PRODUCTS OF CONJUGACY CLASSES IN $SL_2(\mathbb{R})$

S. Yu. Orevkov

Abstract. We compute the product of any n-tuple of conjugacy classes in $SL_2(\mathbb{R})$.

1. Introduction

In this paper we compute the product of any n-tuple of conjugacy classes in the group $SL_2(\mathbb{R})$ (see Theorem 3.1 in §3). The computation is straightforward, the main difficulty being to find a suitable notation for the answer to be readable.

All products of conjugacy classes are computed in [9] for simple finite groups of order less than million and for sporadic simple finite groups. In [12] the same is done for finite unitary groups $GU_3(\mathbb{F}_q)$ and $SU_3(\mathbb{F}_q)$ as well as for finite linear groups $GL_3(\mathbb{F}_q)$ and $SL_3(\mathbb{F}_q)$. Similar questions were studied by several authors, see [2, 5, 8, 10, 15] and references therein.

My special interest in the computation of class products in any kind of linear or unitary groups is motivated by possible applications to plane real or complex algebraic curves, see [3, 11]. Perhaps, the most interesting and non-trivial case when the products of conjugacy classes are completely computed, is the case of the unitary groups $SU(n)$, see [1, 4]. It seems that Belkale's approach [4] could be extended (at least partially) to pseudo-unitary groups $SU(p,q)$ using the techniques developed in [7]. We are going to do this in a subsequent paper. Some products of conjugacy classes in $PU(n,1)$ (especially for $n=2$) are computed in [6, 13, 14].

Note that $SU(1,1)$ is isomorphic to $SL_2(\mathbb{R})$. Indeed,

$$\Phi : SL_2(\mathbb{R}) \to SU(1,1), \qquad A \mapsto \Phi(A) = P^{-1}AP, \qquad P = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \qquad (1)$$

is an isomorphism; recall that $SU(1,1) = \{A \in SL_2(\mathbb{C}) \mid A^*JA = J\}$ where $J = \operatorname{diag}(1,-1)$. So, the main motivation for the computation of class products in $SL_2(\mathbb{R})$ was to get an idea what the answer for $SU(p,q)$ could look like.

2. Conjugacy classes

Let $G = SL_2(\mathbb{R})$. The conjugacy classes in G are given in Table 1. This fact can be easily derived, e. g., from [5, §2].

A more geometric characterization of the conjugacy classes can be given as follows. For $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$, we denote $\vec{x} \wedge \vec{y} = x_1y_2 - x_2y_1$.

Proposition 2.1. *Let $A \in G \setminus \{I, -I\}$ and $0 < \alpha < \pi$. Then:*

(a). $A \in \mathfrak{c}_2^{\varepsilon,\delta}$, $\varepsilon, \delta = \pm 1$, if and only if $\operatorname{tr} A = 2\varepsilon$ and $\delta \vec{x} \wedge A\vec{x} \ge 0$ for any $\vec{x} \in \mathbb{R}^2$;

(b). $A \in \mathfrak{c}_2^{\alpha}$ if and only if $\operatorname{tr} A = 2\cos\alpha$ and $\vec{x} \wedge A\vec{x} > 0$ for any $\vec{x} \in \mathbb{R}^2$. $\square$

class	paramameters	representative	nessessary and sufficient condition on $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$
$\mathfrak{c}_1^{\varepsilon}$	$\{-1,1\}$	$\varepsilon I = \left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{smallmatrix}\right)$	$A = \varepsilon I$
$\mathfrak{c}_2^{\varepsilon,\delta}$	$\{-1,1\}^2$	$\left(\begin{smallmatrix} \varepsilon & 0 \\ \delta & \varepsilon \end{smallmatrix}\right)$	$\operatorname{tr} A = 2\varepsilon \quad \& \quad \delta(c-b) > 0$
$\mathfrak{c}_3^{\alpha}$	$]0,2\pi[\setminus\{\pi\}$	$\left(\begin{smallmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{smallmatrix}\right)$	$\operatorname{tr} A = 2\cos\alpha \quad \& \quad c\sin\alpha > 0$
$\mathfrak{c}_4^{\lambda}$	$\mathbb{R}\setminus[-1,1]$	$\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$	$\operatorname{tr} A = \lambda + \lambda^{-1}$

Table 1. Conjugacy classes in $SL_2(\mathbb{R})$. Note that for $\mathfrak{c}_2^{\varepsilon,\delta}$ (resp. $\mathfrak{c}_3^{\alpha}$) we have $\delta(c-b) > 0 \Leftrightarrow (\delta c > 0$ or $\delta b < 0)$ (resp. $c\sin\alpha > 0 \Leftrightarrow b\sin\alpha < 0$)

We denote the union of all $\mathfrak{c}_i^{\cdot\cdot}$ by $\mathfrak{c}_i$ $(i = 1,2,3,4)$. For $X \subset G$, we denote the complement $G\setminus X$ by X^{c}. We set also

$$\mathfrak{c}_2^{\varepsilon,*} = \mathfrak{c}_2^{\varepsilon,+} \cup \mathfrak{c}_2^{\varepsilon,-}, \qquad \mathfrak{c}_4^{+} = \bigcup_{\lambda>1} \mathfrak{c}_4^{\lambda}, \qquad \mathfrak{c}_4^{-} = \bigcup_{\lambda<-1} \mathfrak{c}_4^{\lambda},$$

$$\overline{\mathfrak{c}_4^{+}} = \mathfrak{c}_4^{+} \cup \{I\} \cup \mathfrak{c}_2^{+*}, \qquad \overline{\mathfrak{c}_4^{-}} = \mathfrak{c}_4^{-} \cup \{-I\} \cup \mathfrak{c}_2^{-*}.$$

If j is an interval (open or closed from either end) contained in $]0,2\pi[\setminus\{\pi\}$, then we set $\mathfrak{c}_3^{j} = \bigcup_{\alpha\in j} \mathfrak{c}_3^{\alpha}$. If the left end of j is "$]0\dots$" or "$]\pi\dots$", then the bracket "$]\dots$" can be replaced by "$[\dots$" or "$\langle[\dots$". Symmetrically, if the right end of j is "$\dots\pi[$" or "$\dots 2\pi[$", then the bracket "$\dots[$" can be replaced by "$\dots]$" or "$\dots]\rangle$", and this means the following:

$$\begin{aligned}
\mathfrak{c}_3^{[0,\dots} &= \mathfrak{c}_2^{++} \cup \mathfrak{c}_3^{]0,\dots}, & \mathfrak{c}_3^{\dots,\pi]} &= \mathfrak{c}_3^{\dots,\pi[} \cup \mathfrak{c}_2^{-+},\\
\mathfrak{c}_3^{[\pi,\dots} &= \mathfrak{c}_2^{--} \cup \mathfrak{c}_3^{]\pi,\dots}, & \mathfrak{c}_3^{\dots,2\pi]} &= \mathfrak{c}_3^{\dots,2\pi[} \cup \mathfrak{c}_2^{+-},\\
\mathfrak{c}_3^{\langle[0,\dots} &= \mathfrak{c}_4^{+} \cup \mathfrak{c}_3^{[0,\dots}, & \mathfrak{c}_3^{\dots,\pi]\rangle} &= \mathfrak{c}_3^{\dots,\pi]} \cup \mathfrak{c}_4^{-},\\
\mathfrak{c}_3^{\langle[\pi,\dots} &= \mathfrak{c}_4^{-} \cup \mathfrak{c}_3^{[\pi,\dots}, & \mathfrak{c}_3^{\dots,2\pi]\rangle} &= \mathfrak{c}_3^{\dots,2\pi]} \cup \mathfrak{c}_4^{+},
\end{aligned}$$

for example, $\mathfrak{c}_3^{\langle[0,\pi]} = \mathfrak{c}_4^{+} \cup \mathfrak{c}_2^{++} \cup \left(\bigcup_{0<\alpha<\pi} \mathfrak{c}_3^{\alpha}\right) \cup \mathfrak{c}_2^{-+}$. Let

$$G^{+} = \overline{\mathfrak{c}_4^{+}} \cup \mathfrak{c}_3^{]0,\pi[}.$$

3. Statement of the main result

Theorem 3.1. *(a). For conjugacy classes contained in G^{+} (see Remark 3.2), their double and triple products are as shown in Tables 2 and 3; in Figure 1 we represent all triples of classes from $\mathfrak{c}_3$ whose product contains I.*

(b). Let $0 < \alpha,\beta,\gamma,\delta < \pi$. Then:

$$\begin{aligned}
&\mathfrak{c}_2^{++}\mathfrak{c}_2^{++}\mathfrak{c}_2^{+-}\mathfrak{c}_2^{+-} = \mathfrak{c}_2^{++}\mathfrak{c}_2^{++}\mathfrak{c}_2^{+-}\mathfrak{c}_3^{\alpha} = \{-I\}^{\mathrm{c}}, &&\\
&\mathfrak{c}_2^{++}\mathfrak{c}_2^{+-}\mathfrak{c}_2^{+-}\mathfrak{c}_3^{\alpha} = \mathfrak{c}_2^{++}\mathfrak{c}_2^{+-}\mathfrak{c}_3^{\alpha}\mathfrak{c}_3^{\beta} = \{I\}^{\mathrm{c}}, &&\\
&\mathfrak{c}_2^{++}\mathfrak{c}_2^{++}\mathfrak{c}_3^{\alpha}\mathfrak{c}_3^{\beta} = \{-I\}^{\mathrm{c}} && \textit{if } \alpha+\beta \ge \pi,\\
&\mathfrak{c}_2^{+-}\mathfrak{c}_2^{+-}\mathfrak{c}_3^{\alpha}\mathfrak{c}_3^{\beta} = \{I\}^{\mathrm{c}} && \textit{if } \alpha+\beta \le \pi,\\
&\mathfrak{c}_3^{\alpha}\mathfrak{c}_3^{\beta}\mathfrak{c}_3^{\gamma}\mathfrak{c}_3^{\delta} = \{-I\}^{\mathrm{c}} && \textit{if } \pi < \alpha+\beta+\gamma+\delta < 3\pi,
\end{aligned}$$

the products of the form $\mathfrak{c}_3^\alpha\mathfrak{c}_3^\beta\mathfrak{c}_3^\gamma\mathfrak{c}_2^{+\pm}$ *are as in Table 3, and for any other four non-scalar conjugacy classes contained in* G^+*, their product is the whole* G*.*

(c). The product of any five non-scalar conjugacy classes is the whole G*.*

Remark 3.2. We see in Table 1 that, for any conjugacy class $\mathfrak{c}$. either $\mathfrak{c} \subset G^+$ or $-\mathfrak{c} \subset G^+$. Thus any product of conjugacy classes can be immediately recovered if one knows all products of classes contained in G^+.

Let $\tilde{G} = PSL_2(\mathbb{R})$. We denote the image of $\mathfrak{c}_i^{\cdot\cdot}$ in $\tilde{G}$ by $\tilde{\mathfrak{c}}_i^{\cdot\cdot}$.

Corollary 3.3. *(a). We have* $\tilde{\mathfrak{c}}_4^\lambda\tilde{\mathfrak{c}}_4^\lambda = \tilde{G}$ *and*

$$\tilde{\mathfrak{c}}_2^{++}\tilde{\mathfrak{c}}_2^{++} = \tilde{\mathfrak{c}}_2^{+-}\tilde{\mathfrak{c}}_2^{+-} = \tilde{\mathfrak{c}}_4^\lambda\tilde{\mathfrak{c}} = \tilde{G}\setminus\{\tilde{I}\}, \qquad \text{where } \tilde{\mathfrak{c}} \neq \tilde{\mathfrak{c}}_4^\lambda, \tilde{I}$$

(see Table 2 for the other double products of classes in $\tilde{G}$*).*

(b). Let $0 < \alpha, \beta, \gamma < \pi$*. The following triple products are equal to* $\tilde{G}\setminus\{\tilde{I}\}$*:*

$$\tilde{\mathfrak{c}}_2^{++}\tilde{\mathfrak{c}}_2^{+-}\tilde{\mathfrak{c}}_3^\alpha, \quad \tilde{\mathfrak{c}}_2^{+-}\tilde{\mathfrak{c}}_3^\alpha\tilde{\mathfrak{c}}_3^\beta \ (\text{if } \alpha+\beta<\pi), \quad \tilde{\mathfrak{c}}_2^{+\pm}\tilde{\mathfrak{c}}_3^\alpha\tilde{\mathfrak{c}}_3^\beta \ (\text{if } \alpha+\beta=\pi),$$
$$\tilde{\mathfrak{c}}_2^{++}\tilde{\mathfrak{c}}_3^\alpha\tilde{\mathfrak{c}}_3^\beta \ (\text{if } \alpha+\beta>\pi), \quad \tilde{\mathfrak{c}}_3^\alpha\tilde{\mathfrak{c}}_3^\beta\tilde{\mathfrak{c}}_3^\gamma \ (\text{if } \pi<\alpha+\beta+\gamma<2\pi).$$

The product of any other three non-trivial conjugacy classes is the whole $\tilde{G}$*.*

(c). The product of any four non-trivial conjugacy classes is the whole $\tilde{G}$*.*

Corollary 3.4. $\mathrm{cn}(\tilde{G}) = \mathrm{ecn}(\tilde{G}) = 4$ *(in the notation of [9]).*

	I	$\mathfrak{c}_2^{++}$	$\mathfrak{c}_2^{+-}$	$\mathfrak{c}_3^\gamma$	$\mathfrak{c}_4^\nu$
$\mathfrak{c}_2^{++}\mathfrak{c}_2^{++}$	$\mathfrak{c}_3^{[0,\pi]\rangle}$	$\{I\}^c$	$\mathfrak{c}_3^{[0,\pi]\rangle} \cup \overline{\mathfrak{c}_4^+}$	$\swarrow$	$\swarrow$
$\mathfrak{c}_2^{++}\mathfrak{c}_2^{+-}$	$\overline{\mathfrak{c}_4^+}$	$\mathfrak{c}_3^{[0,\pi]\rangle} \cup \overline{\mathfrak{c}_4^+}$	$\mathfrak{c}_3^{\langle[\pi,2\pi]} \cup \overline{\mathfrak{c}_4^+}$	$\swarrow$	$\swarrow$
$\mathfrak{c}_2^{+-}\mathfrak{c}_2^{+-}$	$\mathfrak{c}_3^{\langle[\pi,2\pi]}$	$\mathfrak{c}_3^{\langle[\pi,2\pi]} \cup \overline{\mathfrak{c}_4^+}$	$\{I\}^{\mathrm{c}}$	$\swarrow$	$\swarrow$
$\mathfrak{c}_2^{++}\mathfrak{c}_3^\alpha$	$\mathfrak{c}_3^{]\alpha,\pi]\rangle}$	$\left(\{I\} \cup \mathfrak{c}_3^{[0,\alpha]}\right)^{\mathrm{c}}$	$\mathfrak{c}_3^{\langle[0,\pi]\rangle}$	$\swarrow$	$\swarrow$
$\mathfrak{c}_2^{+-}\mathfrak{c}_3^\alpha$	$\mathfrak{c}_3^{\langle[0,\alpha[}$	$\mathfrak{c}_3^{\langle[0,\pi]\rangle}$	$\left(\{-I\} \cup \mathfrak{c}_3^{[\alpha,\pi]}\right)^{\mathrm{c}}$	$\swarrow$	$\swarrow$
$\mathfrak{c}_2^{++}\mathfrak{c}_4^\lambda$	$\mathfrak{c}_3^{\langle[0,\pi]\rangle}$	$\{I\}^{\mathrm{c}}$	$\{-I\}^{\mathrm{c}}$	$\swarrow$	$\swarrow$
$\mathfrak{c}_2^{+-}\mathfrak{c}_4^\lambda$	$\mathfrak{c}_3^{\langle[\pi,2\pi]\rangle}$	$\{-I\}^{\mathrm{c}}$	$\{I\}^{\mathrm{c}}$	$\swarrow$	$\swarrow$
$\mathfrak{c}_3^\alpha\mathfrak{c}_3^\beta$, $\alpha+\beta<\pi$	$\mathfrak{c}_3^{[\alpha+\beta,\pi]\rangle}$	$\left(\{I\} \cup \mathfrak{c}_3^{[0,\alpha+\beta]}\right)^{\mathrm{c}}$	$\mathfrak{c}_3^{\langle[0,\pi]\rangle}$	See Tbl. 3	$\swarrow$
$\mathfrak{c}_3^\alpha\mathfrak{c}_3^\beta$, $\alpha+\beta=\pi$	$\{-I\} \cup \mathfrak{c}_4^-$	$\mathfrak{c}_3^{\langle[\pi,2\pi]\rangle}$	$\mathfrak{c}_3^{\langle[0,\pi]\rangle}$		$\swarrow$
$\mathfrak{c}_3^\alpha\mathfrak{c}_3^\beta$, $\alpha+\beta>\pi$	$\mathfrak{c}_3^{\langle[\pi,\alpha+\beta]}$	$\mathfrak{c}_3^{\langle[\pi,2\pi]\rangle}$	$\left(\{I\} \cup \mathfrak{c}_3^{[\alpha+\beta,\pi]}\right)^{\mathrm{c}}$		$\swarrow$
$\mathfrak{c}_3^\alpha\mathfrak{c}_4^\lambda$	$\mathfrak{c}_3^{\langle[0,\pi]\rangle}$	$\{I\}^{\mathrm{c}}$	$\{-I\}^{\mathrm{c}}$	$\{I\}^{\mathrm{c}}$	G
$\mathfrak{c}_4^\lambda\mathfrak{c}_4^\lambda$	$\{-I\}^{\mathrm{c}}$	G	G	G	G
$\mathfrak{c}_4^\lambda\mathfrak{c}_4^\mu$, $\lambda\neq\mu$	$\{I,-I\}^{\mathrm{c}}$	G	G	G	G

TABLE 2. Double and triple products of conjugacy classes in $SL_2(\mathbb{R})$ ("$\swarrow$" means "see some other cell(s) of this table"). The range of the parameters: $0 < \alpha, \beta, \gamma < \pi$ and $\lambda, \mu, \nu > 1$.

Condition on α,β,γ	$\mathfrak{c}_3^\alpha\mathfrak{c}_3^\beta\mathfrak{c}_3^\gamma$	$\mathfrak{c}_3^\alpha\mathfrak{c}_3^\beta\mathfrak{c}_3^\gamma\mathfrak{c}_2^{++}$	$\mathfrak{c}_3^\alpha\mathfrak{c}_3^\beta\mathfrak{c}_3^\gamma\mathfrak{c}_2^{+-}$
$\alpha+\beta+\gamma<\pi$	$\big(\{I\}\cup\mathfrak{c}_3^{[0,\alpha+\beta+\gamma[}\big)^{\mathrm c}$	G	$G\setminus\{I\}$
$\alpha+\beta+\gamma=\pi$	$\{-I\}\cup\mathfrak{c}_3^{\langle[\pi,2\pi]\rangle}$	$G\setminus\{-I\}$	$G\setminus\{I\}$
$\pi<\alpha+\beta+\gamma<2\pi$	$\mathfrak{c}_3^{\langle[\pi,2\pi]\rangle}$	$G\setminus\{-I\}$	$G\setminus\{I\}$
$\alpha+\beta+\gamma=2\pi$	$\{I\}\cup\mathfrak{c}_3^{\langle[\pi,2\pi]\rangle}$	$G\setminus\{-I\}$	$G\setminus\{I\}$
$\alpha+\beta+\gamma>2\pi$	$\big(\{-I\}\cup\mathfrak{c}_3^{]\alpha+\beta+\gamma-2\pi,\pi]}\big)^{\mathrm c}$	$G\setminus\{-I\}$	G

TABLE 3. Triple and some quadruple products of conjugacy classes in $SL_2(\mathbb{R})$ involving $\mathfrak{c}_3^\alpha\mathfrak{c}_3^\beta\mathfrak{c}_3^\gamma$ with $0<\alpha,\beta,\gamma<\pi$.

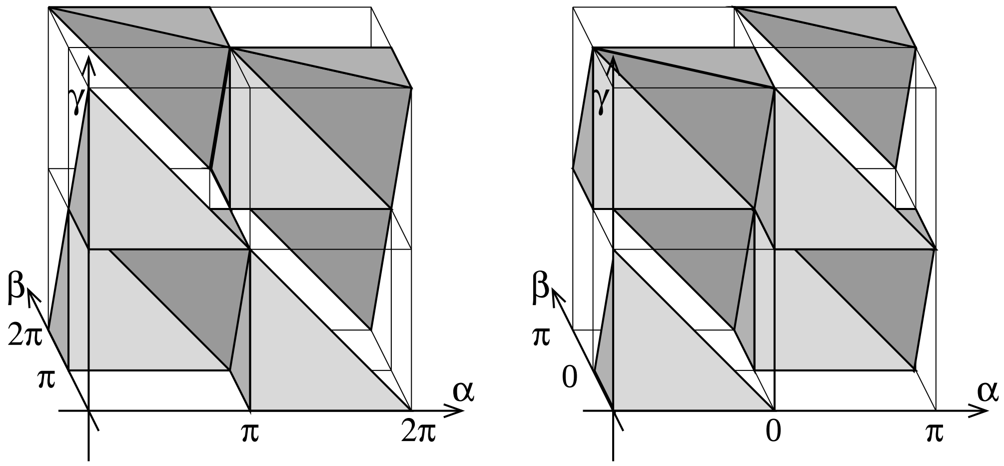

FIGURE 1. The sets $\{(\alpha,\beta,\gamma)\in(\,]0,2\pi[\,\setminus\{\pi\})^3 \mid I\in\mathfrak{c}_3^\alpha\mathfrak{c}_3^\beta\mathfrak{c}_3^\gamma\}$ and $\{(\alpha,\beta,\gamma)\in(\,]-\pi,\pi[\,\setminus\{0\})^3 \mid I\in\mathfrak{c}_3^\alpha\mathfrak{c}_3^\beta\mathfrak{c}_3^\gamma\}$.

4. Range of the trace on the product of two classes

Let Φ be as in (1); see the introduction.

Lemma 4.1. *a).* $SU(1,1)=\{\begin{pmatrix} a & b\\ \bar b & \bar a\end{pmatrix}\mid a,b\in\mathbb{C}, a\bar a-b\bar b=1\}$.

b). $\Phi(\mathfrak{c}_3^\alpha)$ *is the conjugacy class of* $\begin{pmatrix}\lambda & 0\\ 0 & \bar\lambda\end{pmatrix}$, $\lambda=e^{\alpha i}$.

Proof. Let $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$, $ad-bc=1$. Then $A\in SU(1,1)$ if and only if $A^*J=JA^{-1}$. We have:

$$A^*J=\begin{pmatrix}\bar a & \bar c\\ \bar b & \bar d\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}=\begin{pmatrix}\bar a & -\bar c\\ \bar b & -\bar d\end{pmatrix},\quad JA^{-1}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}=\begin{pmatrix}d & -b\\ c & -a\end{pmatrix}\quad\square$$

Lemma 4.2. *Let* $A\in\mathfrak{c}_3^{[0,\pi]}$ *and* $B\in\overline{\mathfrak{c}_4^+}$. *Then* $AB\notin\mathfrak{c}_3^{[\pi,2\pi]}\cup\{I,-I\}$ *unless* A *and* B *both belong to* $\mathfrak{c}_2$ *and have common eigenvector.*

Proof. Let v be a real eigenvector of B that is not an eigenvector of A. The condition $B\in\overline{\mathfrak{c}_4^+}$ implies that the corresponding eigenvalue λ is positive. We have

$v\wedge Av \geq 0$ by Proposition 2.1. Moreover, $v\wedge Av \neq 0$ since v is not an eigenvector of A. Hence $v\wedge ABv = \lambda v\wedge Av > 0$ and the result follows from Proposition 2.1. $\square$

Lemma 4.3. *Let $0<\alpha,\beta<\pi$. Then:*

(a). $\{\operatorname{tr}(AB)\,|\,A\in\mathfrak{c}_3^\alpha,\ B\in\mathfrak{c}_3^\beta\} = \,]-\infty, 2\cos(\alpha+\beta)]$*;*

(b). If $A\in\mathfrak{c}_3^\alpha$, $B\in\mathfrak{c}_3^\beta$, and $\operatorname{tr}(AB)=2\cos(\alpha+\beta)$, then $AB\in\mathfrak{c}_3^{\alpha+\beta}$.

Proof. Let $A\in\mathfrak{c}_3^\alpha$ and $B\in\mathfrak{c}_3^\beta$. We may assume by Lemma 4.1 that $\Phi(A)=\left(\begin{smallmatrix}\lambda & 0\\ 0 & \bar\lambda\end{smallmatrix}\right)$ and $\Phi(B)=Q\left(\begin{smallmatrix}\mu & 0\\ 0 & \bar\mu\end{smallmatrix}\right)Q^{-1}$ with $\lambda=e^{\alpha i}$, $\mu=e^{\beta i}$, and $Q=\left(\begin{smallmatrix}a & b\\ \bar b & \bar a\end{smallmatrix}\right)$ with $a\bar a - b\bar b = 1$. Then we have

$$\begin{aligned}\operatorname{tr}(AB) &= (\lambda\mu+\bar\lambda\bar\mu)a\bar a - (\lambda\bar\mu+\bar\lambda\mu)b\bar b = 2(1+b\bar b)\cos(\alpha+\beta) - 2b\bar b\cos(\alpha-\beta)\\ &= 2\cos(\alpha+\beta) - 4b\bar b\sin\alpha\sin\beta\end{aligned}$$

and the result easily follows. $\square$

Lemma 4.4. *Let $0<\alpha<\pi$. Then* $\{\operatorname{tr}(AB)\,|\,A\in\mathfrak{c}_3^\alpha,\ B\in\mathfrak{c}_2^{++}\} = \,]-\infty, 2\cos\alpha[$ *and* $\{\operatorname{tr}(AB)\,|\,A\in\mathfrak{c}_3^\alpha,\ B\in\mathfrak{c}_2^{+-}\} = \,]2\cos\alpha, \infty[$.

Proof. Let $A\in\mathfrak{c}_3^\alpha$ and $B\in\mathfrak{c}_2^{+\pm}$. Let us fix a quadratic form invariant under A and choose a positively oriented orthonormal base (e_1,e_2) such that e_2 is an eigenvector of B. In this base, the matrices of the corresponding operators are: $A'=\left(\begin{smallmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{smallmatrix}\right)$ and $B'=\left(\begin{smallmatrix}1 & 0\\ \pm p & 1\end{smallmatrix}\right)$ with $p>0$, thus $\operatorname{tr}A'B' = 2\cos\alpha \mp p\sin\alpha$. $\square$

Lemma 4.5. $\{\operatorname{tr}(AB)\,|\,A,B\in\mathfrak{c}_2^{++}\} = \{\operatorname{tr}(AB)\,|\,A,B\in\mathfrak{c}_2^{+-}\} = \,]-\infty, 2]$ *and* $\{\operatorname{tr}(AB)\,|\,A\in\mathfrak{c}_2^{++},\ B\in\mathfrak{c}_2^{+-}\} = [2,\infty[$.

Proof. We consider only the case $A,B\in\mathfrak{c}_2^{++}$ (the other two cases are similar). Let $A=\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)$ and $B=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\mathfrak{c}_2^{++}$. Then $b\leq 0$ (see Table 1) and $\operatorname{tr}AB = a+b+d = b+\operatorname{tr}B = b+2$. Moreover, b can attain any non-positive value. Indeed, consider the matrices $B_1=\left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right)$ $(b<0)$ and $B_2=A$. $\square$

Lemma 4.6. *Let $A\in\mathfrak{c}_4$. Then for any $t_1,t_2\in\mathbb{R}$ there exist matrices $B,C\in G$ such that $\operatorname{tr}B=t_1$, $\operatorname{tr}C=t_2$, and $AB=C$.*

Proof. Without loss of generality we may assume that $A=\operatorname{diag}(\lambda,\lambda^{-1})$, $|\lambda|>1$. Let $B=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$. Then we have $\operatorname{tr}B=a+d$ and $\operatorname{tr}AB=\lambda a+\lambda^{-1}d$. Thus, it is enough to find a and d from the simultaneous equations $a+d=t_1$, $\lambda a+\lambda^{-1}d=t_2$ and then to find b and c such that $bc=ad-1$. $\square$

5. Double products of conjugacy classes

Lemma 5.1. *Let $0<\alpha,\beta<\pi$. Then $\mathfrak{c}_3^\alpha\mathfrak{c}_3^\beta$ is as shown in Table 2.*

Proof. Let $A\in\mathfrak{c}_3^\alpha$, $B\in\mathfrak{c}_3^\beta$, and let $C=AB$.

It follows from Lemma 4.3 that the range of $\operatorname{tr}C$ is as required. So, it remains to show that the conjugacy class of C is uniquely determined by the trace. This is evidently so when $C\in\mathfrak{c}_4$. Let us consider all the other cases.

Case 1. $\alpha+\beta<\pi$. It is clear that $C\notin\{\pm I\}$.

Case 1.1. $C \in \mathfrak{c}_2^{\varepsilon,\delta}$, $\varepsilon,\delta = \pm 1$. We have $\varepsilon = -1$ by Lemma 4.3, so it remains to show that $\delta \neq -1$. Suppose that $\delta = -1$, i.e., $C \in \mathfrak{c}_2^{--}$. We have $B = (-A^{-1})(-C)$ with $-A^{-1} \in \mathfrak{c}_3^{\pi-\alpha}$ and $-C \in \mathfrak{c}_2^{++}$. Hence Lemma 4.4 implies $2\cos\beta = \operatorname{tr} B < \operatorname{tr}(-A^{-1}) = 2\cos(\pi-\alpha)$ which contradicts the assumption that $\alpha+\beta < \pi$.

Case 1.2. $C \in \mathfrak{c}_3$. Let $C \in \mathfrak{c}_3^{\gamma}$, $\gamma \in]0,2\pi[\setminus \{\pi\}$. Then, by Lemma 4.3, we have $\cos\gamma = \frac{1}{2}\operatorname{tr} C \le \cos(\alpha+\beta)$, hence

$$\gamma \ge \alpha + \beta. \tag{2}$$

Thus it suffices to show that γ cannot be $> \pi$.

Suppose that $\gamma > \pi$. Without loss of generality we may assume that $\alpha \ge \beta$. We have $(-A^{-1})(-C) = B$ with $-A^{-1} \in \mathfrak{c}_3^{\pi-\alpha}$, $-C \in \mathfrak{c}_3^{\gamma-\pi}$. The both angles $\pi - \alpha$ and $\gamma - \pi$ are in $]0,\pi[$. Thus, by Lemma 4.3 applied to the matrices $-A^{-1}$, $-C$, and B, we have $\cos\beta = \frac{1}{2}\operatorname{tr} B \le \cos((\pi-\alpha)+(\gamma-\pi)) = \cos(\gamma-\alpha)$. Combining this inequality with (2), we obtain $\gamma - \alpha \ge 2\pi - \beta$. Thus $\gamma \ge 2\pi + \alpha - \beta$ which contradicts our assumptions $\alpha \ge \beta$ and $\gamma < 2\pi$.

Case 2. $\alpha + \beta = \pi$. Follows immediately from Lemma 4.3.

Case 3. $\alpha + \beta > \pi$. We have $C^{-1} = (-B^{-1})(-A^{-1})$ with $-B^{-1} \in \mathfrak{c}_3^{\pi-\beta}$, $-A^{-1} \in \mathfrak{c}_3^{\pi-\alpha}$, and $(\pi-\beta)+(\pi-\alpha) \in]0,\pi[$, thus we reduce this case to Case 1. □

Lemma 5.2. *Let $0 < \alpha < \pi$. Then $\mathfrak{c}_3^{\alpha}\mathfrak{c}_2^{+\pm}$ is as shown in Table 2.*

Proof. Combine Lemma 4.4 with Lemma 4.2. □

Lemma 5.3. *$\mathfrak{c}_2^{++}\mathfrak{c}_2^{++}$, $\mathfrak{c}_2^{++}\mathfrak{c}_2^{+-}$, and $\mathfrak{c}_2^{+-}\mathfrak{c}_2^{+-}$ are as shown in Table 2.*

Proof. $\mathfrak{c}_2^{++}\mathfrak{c}_2^{++}$ is as required due to Lemma 4.5 combined with Lemma 4.2. We obtain $\mathfrak{c}_2^{+-}\mathfrak{c}_2^{+-}$ from $\mathfrak{c}_2^{++}\mathfrak{c}_2^{++}$ by passing to the inverse matrices. The computation of $\mathfrak{c}_2^{++}\mathfrak{c}_2^{+-}$ is immediate from Lemma 4.5 combined with the observation that $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}$ belongs to $\mathfrak{c}_2^{++}$, $\{I\}$, or $\mathfrak{c}_2^{+-}$ according to the sign of $1-t$. □

Lemma 5.4. *The product of $\mathfrak{c}_4^{\lambda}$ with any other conjugacy class is as in Table 2.*

Proof. Let $\lambda > 1$, and $\mathfrak{c}$ be any conjugacy class. Let X be the set that should coincide with $\mathfrak{c}_4^{\lambda}\mathfrak{c}$ according to Table 2. The fact that the product $\mathfrak{c}_4^{\lambda}\mathfrak{c}$ is contained in X, is either evident or follows from Lemma 4.2. Let us prove the inverse inclusion. Let $A \in \mathfrak{c}_4^{\lambda}$, $B_0 \in \mathfrak{c}$, and $C_0 \in X$. We assume that $B_0, C_0 \neq \pm I$ (otherwise everything is evident). We set $t_1 = \operatorname{tr} B_0$ and $t_2 = \operatorname{tr} C_0$. By Lemma 4.6, we can choose matrices B and C such that $\operatorname{tr} B = t_1$, $\operatorname{tr} C = t_2$, and $AB = C$. By passing to inverse matrices if necessary, we may assume that $B \sim B_0$ (note that $A^{-1} \sim A$). We have $\operatorname{tr} C = \operatorname{tr} C_0$ whence C belongs either to the class of C_0 or to the class of C_0^{-1}. However only one of these two classes may be contained in X (see Table 2) which completes the proof. □

All double products of conjugacy classes are computed in Lemmas 5.1–5.4. Using them, one easily computes triple and quadruple products as well; see the subsequent sections.

6. Triple products of conjugacy classes

Due to previous computations we have:

$$\begin{aligned}\mathfrak{c}_2^{++}\mathfrak{c}_2^{++}\mathfrak{c}_2^{++} &= \mathfrak{c}_3^{[0,\pi]\rangle}\mathfrak{c}_2^{++} = \big(\mathfrak{c}_3^{]0,\pi[} \cup \mathfrak{c}_2^{++} \cup -\mathfrak{c}_2^{+-} \cup -\mathfrak{c}_4^{+}\big)\mathfrak{c}_2^{++} \\ &= \mathfrak{c}_3^{]0,\pi]\rangle} \cup \mathfrak{c}_3^{[0,\pi]\rangle} \cup -\overline{\mathfrak{c}_4^{+}} \cup -\mathfrak{c}_3^{\langle[0,\pi]\rangle} = \{I\}^{\mathrm{c}},\end{aligned}$$

$$\begin{aligned}\mathfrak{c}_2^{++}\mathfrak{c}_2^{++}\mathfrak{c}_2^{+-} &= \mathfrak{c}_3^{[0,\pi]\rangle}\mathfrak{c}_2^{+-} = \big(\mathfrak{c}_3^{]0,\pi[} \cup \mathfrak{c}_2^{++} \cup -\mathfrak{c}_2^{+-} \cup -\mathfrak{c}_4^{+}\big)\mathfrak{c}_2^{+-} \\ &= \mathfrak{c}_3^{\langle[0,\pi[} \cup \overline{\mathfrak{c}_4^{+}} \cup -\mathfrak{c}_3^{\langle[\pi,2\pi]} \cup -\mathfrak{c}_3^{\langle[\pi,2\pi]\rangle} = \big(\{-I\} \cup \mathfrak{c}_3^{[\pi,2\pi[}\big)^{\mathrm{c}},\end{aligned}$$

$$\begin{aligned}\mathfrak{c}_3^{\alpha}\,\mathfrak{c}_2^{++}\mathfrak{c}_2^{++} &= \mathfrak{c}_3^{]\alpha,\pi]\rangle}\mathfrak{c}_2^{++} = \big(\mathfrak{c}_3^{]\alpha,\pi[} \cup -\mathfrak{c}_2^{+-} \cup -\mathfrak{c}_4^{+}\big)\mathfrak{c}_2^{++} \\ &= \mathfrak{c}_3^{]\alpha,\pi]\rangle} \cup -\overline{\mathfrak{c}_4^{+}} \cup -\mathfrak{c}_3^{\langle[0,\pi]\rangle} = \big(\{I\} \cup \mathfrak{c}_3^{[0,\alpha]}\big)^{\mathrm{c}},\end{aligned}$$

$$\begin{aligned}\mathfrak{c}_3^{\alpha}\,\mathfrak{c}_2^{++}\mathfrak{c}_2^{+-} &= \mathfrak{c}_3^{]\alpha,\pi]\rangle}\mathfrak{c}_2^{+-} = \big(\mathfrak{c}_3^{]\alpha,\pi[} \cup -\mathfrak{c}_2^{+-} \cup -\mathfrak{c}_4^{+}\big)\mathfrak{c}_2^{+-} \\ &= \mathfrak{c}_3^{\langle[0,\pi[} \cup -\mathfrak{c}_3^{\langle[\pi,2\pi]} \cup -\mathfrak{c}_3^{\langle[\pi,2\pi]\rangle} = \mathfrak{c}_3^{\langle[0,\pi]\rangle},\end{aligned}$$

$$\begin{aligned}\mathfrak{c}_4^{\lambda}\,\mathfrak{c}_2^{++}\mathfrak{c}_2^{++} &= \mathfrak{c}_4^{\lambda}\mathfrak{c}_3^{[0,\pi]\rangle} = \mathfrak{c}_4^{\lambda}\big(\mathfrak{c}_3^{[0,\pi]} \cup -\mathfrak{c}_4\big) = \mathfrak{c}_3^{\langle[0,\pi]\rangle} \cup -\{-I\}^{\mathrm{c}} = \{I\}^{\mathrm{c}}, \\ \mathfrak{c}_4^{\lambda}\,\mathfrak{c}_2^{++}\mathfrak{c}_2^{+-} &= \mathfrak{c}_4^{\lambda}\overline{\mathfrak{c}_4^{+}} = \mathfrak{c}_4^{\lambda}\big(\mathfrak{c}_4^{+} \cup \langle\text{a subset of } {\mathfrak{c}_4}^{\mathrm{c}}\rangle\big) = \{-I\}^{\mathrm{c}}, \\ \mathfrak{c}_4^{\lambda}\,\mathfrak{c}_3^{\alpha}\,\mathfrak{c}_2^{++} &= \mathfrak{c}_4^{\lambda}\mathfrak{c}_3^{]\alpha,\pi]\rangle} = \mathfrak{c}_4^{\lambda}\big(-\mathfrak{c}_4^{+} \cup \langle\text{a subset of } {\mathfrak{c}_4}^{\mathrm{c}}\rangle\big) = \{I\}^{\mathrm{c}}, \\ \mathfrak{c}_4^{\lambda}\,\mathfrak{c}_3^{\alpha}\,\mathfrak{c}_3^{\beta} &= \mathfrak{c}_4^{\lambda}\big(-\mathfrak{c}_4^{+} \cup \langle\text{a subset of } {\mathfrak{c}_4}^{\mathrm{c}}\rangle\big) = \{I\}^{\mathrm{c}}, \\ \mathfrak{c}_4^{\lambda}\,\mathfrak{c}_4^{\mu}\,\mathfrak{c} &= G \qquad\qquad \text{for any non-scalar class } \mathfrak{c}.\end{aligned}$$

If $\alpha+\beta<\pi$, then

$$\begin{aligned}\mathfrak{c}_3^{\alpha}\,\mathfrak{c}_3^{\beta}\,\mathfrak{c}_2^{++} &= \mathfrak{c}_3^{[\alpha+\beta,\pi]\rangle}\mathfrak{c}_2^{++} = \big(\mathfrak{c}_3^{[\alpha+\beta,\pi[} \cup -\mathfrak{c}_2^{+-} \cup -\mathfrak{c}_4^{+}\big)\mathfrak{c}_2^{++} \\ &= \mathfrak{c}_3^{]\alpha+\beta,\pi]\rangle} \cup -\overline{\mathfrak{c}_4^{+}} \cup -\mathfrak{c}_3^{\langle[0,\pi]\rangle} = \big(\{I\} \cup \mathfrak{c}_3^{[0,\alpha+\beta]}\big)^{\mathrm{c}}.\end{aligned}$$

If $\alpha+\beta=\pi$, then

$$\mathfrak{c}_3^{\alpha}\,\mathfrak{c}_3^{\beta}\,\mathfrak{c}_2^{++} = \big(\{-I\} \cup -\mathfrak{c}_4^{+}\big)\mathfrak{c}_2^{++} = -\mathfrak{c}_2^{++} \cup -\mathfrak{c}_3^{\langle[0,\pi]\rangle} = \mathfrak{c}_3^{\langle[\pi,2\pi]\rangle}.$$

If $\alpha+\beta>\pi$, then

$$\begin{aligned}\mathfrak{c}_3^{\alpha}\,\mathfrak{c}_3^{\beta}\,\mathfrak{c}_2^{++} &= \mathfrak{c}_3^{\langle[\pi,\alpha+\beta]}\mathfrak{c}_2^{++} = \big(-\mathfrak{c}_3^{]0,\alpha+\beta-\pi[} \cup -\mathfrak{c}_2^{++} \cup -\mathfrak{c}_4^{+}\big)\mathfrak{c}_2^{++} \\ &= -\mathfrak{c}_3^{]0,\pi]\rangle} \cup -\mathfrak{c}_3^{[0,\pi]\rangle} \cup -\mathfrak{c}_3^{\langle[0,\pi]\rangle} = \mathfrak{c}_3^{\langle[\pi,2\pi]\rangle}.\end{aligned}$$

If $\alpha+\beta+\gamma<\pi$, then

$$\begin{aligned}\mathfrak{c}_3^{\alpha}\,\mathfrak{c}_3^{\beta}\,\mathfrak{c}_3^{\gamma} &= \mathfrak{c}_3^{[\alpha+\beta,\pi]\rangle}\mathfrak{c}_3^{\gamma} = \big(\mathfrak{c}_3^{[\alpha+\beta,\pi-\gamma[} \cup \mathfrak{c}_3^{\pi-\gamma} \cup \mathfrak{c}_3^{]\pi-\gamma,\pi[} \cup -\mathfrak{c}_2^{+-} \cup -\mathfrak{c}_4^{+}\big)\mathfrak{c}_3^{\gamma} \\ &= \mathfrak{c}_3^{[\alpha+\beta+\gamma]\rangle} \cup \big(\{-I\} \cup -\mathfrak{c}_4^{+}\big) \cup \mathfrak{c}_3^{\langle[\pi,\pi+\gamma[} \cup -\mathfrak{c}_3^{\langle[0,\gamma[} \cup -\mathfrak{c}_3^{\langle[0,\pi]\rangle} \\ &= \big(\{I\} \cup \mathfrak{c}_3^{[0,\alpha+\beta+\gamma[}\big)^{\mathrm{c}}.\end{aligned}$$

If $\alpha+\beta+\gamma=\pi$, then

$$\begin{aligned}\mathfrak{c}_3^\alpha\,\mathfrak{c}_3^\beta\,\mathfrak{c}_3^\gamma &= \mathfrak{c}_3^{[\alpha+\beta,\pi]\rangle}\mathfrak{c}_3^\gamma = \big(\mathfrak{c}_3^{\alpha+\beta}\cup\mathfrak{c}_3^{]\alpha+\beta,\pi[}\cup -\mathfrak{c}_2^{+-}\cup -\mathfrak{c}_4^{+}\big)\mathfrak{c}_3^\gamma\\ &= (\{-I\}\cup\mathfrak{c}_4^-)\cup\mathfrak{c}_3^{\langle[\pi,\pi+\gamma[}\cup -\mathfrak{c}_3^{\langle[0,\gamma[}\cup -\mathfrak{c}_3^{\langle[0,\pi]\rangle} = \{-I\}\cup\mathfrak{c}_3^{\langle[\pi,2\pi]\rangle}.\end{aligned}$$

If $\pi<\alpha+\beta+\gamma<2\pi$ and $\alpha+\beta<\pi$, then

$$\begin{aligned}\mathfrak{c}_3^\alpha\,\mathfrak{c}_3^\beta\,\mathfrak{c}_3^\gamma &= \mathfrak{c}_3^{[\alpha+\beta,\pi]\rangle}\mathfrak{c}_3^\gamma = \big(\mathfrak{c}_3^{[\alpha+\beta,\pi[}\cup -\mathfrak{c}_2^{+-}\cup -\mathfrak{c}_4^{+}\big)\mathfrak{c}_3^\gamma\\ &= \mathfrak{c}_3^{\langle[\pi,\pi+\gamma[}\cup -\mathfrak{c}_3^{\langle[0,\gamma[}\cup -\mathfrak{c}_3^{\langle[0,\pi]\rangle} = \mathfrak{c}_3^{\langle[\pi,2\pi]\rangle}.\end{aligned}$$

If $\pi<\alpha+\beta+\gamma<2\pi$ and $\alpha+\beta=\pi$, then

$$\mathfrak{c}_3^\alpha\,\mathfrak{c}_3^\beta\,\mathfrak{c}_3^\gamma = \big(\{-I\}\cup -\mathfrak{c}_4^{+}\big)\mathfrak{c}_3^\gamma = -\mathfrak{c}_3^\gamma\cup -\mathfrak{c}_3^{\langle[0,\pi]\rangle} = \mathfrak{c}_3^{\langle[\pi,2\pi]\rangle}.$$

All other triple products reduce to these by passing to the inverse matrices (and maybe changing the sign). For example, if $\pi<\alpha+\beta+\gamma<2\pi$ and $\alpha+\beta>\pi$, then

$$\pi<(\pi-\alpha)+(\pi-\beta)+(\pi-\gamma)<2\pi,\qquad (\pi-\alpha)+(\pi-\beta)<\pi,$$

hence

$$\mathfrak{c}_3^\alpha\,\mathfrak{c}_3^\beta\,\mathfrak{c}_3^\gamma = (-\mathfrak{c}_3^{\pi-\alpha})^{-1}(-\mathfrak{c}_3^{\pi-\beta})^{-1}(-\mathfrak{c}_3^{\pi-\gamma})^{-1} = \big(-\mathfrak{c}_3^{\langle[\pi,2\pi]\rangle}\big)^{-1} = \mathfrak{c}_3^{\langle[\pi,2\pi]\rangle}.$$

7. Quadruple products. End of proof of Theorem 3.1

We see in Table 2 that any triple product of non-scalar conjugacy classes contains $\mathfrak{c}_4$, and the product of $\mathfrak{c}_4$ with any other non-scalar class contains $\{I,-I\}^{\rm c}$. Thus, to compute the product X of four non-scalar classes, it is enough to check if I and $-I$ belongs to it. In its turn, to decide whether or not $\pm I\in X$, it is enough to check if the inverse of one of these four classes multiplied by ±1 belongs to the product of the others.

For example, let us check that $X:=\mathfrak{c}_3^\alpha\mathfrak{c}_3^\beta\mathfrak{c}_3^\gamma\mathfrak{c}_2^{++}=\{-I\}^{\rm c}$ when $\alpha+\beta+\gamma=\pi$ (see Table 3). Indeed,

$$\begin{aligned}\big(\mathfrak{c}_2^{++}\big)^{-1} &= \mathfrak{c}_2^{+-}\in\mathfrak{c}_3^\alpha\mathfrak{c}_3^\beta\mathfrak{c}_3^\gamma=\{-I\}\cup\mathfrak{c}_3^{\langle[\pi,2\pi]\rangle}, &&\text{hence } I\in X,\\ -\big(\mathfrak{c}_2^{++}\big)^{-1} &= \mathfrak{c}_2^{-+}\notin\mathfrak{c}_3^\alpha\mathfrak{c}_3^\beta\mathfrak{c}_3^\gamma=\{-I\}\cup\mathfrak{c}_3^{\langle[\pi,2\pi]\rangle}, &&\text{hence } -I\notin X.\end{aligned}$$

As another example, let us compute the product $\mathfrak{c}_3^\alpha\mathfrak{c}_3^\beta\mathfrak{c}_3^\gamma\mathfrak{c}_3^\delta$ which we denote by X. We consider only the case when $\alpha+\beta+\gamma+\delta\le2\pi$ because the other case, when this sum is in the range $[2\pi,4\pi[$, is reduced to this one by passing to the inverse matrices. Let $Y=\mathfrak{c}_3^\alpha\mathfrak{c}_3^\beta\mathfrak{c}_3^\gamma$. We have

$$Y=\begin{cases}\Big(\{I\}\cup\mathfrak{c}_3^{[0,\alpha+\beta+\gamma[}\Big)^{\rm c}, & \alpha+\beta+\gamma<\pi,\\ \{-I\}\cup\mathfrak{c}_3^{\langle[\pi,2\pi]\rangle}, & \alpha+\beta+\gamma=\pi,\\ \mathfrak{c}_3^{\langle[\pi,2\pi]\rangle}, & \alpha+\beta+\gamma>\pi.\end{cases}$$

Therefore $\big(\mathfrak{c}_3^\delta\big)^{-1}=-\mathfrak{c}_3^{\pi-\delta}\in Y$ whence $I\in X$, and since $-\big(\mathfrak{c}_3^\delta\big)^{-1}=\mathfrak{c}_3^{\pi-\delta}$, we have

$$-I\in X\ \Leftrightarrow\ \mathfrak{c}_3^{\pi-\delta}\in Y\ \Leftrightarrow\ \pi-\delta\notin\,]0,\alpha+\beta+\gamma[\ \Leftrightarrow\ \pi-\delta\ge\alpha+\beta+\gamma$$

whence $X=G$ if $\alpha+\beta+\gamma+\delta\le\pi$, and $X=\{-I\}^{\rm c}$ if $\pi<\alpha+\beta+\gamma+\delta\le2\pi$.

References

1. S. Agnihotri, C. Woodward, *Eigenvalues of products of unitary matrices and quantum Schubert calculus*, Math. Research Letters **5** (1998), 817–836.
2. Z. Arad, M. Herzog (eds.), *Products of conjugacy classes in groups*, Lecture Notes in Math. 1112, Springer-Verlag, Berlin, Heidelberg, N.Y., Tokyo, 1985.
3. E. Artal Bartolo, J. Carmona Ruber, J. I. Cogolludo Agustín, *Effective invariants of braid monodromy*, Trans. Am. Math. Soc. **359** (2007), no. 1, 165–183.
4. P. Belkale, *Local systems on $\mathbb{P}^1 - S$ for S a finite set*, Compos. Math. **129** (2001), 67–86.
5. J. I. Brenner, *Covering theorems for nonabelian simple groups, IV*, Jñānābha Ser. A **3** (1973), 77–84.
6. E. Falbel, R. A. Wentworth, *On products of isometries of hyperbolic space*, Topology Appl. **156** (2009), 2257–2263.
7. O. García-Prada, M. Logares, V. Muñoz, *Moduli spaces of parabolic $U(p,q)$-Higgs bundles*, Q. J. Math. **60** (2009), no. 2, 183–233.
8. N.L. Gordeev, *Products of conjugacy classes in perfect linear groups. Extended covering number*, Zapiski Nauchn. Semin. POMI **321** (2005), 67–89; Reprinted in:, J. Math. Sci. **136** (2006), 1867–3879.
9. S. Karni, *Covering number of groups of small order and sporadic groups*, Ch. 3 in [2], pp. 52–196.
10. M.W. Liebeck, A. Shalev, *Diameter of finite simple groups: sharp bounds and applications*, Ann. of Math. **154** (2001), 383–406.
11. S. Yu. Orevkov, *Quasipositivity test via unitary representations of braid groups and its applications to real algebraic curves*, J. Knot Theory Ramifications **10** (2001), no. 7, 1005–1023.
12. S. Yu. Orevkov, *Products of conjugacy classes in finite unitary groups $GU(3,q^2)$ and $SU(3,q^2)$*, Ann. Fac. Sci. de Toulouse. Mathématiques (6) **22** (2013), no. 2, 219–251.
13. J. Paupert, *Elliptic triangle groups in $PU(2,1)$, Lagrangian triples and momentum maps*, Topology **46** (2007), 155–183.
14. J. Paupert, P. Will, *Involution and commutator length for complex hyperbolic isometries*, Michigan Math. J. **66** (2017), 699–744.
15. C. Simpson, *Products of matrices*, Differential geometry, global analysis, and topology (Halifax, NS, 1990), CMS Conf. Proc., 12, Amer. Math. Soc., Providence, RI, 1991., pp. 157–185.