# ASYMPTOTIC GROWTH OF THE NUMBER OF CLASSES OF REAL PLANE ALGEBRAIC CURVES WHEN THE DEGREE INCREASES

S.Yu.Orevkov, V.M.Kharlamov

Что мы знаем о лисе? Ничего! И то — не все.
Б. Заходер. *Мохнатая азбука*

## Introduction

Plane algebraic curves and, more generally, projective algebraic hypersurfaces is the subject of the 16th Hilbert problem. They are naturally organized in families which are the spaces of homogeneous polynomials. These spaces are numbered by the number $n+1 \geq 2$ of homogeneous variables and the degree $d \geq 1$ of polynomials. Each one is a projective space $P^N$ with $N = N(n,d) - 1, N(n,d) = \binom{n+d}{d}$ being the number of coefficients in a generic homogenous polynomial of degree $d$ in $n+1$ variables. One associates with it the universal hypersurface $\Gamma \subset P^N \times P^n = \{(p,x) | p(x) = 0\}$, which is a nonsingular variety, and the projection $\mathrm{pr} : \Gamma \to P^N$. The critical locus $D \subset P^N$ of pr is the so-called *discriminant* hypersurface: $p \in D$ if and only if the hypersurface $p = 0$ in $P^n$ is singular, and the induced over $P^N \setminus D$ family $\Gamma^0 = \mathrm{pr}^{-1}(P^N \setminus D) \to P^N \setminus D$ is a deformation family: $\mathrm{pr}|_{\Gamma^0}$ is a proper submersion.

Over $\mathbf{C}$, the space of complex points of $P^N \setminus D$ is connected, which implies that all the nonsingular hypersurfaces of same degree (and same dimension) are diffeomorphic. Over $\mathbf{R}$, the situation is completely different: as soon as $d \geq 2$, $P^N_{\mathbf{R}} \setminus D_{\mathbf{R}}$ (here and further we denote the real point set of a real variety $X$ by $X_{\mathbf{R}}$) is disconnected. There is a tradition to call *rigid isotopy* or *real deformation* a continuous path in $P^N_{\mathbf{R}} \setminus D_{\mathbf{R}}$; respectively, two real hypersurfaces are *rigid isotopic* or *real deformation equivalent* if they are connected by a rigid isotopy. The simplest invariant of the rigid isotopy is the topology of the real part of the hypersurface. By means of it one proves easily the above disconnectedness in any dimension and any degree $\geq 2$.

It is worth to be noted that over $\mathbf{R}$, the definition of $D$ can be interpreted in two different ways: $p \in D$ if $p = 0$ has a real singular point, or $p \in D$ if $p = 0$ has a complex singular point. We prefer the latter, commonly used, definition, which is equivalent to considering the real discriminant as the real point set $D_{\mathbf{R}} \subset P^N_{\mathbf{R}}$ of the complex one (and which gives a discriminant defined over $\mathbf{Z}$). In fact, for our purposes, see below, there is no difference: the complement of the other one is included into $P^N_{\mathbf{R}} \setminus D_{\mathbf{R}}$, and the inclusion establishes a one to one correspondence between the connected components, since the difference between the two discriminants is of dimension $\leq N-2$.

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In his 16th problem Hilbert asks about "the number, form, and position of the sheets" of a nonsingular real algebraic hypersurface of given dimension and degree (in fact, he is mentioning only the plane curves and the surfaces in 3-space). In its extended form the problem can be understood as the study of the deformation equivalence and the deformation invariants.

Let $I(n,d)$ be the number of isotopy types of pairs $(P^n_{\mathbf{R}}, X_{\mathbf{R}})$ where $X_{\mathbf{R}}$ is a nonsingular hypersurface in $P^n_{\mathbf{R}}$ of degree $d$ and let $D(n,d)$ be the number of deformation classes (rigid isotopy classes). It is clear that $I(n,d) \le D(n,d) < \infty$. In what follows we often abbreviate $I(2,d)$ and $D(2,d)$ to $I_d$ and $D_d$. (If $n=2$ the isotopy equivalence coincides with equivalence by homemorphisms of $P^2_{\mathbf{R}}$ and the isotopy classes have a simple encoding, see below.)

The numbers (and the corresponding classes) $I(n,d)$ and $D(n,d)$ are known for some values of $(n,d)$:

$$I(1,d) = D(1,d) = [\frac{d}{2}] + 1, \quad I(n,1) = D(n,1) = 1, \quad I(n,2) = D(n,2) = n,$$

and[1]

| $d:$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $I(2,d):$ | 1 | 2 | 2 | 6 | 8 | 56 | 121 | $\ge 2500$ |
| $I(3,d):$ | 1 | 3 | 5 | 111 | $\ge 3000$ | | | |
| $D(2,d)-I(2,d):$ | 0 | 0 | 0 | 1 | 1 | 8 | $\ge 570$ | |
| $D(3,d)-I(3,d):$ | 0 | 0 | 0 | 55 | | | | |

For higher degrees and dimensions, very few is known. To our knowledge, even the asymptotic behavior of $I(n,d)$ and $D(n,d)$ were never studied systematically, and in this paper we are making an attempt to formulate the principal questions and to give some answers concerning the asymptotics up to weak exponential equivalence (see the definition below). They are more advanced in the case of plane curves, where, in particular, we give the asymptotics for $I(2,d)$ and for the number of isotopy classes of maximal curves realizable by $T$-curves and show that none of the known restrictions is asymptotically valuable.

**Notation.** Beside the usual notion of the *equivalency* of sequences ($a_m \sim b_m$ if $\lim_{m\to\infty} a_m/b_m = 1$), we shall often use *weak equivalency relation*

$$a_m \asymp b_m \qquad \text{if} \quad a_m = O(b_m) \ \text{ and } \ b_m = O(a_m),$$

*exponential equivalency* ($a_m \underset{e}{\sim} b_m$ if $\log a_m \sim \log b_m$) and *exponential weak equivalency* ($a_m \underset{e}{\asymp} b_m$ if $\log a_m \asymp \log b_m$).

[1]The bound $I(3,5) \ge 3000$ was recently announces by B. Chevallier (private communication).

**Example.** $m! \underset{e}{\asymp} m^m$ (by Stirling formula); $(m^2)^{m^2} \underset{e}{\asymp} m^{m^2}$.

The real point set $A_{\mathbf{R}}$ of a real nonsingular plane curve $A \subset P^2$ is a compact nonsingular 1-dimensional smooth submanifold of $P^2_{\mathbf{R}}$, or empty. Any compact nonsingular 1-dimensional submanifold $C$ of $P^2_{\mathbf{R}}$ consists of a finite number of disjoint embedded circles. We call $\mathbf{Z}/2$-degree of $C$ the element realized by $C$ in $H_1(P^2_{\mathbf{R}}; \mathbf{Z}/2) = \mathbf{Z}/2$. If it is zero, all the circles are two-sided. Otherwise, the curve has one and only one one-sided component. The two-sided circles are called *ovals*. In the algebraic case, $C = A_{\mathbf{R}}$, the curve realizes zero if and only if its degree is even.

Any oval decomposes $P^2_{\mathbf{R}}$ in two parts: the internal one, homeomorphic to a disc, and the external one, homeomorphic to a cross-cup (Moebius band). Thus, there appears a natural partial order (tree structure): $a > b$ if the oval $a$ contains $b$ in its interior. This order is invariant under isotopy (which, in our case, is equivalent to invariance under homeomorphisms $P^2_{\mathbf{R}} \to P^2_{\mathbf{R}}$), and, conversely, this order and $\mathbf{Z}/2$-degree determine the curve up to isotopy.

Following tradition, a set of disjoint embedded circles considered up to an isotopy in $P^2_{\mathbf{R}}$ will be called a *real scheme* (of an oval arrangement). Any real scheme can be realized by an algebraic curve. A real scheme is said of degree $d$ if it can be realized by nonsingular algebraic curves of degree $d$.

We represent a real scheme by the rooted tree with $l+1$ vertices, where $l$ is the number of ovals (the root is an additional, the greatest, element; we introduce it to be in accordance with the terminology established in the combinatorics of trees), and, as well, by nested parentheses. In the latter notation, the now traditional rule is such that $\langle 1\langle A\rangle\rangle$ denotes a scheme $\langle A\rangle$ with one additional oval containing $\langle A\rangle$ inside it and $\langle A \sqcup B\rangle$ the disjoint sum of the schemes $\langle A\rangle$ and $\langle B\rangle$. (We omit $\mathbf{Z}/2$-degree, since in what follows it never leads to a confusion.)

Note that in the algebraic case, according to the Harnack inequality, the number of ovals is bounded by $\frac{1}{2}(d-1)(d-2)+1$ if $d$ is even, and by $\frac{1}{2}(d-1)(d-2)$ if $d$ is odd. The real schemes of degree $d$ with such a maximal number of ovals are called $M$-schemes and the corresponding curves are called $M$-curves. They exist for any degree.

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## 1. Real plane curves

**1.1. Curves up to isotopy.** The following statement describes the rough asymptotics of the number of curves considered up to isotopy.

**Proposition 1.** $I_d \underset{e}{\asymp} \exp(d^2)$.

*Proof. Upper bound.* By the Harnack inequality the number of ovals is bounded by $\frac{1}{2}(d-1)(d-2)+1$. Hence, $I_d$ does not exceed the number of rooted trees with $\frac{1}{2}(d-1)(d-2)+2$ vertices. It is known (see, e.g., [17]), that the number of rooted trees with $n$ vertices is bounded by $C^n$ for some constant $C > 0$. (For the precise value of the constant see [13]; the first digits are pointed in Section 1.4 below).

*Lower bound.* Applying Viro's gluing method, we inductively construct for each $k = 0, 1, 2, \dots$ a finite family $\mathcal{C}_k$ of pairwise non-isotopic nonsingular real plane curves of degree $d_k$, where

$$d_0 = 6, \qquad d_{k+1} = 2d_k + 6 \tag{1}$$

in such a way that the real point set $c_{\mathbf{R}}$ of each curve $c$ from $\mathcal{C}_k$ is contained in the affine plane $\mathbf{R}^2$.

To construct $\mathcal{C}_0$, we pick a representative from each isotopy class of curves of degree $d_0 = 6$ (they all can be chosen in $\mathbf{R}^2$ because, as is known, each one is realizable by a perturbation of three suitable ellipses).

As soon as the family $\mathcal{C}_k$ is constructed, associate with every curve $c \in C_k$ a curve which is the union of a round circle lying in the positive quadrant and the image $f(c)$ of $c$ under an affine linear transformation of the plane such that $f(c_{\mathbf{R}})$ is mapped inside the circle. Denote by $\mathcal{C}'_k$ the set of the obtained curves of degree $d_k + 2$. Then, for any non-ordered 4-tuple $c'_1, \dots, c'_4 \in \mathcal{C}'_k$, construct a curve by Viro's method dividing the triangle $\Delta = [(0,0),(0,2d_k+6),(2d_k+6,0)]$ in four triangles with the side $d_k+2$ separated one from another by two ribbons of width 1 and one ribbon of width 2 (three of these four triangles have a vertex and two sides on $\partial\Delta$; the ribbons are necessary to glue the neighboring charts between them). Let $N_k = \operatorname{Card}\mathcal{C}_k$. It is clear from the above construction that

$$N_{k+1} \geq N_k^4/24 \tag{2}$$

We get by induction from (1) and (2) that

$$d_k = 2^k \cdot (d_0+6) - 6, \qquad \log N_k \geq 4^k \cdot \big(\log N_0 - \frac{\log 24}{3}\big) + \frac{\log 24}{3}. \quad \square$$

**1.2. Curves up to deformation.** The following proposition gives a first information on the growth of the number of curves considered up to rigid isotopy.

**Proposition 2.** *There are constants $c_1, c_2$ such that for any $d$*

$$c_1 d^2 \leq \log D_d \leq c_2 d^2 \log d.$$

*Proof.* The lower bound follows from Proposition 1. By Poincaré-Lefschetz duality, to prove the upper bound it is sufficient to get the same bound for the total Betti number of the real point set $D_{\mathbf{R}}$ of the discriminant hypersurface of singular curves of degree $d$. Such a bound is given, e.g., by the Smith-Thom inequality: indeed, the degree $h$ of the discriminant hypersurface is $3(d-1)^2$ and the leading term of this bound (polynomial in $h$ and exponential in $N$) is $h^N$ where $N = N(2,d) - 1 = \binom{d+2}{d} - 1$ (see [15]). $\square$

By a $T$-curve of degree $d$ we mean a (real nonsingular) plane curve obtained by Viro's gluing method applied to a convex triangulation of the Newton triangle $[(0,0),(0,d),(d,0)]$ in primitive (i.e., area $\frac{1}{2}$) triangles with integral vertices (see [16, 9, 6]). Denote by $D_d^T$ the number of deformation classes realized by $T$-curves of degree $d$.

**Proposition 3.** *There is a constant $c > 0$ such that $\log D_d^T \leq cd^2$ for any $d$.*

*Proof.* As it follows from Gelfand-Kapranov-Zelvinskii description of the secondary polytopes (see [5]), a convex triangulation of a (convex) polygon in primitive integral triangles is determined by the multiplicities of the vertices of the triangulation. The total number of vertices is $\frac{1}{2}(d+1)(d+2)$ and the number of edges is $\frac{3}{2}d(d+1)$. Thus, the number of convex triangulations is bounded from above by the number of decompositions of $3d(d+1)-(d+1)(d+2)+3$ in $\frac{1}{2}(d+1)(d+2)$ integral positive summands, which is the binomial coefficient

$$\binom{2d^2}{\frac{1}{2}(d+1)(d+2)-1} \underset{e}{\sim} c^{\frac{1}{2}d^2} \qquad \text{with} \quad c = \frac{4^4}{3^3}.$$

The choice of signs in the vertices can only change the final constant $c$. $\square$

**1.3.** $M$**-curves.** The aim of this section is to recover the asymptotic growth of the number of $M$-schemes which are realizable by $T$-curves.

It is worth noting that similar to $I_d$ the whole number of pairwise non-isotopic $T$-curves of degree $d$ is exponential weak equivalent to $\exp(d^2)$ (one can slightly modify the construction in the proof of the lower bound of Proposition 1). Combined with the upper bound from Proposition 3, this shows that the number of rigid isotopy types of degree $d$ realizable by $T$-curves is also exponential weak equivalent to $\exp(d^2)$.

**Proposition 4.** (see [9]) *If an $M$-scheme of degree $d$ is realizable by a $T$-curve then it satisfies the following condition:*

(*) *The number of ovals $O$ such that there exist ovals $O'$ and $O''$ such that $O'' < O' < O$, does not exceed $\frac{3}{2}m$.* $\square$

**Proposition 5.** *Let $T_d^*$ be the number of different $M$-schemes of degree $d$ satisfying the condition* (*). *Then $T_d^* \underset{e}{\asymp} \exp(d^{3/2})$.*

*Proof.* Let $P(m)$ denotes the set of all non-ordered partitions of $m$, i.e. $P(m) = \{(\lambda_1,\dots,\lambda_k) \mid \lambda_1 \geq \dots \geq \lambda_k > 0,\ \lambda_1+\dots+\lambda_k = m\}$, and let $p(m) = \operatorname{Card} P(m)$. It is known that $p(m) \underset{e}{\asymp} \exp(\sqrt{m})$ (see [1]). For $\lambda = (\lambda_1,\dots,\lambda_k) \in P(m)$, let $S(\lambda)$ denotes the real scheme of oval arrangement $1\langle\lambda_1-1\rangle\rangle \sqcup \dots \sqcup \langle 1\langle\lambda_k-1\rangle$.

*Lower bound.* For any ordered collection $(\lambda^{(1)},\dots,\lambda^{(k)}) \in P(d)^k$ where $k = [3d/2]$, the scheme $S_1 \sqcup 1\langle S_2 \sqcup 1\langle \dots \sqcup 1\langle S_k\rangle \dots\rangle$ where $S_j = S(\lambda^{(j)})$, satisfies the condition (*). Hence, $T_d^* \geq p(d)^k \underset{e}{\asymp} \exp(d\sqrt{d})$.

*Upper bound.* Let $S$ be a real $M$-scheme of degree $d$ satisfying the condition (*). Denote by $S^*$ the subscheme of $S$ consisting of the ovals which are the outermost ovals of nests of the depth $\geq 3$ (by the condition (*), we have $\operatorname{Card} S^* \leq \frac{3d}{2}$). Let $T(S^*)$ be the rooted tree of $S^*$.

Let us choose a representative in each class of isomorphic trees and let us fix (arbitrarily) a numbering of its vertices. Let $v_1,\dots,v_k$ ($k \leq \frac{3d}{2}$) be the ovals of $S^*$ numbered in accordance with the chosen order of the vertices of $T(S^*)$. For any $j$ with $1 \leq j \leq k$, let us consider all the ovals of $S \setminus S^*$ lying inside $v_j$ which are not separated from $v_j$ by other ovals of $S^*$. These ovals constitute a real subscheme of the form $S(\lambda^{(j)})$ for some partition $\lambda^{(j)} \in P(m_j)$, and $m_1+\dots+m_k = l(d)-k$ where $l(d) \sim d^2/2$ is the number of ovals in an $M$-scheme. Thus,

$$T_d^* \leq \sum_{k=1}^{[3d/2]} T_k \sum_{m_1+\dots+m_k=l(d)-k} p(m_1)\dots p(m_k)$$

where $T_k$ is the number of all rooted trees with $k$ vertices. For some constant $a$ we have $\log p(m_j) < a\sqrt{m_j}$. Hence, by the Cauchy-Buniakowski inequality, one can uniformly estimate each summand in the interior sum:

$$\log\big(p(m_1)\dots p(m_k)\big) < a\sum\sqrt{m_j} \le a\sqrt{m_1+\dots+m_k}\sqrt{k} < a\sqrt{l(d)\cdot 3d/2},$$

the number of the summands being $\binom{l(d)}{k} < \binom{d^2}{3d/2} \underset{e}{\asymp} d^d$ which implies

$$T_d^* < (3d/2)\,T_d\binom{d^2}{3d/2}\exp(bd^{3/2}) \underset{e}{\asymp} \exp(d^{3/2}). \quad \square$$

Let $I_d^{TM}$ denotes the number of $M$-schemes of degree $d$ which are realizable by $T$-curves.

**Proposition 6.** $I_d^{TM} \underset{e}{\asymp} \exp(d^{3/2})$.

*Proof. Upper bound.* Follows from Propositions 4 and 5.

*Lower bound.* Let $d$ be a positive integer. Set $k=[\frac{1}{6}d]-1$ and denote the triangle $[(0,0),(d,0),(0,d)]$ by $\Delta_d\subset\mathbf{R}^2$. For any collection of partitions $(\lambda_0,\dots,\lambda_{k-1})$, $\lambda_j\in P(2(k-j))$ we shall construct an $M$-curve by Haas' method [6].

Let us consider the following points in $\Delta_d$:

$$A_j=(5j,0),\ \ j=0,\dots,k+1;\quad B_0=(0,d),\ \ B_j=(5j+1,6(k-j)-2),\ \ j\ge 1;$$
$$C_j=(6k-j,0),\ \ D_j=(5j+1,0),\ \ E_j=(5j+2,6(k-j)-4),\ \ j=0,\dots,k.$$

Let us cut $\Delta_d$ by distinct two-segment broken lines

$$A_jB_jC_j\ \ (j=1,\dots,k)\qquad\text{and}\qquad D_jE_jA_{j+1}\ \ (j=0,\dots,k).$$

Now let us cut each triangle $D_jE_jA_{j+1}$ $(j=0,\dots,k-1)$ as follows. Let $\lambda_j=(\lambda_j^{(1)},\lambda_j^{(2)},\dots)$. Set $s_j^{(\nu)}=\lambda_j^{(1)}+\dots+\lambda_j^{(\nu)}$, $y_j^{(\nu)}=2+2\cdot[3s_j^{(\nu)}/2]$, and $E_j^{(\nu)}=(5j+2,y_j^{(\nu)})$. Let us cut each triangle $D_jE_jA_{j+1}$ into domains $Z_j^{(1)},Z_j^{(2)},\dots$ by the broken lines $D_jE_j^{(1)}A_{j+1}$, $D_jE_j^{(2)}A_{j+1},\dots$. Then the interior of the segment which is cut by the domain $Z_j^{(\nu)}$ from the line $x=5j+3$, contains $\lambda_j^{(\nu)}$ integral points with an even $y$-coordinate (totally, the segment which is cut by the triangle $D_jE_jA_{j+1}$ from the line $x=5j+3$, contains exactly $2(k-j)$ points with an even $y$-coordinate).

Thus, we have cut $\Delta_d$ into $k+s+1$ polygons where $s$ is the total number of elements of the partitions $\lambda_1,\lambda_2,\dots$. It is not difficult to check that any side of any of the polygons contains no other integral points except the ends, and one of the ends is on the ends lies on a side of $\Delta_d$. Following [6], we shall call these polygons *zones*. Let us attach a sign to each zone so that the signs of adjacent zones are opposite. Then, extend the zone decomposition to a convex triangulation and define a sign distribution $\delta:\Delta_d\cap\mathbf{Z}^2\to\{\pm1\}$ setting $\delta(x,y)=(-1)^{xy+1}\zeta^y$ where $\zeta$ is the sign of the zone which contains the point $(x,y)$ (since all integral points on the zone boundaries have even $y$-coordinates, the definition is coherent on the intersections of the zones).

By Haas' theorem [6], the corresponding $T$-curve is an $M$-curve and it is not difficult to check that its real scheme has form

$$a\sqcup 1\langle a_0\sqcup S_0\sqcup 1\langle a_1\sqcup S_1\sqcup 1\langle\quad\dots\quad\sqcup 1\langle a_{k-1}\sqcup S_{k-1}\sqcup 1\langle 1\rangle\rangle\dots\rangle\rangle\rangle$$

for some integers $a, a_0, a_1, \dots$, where $S_j = 1\langle\lambda_j^{(1)}\rangle \sqcup 1\langle\lambda_j^{(2)}\rangle \sqcup \dots$ for $\lambda_j = (\lambda_j^{(1)}, \lambda_j^{(2)}, \dots)$. The nest of the depth $k+1$ corresponds to the broken lines $A_j B_j C_j$, and the subschemes $S_j$ correspond to cuttings of the triangles $D_j E_j A_{j+1}$. The nests $1\langle\lambda_j^{(1)}\rangle$, $1\langle\lambda_j^{(3)}\rangle, \dots$ are above the axis $y = 0$, and the nests $1\langle\lambda_j^{(2)}\rangle$, $1\langle\lambda_j^{(4)}\rangle, \dots$ are beneath the axis $y = 0$.

Thus, for any $d$, we constructed

$$\prod_{j=1}^{k} p(2j) \underset{e}{\asymp} \exp\Big(\sum_{j=1}^{k} \sqrt{j}\Big) \underset{e}{\asymp} \exp(d^{\frac{3}{2}})$$

different $M$-schemes. $\square$

**1.4. A coefficient of $d^2$ in the asymptotic of the number of real schemes.** In Proposition 1, we used the number of rooted trees as an upper bound for $I_d$. Due to Otter [13] (see also [8; Section 9.5]), the following exponential equivalence for the number $T_n$ of rooted trees holds

$$T_n \underset{e}{\sim} C^n, \qquad C = 2.95576 \tag{3}$$

Hence,

$$\log I_d \le Ad^2 + o(d^2), \qquad A = (\log C)/2. \tag{4}$$

It happens, that none of the known restrictions for the arrangement of ovals allows us to reduce the coefficient $A$ in the estimate (4). The results of this sections do not serve to improve the upper bound for $I_d$ (which is one of the goals of the paper) but in contrary, they show that something is useless for this purpose. This is why we do not give proofs in all detail.

The only known restrictions which could be asymptotically valuable, are the restrictions coming from Bézout's theorem (for instance, the absence of nests deeper than $d/2$). Other restrictions (Petrovsky inequality, Arnold inequality etc.) provide corrections in (4) of order $O(1)$ or $O(\log d)$).

To give a strict sense to the statement of non-improvability of the estimate (4), let us formalize the notion of a restriction coming from Bézout's theorem. Let us say that *a real scheme $S$ satisfies Bézout's theorem for the degree $d$* if for any triple of positive integers $(k, d', c)$, $d' < d$ satisfying the condition

(**) *Through any $k$ points in $\mathbf{RP}^2$ in general position, there exists a real curve of the degree $d'$ which has at most $c$ connected components*

and for any choice of $k$ points there exist $c$ smoothly embedded circles passing through the chosen points which intersect $S$ at most in $dd'$ points.

*Remark.* As it follows from Harnack's proof of Harnack's inequality [7], any real scheme satisfying Bézout's theorem for a degree $d$ has at most $\frac{1}{2}(d-1)(d-2)+1$ connected components.

Let $B_d$ denotes the number of different schemes satisfying Bézout's theorem for the degree $d$.

**Proposition 7.** $\log B_d \sim Ad^2$ *where $A$ is the constant in (3) and (4).*

A proof follows immediately from Lemmas 8 and 9 below.

**Definition.** A *nest* is a sequence of disjoint ovals $v_1, \dots, v_p$ in $P^2_{\mathbf{R}}$ such that $v_1 > \dots > v_p$. The number of ovals $p$ is called the *depth* of the nest. The maximal depth of a nest of a real scheme (of a rooted tree) $S$ is called the *depth* of $S$.

A triple $(k, d', c)$ with $d' < d$ satisfies the condition (**) only if (conjecturally, if and only if) $k \le \frac{1}{2}d'(d'+3)$ and $k \le 3d' + c - 2$, cf. [3]. Using this fact, one can prove the following statement.

**Lemma 8.** *Let $\alpha$ be a fixed number such that $0 < \alpha < \frac{1}{6}$. Let $d$ be a sufficiently large integer. Suppose that a real scheme $S$ has at most $l(d) = \frac{1}{2}\big((d-1)(d-2) + (1+(-1)^d)\big)$ ovals and $S$ is of depth $< [\alpha d]$. Then $S$ satisfies Bézout's theorem for degree $d$.* $\square$

Let us denote by $T_n^{[h]}$ the number of rooted trees of depth $\le h$ with $n$ vertices.

**Lemma 9.** *Let $h_n$ be a sequence such that*

$$\lim_{n\to\infty} \frac{\log n}{h_n} = 0. \qquad \textit{Then } \; T_n^{[h_n]} \underset{e}{\sim} T_n \underset{e}{\sim} C^n.$$

*Proof.* Let us number arbitrarily the vertices of each tree. Pick a rooted tree $T$ of depth $> h$ with $n$ vertices. Choose the highest vertex $v$. If it is not unique, choose the one whose number is minimal. Consider the branch of the depth $h-1$ which contains $v$ (by definition, a *branch* of a tree is one of two components obtained by deleting an edge). Cutting this branch, we obtain another rooted tree. If its depth is still $> h$, we iterate this process. Since each time we cut out $\ge h$ vertices, we perform not more than $n/h$ iteration. As the result, we obtain $\le n/h$ rooted trees of depth $\le h-1$. Connecting all of them to a new common root, we obtain a rooted tree with $n+1$ verices of depth $\le h$. To reconstruct $T$, it suffices to indicate the ends of edges which were deleted at each step. Thus, $T_n \le T_{n+1}^{[h]} \cdot n^{\frac{n}{h}}$. $\square$

*Remarks.* **1.** Our proof of Proposition 1 provides the lower bounds

$$I_d \ge C_6^{d^2}, \quad C_6 = 1.02081 \qquad (d_0 = 6,\ N_0 = 56)$$
$$I_d \ge C_8^{d^2}, \quad C_8 = 1.03511 \qquad (d_0 = 8,\ N_0 \ge 2500).$$

We see that both constants $C_6$ and $C_8$ are very far from $C$.

**2.** The numbers $T_n^{[h]}$ can be computed by recurrent formula for the generating functions [14] (see also [8; Section 3.1]):

$$T^{[h+1]}(x) = x \exp\Big(\sum_{k=1}^{\infty} \frac{T^{[h]}(x^k)}{k}\Big) \qquad \text{where} \quad T^{[h]}(x) = \sum_{n=1}^{\infty} T_n^{[h]} x^n$$

**3.** To prove Proposition 7 we used estimates $\log B_d < \log T_n^{[a\sqrt{n}]} = \log T_n - O(\sqrt{n}\log n)$ where $n = l(d) \sim \frac{1}{2}d^2$. A numerical experiment (using the above recurrent formula) allows us to surmise a better estimate $\log T_n^{[a\sqrt{n}]} = \log T_n - O(\log n)$.

## 2. Generalizations

Some of the preceding results can be easily extended to hypersurfaces in nonsingular varieties of any dimension. To state these generalizations let us introduce an additional notation. For a very ample divisor $L$ on a nonsingular variety $X$ of dimension $n$ denote by $I(n, dL)$, respectively by $D(n, dL)$, the number of real nonsingular hypersurfaces in the linear system $|dL|$ considered up to isotopy, respectively up to deformation equivalence. As in the case of projective hypersurfaces, the number $D(n, dL)$ is the number of connected components of the complement of the corresponding discriminant $\Delta(dL) \subset |dL|$ of the linear system $|dL|$. Let us call the system *generic* if generic points of the discriminant are the hypersurfaces $H \in |dL|$ with a nondegenerate quadratic double point which is the only its singular point. Since for sufficiently large $d$ the projective image of $X$ by $dL$ contains no linear subspaces, the system $|dL|$ is generic for large $d$ (see [4]).

**2.1. Hypersurfaces up to isotopy.** The lower bound in Proposition 1 extends to hypersurfaces in any toric variety of any dimension.

**Proposition 10.** *For any ample divisor $L$ on a toric variety $X$ of dimension $n$, there is a constant $c = c(L, X)$ such that $I(n, dL) \geq c \exp(d^n)$ for any $d$.*

The proof is essentially the same as that of the lower bound in Proposition 1.

**2.2. Hypersurfaces up to deformation.** The upper bound from Proposition 2 is easily generalized to ample divisors on nonsingular varieties of any dimension. Via evident $I(n, dL) \leq D(n, dL)$ it gives the best upper bound for $I(n, dL)$ we know.

**Proposition 11.** *If $L$ is an ample divisor on a nonsingular variety of dimension $n$, then, there is a constant $c'_n$ such that*

$$\log I(n, dL) \leq \log D(n, dL) \leq c'_n d^n \log d.$$

*Proof.* As in the proof of Proposition 2, it is sufficient to count the leading terms in the polynomial expansion of the degree of the discriminant and of the dimension of the linear system. The first is counted through integrating the Euler characteristic over a Lefschetz pencil: it gives $a = (n+1)d^n L^n$. The second is counted by means of the asymptotic Riemann-Roch formula: it gives $b = \frac{1}{n!} d^n L^n$. Thus, it remains to apply the Smith-Thom inequality, which gives $a^b$ as the leading term of the bound. □

**Remark.** There are two other approaches to bounding from above $I$ and $D$. The first one consists in constructing of an algorithm recognizing the topological or $C^\infty$ type of a hypersurface and bounding the number of classes by the complexity of the algorithm. Unfortunately, we do not know algorithms which provide a better bound than the one from Proposition 10. The other one would work for hypersurfaces on toric varieties and consists in studying of the Gelfand-Kapranov-Zelvinsky secondary polytopes, which are the Newton polytopes of the discriminants. We collected some information concerning this approach in the next subsection.

**2.3. Number of $T$-hypersurfaces.** Denote the number of deformation classes of $T$-hypersurfaces in $|dL|$ on a given toric variety of dimension $n$ by $D'(n, dL)$.

**Proposition 12.** *There is a constant* $c = c(n, L)$ *such that* $\log D'(n, dL) \le cd^n$.

The proof is similar to that of Proposition 3.

**2.4. Newton polytopes of the discriminants.** Let consider for simplicity the case of hypersurfaces of projective space of dimension $n$. In several proofs above there appear the discriminant hypersurface $\Delta(d) \subset P^N$ and its Newton polyhedron $Q(d) \subset \mathbf{R}^n$. The Betti numbers of $\Delta(d)$ bound the Betti numbers of $P^N_{\mathbf{R}} \setminus \Delta(d)_{\mathbf{R}}$. The vertices of $Q(d)$ are in one-to-one correspondence with convex integral triangulations of the simplex $\{(i_1, \dots, i_n) | i_k \ge 0, \sum i_k \le d\}$ (see [5]). In the proofs above, to bound the Betti numbers we used only the degree of $\Delta(d)$ and to bound the number of triangulations (and thus the number $D^T(n, d)$ of deformation classes of $T$-hypersurfaces) the description, due to Gelfand-Kapranov-Zelvinsky, of the coordinates of the vertices of $Q(d)$.

Certainly, these two objects, $\Delta(d)$ and $Q(d)$, contain more information which is still to be recovered. For example, $\Delta(d)$ is a very singular variety, which is not taken into account in our proofs. One could expect to get a better bound on its total Betti number through the number of critical values of the discriminant polynomials, which is equivalent to bounding of the volume of $Q(d)$. Unfortunately, as it is shown in [12], it is not the case. So, some finer analysis of the topology of $\Delta(d)$ is needed.

It is worth noting that there is a related problem of bounding the number of integral triangulations. Above we used already the observation that the number of convex integral triangulations grows as $\exp(cd^n)$. The nonconvex triangulations are related somehow with the internal integral points of $Q(d)$ and one may expect that the number of nonconvex triangulations has also the growth $\exp(cd^n)$ with, probably, another constant $c$. For the moment such a result is proven only for $n = 2$, and without appealing to $Q(d)$, see [11].

A natural question is *how many (asymptotically) convex triangulations are among all integral triangulations?* This is not clear even for $n = 2$. As we already mentioned, the both quantities are $\underset{e}{\asymp} \exp(d^2)$ but do the coefficients of $d^2$ coincide? As a lower bound for the number $N_d$ of all integral triangulations of a square $d \times d$, one can use $(N_{k,m}^{1/km})^{d^2}$ where $N_{k,m}$ is the number of integral triangulations of a rectangle $k \times m$ for some fixed $k$, $m$ (if $k$ and $m$ are fixed then $N_{k,m}$ can be computed explicitely). In all the cases where we computed $N_{k,m}$, this bound was much less than the upper bound $(4^4/3^3)^{d^2}$ for the number convex triangulations provided by the proof of Proposition 3.

Probably, the constant $4^4/3^3$ can be reduced as follows. To get this constant, we estimated the number of vertices of the secondary polytope via the number of all integral points in the simplex containing it. One can expect to obtain better estimates for the number of vertices of a polytope if all the vertices are integral (compare with [2], [10]).

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Steklov Math.Inst. (Moscow, Russia)

Université Paul Sabatier (Toulouse, France)

IRMA (CNRS) et Université Louis Pasteur, Strasbourg, France