a_1, \dots, a_p$ be non-negative integers such that*

$$(1, s_1, \dots, s_p, 1, -s_p, \dots, -s_1) = (\underbrace{1, \dots, 1}_{a_1}, -1, \underbrace{1, \dots, 1}_{a_2}, -1, \dots, \underbrace{1, \dots, 1}_{a_p}, -1).$$

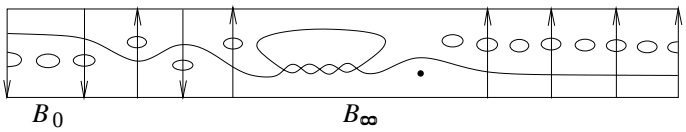


FIG. 16

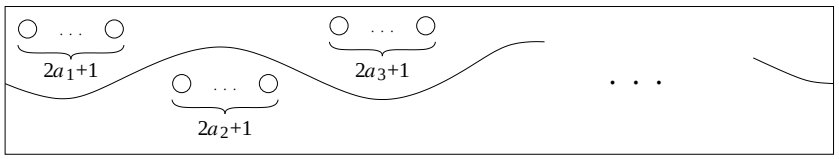


FIG. 17

Then there exists a curve of bidegree $(3,0)$ *on* F_{p+2} *arranges as in Fig. 17.*

Proof. Set $B_0 = B_\infty =$ [the block in Fig. 13.1] in Lemma 1. □

Corollary 2. *There exist curves of bidegree* $(3,0)$ *arranged on* F_7 *as in Figure 17 with* $p = 5$ *and* $(a_1,\dots,a_5) = (15,1,1,1,1)$, $(11,5,1,1,1)$, $(9,5,3,1,1)$, $(9,1,7,1,1)$, $(7,5,1,1,5)$, $(7,3,5,3,1)$, $(7,1,3,1,7)$, $(5,3,5,3,3)$, *or* $(5,5,5,1,3)$. □

Lemma 2. *Let* $A \subset F_{p+2}$ *be a curve constructed in Lemma 1 (or in Corollaries 1 and 2). Then there exist a nodal curve* $A' \subset F_{p+2}$ *of the same bidegree obtained from A by applying of any number of transformations shown in Figure 18.*

FIG. 18

FIG. 19

Proof. Apply the transformations in Figure 19 to the graph Γ. □

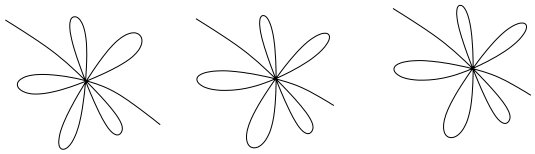
Remark 4. Corollary 1 can be easily proved by Viro's method using the subdivision of the triangle $(0,0)$-$(3p+6,0)$-$(0,3)$ into $n+2$ triangles $(3p,0)$-$(3p+3,0)$-$(0,3)$. However, it is not clear how to prove Lemma 2 in this way.

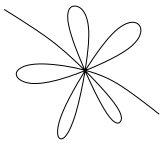
Corollary 3. *There exist curves of degree 9 with a simple* 6*-fold singular point and 13 ovals distributed as in Figure 20 where a and b are the number of ovals in the corresponding regions, the exterior ovals are not shown, and* $S_1 = \{(a,b) \mid a+b \le 10$ *and a, b are odd*$\}$, $S_2 = S_1 \cup \{(1,11)\} \setminus \{(5,5)\}$.

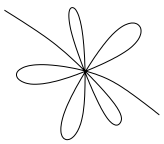
Proof. Apply Lemma 2 and the transformation $\beta_1(p_1)\dots\beta_6(p_6)$ (see Sect. 2) to the curves from Lemma 1. The curves from Corollary 2 provide the upper two rows of Fig. 20.1. In the other cases, one should choose the central blocks in Lemma 1 in the following way.

The curves in the lower row in Fig. 20.1. The left: (13.1 and 13.3) or (13.1 and 13.5); the middle: (13.1 and 13.4); the right: (13.2 and 13.2).

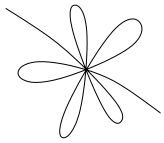
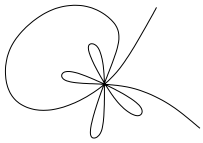
The curves in Fig. 20.2 and 20.3. The upper left curve in each of Fig. 20.2 and 20.3: (13.1 and 13.4); the other curves: (13.1 and 13.2). □







(a,b) $\in S_1$



(a,b) $\in S_2$