COMPLEX ORIENTATIONS OF M-CURVES OF DEGREE 7

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A real algebraic non-singular curve A in $\mathbf{RP}^2$ is called an *M-curve* if the set of its real points $\mathbf{R}A$ has the maximal possible number of connected components $(m-1)(m-2)/2+1$ where m is the degree of A. The complexification of an M-curve A is divided by $\mathbf{R}A$ into 2 halves. Each half induces the boundary orientation on $\mathbf{R}A$ which is called the *complex orientation*. Suppose $\deg A$ is odd. An oval O of $\mathbf{R}A$ is called *positive* if $[O] = -2[J] \in H_1(\mathbf{RP}^2 \setminus \operatorname{Int} O)$ where J is the odd branch of $\mathbf{R}A$, otherwise O is called *negative*.

The *real scheme of A* is the isotopy type of $(\mathbf{RP}^2, \mathbf{R}A)$, the *complex scheme of A* is the isotopy type of $(\mathbf{RP}^2, \mathbf{R}A)$ where $\mathbf{R}A$ is considered together with the complex orientation. We shall use the notation of real and complex schemes proposed by Viro (see [11]). For instance,

$$\langle J \sqcup \beta_+ \sqcup \beta_- \sqcup 1_\varepsilon \langle \alpha_+ \sqcup \alpha_- \rangle\rangle, \qquad \varepsilon = \pm, \tag{1}$$

denotes the complex scheme in Figure 1. It consists of the odd branch J, an oval O (the *non-empty oval*) which is positive for $\varepsilon = +$ and negative for $\varepsilon = -$, there are α_+ positive and α_- negative ovals inside O (*interior ovals*), and β_+ positive and β_- negative ovals outside O (*exterior ovals*).

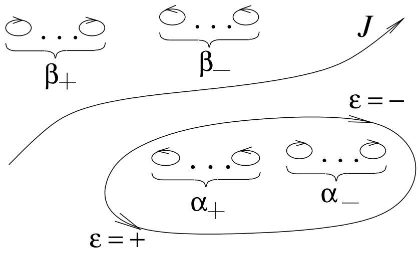

Fig. 1

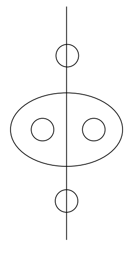

Fig. 2

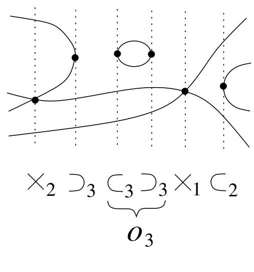

Fig. 3

The classification up to isotopy of non-singular real curves of degree 7 was obtained by Viro [10]. The complete list of all realisable real schemes is $\langle J \rangle$, $\langle J \sqcup 1\langle 1\langle 1\rangle\rangle\rangle$, and $\langle J \sqcup \beta \sqcup 1\langle\alpha\rangle\rangle$ where $\alpha + \beta \le 14$, $\alpha < 14$. Here we study complex M-schemes of degree 7. All they are of the form (1) with $\alpha_+ + \alpha_- + \beta_+ + \beta_- = 14$, $\alpha_+ + \alpha_- < 14$ but not every such a scheme is realisable. The first restriction is provided by Rokhlin–Mishachev formula for complex orientations: $(\beta_+ - \beta_-) + (1 - 2\varepsilon)(\alpha_+ - \alpha_-) + \varepsilon = 3$. Another formula for complex orientations [5; Sect. 1.5] yields $\beta_+ - \beta_- = \alpha_- - \alpha_+ = 1$ if $\varepsilon = +1$. However, these conditions are not sufficient for a complex scheme to be realisable.

A very detailed study of complex M-schemes of degree 7 was performed by S. Fiedler [1]. By investigating the behaviour of a curve with respect to certain pencils of cubics, she eliminated almost all complex schemes for whom realisability was open. One more complex scheme (namely, $\langle J \sqcup 10_+ \sqcup 3_- \sqcup 1_-\langle 1_-\rangle\rangle$) is excluded in [6]. Here we prove the non-realisability of the remaining two complex schemes. Following [2], we say that a curve of degree 7 has a *jump* if it contains 5 ovals arranged as in Figure 2.

Theorem 1. *There does not exist an M-curve of degree 7 without jumps whose complex scheme is either*

$$\langle J \sqcup 2_+ \sqcup 1_- \sqcup 1_-\langle 6_+ \sqcup 5_-\rangle\rangle \tag{2}$$

or

$$\langle J \sqcup 8_+ \sqcup 4_- \sqcup 1_-\langle 1_+ \sqcup 1_-\rangle\rangle. \tag{3}$$

The non-realisability of (2), (3) by curves of degree 7 with a jump was proved in [1]. Thus, we have

Corollary. *There is no curve of degree 7 with complex scheme (2) or (3).*

Theorem 2. *For $1 \le k \le 5$ there exist an M-curves of degree 7 with a jump whose complex scheme is $\langle J \sqcup (7-k)_+ \sqcup (6-k)_- \sqcup 1_+\langle k_+ \sqcup (k+1)_-\rangle\rangle$.*

In addition to the classification of complex 7th degree M-schemes, for any scheme S. Fiedler also studied whether it is realisable by a curve with or without jumps. Theorems 1 and 2 together with the results of [1], [6] and the construction [7] of the scheme $\langle J \sqcup 5_+ \sqcup 4_- \sqcup 1_+\langle 2_+ \sqcup 3_-\rangle\rangle$ without a jump completes this classification also.

For the proof of Theorem 1 we use the method [5] based on application of Murasugi–Tristram inequality to certain link in $L \subset S^3$ and a surface $N \subset B^4$ bounded by L. This link is determined by the arrangement of $\mathbf{R}A$ with respect to a pencil of line (it is the intersection of $\mathbf{C}A$ with the union of the complex lines of the pencil parametrised by a circuit around one of the halves of $\mathbf{CP}^1 \setminus \mathbf{RP}^1$). We improve this method using vanishing cycles coming from other pencils of lines (see Sect. 4).

In some cases we apply Murasugi–Tristram inequality to the double covering of B^4 branched along a component of N which is an unknotted disk. We applied this idea in [5; Sect. 5.8] but it was another proof of a result proved already with Murasugi–Tristram inequality for twisted signatures. Here we use the double covering only when this is really necessary.

Theorem 2 is proved in Sect. 6 using Viro's method [12]. In Appendix 1 we apply the same methods to the 8th degree schemes $\langle 4 \sqcup 1\langle 2 \sqcup 1\langle 14\rangle\rangle$ and $\langle 14 \sqcup 1\langle 2 \sqcup 1\langle 4\rangle\rangle$. These are two of the 9 real M-schemes for whom realizability is still open. For each scheme we leave only 3 possible arrangements of ovals with respect real lines and we show that 2 of them are realizable by flexible curves.

In Appendix 2 we illustrate how the using of Göritz matrix instead of Seifert matrix may reduce the volume of computations so that everything can be checked without a computer. We prove non-realizability of certain $(M-1)$-perturbation of the singularity X_{21} (four smooth real branches with simple tangency). The question of its realizability was open.

1. The method of prohibition

For the reader's convenience, we first describe briefly the method proposed in [5]. Let A be a nodal real algebraic curve of degree m and $\mathcal{L}_p$ be the pencil of lines through a point $p \in \mathbf{RP}^2 \setminus \mathbf{R}A$ in general position. We shall suppose that any real line $l \in \mathcal{L}_p$ meets A at least at $m-2$ real points (counting the multiplicities)[1] and there exists a real line $l_\infty \in \mathcal{L}_p$ meeting A transversally at m distinct real points.

We encode the arrangement of $\mathbf{R}A$ with respect to $\mathcal{L}_p$ (the *$\mathcal{L}_p$-scheme of* $\mathbf{R}A$) as follows. Choose affine coordinates so that the line l_∞ is infinite and the other lines of $\mathcal{L}_p$ are vertical. Let us move a vertical ruler from left to the right. Each time it is tangent to $\mathbf{R}A$ at a point x we write the symbol $\supset_{k+1}$ (if $\mathbf{R}A$ is to the left of x) or $\subset_{k+1}$ (if $\mathbf{R}A$ is to the right of x). When the ruler passes through a double point x, we write $\times_{k+1}$. In all the cases k is the number of the intersection points below x (see Figure 3). We abbreviate $\subset_k \supset_k$ to o_k (an empty oval in the k-th band).

If l_t, $t \in [0,1]$, is a generic closed path in $\mathcal{L}_p$, it defines a braid on m strings $b \in B_m$. Indeed, the m points $\mathbf{C}l_t \cap \mathbf{C}A$ travel on the complex plane (we may identify all l_t using the chosen affine coordinates). Let b be the braid corresponding to a simple closed path surrounding all the lines from the upper half-plane of $\mathcal{L}_p$ which are tangent to A (when we say of the upper half-plane, we identify $\mathcal{L}_p$ with $\mathbf{CP}^1$).

A braid b obtained in this way from a real algebraic curve is *quasipositive*, i.e. $b = \prod w_i \sigma_{j_i} w_i^{-1}$, $w_i \in B_m$. A necessary condition for the quasipositivity is provided by Murasugi–Tristram inequality which can be rewritten in this case as

$$n_\zeta(\hat{b}) \geq |\sigma_\zeta(\hat{b})| + m - e(b) \tag{4}$$

where $\zeta \in \mathbf{C}$, $|\zeta| = 1$, the link $\hat{b}$ is the closure of the braid b in the 3-sphere, $e : B_n \to \mathbf{Z}$ is the homomorphism which take each generator σ_i, $i = 1, \dots, m-1$ into 1, and for a link L we denote $\sigma_\zeta(L) = \operatorname{signature}(V_\zeta)$, $n_\zeta(L) = 1 + \operatorname{nullity}(V_\zeta)$ where $V_\zeta = (1-\zeta)V + (1-\bar{\zeta})V^T$ for a Seifert matrix V of L corresponding to a connected Seifert surface (see [5]). The link $L = \hat{b}$ is considered here with the orientation induced by the projection from p onto $\mathbf{RP}^1$ (we call it the *braid orientation*).

Under the assumptions about $\mathcal{L}_p$ formulated in the beginning of this section, the braid b is determined by the $\mathcal{L}_p$-scheme of $\mathbf{R}A$ and it can be easily computed as follows. Put $\pi_{k,l} = \sigma_k \dots \sigma_l$ and $\Delta_m = \pi_{1,m-1}\pi_{1,m-2}\dots\pi_{1,1}$. Then $b = b_{\mathbf{R}}\Delta_m$ where $b_{\mathbf{R}}$ is obtained from the encoding word by the following algorithm:

(i). Replace each subword $\supset_k \times_{i_1} \dots \times_{i_n} \subset_l$ with $\sigma_k^{-1}\delta_1 \dots \delta_n \tau_{k,l}$ where

$$\delta_j = \begin{cases} \sigma_{i_j}^{-1}, & i_j < k-1, \\ \sigma_{i_j+2}^{-1}, & i_j > k-1, \\ \tau_{k,k+1}\sigma_{k+1}^{-1}\tau_{k+1,k}, & i_j = k-1; \end{cases} \qquad \tau_{k,l} = \begin{cases} \pi_{l,k+1}^{-1}\pi_{k,l-1}, & k < l, \\ \pi_{l,k-1}^{-1}\pi_{k,l+1}, & k > l, \\ 1, & k = l. \end{cases}$$

(ii). Replace each $\times_k$ (which was not replaced in the step (i)) with σ_k^{-1}.

To compute $\sigma_\zeta(\hat{b})$ and $n_\zeta(\hat{b})$, I have implemented a simple computer program whose input is an $\mathcal{L}_p$-scheme in the above encoding. This program was used for computations in [5]. The algorithms can be found in [5].

[1] Sometimes this method works with $m-4$ real intersections (see [5]).

2. Non-realisability of the complex scheme (2)

Suppose, there exists a real algebraic 7th degree curve A without a jump whose complex scheme is (2). Let V be the non-empty oval and p_0 a point in an empty exterior oval v_0. Let $v_1,\dots,v_{11}$ be the interior ovals numbered in the natural order when viewed from p_0 (this means that $v_1,\dots,v_{i-1}$ are separated from $v_{i+1},\dots,v_{11}$ by $V\cup\ell$ where ℓ is a line through p_0 and v_i). Let us choose p inside v_2 and let l_∞ pass through p and v_1.

Using the fact that one can draw a conic through any 5 empty ovals and this conic meets the curve not more than in 14 points, one can show (one uses here that the curve has no jump) that the $\mathcal{L}_p$-scheme has form $[\supset_2 o_{i_1} o_{i_2} o_{i_3} o_2^9 \subset_5]$ where $i_j\in\{3,4\}$.

The ovals $v_1,v_3,\dots,v_{11}$ are positive and $v_2,v_4,\dots,v_{10}$ are negative. This follows from the alternating of orientations in $\mathcal{L}_{p_0}$ (see [2]) and the absence of a jump. The alternating of orientations in $\mathcal{L}_p$ implies that (i_1,i_2,i_3) is either $(3,3,4)$ or $(4,3,3)$. Let b be the corresponding braid (see Sect. 1). In the both cases we have $e(b)=8$ and $\sigma_\zeta(\hat b)=-4$, $n_\zeta(\hat b)=1$ for $\zeta=\exp(10\pi i/19)$. This contradicts (4) with $m=7$.

Remarks. **1.** The choice of p inside v_1 does not lead to a contradiction with (4).

2. The minimal possible denominator of a rational θ such that $\sigma(\theta):=\sigma_{\exp(2\pi i\theta)}(\hat b)=-4$, is 19. However, to prove the existence of such θ, it suffices to check that $\zeta=i$ is a simple root of the Alexander polynomial of $\hat b$ and $\sigma(1/4)=\sigma_i(\hat b)=-3$. This implies $\sigma(1/4\pm\varepsilon)=-2$, $\sigma(1/4\mp\varepsilon)=-4$ for $0<\varepsilon\ll 1$ (as it happened, in particular, for $\varepsilon=1/76$).

3. Non-realisability of (3): step 1 of the proof of Theorem 1

Suppose, there exists a real algebraic 7th degree curve A without a jump whose complex scheme is (3). Let v_+ and p_+ (resp. v_- and p_-) be the positive (resp. negative) interior oval and a point inside it. Let $l_\infty=(p_+p_-)$.

The $\mathcal{L}_{p_+}$-scheme of $\mathbf{R}A$ has form $[\supset_2 o_{i_1}\dots o_{i_{12}}\subset_5]$, $i_j\in\{3,4\}$. If we move the point p from p_+ to p_- along the line segment which does not meet the non-empty oval, the order of exterior ovals in the pencil $\mathcal{L}_p$ does not change (otherwise, we would get the position in Figure 2 at some moment). Hence, the $\mathcal{L}_{p_-}$-scheme of $\mathbf{R}A$ is $[\supset_2 o_{7-i_1}\dots o_{7-i_{12}}\subset_5]$.

It follows from the orientations alternating rule that the j-th exterior oval is positive if j is odd and $i_j=4$ or if j is even and $i_j=3$. Otherwise it is negative. Let $b_\pm$ be the braid constructed in Sect. 1 for the pencil $\mathcal{L}_{p_\pm}$. We have $e(b_\pm)=8$. Verifying (4) with $\zeta=-1$ for b_+ and for all sequences $(j_1,\dots,j_{12})$ with 8 positive ovals, we see that it is satisfied only in 10 cases (up to the evident symmetry) where the $\mathcal{L}_{p_+}$-schemes are $[\supset_2 o_3^{\beta_1} o_4^{\beta_2} o_3^{\beta_3} o_4^{\beta_4}\subset_5]$ with $[\beta_1\dots\beta_4]$ being one of

[1173] [1155]* [1137] [1119] [1353]* [1551]* [1533]** [1731] [1713] [1911]

(the corresponding $\mathcal{L}_{p_-}$-schemes are $[\supset_2 o_4^{\beta_1} o_3^{\beta_2} o_4^{\beta_3} o_3^{\beta_4}\subset_5]$ with the same $[\beta_1\dots\beta_4]$). However, in the cases neither marked by * nor by **, b_+ does not satisfy (4) with $\zeta=\exp(2\pi i/3)$: we have $\sigma_\zeta(\hat b_+)=-4$, $n_\zeta(\hat b_+)=1$. In the other 4 cases marked by * or **, (4) holds for the both braids $b_\pm$ and for all ζ on the circle $|\zeta|=1$.

Now let us put $p=p_-$, $b=b_-$ and forget the pencil $\mathcal{L}_{p_+}$. For the remaining 4 cases we apply the method described in [5; Sect. 5.8]. If an $\mathcal{L}_p$-scheme is realisable then the link $L = \hat{b}$ bounds a smooth surface $N := \mathbf{C}A \cap B^4$ in a 4-ball $B_4 \in \mathbf{CP}^2$ such that $L = \partial N$ and $\chi(N) = m - e(b)$ $(= -1$ in our case) where χ is the Euler characteristic (see details in [5]).

Since A is an M-curve, $\mathbf{R}A$ divides $\mathbf{C}A$ into two conjugated halves A^+ and A^-. Put $N^\pm = N \cap A^\pm$ and $L^\pm = \partial N^\pm$. Thus, L^+ (resp. L^-) is the part of L where the complex orientation coincides (resp. does not coincide) with the braid orientation. Five strings of b comes from the odd branch of $\mathbf{R}A$ and from the two ovals surrounding p (big ovals). The other two strings comes from the chain of the other 13 (small) ovals. Since the both big ovals are negative, only one string (coming from the chain of the small ovals) belongs to L^+. Hence, N^+ is an unknotted disk in B^4 and we have $\chi(N^+) = 1$, $\chi(N^-) = -2$.

Let $\xi : \tilde{B}^4 \to B^4$ be the double covering branched along N^+. Since N^+ is unknotted, $\tilde{B}^4$ is also a 4-ball. Put $\tilde{N} = \xi^{-1}(N)$, $\tilde{L} = \xi^{-1}(L)$, etc. Then

$$\chi(\tilde{N}) = \chi(\tilde{N}^+) + \chi(\tilde{N}^-) = \chi(N^+) + 2\chi(N^-) = -3. \tag{5}$$

By Murasugi–Tristram inequality, for any ζ with $|\zeta| = 1$ we must have

$$|\sigma_\zeta(\tilde{L})| - n_\zeta(\tilde{L}) \le -\chi(\tilde{N}). \tag{6}$$

It is not difficult to find a braid representing $\tilde{L}$ starting with b (see, e.g. [5]). Computing the signatures and the nullities, we obtain $\sigma_\zeta(\tilde{L}) = -6$, $n_\zeta(\tilde{L}) = 1$, $\zeta = \exp(2\pi i\theta)$ where $\theta = 2/5$ for $[\beta_1 \dots \beta_4] = [1155]$ and $\theta = 5/14$ for $[\beta_1 \dots \beta_4] = [1551]$, $[1353]$. This contradicts (5), (6).

Thus, only the case $[\beta_1 \dots \beta_4] = [1533]$ remains open (the case marked above by **). In this case (6) holds for any ζ with $|\zeta| = 1$.

4. Vanishing cycles

Suppose we want to exclude an $\mathcal{L}_p$-scheme $[w]$ using the method described and applied above (w is a word in $\subset_k$, $\supset_k$, $\times_k$). If we can prove that every curve realising $[w]$ admits a degeneration of the form $[w] \to [w\times_k]$ then we have increased chances to obtain a contradiction. Indeed, we have $b \to b' = b\sigma_k^{-1}$ for the corresponding braids, hence, it may happen that b' is quasipositive but b is not. If we use Murasugi–Tristram inequality then also it is easier to obtain a contradiction for b' because $e(b') = e(b) - 1$ while $|\sigma_\zeta| - n_\zeta$ may change either by $+1$ or by -1, hence, it is possible that b satisfies (4) while b' does not.

However, it is very difficult to prove the existence of degenerations with desired properties. For example, this is the point where Hilbert made a mistake (in the text of the 16-th problem, he erroneously claimed that the sextic $\langle 5 \sqcup 1\langle 5\rangle\rangle$ does not exist). Here we show how sometimes one can pass from b to b' without requiring the existence of such a degeneration. If a curve A admits the above degeneration then a vanishing cycle appears on $\mathbf{C}A$. It is a circle on $\mathbf{C}A$ which bounds a smooth totally real (i.e. no tangent plane is a complex line) disk whose interior does not meet $\mathbf{C}A$. In fact, one needs only such a disk to apply the above arguments. Its existence sometimes can be easily proved by considering another pencil of lines (this construction was proposed by Viro [11; Sect. (4.12)], [4; Sect. 5]; he used it to construct 2-cycles on the double covering).

Let A be an algebraic curve in $\mathbf{RP}^2$, $p \in \mathbf{RP}^2$ a point in general position (in particular, $p \notin A$), and U a sufficiently small ball around p. Denote by π_p :

$\mathbf{CP}^2 \setminus \{p\} \to \mathbf{CP}^1$ the projection from p. Let H_0 be one of the halves of $\mathbf{CP}^1 \setminus \mathbf{RP}^1$ and $H_\varepsilon = H_0 \setminus (\varepsilon\text{-neighbourhood of } \mathbf{RP}^1)$. Then $\pi_p^{-1}(H_\varepsilon) \setminus U$ is a 4-ball. Denote it by B^4_ε and its boundary by S^3_ε. Let $L_\varepsilon = S^3_\varepsilon \cap \mathbf{C}A$ and $N_\varepsilon = B^4_\varepsilon \cap \mathbf{C}A$. If $0 < \varepsilon \ll 1$ then $(B^4_\varepsilon, N_\varepsilon)$ does not depend on ε and we shall omit the subscript $_\varepsilon$ in this case. We have $L_0 = \mathbf{R}A \cup S$ where S is a union of smooth circles invariant under the complex conjugation. $\mathbf{R}A$ meets S at points where the tangent belongs to $\mathcal{L}_p$. The link (S^3, L) is obtained from (S^3_0, L_0) by the smoothing of double points as it is shown in Figure 4. It is the closure of the braid b discussed in Sect. 1.

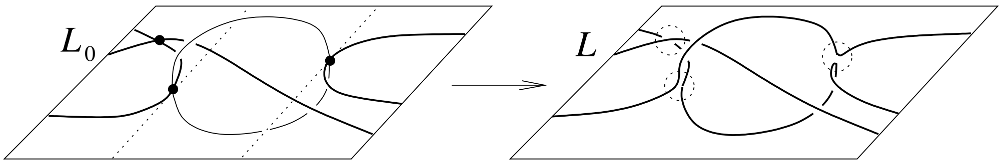

FIG. 4

Let $a, b \in \mathbf{R}A \setminus S$ and let γ_0 be a smooth simple path from a to b such that $\mathbf{R}A \cap \gamma_0 = \{a, b\}$ and γ_0 is transverse to $\mathbf{R}A$. Suppose there exists a totally real embedded half-disk $D \subset B^4_0$ such that $\gamma_0 \subset \partial D$, $((\partial D) \setminus \gamma) \subset (\mathbf{C}A \setminus \mathbf{R}A)$, and D is transverse to $\mathbf{C}A \cup \mathbf{RP}^2$. Let $\dot{B}^4_\varepsilon = B^4_\varepsilon \setminus V$ where V is a sufficiently small neighbourhood of D. This is also a 4-ball. Put $\dot{S}^3_\varepsilon = \partial \dot{B}^4_\varepsilon$, $\dot{N}_\varepsilon = \dot{B}^4_\varepsilon \cap \mathbf{C}A$, $\dot{L}_\varepsilon = \dot{S}^3_\varepsilon \cap \mathbf{C}A$. As above, if $\varepsilon > 0$ is so small that the object does not depend on it, we shall omit the subscript from the notation.

Let $L'_0 \subset S^3_0$ be obtained from L_0 by the surgery along γ_0 with the framing which makes one negative half-twist with respect to the framing defined by $\mathbf{RP}^2$. This means that $L'_0 = (L_0 \setminus f(I \times \partial I)) \cup f(\partial I \times I)$ where $I = [0, 1]$ and $f : I^2 \to S^3$ is an embedding (1-handle) such that $f(I^2) \cap L_0 = f(I \times \partial I)$, $f(\{1/2\} \times I) = \gamma_0$, and the band $f(I^2)$ makes one negative half-twist with respect to a neighbourhood of γ_0 in $\mathbf{RP}^2$. Finally, let us define L' as the link obtained from L'_0 by the smoothing of double points, the same way as L was obtained from L_0 (see Figure 5).

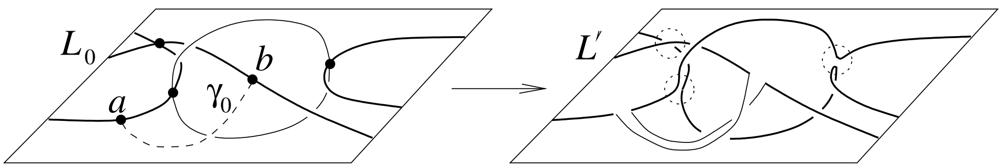

FIG. 5

Proposition. *(a).* $(\dot{S}^3, \dot{L})$ *is isotopic to* (S^3, L'). *(b).* $\chi(\dot{N}) = \chi(N) + 1$.

Proof. (a). Since the smoothing $\dot{L}_0 \to \dot{L}$ is the same as $L_0 \to L$, it suffices to prove that $(\dot{S}^3_0, \dot{L}_0)$ is isotopic to (S^3_0, L'_0). Consider an isotopy $\{\gamma_t\}$, $t \in [0, 1]$, such that $\gamma_1 = \partial D \setminus \gamma_0$. Let N_t be a smooth surface defined in a neighbourhood of γ_t which is tangent to the vector field $(\sqrt{-1}) \cdot v_t$ where v_t is a vector field tangent to γ_t. We can choose the surfaces N_t so that they depend smoothly on t and N_1 is a neighbourhood of γ_1 in $\mathbf{C}A$. Since D is totally real, all N_t are transverse to D. Let D_t be the part of D bounded by $\gamma_0 \cup \gamma_t$ and let V_t be a neighbourhood of D_t, small with respect to N_t. Put $S^3_t = \partial(B^4_0 \setminus V)$. Then $(S^3_t, S^3_t \cap (N_t \cup N)$ defines the required isotopy everywhere outside small neighbourhoods of the points a and b (the common endpoints of all γ_t). It is not difficult to modify the isotopy in these neighbourhoods to complete the proof.

(b). $\dot N = N \setminus h$ where h is a tubular neighbourhood of $N \cap D$. Thus, $\dot N$ is N cut along a simple path with the both ends on ∂N. $\square$

To construct the totally real disk D, we use another pencil of lines $\mathcal{L}_q$. Suppose, there exists a real segment $[l_1, l_2]$ of the pencil $\mathcal{L}_q$ such that l_j is tangent to A at a point a_j, l_j has $m-1$ real intersection points with A ($j = 1, 2$), and any other line of $[l_1, l_2]$ has $m-2$ real intersections with A. Suppose also a_1 is connected with a_2 by a path $\gamma \subset \mathbf{RP}^2$ which does not meet A (except the ends a_j) and which meets each line of $[l_1, l_2]$ at a single point. Let $l \in [l_1, l_2]$, $\mathbf{C}l \setminus \mathbf{R}l = l^+ \cup l$. It is clear that $\mathbf{C}l \cap S^4_0 = \mathbf{R}l$. Hence, one of $l^\pm$ (let it be l^+) lies in B^4_0. Since $\mathbf{R}l \cap \mathbf{R}A$ has $m-2$ points, $l^+ \cap \mathbf{C}A$ is one point. Connect it by a straight segment to the point $\mathbf{R}l \cap \gamma$. These segments form a totally real disk with the required properties.

Example. Suppose that q is very close to p. Then the links L and L' are shown in Figure 6. We see that L' is obtained from L by adding a meridian. A Seifert surface for L' can be obtained from a Seifert surface of L by attaching a twisted band (see Figure 6). Hence, we have $Q' = Q \oplus \langle -1 \rangle$ for the Seifert forms. Thus, the both sides of Murasugi–Tristram inequality change by 1. As one could expect, such a trivial construction can not lead to a non-trivial result.

A non-trivial example will be given in the next section.

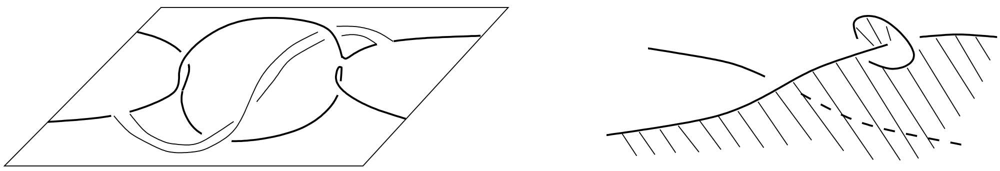

FIG. 6

5. NON-REALISABILITY OF (3): COMPLETING THE PROOF OF THEOREM 1

Let the notation be as in Sect. 3 and $p = p_-$. It remains to prove that the $\mathcal{L}_p$-scheme of A can not be $[\supset_2 o_4 o_3^5 o_4^3 o_3^3 \subset_5]$. We are going to give two different proofs. The first one is based on the observations from Sect. 4. The second proof uses an auxiliary line. In the both proofs we apply the double covering as we did this in the end of Sect. 3.

Proof 1. Let q be a point in any exterior oval. Since the curve has no jump, the Viro construction described in the end of Sect. 4 yields us a vanishing cycle between v_+ and v_-. If we modify the braid b (corresponding to $\mathcal{L}_p$) using the vanishing cycle coming from $\mathcal{L}_q$, we obtain the braid b' coinciding with the braid that would be obtained from the $\mathcal{L}_p$-scheme $[\times_1 \supset_2 o_4 o_3^5 o_4^3 o_3^3 \subset_5]$ (the curve obtained from A by replacing $v_+ \cup v_-$ with the figure "8").

Let b' be the corresponding braid, $L' = \hat b'$, and let $L' = (L')^+ \sqcup (L')^-$ be the partition of L', analogous to the partition $L = L^+ \sqcup L^-$ of L described in Sect. 3 (see Figure 7 where the complex orientations are indicated and $(L')^+$ is thicker than $(L')^-$). Like in Sect. 3, $(L')^+$ must bound an unknotted disk $(N')^+ \subset B^4$. Let $\xi : \tilde B^4 \to B^4$ be the covering branched along $(N')^+$, and $\tilde L' = \xi^{-1}(L')$.

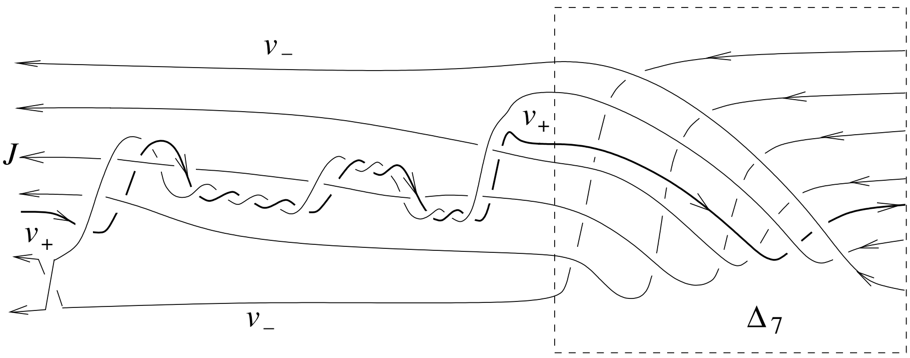

FIG. 7

Similar arguments show that we must have $|\sigma_\zeta(\tilde{L}')| - n_\zeta(\tilde{L}') \le \chi(\tilde{N}') = -1$ but this contradicts the result of computations: $\sigma_\zeta(\tilde{L}') = -4$, $n_\zeta(\tilde{L}') = 1$ for $\zeta = \exp(3\pi i/4)$. $\square$

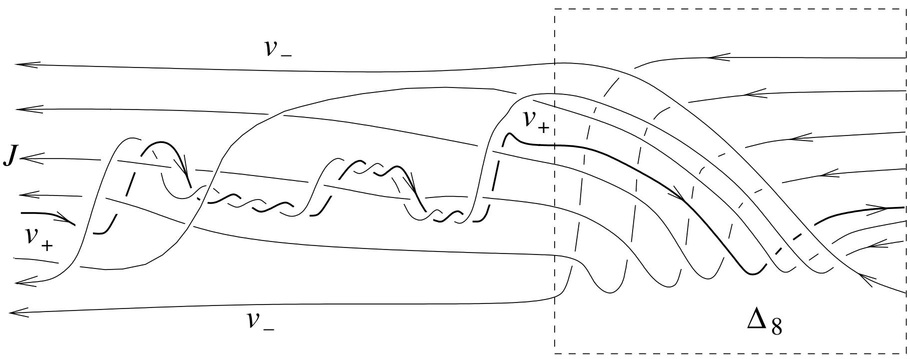

FIG. 8

Proof 2. Let ℓ be a line through v_+ and any exterior oval. Let b'' be the braid constructed for the curve $A \cup \ell$ using the same pencil $\mathcal{L}_p$ with $p = p_-$ and let $L'' = \hat{b}''$. Then $L'' = L \sqcup L_\ell$ where $L = L^+ \sqcup L^-$ is the same as in Sect. 3 and L_ℓ is the component corresponding to ℓ (see Figure 8 where L^+ is thicker and the complex orientations are indicated only on L). The same arguments with the double covering branched along L^+ and $\zeta = \exp(3\pi i/4)$ lead to a contradiction. $\square$

6. CONSTRUCTIONS (PROOF OF THEOREM 2)

Let C be an algebraic curve and $p \in C$ a point which is not an inflection point. Let us define the birational quadratic transformation $f_{C,p}$ of $\mathbf{P}^2$ as follows. Let (x, y) be affine coordinates so that p is the infinite point of the axis $y = 0$ and the line at infinity is tangent to C at p. The equation of the germ of C at p has the form $y = ax^2 + o(x^2)$, $x \to \infty$. Then put $f_{C,p}(x, y) = (x, y - ax^2)$.

Let A be an M-quintic arranged with respect a line L as in Figure 9 (see [8]). Denote by $q_1, \dots, q_4$ points where lines through p are tangent to A (see Figure 9). Let B_i be the proper transform of A under f_{A,q_i}, $i = 1, \dots, 4$ (see Figure 10). This is a curve of degree 7 with a singular point which can be written in suitable local analytic coordinates as $v(v - au^3)(v - bu^3) = 0$, $0 < a < b$. All dissipations of this singularity up to a symmetry are shown in Figure 11 (see [12] for dissipations

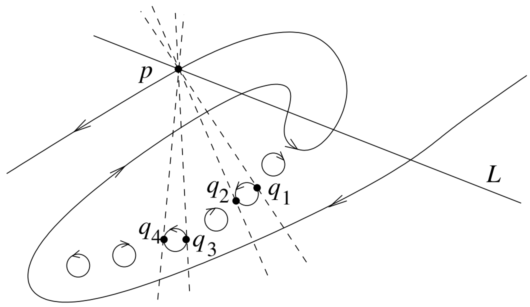

FIG. 9

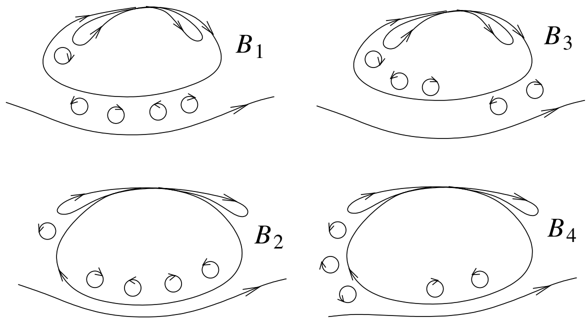

FIG. 10

of J_{10}^-). This picture correspond to dividing the triangle (0,0)-(9,0)-(3,0) by the segment (6,0)-(0,3). By a result of Shustin [9; Lemma], all these dissipations can be applied to $B_1, \dots, B_4$ and we obtain the all the complex schemes in Theorem 2.

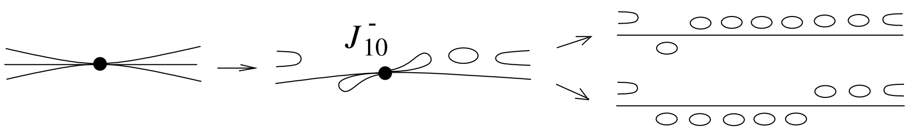

FIG. 11

APPENDIX 1. SOME COMPUTATIONS FOR M-CURVES OF DEGREE 8

Let us denote by BL_m (resp. BL_m^+) the space of all smooth non-oriented (resp. oriented) one-dimensional submanifolds A of $\mathbf{RP}^2$ such that any projective line meets A at most at m points. In particular, any real algebraic curve of degree m belongs to BL_m (the notation BL means B̲ézout theorem for L̲ines). We say that two curves are $BL_m^{(+)}$-*isotopic* if they belongs to the same connected component of $BL_m^{(+)}$. When we speak of BL_m^+-isotopy of a separating real algebraic curve, we mean the complex orientations. If two real algebraic curves of degree m are rigid isotopic then they are BL_m-equivalent.

Question. *Do there exist two real algebraic curves of the same degree m which are BL_m-isotopic but not rigid isotopic?*

Proposition. *Let A be an M-curve of degree 8 realising the real scheme $\langle \beta \sqcup 1\langle 2 \sqcup 1\langle \alpha\rangle\rangle$ whith even $\alpha > 0$ and $\beta = 18 - \alpha$. Then:*

a). $\mathbf{R}A$ is BL_8^+-isotopic to a curve such that all the empty ovals are ranged along some convex curve (the dashed curve in Figure 12) and all the numbers $\alpha_1, \beta_1, \beta_2, \alpha_2$ are odd.

b). If (α, β) is $(4, 14)$ or $(14, 4)$ then the only possible values of $[\alpha_1, \beta_1, \beta_2, \alpha_2]$ are $[1, 5, 9, 3], [1, 7, 7, 3], [1, 13, 1, 3], [5, 1, 3, 9], [7, 1, 3, 7], [3, 1, 13, 1],$

Proof. a). It is proved in [5; Corollary 1.7] that the complex scheme of A is as in Figure 12. Hence, the interior ovals must be divided into two chains if viewed from the intermediate empty oval which is negative with respect to the outer oval (v_- in Figure 12). These two chains must be separated by at least one negative oval because of the complex orientations. Applying Bézout theorem for auxilary lines and conics, one can complete the proof.

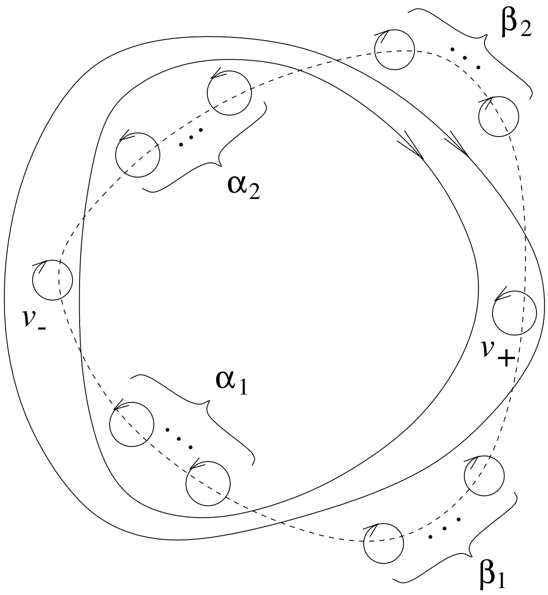

FIG. 12

b). Let us choose a point p inside v_+ and consider the braid b corresponding to the pencil $\mathcal{L}_p$. It has form

$$b = \pi_{6,3}^{-1}\pi_{5,2}^{-1}\pi_{3,4}\pi_{2,3}\sigma_6^{1-k}\sigma_4^{1+k}\sigma_5^{-\alpha_1}\tau_{5,3}\sigma_3^{-\beta_1}\tau_{3,4}\sigma_4^{-1}\tau_{4,3}\sigma_3^{-\beta_2}\tau_{3,5}\sigma_5^{1-\alpha_2}\,\Delta_8$$

where k is an integer parameter (the number of twists in the segment of $\mathcal{L}_p$ where a line of $\mathcal{L}_p$ has 4 real intersections with A). We have $e(b) = 8$. Let $L = \hat{b} \subset S^3$ and $N \subset B^4$, $\partial N = L$ be as above. The complex orientations imply that k is even. Hence, the image of b in the symmetric group is the permutation (18)(2)(3)(4)(5)(67). Its cycles correspond to components of L. Denote them respectively by $L_1, \dots, L_6$. Define the partition $N = N^+ \sqcup N^-$, $L^\pm = \partial N^\pm$ as in Sect. 3. Then $L^+ = L_1 \cup L_3 \cup L_5 \cup L_6$ and $L^- = L_2 \cup L_4$. The linking numbers $l_{ij} = L_i \cdot L_j$ are $l_{12} = l_{13} = l_{14} = l_{15} = -l_{35} = 1$, $l_{16} = -l_{36} = l_{56} = 2$, $l_{24} = l_{25} = l_{26} = 0$, $l_{23} = (2-k)/2$, $l_{34} = -9$, $l_{45} = (4+k)/2$, $l_{46} = 4$. Since $\chi(N) = m - e(b) = 0$ and the genus of N is zero (because A is an M-curve) N has three connected components. Hence, one of $N^\pm$ is conenected and the other has two components. Suppose $N^+ = N' \sqcup N''$. The only partition $L = L' \sqcup L''$ such that $L' \cdot L'' = 0$ is $L' = L_1 \cup L_3$, $L'' = L_5 \cup L_6$. But in this case $N'' \cup \operatorname{conj} N''$ would be disconnected from the rest of the curve. Thus, N^+ is connected and $N^- = N_2 \sqcup N_4$ with $\partial N_j = L_j$. It is clear that the both N_2 and N_4 are unknoted disks in B^4. The condition $L_2 \cdot L^+ = 0$ implies $k = 4$.

Up to symmetry, we have 14 cases: $\alpha_1 = 1$, $\beta_1 = 2a+1$, $\beta_2 = 13-2a$, $\alpha_2 = 3$ (denote the corresponding braids by b_a, $a = 0, \dots, 6$) and $\alpha_1 = 2a+1$, $\beta_1 = 1$, $\beta_2 = 3$, $\alpha_2 = 13 - 2a$ (denote the corresponding braids by c_a, $a = 0, \dots, 6$).

Let us show that the braids b_a and c_a are not quasipositive for $a = 0, 1, 4, 5$. Put $\zeta = \exp(2\pi i\theta)$. Then we have $\sigma_\zeta(\hat{b}_a) = \sigma_\zeta(\hat{c}_a) = -3$, $n_\zeta(\hat{b}_a) = n_\zeta(\hat{c}_a) = 1$ for $(a, \theta) = (1, 1/6), (4, 3/16), (5, 1/8)$. This contradicts (4). To prohibit b_0 and c_0, let us consider the double covering $\xi : \tilde{B}^4 \to B^4$ branched along N_4. Let $\tilde{N} = \xi^{-1}(N)$ and $\tilde{L} = \xi^{-1}(L)$ (here $L = \hat{b}_0$ or $\hat{c}_0$). Then $\chi(\tilde{N}) = \chi(N_4) + 2\chi(N \setminus N_4) = -1$ but a computation shows that $\sigma_\zeta(\tilde{L}) = -4$ and $n_\zeta(\tilde{L}) = 1$ for $\zeta = \exp(\pi i/3)$. This contradicts (6). $\square$

Note that all the invariants that we computed, are the same for $\hat{b}_a$ and $\hat{c}_a$. Maybe, these links are equivalent.

Now, let us show that at least 4 of the non-prohibited BL_8^+-isotopy types: $[1,7,7,3]$, $[1,13,1,3]$, $[7,1,3,7]$, $[3,1,13,1]$ are realizable by flexible curves in the sense of [5] (i.e. the corresponding braids are quasipositive, in particular, this implies the existence of J-holomorphic curves for some tame almost complex structure J on $\mathbf{CP}^2$ invariant under the complex conjugation).

All the 4 flexible curves are obtained as (flexible) smoothings of the union of 4 ellipses which are tangent to each other with order 4 at a single point. We perform the smoothings in two steps. At the 1st step we simplify the singular point according Figure 13 where 3 smooth branches touch each other with order 3 at the triple point. At the 2nd step we dissipate the triple point as in Figure 11 (see Section 6).

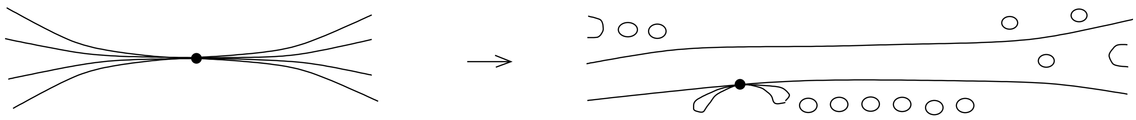

FIG. 13

The braid corresponding to Figure 13 is

$$b = \sigma_3^{-3}\tau_{3,1}\Delta_3^{-3}\sigma_1^{-7}\tau_{1,3}\sigma_3^{-1}\tau_{3,2}\sigma_2^{-1}\tau_{2,3}\sigma_3^{-1}\tau_{3,2}\Delta_4^4.$$

One can check that b is quasipositive: $b = (b_1^{-1}\sigma_1 b_1)(b_2^{-1}\sigma_1 b_2)$ where

$$b_1 = \sigma_2\sigma_3^{-2}\alpha, \qquad b_2 = (\sigma_2\sigma_3^2)^3\sigma_1\alpha, \qquad \alpha = \sigma_3^3\sigma_2^{-1}\sigma_3^{-2}\sigma_2^{-1}.$$

APPENDIX 2. AN $(M-1)$-SMOOTHING OF FOUR TANGENT BRANCHES

The method of prohibition described in Sect. 1 can be also applied to smoothings of singularities. Following [12], we say that a singular point is of the type X_{21} if four smooth real branches have simple tangency at this point. Smoothings (dissipations) of X_{21} were discussed in [12]. The classification of M-smoothings (those which have 9 ovals) was completed by Shustin (see [12]). Among $(M-1)$-smoothings (with 8 ovals) three cases are open. Here we prohibit one of them.

Proposition. *A singularity X_{21} has no smoothing of the form Figure 14.*

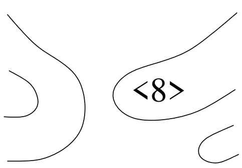

FIG. 14

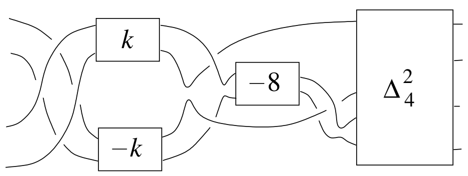

FIG. 15

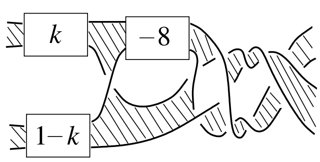

FIG. 16

Let A_0 be a curve with the singularity X_{21} at the origin such that the line $y=0$ is the common tangent to the four branches and let A be a smoothing of A_0. Let $\mathcal{L}$ be the pencil of lines $x =$const. We use the same construction as above. Fix

$r_x, r_y > 0$ such that A_0 looks like 4 parabolas in the bidisk $D = \{|x| < r_x,\ |y| < r_y\}$ and suppose that A is close to A_0 with respect to r_x and r_y. The set $D \cap \{\operatorname{Im} x > \varepsilon\}$ for $0 < \varepsilon \ll r_x$ is a 4-ball (denote it by B^4) and $\mathbf{C}A$ cuts on $S^3 = \partial B^4$ a link L which is the closure of a quasipositive braid $b \in B_4$. The only difference is that we must consider Δ_4^2 instead of Δ_4, i.e. $b = b_{\mathbf{R}} \Delta_4^2$ where $b_{\mathbf{R}}$ is determined by the $\mathcal{L}$-scheme of $\mathbf{R}A$. If $\mathbf{R}A$ is like in Figure 14 then

$$b = \sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{1-k}\sigma_3^{1+k}\sigma_2^{-8}\tau_{2,1}\,\Delta_4^2$$

(see Figure 15). Here k is the number of twists which make the two points of $\ell \cap \mathbf{C}A$ with $\operatorname{Im} y > 0$ when ℓ runs the segment of $\mathcal{L}$ where there are no real intersections.

Let us show that b does not satisfy (4) with $\zeta = -1$ for any integer k. We have $m = 4$, $e(b) = 2$. Hence, to obtain a contradiction, it suffices to show that $n_{-1}(\hat{b}) > 1$ but this means $\det \hat{b} \neq 0$. To compute $\det \hat{b}$, we transform L as in Figure 16 and compute the Göritz matrix G (see [3]) corresponding to the non-orientable surface shown in Figure 16. It is a 4×4-matrix and we have $\pm \det G = 15k^2 - 9k - 106$. Hence, $\det \hat{b} = \det G \neq 0$ for any $k \in \mathbf{Z}$.

Remark. The matrix $W_{\mathbf{i}}^J(\mathbf{e})$ defined in [5; Sect. 2.6] and used in [5; Sect. 8.1] in a similar context, can be also interpreted as the Göritz matrix of a suitable non-oriented surface.

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