REALIZABILITY OF A BRAID MONODROMY BY AN ALGEBRAIC FUNCTION IN A DISK

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Abstract. A braid is called *algebraic* if it is conjugated to the local braid of an algebraic function at a singular point. It is shown that any homomorphism of a free group into a braid group which takes each generator to an algebraic braid, can be realized as the braid monodromy of an algebraic function in a disk.

Réalisabilité d'une monodromie des tresses par une fonction algébrique dans le disque.

Résumé. Une tresse appelée *algébrique* si elle est conjuguée à la tresse locale d'une fonction algébrique en un point singulier. On démontre que tout homomorphisme du groupe libre dans le groupe des tresses, qui envoie chaque générateur sur une tresse algébrique, peut être réalisé comme monodromie des tresses d'une fonction algébrique dans le disque.

Version française abrégée

§1. Monodromie des tresses. Soit $p(x,y)\in\mathbf{C}[z,w]$ un polynôme sans facteurs multiples de deux variables complexes. Supposons que p soit unitaire par rapport à w, c.-à-d. $p = w^n + a_1(z)w^{n-1} + \dots + a_n(z)$. Alors, pour chaque domaine $D\subset\mathbf{C}$, le polynôme p définit une fonction multivoque $F(z)=\{w\,|\,p(z,w)=0\}$. Cette fonction a n valeurs en un point générique, et moins que n valeurs dans un ensemble fini $S=S(F)=\{z_1,\dots,z_s\}$. Le fait que p soit unitaire implique que F n'a pas des pôles.

Notons $V_n\subset\mathbf{C}[w]$ la variété des polynômes unitaires de degré n et soit $\Delta_n\subset V_n$ l'hypersurface du discriminant, constituée des polynômes qui ont des racines multiples. Evidemment, $V_n\simeq\mathbf{C}^n$. On considère *le groupe des tresses à n brins* $B_n=\pi_1(V_n-\Delta_n,p_0)$, où p_0 est un polynôme fixé.

Soit $\tilde p:\mathbf{C}\to V_n$ l'application définie par $\tilde p(z_0)=p(z_0,\cdot)$. Par définition de S, le polynôme $\tilde p(z)\notin\Delta_n$ si $z\notin S$. Fixons un point de base $z_0\in D-S$ et un chemin dans $V_n-\Delta_n$ entre $\tilde p(z_0)$ et p_0. On peut définir l'homomorphisme $\tilde p_*:\pi_1(D-S)\to\pi_1(V_n-\Delta_n)=B_n$, appelé *monodromie des tresses de la fonction algébrique* $F(z)$. Etant donné un lacet $\gamma:[0,1]\to D-S$, la classe de conjugaison de la tresse $\tilde p_*([\gamma])$ ne dépend que de F et γ.

Considérons un germe en $z=0$ de fonction n-valuée $w=F(z)$ définie par un polynôme unitaire de w dont les coefficients dépendent analytiquement de z. Pour un ε assez petit, la tresse associée au chemin $t\mapsto\varepsilon e^{it}$, $t\in[0,2\pi]$ ne dépend pas de ε à conjugaison près. On dit qu'une tresse est une *tresse locale de F en $z=0$* si elle

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est conjuguée à la tresse, construite de cette façon. Une tresse est dite *algébrique* si elle est une tresse locale d'une fonction multivoque en un point singulier.

Soient $S_0 = \{1, \ldots, s\}$ et $D_0 \subset \mathbf{C}$ un disque contenant S_0. Fixons une système de générateurs $\gamma_1, \ldots, \gamma_s$ du $\pi_1(D_0 - S_0)$ tel que γ_i soit représenté par un petit cercle autour du point j.

Proposition 1. *Soient $b_1, \ldots, b_s \in B_n$ des tresses algébriques et soit $D \subset \mathbf{C}$ un disque. Alors, il existent un difféomorphisme $\varphi : (D_0, S_0) \to (D, S)$ et une fonction algébrique $w = F(z)$ définie par un polynôme $p(z, w)$, unitaire par rapport à w, tels que $S(F) = S$ et la monodromie des tresses $\tilde{p}$ envoie les générateurs $\varphi_*(\gamma_1), \ldots, \varphi_*(\gamma_s)$ de $\pi_1(D - S)$ sur $b_1, \ldots, b_s$.*

Exemple. Toute tresse conjuguée à un générateur standard de B_n, est algébrique (elle est une tresse locale de $F(z) = \{\pm\sqrt{z}, 3, 4, \ldots, n\}$). Donc, si on choisi les γ_i tels que ∂D soit librement homotope à $\prod \gamma_j$, la proposition 1 donne le théorème de L. Rudolph [1] qui dit que toute tresse quasi-positive peut être réalisée par une fonction algébrique dans un disque (une tresse est dite *quasi-positive* si elle est un produit de tresses conjuguées aux générateurs standards du groupe des tresses).

La proposition 1 est une conséquence immédiate de la proposition 2.

§2. Lacets analytiques dans $\mathbf{C}^n - \Delta$. Soit Δ une hypersurface algébrique dans $\mathbf{C}^n$. Etant donné un voisinage $U \subset \mathbf{C}$ de 0 et une application analytique $f : U \to \mathbf{C}^n$ telle que $f(U - 0) \subset \mathbf{C}^n - \Delta$, on dit que la *classe d'homotopie libre locale de f en* 0 est la classe de conjugaison dans $\pi_1(\mathbf{C}^n - \Delta)$, qui correspond au lacet $[0, 2\pi] \to \mathbf{C}^n$, $t \mapsto f(\varepsilon e^{it})$ pour $\varepsilon \ll 1$. Soient (D_0, S_0) et $\gamma_1, \ldots, \gamma_s$ comme ci-dessus.

Proposition 2. *Soit $U \subset \mathbf{C}$ un voisinage de 0. Pour chaque $j = 1, \ldots, s$, soit $f_j : (U, 0) \to (\mathbf{C}^n, \Delta)$, une application analytique telle que $f_j(U) \subset \mathbf{C}^n - \Delta$ et soit $b_j \in \pi_1(\mathbf{C}^n - \Delta)$ un représentant arbitraire de la classe d'homotopie libre locale de f_j en 0. Soit $D \subset \mathbf{C}$ un disque. Alors, il existe un difféomorphisme $\varphi : (D_0, S_0) \to (D, S)$ et une application polynômiale $p : D \to \mathbf{C}^n$ tels que $p^{-1}(\Delta) \cap D = S$ et $p_*(\varphi_*(\gamma_j)) = b_j$, $j = 1, \ldots, s$.*

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§1. Braid monodromy. Let $p(z, w) \in \mathbf{C}[z, w]$ be a reduced (i.e. without multiple factors) polynomial in two complex variables. Suppose that p is monic with respect to w, i.e. $p = w^n + a_1(z)w^{n-1} + \cdots + a_n(z)$. Then in any domain $D \subset \mathbf{C}$, the polynomial p defines a multi-valued function $F(z) = \{w \,|\, p(z, w) = 0\}$. This function takes n values in a generic point of D and takes less than n values in a finite set $S = S(F) = \{z_1, \ldots, z_s\}$. The fact that p is monic means that F has no poles.

Denote by $V_n \subset \mathbf{C}[w]$ the variety of all monic polynomials of degree n, and let $\Delta_n \subset V_n$ be the discriminant hypersurface consisting of polynomials with multiple roots. Clearly that $V_n \simeq \mathbf{C}^n$. We define *the braid group on n strings* as $B_n = \pi_1(V_n - \Delta_n, p_0)$, where p_0 is a fixed polynomial.

Let $\tilde{p} : \mathbf{C} \to V_n$ be the mapping defined by $\tilde{p}(z_0) = p(z_0, \cdot)$. By the definition of S, we have $\tilde{p}(z) \notin \Delta_n$ for $z \notin S$. Chose a base point $z_0 \in D - S$ and a path in $V_n - \Delta_n$ between $\tilde{p}(z_0)$ and p_0. Then we can define the homomorphism $\tilde{p}_* : \pi_1(D - S) \to \pi_1(V_n - \Delta_n) = B_n$ which is called *the braid monodromy of the algebraic function* $F(z)$. (We do not speak of braid monodromy of a curve, because it depends on the choice of coordinates). The braid monodromy is defined by F up to an inner automorphism of B_n and it determines $(D \times \mathbf{C}, \mathrm{graph}(F))$ up to an ambient isotopy of $D \times \mathbf{C}$ preserving the projection onto D. Given a closed path $\gamma : [0, 1] \to D - S$, the conjugacy class of the braid $\tilde{p}_*([\gamma])$ depends only on F and γ and does not depend on the choice of base points and paths between them.

Consider a germ at $z = 0$ of an n-valued function $w = F(z)$ defined by a monic reduced polynomial in w with coefficients analytically depending on z. For ε sufficiently small, the braid associated with the path $t \mapsto \varepsilon e^{it}$, $t \in [0, 2\pi]$ does not depend on ε up to conjugacy. A braid is called *a local braid of* F *at* 0 if it is conjugated to a braid which appears in this way. A braid is called *algebraic* if it is a local braid of a multi-valued function at a singular point.

L. Rudolph [1] has proved that if γ is a simple closed positively oriented path then the braid $\tilde{p}_*([\gamma])$ is *quasipositive* (equivalent to a product of braids conjugated to standard generators of the braid group) and that any quasipositive braid can be realized as the boundary braid of some algebraic function on a disk without poles. We generalize the Rudolph's theorem showing (Proposition 1 below) that any homomorphism $\pi_1(D - S) \to B_n$ which takes small loops around points of S to algebraic braids, can be realized as the braid monodromy of an algebraic function without poles in a disk.

Let $S_0 = \{1, \ldots, s\}$ and let $D_0 \subset \mathbf{C}$ be a disk containing S_0. Fix a system of generators $\gamma_1, \ldots, \gamma_s$ of $\pi_1(D_0 - S_0)$ such that γ_j is represented by a small positively oriented loop around the point j.

Proposition 1. *Let* $b_1, \ldots, b_s \in B_n$ *be any algebraic braids and* $D \subset \mathbf{C}$ *a disk. Then there exist a diffeomorphism* $\varphi : (D_0, S_0) \to (D, S)$ *and an algebraic function* $w = F(z)$ *on* D *defined by a monic in* w *polynomial* $p(z, w)$ *such that* $S(F) = S$ *and the braid monodromy* $\tilde{p}_*$ *takes the generators* $\varphi_*(\gamma_1), \ldots, \varphi_*(\gamma_s)$ *of* $\pi_1(D - S)$ *to* $b_1, \ldots, b_s$.

Example. Any braid conjugated to a standard generator of B_n, is algebraic (it is a local braid of $F(z) = \{\pm\sqrt{z}, 3, 4, ..., n\}$). Thus, if we chose γ_j's so that ∂D is freely homotopic to $\prod \gamma_j$ then Proposition 1 gives the Rudolph's theorem mentioned above.

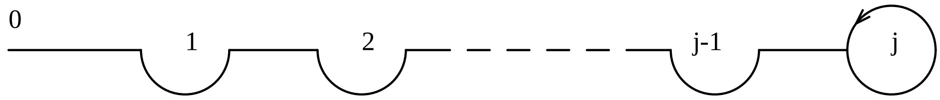
Remark. Proposition 1 remains true if we omit in it (and in the definition of algebraic braids) the condition that the polynomials in w are monic. Proposition 1 as well as its non-monic version are direct consequences of Proposition 2 below.

§**2. Analytic loops in** $\mathbf{C}^n - \Delta$**.** Let Δ be any algebraic hypersurface in $\mathbf{C}^n$ given by $\Phi = 0$. If $U \subset \mathbf{C}$ is a neighborhood of 0 and $f : U \to \mathbf{C}^n$ is an analytic mapping such that $f(U - 0) \subset \mathbf{C}^n - \Delta$ then we define the *local free homotopy class of* f *at* 0 as the conjugacy class in $\pi_1(\mathbf{C}^n - \Delta)$ corresponding to the loop $[0, 2\pi] \to \mathbf{C}^n$, $t \mapsto f(\varepsilon e^{it})$ for a sufficiently small ε. Let (D_0, S_0) and $\gamma_1, \ldots, \gamma_s$ be as above.

Proposition 2. *Let $U \subset \mathbf{C}$ be some neighborhood of 0. For $j = 1, \dots, s$, let $f_j : (U, 0) \to (\mathbf{C}^n, \Delta)$ be an analytic mapping such that $f_j(U - 0) \subset \mathbf{C}^n - \Delta$, and let $b_j \in \pi_1(\mathbf{C}^n - \Delta)$ be any representative of the local free homotopy class of f_j at 0. Let $D \subset \mathbf{C}$ be a disk. Then there exist a diffeomorphism $\varphi : (D_0, S_0) \to (D, S)$ and a polynomial mapping $p : D \to \mathbf{C}^n$ such that $p^{-1}(\Delta) \cap D = S$ and for the induced mapping $\pi_1(D_0 - S_0) \to \pi_1(D - S) \to \pi_1(\mathbf{C}^n - \Delta)$ one has $p_*(\varphi_*(\gamma_j)) = b_j$, $j = 1, \dots s$.*

The rest of the paper is devoted to the proof of Proposition 2. Introduce the following notation. For paths $\alpha_k : [a_k, b_k] \to X$, $k = 1, 2$ we denote by $\alpha_1 * \alpha_2$ their product and by α_k^{-1} the inverse in the sense of the multiplication in the fundamental gruppoid of X ($\alpha_1 * \alpha_2$ is defined only when $\alpha_2(a_2) = \alpha_1(b_1)$). The class of a loop α in $\pi_1(X)$ we denote by $[\alpha]$. Thus, $[\alpha^{-1}] = [\alpha]^{-1}$ and $[\alpha * \beta] = [\alpha][\beta]$. In other aspects we treat paths as functions, for instance, if $\alpha : [a, b] \to \mathbf{C}$ is a path and $c, d \in \mathbf{C}$ then $c\alpha + d$ is just $t \mapsto c\alpha(t) + d$. We use the same notation for a segment $[a, b] \subset \mathbf{C}$ and for a path which parameterizes it. Denote the path $[0, 2\pi] \to \mathbf{C}$, $t \mapsto -e^{it}$ by θ and let $\theta_- = \theta|_{[0,\pi]}$. For $j = 1, 2, ...$ and $0 < \varepsilon < 1/2$ define the path $\gamma_{j,\varepsilon}$ as $\alpha * (j + \varepsilon\theta) * \alpha^{-1}$, where $\alpha = [0, 1 - \varepsilon] * (1 + \varepsilon\theta_-) * [1 + \varepsilon, 2 - \varepsilon] * (2 + \varepsilon\theta_-) * \dots * [j - 1 + \varepsilon, j - \varepsilon]$.

$\gamma_{j,\varepsilon}$:



Suppose that a (non-closed) path $f : ([0, a], S_0) \to (\mathbf{C}^n, \Delta)$ is analytic near S_0. For $0 < j < a$ and ε sufficiently small the element $[f \circ \gamma_{j,\varepsilon}] \in \pi_1(\mathbf{C}^n - \Delta)$ does not depend on ε and we denote it by $[[f]]_j$.

Lemma 1. *Let $f_0, f_1 : ([0, a], S_0) \to (\mathbf{C}^n, \Delta)$ be two paths which are analytic near S_0. Let $N_j = 1 + \mathrm{ord}_{x=j}(\Phi \circ f_0)$. Suppose that (i) $f_0(0) = f_1(0)$, (ii) N_j-jets of f_0 and f_1 at x_j coincide, $j = 1, ..., s$ and (iii) $||f_1(x) - f_0(x)|| < \mathrm{dist}(f_0(x), \Delta)$ for any $x \in [0, a] \setminus S_0$. Then $[[f_1]]_j = [[f_0]]_j$.*

Proof. Continue f_0, f_1 into some complex neighborhoods of points $1, 2, ...$ Let $f_u = f_0 + u(f_1 - f_0)$. For any j we have $\Phi(f_0(x)) = \alpha_j \cdot (x - j)^{N_j - 1} + (x - j)^{N_j} g_j(x)$ and $f_1 - f_0 = (x - j)^{N_j} h_j(x)$ where g_j and h_j are analytic near $x = j$. Hence, $\Phi(f_u(x)) = \alpha_j \cdot (x - j)^{N_j - 1} + (x - j)^{N_j} H_j(x, u)$ where H_j is analytic near the line $x = j$ in $\mathbf{C}^2$. Hence, there exists $\delta > 0$ such that for any j, H_j is continuous in $V_{j,\delta} = \{x \in \mathbf{C} : |x - j| < \delta\} \times [0, 1]$. Let $M = \max_j \max_{V_{j,\delta}} |H_j / \alpha_j|$. Then for $\varepsilon < \min(\delta, 1/M)$ we have a homotopy $f_u \circ \gamma_{j,\varepsilon}$ in $\mathbf{C}^n - \Delta$. □

Without loss of generality we may assume that $\gamma_j = [\gamma_{j,\varepsilon}]$ for some ε. Otherwise, for each j we replace γ_j with $[\gamma_{j,\varepsilon}]$ and b_j with $W_j(b_1, ..., b_s)$ where W_j is the word such that $[\gamma_{j,\varepsilon}] = W_j(\gamma_1, ..., \gamma_s)$.

Lemma 2. *In the hypothesis of Proposition 2 there exists a path $f : ([0, a], S_0) \to (\mathbf{C}^n, \Delta)$, analytic near S_0, such that $[[f]]_j = b_j$ for any $j = 1, ..., s$.*

Proof. Let ε be so small that $b_j = c_j * (f_j \circ \varepsilon\theta) * c_j^{-1}$ for some c_j's. If $|x - j| < \varepsilon$, we put $f(x) = f_j(x)$. Define f on $[0, 1 - \varepsilon]$ as a reparameterization of c_1 and on $[j + \varepsilon, j + 1 - \varepsilon]$ for $j \geq 1$ as a reparameterization of $(f_j \circ \varepsilon\theta_-)^{-1} * c_j^{-1} * c_{j+1}$. □

Lemma 3. *Let N be an integer and $S = \{x_1, ..., x_s\} \subset [a, b] \subset \mathbf{R}$. Let f and ρ be real continuous functions on $[a, b]$ such that f is analytic near S and ρ is piecewise algebraic near S. Suppose that $\rho \geq 0$ on $[a, b]$ and $\rho(x) > 0$ for $x \notin S$. Then there exists a polynomial p such that for $x \notin S$ one has $|f(x) - p(x)| < \rho(x)$ and the N-jets of f and p at x_j coincide for any $j = 1, \dots, s$.*

Proof. Denote by $\alpha_j \in \mathbf{Q}$ the order of zero of ρ at x_j and let $\alpha \in 2\mathbf{Z}$, $\alpha > \max_j \alpha_j$. Put $q_1 = \prod_j (x - x_j)^\alpha$. Then q_1/ρ is continuous on $[a, b]$. Denote by M its maximum and let $q = q_1/M$. Then on $[a, b]$ we have $0 \leq q \leq \rho$. Put $N_1 = \max(N, \alpha)$. Let p_1 be a polynomial whose N_1-jets at $x_1, \dots, x_s$ are the same as those of f and put $g = (f - p_1)/q$. The function g is continuous and hence, by Weierstrass approximation theorem, there exists a polynomial p_2 such that $|g - p_2| < 1$. Then for $p = p_1 + p_2 q$ and $x \in [a, b] - S$ we have $|f - p|/\rho \leq |f - p|/q = |g - p_2| < 1$. □

Proof of Proposition 2. Let f be as in Lemma 2. Put $a = s + 1/2$, $\rho(x) = \operatorname{dist}(f(x), \Delta)$ and $N = 1 + \max_j \operatorname{ord}_{x=j} \Phi \circ f$. Applying Lemma 3 to Re and Im of each coordinate, we can approximate f by a polynomial mapping $p_1 : [0, a] \to \mathbf{C}^n$ with the same N-jets at S_0, so that $||p_1 - f|| < \rho$ on $[0, a] - S_0$. Hence, $[[p_1]]_j = [[f]]_j = b_j$ by Lemmas 1, 2. Chose $\varepsilon < \operatorname{dist}([0, a], p_1^{-1}(\Delta) - S_0)$ and let D_1 be the ε-neighborhood of $[0, a]$ in $\mathbf{C}$. Let $\varphi_1 : D \to D_1$ be a polynomial approximation of a conformal isomorphism $D \to D_1$ and let $\psi_t : (\varphi_1(D), S_0) \to (D_0, S_0)$ be a homotopy such that $\psi_0 = \mathrm{id}$ and ψ_1 is a diffeomorphism. To complete the proof, put $p = p_1 \circ \varphi_1$, $\varphi = \varphi_1^{-1} \circ \psi_1^{-1}$ and note that $p_*(\varphi_*(\gamma_j)) = [p_1 \circ \psi_1^{-1} \circ \gamma_{j,\varepsilon}] = [p_1 \circ \gamma_{j,\varepsilon}] = [[p_1]]_j = b_j$. □

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