ACYCLIC ALGEBRAIC SURFACES BOUNDED BY SEIFERT SPHERES

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Abstract. Main result: let $Y = X - D$ where X is a smooth projective variety and D a curve. Suppose that $H_q(Y;\mathbf{Q}) = 0$ for $q > 0$ and $\bar\kappa(Y) = 2$. If the boundary of a "tubular neighbourhood" of D is a Seifert $\mathbf{Q}$-homology sphere with r multiple fibers then $r \le 16$.

Let Y be a complex algebraic surface. We say that it is $\mathbf{Z}$-*acyclic* (respectively $\mathbf{Q}$-*acyclic*) if its reduced homology with coefficients in $\mathbf{Z}$ (resp. in $\mathbf{Q}$) vanishes. Topologically one can represent Y as a compact 4-manifold with boundary (denote the boundary by S), attached by a collar $S \times [0,1)$. Call S *the boundary of* Y. If Y is an affine surface in $\mathbf{C}^m$ then S is the intersection of Y with a sufficiently large sphere. We say that Y is A-*acyclic at infinity* if S is an A-homology 3-sphere. ($A = \mathbf{Z}, \mathbf{Q}$). If Y is A-acyclic then it is A-acyclic at infinity. If Y is $\mathbf{Q}$-acyclic and $\mathbf{Z}$-acyclic at infinity, then it is $\mathbf{Z}$-acyclic.

In the paper [**R**] Ramanujam proved that the only $\mathbf{Z}$-acyclic surface bounded by a homotopy 3-sphere is $\mathbf{C}^2$, and he also constructed there the first example of a non-trivial $\mathbf{Z}$-acyclic (and even contractible) surface. Later on Gurjar and Shastri [**GS**] proved that all $\mathbf{Z}$-acyclic surfaces are rational. Tom Dieck and Petri [**DP**] classified all acyclic surfaces which rise out of line configurations on $\mathbf{P}^2$. Fujita [**F**] (resp. Miyanishi, Tsunoda [**MT**] and Gurjar, Miyanishi [**GM**]) classified acyclic surfaces with $\bar\kappa = 0$ (resp. $-\infty$ and 1), where $\bar\kappa$ denotes the log-Kodaira dimension. Zaidenberg [**Z1**] pointed out the connection of $\mathbf{Z}$-acyclic surfaces with exotic algebraic and analytic structures on $\mathbf{C}^n$, $n \ge 3$. Flenner and Zaidenberg [**FZ**] studied deformations of acyclic surfaces.

A *Seifert fibration* (see [**S**], [**O**]) on a smooth compact 3-manifold M is a mapping onto a 2-manifold $\pi : M \to B$, which is a locally trivial fibration with fiber S^1 over $B - \{p_1, ..., p_r\}$ and which looks near p_j like $D^2 \times S^1 \to D^2$, $(z_1, z_2) \mapsto z_1^{\mu_j}/z_2^{\nu_j}$, where $D^2 = \{|z|^2 < 1\} \subset \mathbf{C}$, $S^1 = \partial D^2$ and μ_j, ν_j are coprime integers, $\mu_j \ge 2$. The $\pi^{-1}(p_j)$ are called *multiple fibers*; M is called *Seifert manifold* if it admits a Seifert fibration. *Seifert A-homology sphere* (A stands for $\mathbf{Z}$ or $\mathbf{Q}$) is a Seifert manifold M with $H_*(M;A) = H_*(S^3;A)$. In this case the base B is a 2-sphere. The question, when a Seifert homology sphere bounds an acyclic 4-manifold, was studied, for instance, in [**FS**], [**NZ**].

Our main result is:

Theorem 1. *Let Y be a smooth algebraic $\mathbf{Q}$-acyclic surface of logarithmic Kodaira dimension 2, bounded by a Seifert $\mathbf{Q}$-homology sphere with r multiple fibers. Then:*

(a). Y can not be $\mathbf{Z}$-acyclic.

(*b*). $r \le 16$.

Let Y be a $\mathbf{Q}$-acyclic surface. Consider an algebraic compactification X of Y such that $Y = X - D$, where D is a reduced curve with simple normal crossings (an SNC-curve). Then all irreducible components of D are rational, and the dual weighted graph of D (denote it by Γ_D) is a tree (see [**Mu**]). (The *dual graph* of a curve is the weighted graph, whose vertices correspond to the irreducible components, edges correspond to their intersection points and the weight of a vertex is the self-intersection number.) A tree is called *r-fork* if it has one vertex of valence r and other valences are ≤ 2. Suppose that D is *minimal*, i.e. it contains no (-1)-curve intersecting one or two others. A $\mathbf{Q}$-acyclic surface Y with $\bar\kappa(Y) = 2$ is bounded by a Seifert sphere if and only if Γ_D (with minimal D) is a fork.[1] Thus, we can reduce Theorems 1 to:

Theorem 1′. *Let D be a minimal SNC-curve on a smooth projective surface X. Suppose that $Y = X - D$ is $\mathbf{Q}$-acyclic, $\bar\kappa(Y) = 2$ and the dual graph Γ_D is an r-fork. Then:*

(a). Y can not be $\mathbf{Z}$-acyclic.
(b). $r \le 16$.

Remark 1. As we mentioned above, acyclic surfaces with $\bar\kappa < 2$ are classified [**F**], [**MT**], [**GM**]. Using this classification and the classification of Seifert homology spheres [**O**], one can see that if Y is a $\mathbf{Z}$-acyclic surface which is bounded either by a Seifert sphere or by a fork, then $Y = \mathbf{C}^2$. If Y is $\mathbf{Q}$-acyclic and $\bar\kappa(Y) < 2$ then all the possible values for r are shown in the following table:

$\bar\kappa(Y)$	$-\infty$	0	1
∂Y is a Seifert sphere with r mult. fibers	$\{0,1,2,3\}$	$\{3,4,5\}$	$\{4,5,\dots\}$
Γ_D is an r-fork	$\{0,1,\dots\}$	$\{3\}$	$\varnothing$

This fact can be easily deduced from the results in [**F**], [**GM**] and [**MS**]. Note only that the cases with $\bar\kappa = 0, 1$ and $r \ge 4$ correspond to the surfaces $X - D$ with Γ_D of the form $\vdots\ {>}\!\circ\!\!-\!\!-\!\!\overset{0}{\circ}\!\!-\!\!-\!\!\circ\!{<}\ \vdots$. Such a surface is bounded by a Seifert sphere because Γ_D becomes a fork after a 0-absorption (see [**EN**], [**N**]).

Remark 2. Zaidenberg asked [**Z2**; Question 1.6] if there is only a finite list of possibilities for the topological type of the dual graph at infinity of an acyclic (resp. contractible) surface with $\bar\kappa = 2$. Theorem 1′ can be considered as a very first step toward the positive answer to this question.

Remark 3. The proof of the part (b) of Theorem 1′ is based on the logarithmic Bogomolov–Miyaoka–Yau (log-BMY) inequality [**Mi**], strengthened by Kobayashi–Nakamura–Sakai [**KNS**], and Fujita's computation [**F**] of the Zariski decomposition of $K + D$. The part (a) also can be obtained as a direct consequence of the elementary formulas from §§1–3 (most of them needed for the part (b)) using the rationality of $\mathbf{Z}$-acyclic surfaces [**GS**] and

[1] It is so, because when $\bar\kappa = 2$, the tree Γ_D satisfies so called Negative Chains Condition: If the valence of a vertex is ≤ 2 then its weight is ≤ -2. When $\bar\kappa < 2$, the both assertions "if" and "only if" are wrong.

the log-BMY inequality[2]. However, these two results are quite non-trivial, while, as the referee of the first version of the paper has pointed out,

"... a very elementary proof is possible. Using Lemma 4.1 in part I of [**GS**], we can show:
Write $K_X \sim a_0D_0 + \sum_{i\geq 1} a_iD_i$ where D_0 is the central curve. Then $a_0 \geq 0 \implies$ all $a_i \geq 0$ and $a_0 < 0 \implies$ all $a_i < 0$. But if all $a_i \geq 0$, then $p_g(X) > 0$. This is not possible. Hence all $a_i < 0$. But then $K + D$ is either trivial or a strictly negative divisor. In the latter case, $\bar{\kappa}(Y) = -\infty$. If $K + D \sim 0$, then $(K + D) \cdot D_0 = -2 + r = 0 \implies r = 2$. Hence Γ_D is linear. This completes the proof."

In fact, only the implication "$a_0 \leq 0 \implies$ all $a_i \leq 0$" is proven in [**GS**, Lemma 4.1]. However, the proof can be easily completed to derive the implication "$a_0 < 0 \implies$ all $a_i < 0$" as well. Indeed, if $a_0 < 0$ then by [**GS**, (4.1)] all $a_i \leq 0$. If some of them were $= 0$ then (due to connectedness of D) would exist two components D_i and D_j such that $a_i = 0$, $a_j \neq 0$ and $D_i \cdot D_j = 1$. Then, since $D_i^2 + 2 \leq 0$, one would have $0 = g(D_i) = KD_i + D_i^2 + 2 \leq KD_i = a_j + \sum_{k\neq i,j} a_kD_kD_i \leq a_j < 0$.

Remark 4. After the old proof of Theorem $1'$(a) was omitted, the propositions 1.4 – 1.6 remained without applications. However, we decided to leave them because they are simple but maybe they are of some independent interest.

Remark 5. The estimate $r \leq 16$ in Theorem $1'$, requires messy calculations (see §8). However, the fact that r is bounded from above, can be obtained without them. Therefore, we presented in §7 a shorter proof of Theorem $1'$ with a weaker estimate for r.

Remark 6. The estimate $r \leq 16$ still does not seem to be the best possible. However, a stronger estimate needs other techniques, because an attempt to prove it by the methods of this paper leads to so huge volume of calculations that the result does not worth them.

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§1. Weighted trees and their discriminants

We list in this section some well-known elementary facts about discriminants of weighted trees. A *weighted tree* is a finite tree (finite graph without cycles) with an integer weight $w(v)$ assigned to each vertex v. Let Γ be a weighted tree and $v_1, ..., v_n$ be its vertices. The *incidence matrix* of Γ is $A_\Gamma = (a_{ij})$, where

$$a_{ij} = \begin{cases} w(v_i) & \text{if } i = j, \\ 1 & \text{if } v_i \text{ is connected to } v_j \text{ by an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

[2] see the preliminary version of this paper in *"Mathematica Gottingensis"*, **38**(1995).

The *discriminant* of Γ is defined as $d(\Gamma) = \det(-A_\Gamma)$. By convention, $d(\varnothing) = 1$. Clearly, this definition is independent of the order of the vertices and that the discriminant of a disjoint union is the product of the discriminants of the connected components.

The following lemma can be easily obtained, using the Cramer rule (see, e.g. [**EN**] for details).

Lemma 1.1. *Let Γ be a weighted tree with $d(\Gamma) \neq 0$. Let $B_\Gamma = (b_{ij}) = A_\Gamma^{-1}$ be the inverse matrix. Then*

$$b_{ij} = -d(\Gamma - [v_i, v_j])/d(\Gamma),$$

where $[v_i, v_j]$ is the minimal connected subgraph of Γ, which contains v_i and v_j. $\square$

Lemma 1.2. *Let Γ be a weighted tree, v a vertex of Γ and $w(v)$ the weight of v. Denote by $\Gamma_1, ..., \Gamma_r$ the connected components of $\Gamma - v$, and let $\Gamma'_j = \Gamma_j - v_j$, $j = 1, ..., r$, where v_j is the vertex of Γ_j, connected by an edge to v. Then (remind that $d(\varnothing) = 1$)*

$$d(\Gamma) = -w(v) \prod_{j=1}^{r} d(\Gamma_j) - \sum_{j=1}^{r} \Big(d(\Gamma'_j) \prod_{k \neq j} d(\Gamma_k) \Big).$$

Proof. Expand the determinant of A_Γ according to the row, corresponding to v. $\square$

The *valence* of a vertex v of a graph is the number of edges, incident to v. A graph is called a *linear chain* if its vertices $v_1, ..., v_n$ can be orders so, that v_i is connected to v_j iff $|i - j| = 1$.

Corollary 1.3. *Let T be a linear chain with all weights ≤ -2.*
a). If v is one of the ends of T then $d(T) > d(T - v) > 0$.
b). Let u be any vertex of T. Denote by T_1 and T_2 the connected components of $T - u$, and let $a = d(T)$, $b = d(T_1)$, $c = d(T_2)$. Then $a \geq b + c$.

Proof. a). Induction by the number of vertices, using Lemma 1.2.

b). For $i = 1, 2$ let u_i be the vertex of T_i, nearest to u, and $T'_i = T_i - u_i$. Put $b' = d(T'_1)$, $c' = d(T'_2)$ (if $T'_1 = \varnothing$, put $b' = 0$). Let w be the weight of u. Then by Lemma 1.2 we have $a = -wbc - bc' - b'c = (-w - 2)bc + b(c - c') + c(b - b') \geq b + c$, because $-w - 2 \geq 0$, and by (a), $c - c' \geq 1$, $b - b' \geq 1$. $\square$

The following three propositions will not be used in the rest of the paper.

Proposition 1.4. *Let Γ be a weighted tree; u and v two its vertices. Let $A_0, ..., A_k$ be the connected components of $\Gamma - u$, and $B_0, ..., B_m$ be those of $\Gamma - v$, indexed in such a way that $v \in A_0$, $u \in B_0$. Denote: $a_i = d(A_i)$, $b_i = d(B_i)$, $a = a_1 ... a_k$, $b = b_1 ... b_m$, $\Delta = d(\Gamma)$, $\delta = d(A_0 \cap B_0)$, $c = d((A_0 \cap B_0) - [u, v])$. Suppose that $a \neq 0$, $b \neq 0$, $\Delta \neq 0$. Then*

$$\delta \Delta = a_0 b_0 - abc^2. \tag{1}$$

Proof. Let M be the minor of A_Γ obtained by deleting the two rows and the two columns, corresponding to u and v. Clearly, $M = \delta ab$. On the other hand, by Jacobi formula for the minor of the inverse matrix,

$$\frac{M}{\Delta} = \begin{vmatrix} b_{uu} & b_{uv} \\ b_{vu} & b_{vv} \end{vmatrix},$$

where, by Lemma 1.1, $b_{uu} = aa_0/\Delta$, $b_{uv} = b_{vu} = abc/\Delta$, $b_{vv} = bb_0/\Delta$. $\square$

Remarks. **1.** If γ is a linear chain and $d(\Gamma) = \pm 1$ then (1) is the formula for the "edge determinant" due Eisenbud–Neumann.

2. In fact, (1) is still true even if any of its ingredients are zeros.

A tree Γ is called *r-fork*, if it contains a vertex v_0 of valence r and the valences of other vertices are ≤ 2.

Proposition 1.5. *Let Γ be an r-fork and v_0 the vertex of valence r. Suppose that the weights of the other vertices are ≤ -2. Let Q_Γ be the quadratic form, defined by A_Γ. Then:*
(i) if $d(\Gamma) > 0$ then Q_Γ is negatively definite;
(ii) if $d(\Gamma) < 0$ then Q_Γ has the signature $(+, -, ..., -)$.

Proof. Apply the Sylvester criterium, choosing an increasing sequence $1 = M_0$, $M_1, ...$, $M_n = d(\Gamma)$ of principal minors of the matrix $-A$, where M_{n-1} is obtained from M_n by deleting the row and the column, which correspond to v_0. It follows from Corollary 1.3, that $M_i > 0$ for $i < n$. $\square$

Proposition 1.6. *In the hypothesis of Proposition 1.5 if $d(\Gamma) = -1$ then all the entries b_{ij} of $B_\Gamma = A_\Gamma^{-1}$ are non-negative.*

Proof. Denote by $T_1, ..., T_r$ the connected components of $\Gamma - v_0$, and by v_j the end vertex of T_j (the vertex of T_j, whose valence in Γ is 1). Denote also: $\Gamma'_j = \Gamma - v_j$, $T'_j = T - v_j$, $\Delta'_j = d(\Gamma'_j)$, $a_j = d(T_j)$, $a'_j = d(T'_j)$, $e_j = a'_j/a_j$ $(j = 1, ..., r)$, and $p = a_1 ... a_r$.

By Lemma 1.1 it is enough to show that the discriminant of any connected proper (i.e. $\neq \Gamma$) subgraph of Γ is non-negative. First, we prove this for the subgraphs Γ'_j. Indeed, applying 1.4 with $u = v_0$ and $v = v_j$, we obtain $a'_j \cdot (-1) = \Delta'_j a_j - p/a_j$, or, dividing by a_j, $\Delta'_j = p/a_j^2 - e_j$. But $p/a_j^2 > 0$ and $e_j < 1$. Hence, $\Delta'_j > -1$, but $\Delta'_j \in \mathbf{Z}$, so, $\Delta'_j \geq 0$.

Let Γ'' be any proper connected subgraph of Γ. It is contained in some Γ'_j. Chose an increasing sequence of principal minors which involves $d(\Gamma'')$ as well as $d(\Gamma'_j)$, and estimate the signature of Q_Γ, by Sylvester criterium. Clearly, the inequality $d(\Gamma'') < 0$ contradicts Proposition 1.5. $\square$

§2. Some elementary linear algebra on dual graphs

Let X be a smooth projective algebraic surface and D a reduced SNC-curve on X. Denote by V_D the subspace of $H^2(X; \mathbf{Q})$ generated by the irreducible components $D_1, ..., D_n$ of D. We shall call elements of V_D by $\mathbf{Q}$-*divisors*.

Denote by $A_D = (D_i \cdot D_j)_{ij}$ the intersection matrix of D. Let Γ_D be the dual weighted graph of D. Clearly that A_D is the incidence matrix (see §1) of Γ_D. Define the *discriminant* of D as $d(D) = d(\Gamma_D) := \det(-A_D)$.

Suppose that $d(D) \neq 0$ (in particular D_i's are linearly independent), and let $B_D = A_D^{-1}$.

Lemma 2.1. *For C_1, $C_2 \in V_D$ one has $C_1 \cdot C_2 = \sum_{i,j} b_{ij}(C_1 \cdot D_i) \cdot (C_2 \cdot D_j)$*

Proof. Any bilinear form defines a homomorphism to the dual space. One can interpret A_D as the matrix of that for the intersection form. Then the required equality is just $C_1 \cdot C_2 = \langle A_D C_1, C_2 \rangle = \langle Z_1, B_D Z_2 \rangle$ for $Z_k = A_D C_k$, $k = 1, 2$. $\square$

Let K_X be the canonical class of V and let $K = K_D$ be its orthogonal projection onto V_D. Actually, for the main purpose of this paper we need only the case, when $V_D = \operatorname{Pic} X \otimes \mathbf{Q}$, and hence $K_D = K_X$ (it is so if $X - D$ is $\mathbf{Q}$-acyclic). However, this assumption does not simplify the statements (nor the proofs), in this and next §§, so we do not restrict ourselves by this case here.

For an irreducible component C of D denote by $\nu_D(C)$ its valence in Γ_D, i.e. $\nu_D(C) = C \cdot (D - C)$, and put $\nu_i = \nu_D(D_i)$. Let χ_i be the Euler characteristic of D_i.

Lemma 2.2. $(K + D) \cdot D_i = \nu_i - \chi_i$.

Proof. Apply adjunction formula: $D_i \cdot (K + D) = D_i \cdot (K + D_i) + \nu_i = \nu_i - \chi_i$. $\square$

Corollary 2.3. (cf. [**OZ**]) $(K + D)^2 = \sum_{i,j} b_{ij}(\nu_i - \chi_i)(\nu_j - \chi_j)$. $\square$

Following Fujita [**F**], define a *twig* of D as a maximal linear rational branch. It means that T is a twig, if $T = C_1 \cup ... \cup C_k$, where each C_i is a rational irreducible component of D; $\nu_D(C_k) = 1$; $\nu_D(C_i) = 2$ and $C_i \cdot C_{i+1} = 1$ for $1 \le i < k$; and if we denote by C_0 the component of $D - T$, which intersects C_1, then either C_0 is not rational or $\nu_D(C_0) \ne 2$. In this case C_0 is called the *root* of the twig T (it is not contained in T); C_k is called the *tip* of T. The rational number $d(T - C_k)/d(T)$ is called *inductance* of T and is denoted by $e(T)$ (we use the convention: $d(\varnothing) = 1$, $e(\varnothing) = 0$). The twig is called *admissible* if $C_i^2 < -1$ for all $i = 1, ..., k$. Clearly, that if a twig T is admissible then $d(T) > 0$ and $0 < e(T) < 1$ (see Corollary 1.3)

For a twig T of D with $d(T) \ne 0$ we define the *bark* of T (see [**F**]) as the unique $\mathbf{Q}$-divisor $\mathrm{Bk}(T)$ in V_T (i.e. $\mathrm{Supp}(\mathrm{Bk}(T)) \subset T$), such that $\mathrm{Bk}(T) \cdot \mathrm{tip}(T) = -1$, $\mathrm{Bk}(T) \cdot C = 0$ for a component C of T, which is not the tip. The following lemma is an immediate consequence of Lemmas 1.1 and 2.1, applied to the matrix B_T.

Lemma 2.4. (Fujita, [**F**, (6.16)]). *Let T be a twig of D, and $d(T) \ne 0$. Then*
(i). $\mathrm{Bk}(T)^2 = -e(T)$.
(ii). If C is a vertex of a twig T then the coefficient of C in $\mathrm{Bk}(T)$ *is equal to $d(T_C)/d(T)$, where T_C is the connected component of $T - C$ which is between C and the root of T.*
(iii). In particular, if C is the vertex, nearest to the root, then the coefficient of C is equal to $1/d(T)$.

§3. Local Zariski–Fujita decomposition

Let, as in §2, D be an SNC-curve on a smooth projective algebraic surface X, $K = K_D$ be the projection of K_X onto V_D, and suppose that D is not a linear chain of rational components, and that all the twigs of D are admissible.

In this case we define the *local Zariski–Fujita decomposition of $K + D$ near D* as $K + D = H + N$, where $N = N_D$ is the sum of the barks of all the twigs of D. The $\mathbf{Q}$-divisors $H = H_D$ and N_D are called respectively *positive* and *negative parts of $K_D + D$ near D*. From Lemma 2.2 and the definition of bark we obtain immediately the following properties of the local Zariski–Fujita decomposition:

Lemma 3.1. (Fujita, [**F**, (6.12)]).
(i) $K + D = H + N$, where H, $N \in V_D$;

(ii) $\mathrm{Supp}(N)$ *is contained in the union of all twigs of* D*;*
(iii) H *is orthogonal to each irreducible component of* N.

Remark. It is proved in [**F**] (we do not use this here), that H and N are uniquely defined by the conditions (i)–(iii) in Lemma 3.1. Fujita has also proved (see [**F**, (6.20–6.24)]) that under certain conditions Zariski decomposition of $K+D$ coincides with the local one (see Theorem 5.2 below). Even if this is not the case, it is much more convenient to calculate separately H^2 and N^2 in order to calculate $(K+D)^2$ in terms of discriminants of subgraphs (i.e. via the inverse matrix $B_D = A_D^{-1}$).

Denote by $\mathrm{br}(D)$ the set of all irreducible components C of D which have either positive genus or $\nu_D(C) > 2$, and put

$$h_i = \begin{cases} \nu_i - \chi_i - \sum \frac{1}{d(T)} & \text{for } i \in \mathrm{br}(D) \\ 0 & \text{otherwise.} \end{cases}$$

where T runs through all twigs, rooted by D_i

Lemma 3.2. *If all the twigs of* D *are admissible, then* $H_D \cdot D_i = h_i$ *for any* i.

Proof. By Lemma 2.2 we have $(K+D)\cdot D_i = \nu_i - \chi_i$. By Lemma 2.4(iii) and the definition of bark we have

$$N_D \cdot D_i = \begin{cases} \sum \frac{1}{d(T)} & \text{for } i \in \mathrm{br}(D) \\ 2-\nu_i & \text{otherwise.} \end{cases}$$

It remains to subtract the latter equality from the former one. □

Corollary 3.3. [**OZ**] *If all the twigs of* D *are admissible, then* $H_D^2 = \sum_{i,j\in \mathrm{br}(D)} b_{ij} h_i h_j$.

Proof. Apply Lemmas 2.1 and 3.2.

§4. The formulas from §§2,3 for the case of a fork

Let D be a *rational* r*-fork* on a smooth projective algebraic surface X. This means that D is an SNC-curve with rational components, and the dual graph of D is an r-fork. Introduce the following notation. Denote by $D_0, \dots, D_n$ the irreducible components of D and by $\nu_i = \nu(D_i)$ their valences. Without loss of generality we may assume that $\nu_0 = r$ (and hence, $\nu_i \le 2$ for $i > 0$). Let $T_1, \dots, T_r$ be the twigs of D, i.e. the connected components of $D - D_0$, and $d_1, \dots, d_r$ their discriminants. For $i = 1, \dots, n$ put

$$a_i = d_j, \qquad b_i = d(T_{j,i}^+), \qquad c_i = d(T_{j,i}^-),$$

where T_j is the twig containing D_i and $T_{j,i}^+$ (resp., $T_{j,i}^-$) is the connected component of $T_j - D_i$, which does not intersect (resp., does intersect) the "central" curve D_0 (see Fig. 1). Extend this notation for $i = 0$, putting $a_0 = b_0 = 1$, $c_0 = 0$.

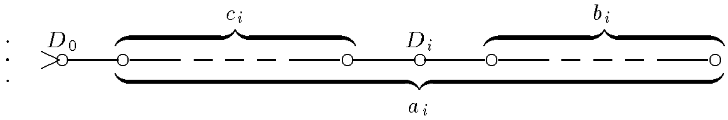


Fig. 1.

Let V_D be the $\mathbf{Q}$-vector space generated by $D_0,...,D_n$. Denote by V_j, $j=1,...,r$ the subspace of V_D generated by the irreducible components of T_j, and let V_H be the orthogonal complement of $\bigoplus_{j=1}^r V_j$. Denote by $\mathrm{pr}_1,\dots,\mathrm{pr}_r$ and pr_H the orthogonal projections onto $V_1,\dots,V_r$ and V_H respectively. Let $K+D=H+N$ be the local Fujita decomposition of $K+D$ near D. Since V_H is one-dimensional, it is generated by H unless $H=0$. Let $N_j=\mathrm{Bk}(T_j)$ (clearly, that $\mathrm{pr}_j(N)=N_j$, $\mathrm{pr}_H(N)=0$ and $N=\sum N_j$). Denote:

$$p=\prod_{j=1}^r d_j;\qquad \Delta=d(D);\qquad h=r-2-\sum_{j=1}^r\frac{1}{d_j};\qquad \varepsilon=-ph/\Delta. \tag{2}$$

Lemma 4.1. *Let C be a $\mathbf{Q}$-divisor in V_D. Put $x_i=C\cdot D_i$, $i=0,...,r$ and $C_H=\mathrm{pr}_H(C)$. Then*

$$\begin{array}{llll}
a). & H^2=\varepsilon h; & d). & C\cdot D=\sum_{i=0}^n x_i;\\
b). & C\cdot H=\varepsilon\sum_{i=0}^n x_i\dfrac{b_i}{a_i}; & e). & C\cdot K=\sum_{i=0}^n x_i\Big(\dfrac{c_i+\varepsilon b_i}{a_i}-1\Big);\\
c). & C\cdot N=\sum_{i=0}^n x_i\dfrac{c_i}{a_i}; & f). & C_H^2=\dfrac{(C\cdot H)^2}{\varepsilon h}=\dfrac{\varepsilon}{h}\Big(\sum_{i=0}^n x_i\dfrac{b_i}{a_i}\Big)^2.
\end{array}$$

Proof. (a) is an immediate consequences of Corollary 3.3. By Lemma 1.1, the entry b_{0i} of the matrix B_D is equal to $-(b_i\cdot(p/a_i))/\Delta$. Thus, (b) follows from Lemmas 2.1 and 3.2. (c) follows from Lemma 2.4(ii); (d) is trivial; (e) follows from (b,c,d) since $K=H+N-D$; (f) follows from (b) and (a). □

Corollary 4.2. *If $r\ge 4$ and all twigs of D are admissible then there exists no smooth rational (-1)-curve C on X such that $C\cdot D=1$ and $C\not\subset D$.*

Proof. Suppose that such a curve C exists. Then $C\cdot K=-1$ and $C\cdot D=1$ implies that for some i we have $x_i=1$, $x_k=0$ for $k\ne i$. Hence, by Lemma 4.1(e) we have $-1=C\cdot K=(c_i+\varepsilon b_i)/a_i-1$ But if $r>3$ then $\varepsilon>0$. Contradiction. □

§5. Zariski decomposition and refined log-BMY inequality

Let D be an SNC-curve on a smooth projective surface X, and $Y=X-D$. Remind the following definition (see e.g. [**F**], [**Ii**]). If $\bar\kappa(Y)\ge 0$, then there exists *the Zariski decomposition* $K+D=H+N$, where H, N are $\mathbf{Q}$–divisors in X such that

(i) the intersection form is negatively definite on the subspace V_N generated by the irreducible components of N (in particular, $N^2\le 0$);
(ii) $HC\ge 0$ for any complete irreducible curve $C\subset X$;
(iii) H is orthogonal to V_N (and hence, $(K+D)^2=H^2+N^2$).

The main tool, used in the proof of Theorem 1′, is the following refined version of the log-BMY inequality.

Theorem 5.1. (Kobayashi–Nakamura–Sakai [**KNS**]) *If $\bar\kappa(Y)=2$, then $H^2 \le 3e(Y)$, where e is the topological Euler characteristic.*

The following theorem is a partial case of [**F**, (6.20)].

Theorem 5.2. (Fujita) *Let $Y = X - D$ be a smooth projective surface with $\bar\kappa(Y) \ge 0$ and D a connected SNC-curve on it. Suppose that all twigs of D are admissible and D is neither a linear rational chain, nor a 3-fork. Then the (global) Zariski decomposition of $(K + D)$ coincides with the local Zariski–Fujita decomposition near D unless there exists a smooth rational (-1)-curve C on X, which is not contained in D and which satisfies one of the following conditions.*
(i). $D \cdot C = 0$, i.e. $D \cap C = \varnothing$.
(ii). $T \cdot C = 1$ for some twig T of D.

Corollary 5.3. *Let $Y = X-D$ be a $\mathbf{Q}$-acyclic surface with $\bar\kappa(Y) = 2$, and D be a minimal rational r-fork with $r \ge 4$. Then Zariski decomposition of $K + D$ coincides with its local Zariski–Fujita decomposition near D.*

Proof. Let C be some smooth rational (-1)-curve on X. Since $\bar\kappa(X) = 2$, according to [**F**, (6.13)], all the twigs are admissible, so, according to the Theorem 5.2 it suffices to check that C does not satisfies (i), (ii) of 5.2. The condition (i) evidently contradicts to $H_2(Y) = 0$. The condition (ii) contradicts Corollary 4.2. □

§6 Beginning of the proof of Theorem 1′

Let D be a minimal SNC-curve on a smooth projective X, such that Γ_D is an r-fork with $r \ge 4$, $Y = X - D$ is a $\mathbf{Q}$-acyclic surface and $\bar\kappa(Y) = 2$. Introduce the notation as in §4. Since $\bar\kappa(Y) = 2$, it follows from [**F**, (6.13)], that all twigs are admissible, so, all a_i, b_i, c_i are positive for $i > 0$.

Lemma 6.1. $r \le 2h + 4$.

Proof. By (2), $h = r - 2 - 1/d_1 - \ldots - 1/d_r \ge r - 2 - {}^1\!/_2 - \ldots - {}^1\!/_2 = (r/2) - 2$. □

Due to the refined log-BMY inequality (Theorem 5.1) and Corollary 5.3, we have (see Lemma 4.1(a))

$$\varepsilon h \le 3. \tag{3}$$

Thus, by Lemma 6.1 we must estimate h from above, or, equivalently, ε from below.

Lemma 6.2. *If $D_0^2 \le 0$ then $h < (3 + \sqrt{33})/2 \approx 4.3722...$*

Proof. Denote: $d_j = d(T_j)$, $d'_j = d(T'_j)$, $j = 1, ..., r$, where T'_j is obtained from the twig T_j by deleting the component, nearest to D_0. Then, by Lemma 1.2, if $D_0^2 \le 0$, we have

$$-\Delta = p \cdot \Big(D_0^2 + \sum_{j=1}^{r} \frac{d'_j}{d_j}\Big) \le p \cdot \Big(0 + \sum_{j=1}^{r} \frac{d_j - 1}{d_j}\Big) = p \cdot (h + 2).$$

Thus, (3) implies $3 \ge h\varepsilon = -ph^2/\Delta \ge h^2/(h + 2)$, hence $h^2 - 3h - 6 \le 0$. □

Corollary 6.3. *If $r > 12$ then X is rational.*

Proof. If $r > 12$ then by 6.1 and 6.2 we have $D_0^2 > 0$. Hence, [**Sh**; Ch. II, §4, Theorem 2] implies that X is rational. $\square$

From now on we suppose that $r > 12$, hence by 6.3, X is rational, and there exists a smooth rational (-1)-curve C on X. Hence,

$$C^2 = -1; \qquad C \cdot K = -1 \tag{4}$$

Like in Lemma 4.1, put $x_i = C \cdot D_i$, $i = 0, \dots, n$ and $C_H = \mathrm{pr}_H(C)$. Put also $C_j = \mathrm{pr}_j(C)$, $j = 1, \dots, r$, $C_N = \sum_{j=1}^{r} C_j$. By Lemma 6.2, $C \neq D_0$, and from minimality of D we know that $C \neq D_i$, $i > 0$. So, $C \not\subset D$, hence, all x_i are ≥ 0.

Lemma 6.4. $-C_N^2 \geq CN$.

Proof. Let $I_j = \{i \,|\, D_i \subset T_j\}$. Then by Lemma 2.1 and Lemma 4.1(c)

$$-C_j^2 = \sum_{i \in I_j} x_i^2 \frac{c_i b_i}{a_i} + 2 \sum_{i,k \in I_j; i<k} x_i x_k \frac{c_i b_k}{a_i} \geq \sum_{i \in I_j} x_i^2 \frac{c_i b_i}{a_i} \geq \sum_{i \in I_j} x_i \frac{c_i}{a_i} = CN_j.$$

Lemma 6.5. *If $C \cdot D > 2$ then $h \leq (9 + \sqrt{21})/2 \approx 6.7912...$*

Proof. By Corollary 1.3(b) we have $b_i/a_i + c_i/a_i \leq 1$, hence, by Lemma 4.1(b,c,d), $(CH)/\varepsilon + CN \leq CD$. Therefore, by (4),

$$1 = -CK = -CH - CN + CD \geq -CH + \frac{CH}{\varepsilon} = CH \frac{1-\varepsilon}{\varepsilon}.$$

Thus, $CH \leq \varepsilon/(1-\varepsilon)$, hence, by Lemma 4.1(f), $C_H^2 \leq \varepsilon/((1-\varepsilon)^2 h)$, and by (4) and Lemma 6.4, $2 = -C^2 - CK = (-C_N^2 - CN) - (C_H^2 + CH) + CD \geq CD - \varepsilon_1$, where

$$\varepsilon_1 = \frac{\varepsilon}{1-\varepsilon}\Big(1 + \frac{1}{(1-\varepsilon)h}\Big).$$

Since CD is integer, $CD > 2$ implies $\varepsilon_1 \geq 1$, hence $2\varepsilon^2 - (3 + \frac{1}{h})\varepsilon + 1 \leq 0$, hence $\varepsilon \geq \frac{1}{4h}\big(3h + 1 - \sqrt{h^2 + 6h + 1}\big)$, and by (3) it implies $h^2 - 9h + 15 \leq 0$. $\square$

§7. Proof of Theorem 1′ with a weaker estimate

Let all the notation be like in §§4,6, but in this section we shall suppose, that $CD = 2$. Let i and k be such indices that $CD_i + CD_k = CD = 2$. Thus, if $i = k$ then $x_i = 2$, $x_l = 0$ for $l \neq i$, and if $i \neq k$ then $x_i = x_k = 1$, $x_l = 0$ for $l \neq i, k$. In any case we rewrite the last two formulas of Lemma 4.1 as

$$e'). \quad CK = \Big(\frac{c_i}{a_i} + \frac{c_k}{a_k}\Big) + \varepsilon\Big(\frac{b_i}{a_i} + \frac{b_k}{a_k}\Big) - 2; \qquad f'). \quad C_H^2 = \frac{\varepsilon}{h}\Big(\frac{b_i}{a_i} + \frac{b_k}{a_k}\Big)^2. \tag{5}$$

Denote by Q_{ik} "the predicate of belonging D_i and D_k to the same twig", i.e. $Q_{ik}=1$ if $D_i\cup D_k\subset T_j$ for some j, and $Q_{ik}=0$ otherwise. When $Q_{ik}=1$, without loss of generality we can assume that D_i is between D_0 and D_k. In this notation we have

$$-C_N^2=\frac{b_ic_i}{a_i}+\frac{b_kc_k}{a_k}+2Q_{ik}\frac{c_ib_k}{a_i}. \tag{6}$$

Using (5), (6) and the fact that $C^2=C_H^2+C_N^2$, we rewrite (4) as

$$\Big(\frac{c_i}{a_i}+\frac{c_k}{a_k}\Big)+\varepsilon\Big(\frac{b_i}{a_i}+\frac{b_k}{a_k}\Big)=1, \tag{7}$$

$$\Big(\frac{b_ic_i}{a_i}+\frac{b_kc_k}{a_k}\Big)+2Q_{ik}\frac{c_ib_k}{a_i}-\frac{\varepsilon}{h}\Big(\frac{b_i}{a_i}+\frac{b_k}{a_k}\Big)^2=1. \tag{8}$$

Lemma 7.1. *Suppose that one of the following conditions holds:*
(i) $x_0>0$; (ii) $x_0=0$ (i.e. $i\neq 0$ and $k\neq 0$) and $b_i\geq 2$, $b_k\geq 2$.
Then there exists a constant A_1 such that $h<A_1$.

Proof. In the case (i) without loss of generality we suppose that $k=0$, and, putting $a_k=b_k=1$, $c_k=Q_{ik}=0$, into (8), and using $c_i/a_i<1$, we see that $b_i>1$, hence, $b_i\geq 2$. Thus, in the both cases (i) and (ii) we have $(c_\nu/a_\nu)\cdot(b_\nu-2)\geq 0$ for $\nu=i,k$. Hence, subtracting (7) multiplied by 2 from (8), we obtain

$$\frac{\varepsilon}{h}u^2+2\varepsilon u-1=\sum_{\nu=i,k}\frac{c_\nu}{a_\nu}\cdot(b_\nu-2)+2Q_{ik}\frac{c_ib_k}{a_i}\geq 0,\qquad\text{where}\quad u=\frac{b_i}{a_i}+\frac{b_k}{a_k}.$$

Since $u<2$ and $\varepsilon\leq 3/h$, we see that h can not be arbitrary big. $\square$

Lemma 7.2. *If $x_0=0$ (i.e. $i\neq 0$ and $k\neq 0$), $b_k=1$ and $Q_{ik}=1$ then $h<(3+\sqrt{21})/2\approx$* 3.791...

Proof. Putting $b_k=Q_{ik}=1$, $a_i=a_k=a$ into (7) and (8), subtracting (7) from (8) and multiplying the result by $a/(b_i+1)$, we see that $c_i-\varepsilon-(\varepsilon/h)\cdot(1+b_i)/a=0$. Hence, using the estimates $c_i\geq 1$ and $(b_i+1)/a\leq 1$, we get $1-\varepsilon-(\varepsilon/h)\leq 0$, and applying (3), we obtain $h^2-3h-3<0$. $\square$

Lemma 7.3. *Let $Q_{ik}=0$ and $b_k=1$. Then $b_i\geq 2$.*

Proof. If $b_i=1$, then subtracting (7) from (8) we would obtain $\varepsilon=0$. $\square$

Lemma 7.4. *If $x_0=0$ (i.e. $k\neq 0$ and $i\neq 0$), $b_k=1$ and $Q_{ik}=0$ then there exists a constant A_2 such that $h<A_2$.*

Proof. Putting $b_k=1$, $Q_{ik}=0$ into (7) and (8), subtracting (7) from (8) and multiplying the result by a_i, we see that

$$b_ic_i-c_i=\Big(b_i+\frac{a_i}{a_k}\Big)\varepsilon_1,\qquad\text{where}\quad \varepsilon_1=\varepsilon\cdot\Big(1+\frac{1}{h}\big(\frac{b_i}{a_i}+\frac{1}{a_k}\big)\Big)=O(\varepsilon)$$

or, equivalently,

$$\frac{a_i}{a_k} = \frac{b_i c_i - c_i}{\varepsilon_1} - b_i. \tag{9}$$

On the other hand, applying the estimate $c_k \le a_k - 1$ (see 1.3(a)) to (7), putting $b_k = 1$ and multiplying the obtained inequality by a_i, we see that

$$c_i + \varepsilon b_i \ge \frac{a_i}{a_k}(1-\varepsilon). \tag{10}$$

Substituting (9) into (10), we obtain $(1-\varepsilon)b_i c_i \le \varepsilon_1 b_i + (1+\varepsilon_1-\varepsilon)c_i$. Replacing b_i with $b'+1$, this inequality can be transformed into $(b'-\varepsilon_2)(c_i-\varepsilon_3) \le \varepsilon_4$ where ε_2, ε_3 and ε_4 are $O(\varepsilon)$. Since $b' \ge 1$ (by 7.3) and $c_i \ge 1$, we see that ε can not be arbitrary small. $\square$

Proposition 7.5. *Under the hypothesis of Theorem 1′ one has $r \le 30$.*

Proof. Lemmas 6.2 – 7.4 imply $h < \max(A_1, A_2)$. Easy to see that these constants can be chosen to be less than $13\,{}^1\!/_2$. Hence, by 6.1, we have $r \le 2h + 4 < 31$.

§8. More precise estimates for the case $C \cdot D = 2$

In this and the next section we are going to prove Theorem 1′ in full volume (with the estimate $r \le 16$). To this end we strengthen here the estimates for h given in §7. Thus, let C be a smooth rational (-1)-curve on X, where $X - D$ is a $\mathbf{Q}$-acyclic surface with $\bar\kappa = 2$, and $CD = 2$. Let the notation be like in §§4,6,7. Denote also $h + (1/a_i) + (1/a_k)$ by h^+. We shall need the following evident identity:

$$b(x-y)^2 = (x^2+y^2)b + xy\big((b-1)^2 - b^2 - 1\big) = (y-bx)(by-x) + xy(b-1)^2. \tag{11}$$

Lemma 8.1. *Let $k \ne 0$, $Q_{ik} = 0$, $b_k = 1$ and $h^+ \ge 7\frac{1}{2}$. Then $h^+ = 8$, $b_i = 5$, $c_i = 1$, $c_k = a_k - 1$, $a_i = 5a_k - 1$ and $a_k = 2, 3$ or 4.*

Proof. Denote $a_k - c_k$ by c'_k. Putting $Q_{ik} = 0$, $b_k = 1$, $c_k = a_k - c'_k$ into (7), (8) and resolving the obtained simultaneous equations with respect to ε and h, we see that

$$\varepsilon = \frac{c'_k a_i - c_i a_k}{a_i + b_i a_k}, \qquad h = \frac{(c'_k a_i - c_i a_k)(b_i a_k + a_i)}{a_i a_k u}, \tag{12}$$

where $u = c_i b_i a_k - c'_k a_i > 0$. Hence,

$$h^+ = (b_i - 1)(c_i + c'_k)/u; \tag{13}$$

$$\begin{aligned}
3 \geq \varepsilon h &= \frac{(c'_k a_i - c_i a_k)^2}{a_i a_k u} = \frac{b_i(c'_k a_i - c_i a_k)^2}{b_i a_i a_k u} && \text{by (3), (12)} \\
&= \frac{(c_i a_k - c'_k b_k a_i)u + c_i c'_k a_i a_k (b_i - 1)^2}{b_i a_i a_k u} && \text{by (11)} \\
&= \frac{c_i}{b_i a_i} - \frac{c'_k}{a_k} + \frac{c_i c'_k (b_i - 1)^2}{b_i u} && && (14) \\
&> -\frac{c'_k}{a_k} + \frac{c_i c'_k (b_i - 1)^2}{b_i u} && \text{omit } \frac{c_i}{b_i a_i} && (15) \\
&> -\frac{c'_k}{c'_k + 1} + \frac{c_i c'_k (b_i - 1)^2}{b_i u}; && \text{use } a_k \geq c'_k + 1 && (16) \\
u &> \frac{c_i c'_k (c'_k + 1)(b_i - 1)^2}{(4c'_k + 3) b_i}; && \text{by (16)} && (17) \\
h^+ &< \frac{(c_i + c'_k)(4c'_k + 3) b_i}{c_i c'_k (c'_k + 1)(b_i - 1)}. && \text{by (13), (17), 7.3} && (18)
\end{aligned}$$

Denote the right hand side of (18) by $\eta^+(b_i) = \eta^+_{c_i, c'_k}(b_i)$. Easy to check that η^+ is decreasing with respect to each variable when $b_i \geq 2$, $c_i \geq 1$, $c'_k \geq 1$.

In the Table 1 we show the values of c_i, c'_k, b_i, for which $\eta^+(b_i) \leq 7\frac{1}{2}$ and hence, the inequality $h^+ < 7\frac{1}{2}$ follows from (18).

Table 1.

:	$c_i = 1$	$c_i = 2$	$c_i \geq 3$	$c_i \geq 14$
$c'_k = 1$:	$b_i \geq 15$	$b_i \geq 4$	$b_i \geq 3$	$b_i \geq 2$
$c'_k = 2$:	$b_i \geq 4$	$b_i \geq 2$	$b_i \geq 2$	$b_i \geq 2$
$c'_k \geq 3$:	$b_i \geq 3$	$b_i \geq 2$	$b_i \geq 2$	$b_i \geq 2$

Table 2.

:	$c_i = 1$	$c_i = 2$	$c_i \leq 6$
$c'_k = 1$:	$b_i \leq 4$	$b_i \leq 3$	$b_i = 2$
$c'_k = 2$:	$b_i \leq 3$		
$c'_k \leq 6$:	$b_i = 2$		

To see this, it is enough to verify that

$$\begin{aligned}
&\eta^+_{1,1}(15) = 7\tfrac{1}{2}, \quad \eta^+_{2,1}(4) = 7, \quad \eta^+_{3,1}(3) = 7, \quad \eta^+_{14,1}(2) = 7\tfrac{1}{2}, \\
&\eta^+_{1,2}(4) \ = 7\tfrac{1}{3}, \quad \eta^+_{2,2}(2) = 7\tfrac{1}{3}, \\
&\eta^+_{1,3}(3) \ = 7\tfrac{1}{2},
\end{aligned}$$

In the Table 2 we show the values of c_i, c'_k, b_i, for which the inequality $h^+ < 7\frac{1}{2}$ follows from (13), using the evident estimate $u \geq 1$.

Comparing the two tables (note that $b_i \geq 2$ by 7.3) shows that the only cases which are not covered by them, are:

$$7 \leq c_i \leq 13,\ c'_k = 1,\ b_i = 2; \qquad c_i = 1,\ c'_k \geq 7,\ b_i = 2; \qquad c_i = c'_k = 1,\ 5 \leq b_i \leq 14.$$

Consider these three cases separately:

Case 1. ($7 \leq c_i \leq 13$, $c'_k = 1$ and $b_i = 2$). It follows from (17) that $u > c_i/7 \geq 1$. Hence, $u \geq 2$ and (13) implies $h^+ \leq (c_i + 1)/u \leq (13 + 1)/2 = 7$.

Case 2. ($c_i = 1$, $c'_k \geq 7$ and $b_i = 2$).

Subcase 2.1. ($c'_k = 7$). Suppose that $u = 1$. Then by definition of u we have

$$2a_k - 7a_i = 1. \tag{19}$$

We know that $a_i \geq b_i + 1 = 3$. If a_i were equal to 3, then by (19) one would have $a_k = 11$, and hence, (14) would imply $3 \geq \varepsilon h = \frac{100}{33}$. Therefore, $a_i > 3$, but a_i is odd by (19), hence, $a_i \geq 5$. Thus, by (19) we have $a_k = (7a_i + 1)/2 \geq 18$. Hence, (14) implies $3 \geq 1/2a_i - 7/a_k + {}^7\!/_2 > -7/a_k + {}^7\!/_2 \geq -{}^7\!/_{18} + {}^7\!/_2 > 3$.

The obtained contradiction shows that $u \geq 2$. Hence, (13) implies $h^+ = \frac{8}{u} \leq 4$.

Subcase 2.2. ($c'_k \geq 8$). It follows from (15) that $3 > -(c'_k/a_k) + (c'_k/2u) > -1 + (c'_k/2u)$. Hence, $u > c'_k/8 \geq 1$. Subtracting (14) multiplied by 2 from (13), we see that $h^+ - 6 \leq 1/u - 1/a_i + 2c'_k/a_k$. But $0 < u = 2a_k - c'_k a_i$ implies $2c'_k/a_k < 4/a_i$, hence, $h^+ - 6 < 1/u + 3/a_i \leq {}^1\!/_2 + {}^3\!/_3$.

Case 3. ($c_i = 1$, $c'_k = 1$ and $5 \leq b_i \leq 14$). By (17) we have $u > \frac{2}{7}(b_i - 1)^2/b_i > \frac{2}{7}(b_i - 2)$. Hence, $b_i < (7u + 4)/2$ and this implies

$$b_i \leq \begin{cases} (7u+2)/2 & \text{if } u \text{ is even} \\ (7u+3)/2 & \text{if } u \text{ is odd} \end{cases} \tag{20}$$

Thus, for $u > 1$ by (13) we have $h^+ = 2(b_i - 1)/u \leq 7\,{}^1\!/_3$.

Suppose that $u = 1$. Then (20) implies $b_i = 5$. By (15) we obtain $3 > -(1/a_k) + {}^{16}\!/_5$. Since $a_k \geq 2$, we have only three solutions: $a_k = 2, 3, 4$. For them $a_i = b_i a_k - u = 5a_k - 1$, and by (13) we have $h^+ = 2(b_i - 1)/u = 8$. This is the only case when $h^+ \geq 7\frac{1}{2}$. $\square$

Lemma 8.2. *Let $k = 0$ and $h^+ \geq 8$. Then $h^+ = 8$ and $(a_i, b_i, c_i) = (13, 2, 7)$.*

Proof. The proof is similar to that of Lemma 8.1. The beginning of the proof of 8.1 including the formulas (12), (13), (14) and (15) is valid in the case $k = 0$ without changes. However, the implication (15) $\Rightarrow$ (16) does not work in this case. Since we have $a_k = b_k = c'_k = 1$, let us denote a_i, b_i and c_i simply by a, b and c till the end of the proof. Then $u = bc - a$.

First, note that $c > 1$ because otherwise u would be negative. Eliminating u from (13) and (15), we see that

$$h^+ < \eta^+(b, c), \qquad \text{where} \quad \eta^+ = \frac{4(c+1)b}{c(b-1)}. \tag{21}$$

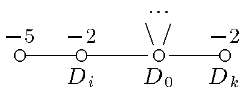
Case 1. $b \geq 4$. Since $c \geq 2$, by (21) we have $h^+ < \eta^+(4, 2) = 8$.

Case 2. $b = 3$. If $c \geq 4$ then $h^+ < \eta^+(3, 4) = 7\frac{1}{2}$ by (21). If $c \leq 3$ then (13) implies $h^+ = 2(c+1)/u \leq 2(c+1) \leq 8$, hence $h^+ < 8$ unless $c = 3$ and $u = 1$. But in this case $a = bc - u = 8$ which contradicts (14).

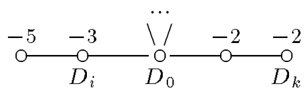
Case 3. $b = 2$. By (14) we have $3 \geq \frac{c}{2a} - 1 + \frac{c}{2u} > -1 + \frac{c}{2u}$. Hence, $c < 8u$ and being integer, $c \leq 8u - 1$. Putting this estimate into (13), we see that $h^+ = (c+1)/u \leq 8$ and $h^+ < 8$ unless $c = 8u - 1$. If $h^+ = 8$, then putting $c = 8u - 1$, $a = 2c - u = 15u - 2$ into (14), we obtain $u = 1$. Hence $(a, b, c) = (13, 2, 7)$. $\square$

Corollary 8.3. *(a). Under the hypothesis of Lemma 8.1 the graph Γ_D has one of the following forms:*

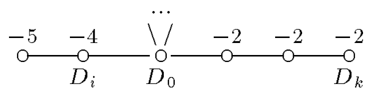
$a_i = 9,\ a_k = 2:$



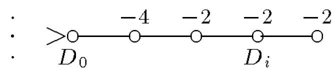
$a_i = 14,\ a_k = 3:$



$a_i = 19,\ a_k = 4:$



(b). Under the hypothesis of Lemma 8.2 the graph Γ_D has the form:



Lemma 8.4. *Let $b_i \ge b_k \ge 2$. Then*

$$h < \Big(\frac{b_k}{a_k} + \frac{b_i}{a_i}\Big)\cdot\Big(\frac{2}{b_k} + \frac{3b_k}{q}\Big), \qquad \text{where} \quad q = (b_k - 1) + (b_i - b_k)\frac{c_i}{a_i}. \tag{22}$$

Proof. Denote $(b_i/a_i) + (b_k/a_k)$ by u. Multiplying (7) by b_k, subtracting the result from (8) and using the estimate $Q_{ik}c_ib_k/a_i \ge 0$, we obtain the inequality $(\varepsilon/h)u^2 + b_k\varepsilon u - q \ge 0$, where q denotes the same as in (22). Therefore, we have

$$u \ge \frac{hb_k}{2}\Big(-1 + \sqrt{1 + \frac{4q}{\varepsilon h b_k^2}}\Big) \overset{\text{by (3)}}{\ge} \frac{hb_k}{2}\Big(-1 + \sqrt{1 + \frac{4q}{3b_k^2}}\Big) = \frac{hb_k}{2}\big(-1 + \sqrt{1+v}\big),$$

where $v = 4q/(3b_k^2)$. It remains to apply the evident estimate
$-1 + \sqrt{1+v} = -1 + (1+v)/\sqrt{1+v} > -1 + (1+v)/(1 + (v/2)) = v/(2+v)$. $\square$

Lemma 8.5. *Let $b_i \ge b_k \ge 10$. Then $h < 6\frac{41}{55} \approx 6.745\ldots$.*

Proof. Applying the estimates $(b_k/a_k) + (b_i/a_i) < (b_k/(b_k+1)) + 1$ and $q \ge b_k - 1$ to the inequality (22), we see that $h < f(b_k)$ where

$$f(b) = \Big(1 + \frac{b}{b+1}\Big)\cdot\Big(\frac{2}{b} + \frac{3b}{b-1}\Big) = 6 + \frac{2}{b} + \frac{3}{b-1} + \frac{2}{b+1} + \frac{3}{b^2-1}.$$

f decreases when $b > 1$. Hence, $h < f(b_k) \le f(10) = 6\frac{41}{55}$. $\square$

Lemma 8.6. *Let $b_i \ge b_k \ge 2$. Suppose also that $b_k \le 9$ and $a_k \ge 20$. Then $h \le 5\frac{113}{120}$.*

Proof. Case 1. $(3 \le b_k \le 9)$. Apply to (22) the estimates $b_i/a_i < 1$, $a_k \ge 20$ and $q \ge b_k - 1$. We obtain the inequality

$$h < f(b_k), \qquad \text{where} \quad f(b) = \Big(1 + \frac{b}{20}\Big)\Big(\frac{2}{b} + \frac{3b}{b-1}\Big).$$

Direct calculation shows that $f(b) \le 5\frac{113}{120}$ for $b = 3, 4, ..., 9$.

Case 2. ($b_k = 2$). Substituting $b_k = 2$ into (22) and applying the estimates $a_k \ge 20$, $c_i \ge 1$, we obtain $h < f(a_i, b_i)$ where

$$f(a,b) = \Big(\frac{1}{10}+\frac{b}{a}\Big)\Big(1+\frac{6a}{a+b-2}\Big) \quad \text{and} \quad \frac{\partial f}{\partial b} = \frac{5b^2+\gamma_1 b+\gamma_2}{5a(a+b-2)^2}, \quad \begin{array}{l}\gamma_1 = 10a-20,\\ \gamma_2 = 32a^2-80a+20.\end{array}$$

If $a \ge 3$ then $\gamma_1, \gamma_2 > 0$, hence $f'_b > 0$. Therefore, since $b_i \le a_i - 1$, we have $h < f(a_i, b_i) \le f(a_i, a_i - 1) = g(a_i)$, where $g(a) = f(a, a-1)$. Easy to calculate that $g'(a) < 0$ when $a > 1$. Recall that $a_i \ge b_i + 1 \ge b_k + 1 = 3$. Hence, $h < g(a_i) \le g(3) = 5\frac{11}{30}$. $\square$

Lemma 8.7. *Let $b_i \ge b_k \ge 2$. Suppose also that $b_k \le 9$ and $a_i \ge 40$. Then $h < 6.8$.*

Proof. From (22) and the estimates $a_k \ge b_k + 1$ and $c_i \ge 0$, we obtain the inequality

$$h < f_{b_k}(a_i, b_i), \qquad \text{where} \quad f_m(a,b) = \Big(\frac{m}{1+m}+\frac{b}{a}\Big)\cdot\Big(\frac{2}{m}+\frac{3ma}{(m-1)a+b-m}\Big).$$

If $a \ge 6$, $b \ge 2$, $m \ge 2$ then f_m is monotonically increasing with respect to b. Indeed, one can check that

$$\frac{\partial f_m}{\partial b} = \frac{2}{ma} + \frac{3m}{m+1}\cdot\frac{\gamma_1 a - \gamma_2}{((m-1)a+b-m)^2}, \qquad \begin{array}{l}\gamma_1 = m^2-m-1,\\ \gamma_2 = m^2+m.\end{array}$$

$m \ge 2$ implies $\gamma_1 > 0$, hence, for $a > 6$ we have $\gamma_1 a - \gamma_2 > 6\gamma_1 - \gamma_2 = 5m^2 - 7m - 6 \ge 0$, thus, $\frac{\partial f_m}{\partial b} > 0$. Obviously, for $b \ge 2$ the denominator is non-zero.

We know that $b_i \le a_i - 1$ and $a_i \ge 40$. Hence, $h < f_{b_k}(a_i, a_i - 1) < g_{b_k}(a_i)$, where

$$g_m(a) := f_m(a, a-1) + \frac{2}{ma} = 6 + \frac{m+2}{m^2+m} + \frac{3(m+1)}{ma-m-1}.$$

Clearly, g_m is monotonically decreasing with respect to a when $a \ge 2$. Thus, it suffices to check that $g_m(40) < 6.8$ for $m = 2, ..., 9$. $\square$

Lemma 8.8. *Suppose that $b_i \ge b_k \ge 2$ and $a_k < 20$, $a_i < 40$. Then $h \le 6.023810...$*

Proof. Since $b_\nu < a_\nu$ and $c_\nu < a_\nu$, it suffices to check only finitely many possibilities for the values of Q_{ik}, a_ν, b_ν and c_ν (where $\nu = i, k$). In each case we can find ε and h from the equations (7), (8) and search the maximum of h under the restrictions $\varepsilon > 0$, $h > 0$, $\varepsilon h \le 3$. These calculations were performed with a computer. The corresponding C-program is shown on the Fig. 2. $\square$

Corollary 8.9. *Let $b_i \ge b_k \ge 2$. Then $h < 6.8$.*

Proof. For $b_k \ge 10$ see 8.5; for $b_k \le 9$ see 8.6 – 8.8 $\square$

```
#include <stdio.h>
main(){ int ak,bk,ck, ai,bi,ci, Q; double B,C,BC,h, hmax=0;
   for( Q=0; Q<=1; Q++ ){
    for( bk=2; bk<=9; bk++ ){
     for( ak=bk+1; ak<=21; ak++ ){
      for( bi=bk; bi<=40; bi++ ){
       for( ai=bi+1; ai<=41; ai++ ){
        for( ck=1; ck<=ak-bk; ck++ ){
         for( ci=1; ci<=ai-bi; ci++ ){
          B=(double)bi/ai + (double)bk/ak;
          C=(double)ci/ai + (double)ck/ak;
          BC=(double)(bi*ci)/ai + (double)(bk*ck)/ak;
          if( ai==ak ) BC=BC+(double)(2*Q*ci*bk)/ai;
          if( 1-C <= 0 )continue;                         /* eps>0 */
          if( BC-1 <= 0 )continue;                        /*  h>0  */
          if( (1-C)*(1-C) > 3*(BC-1) )continue;           /*  BMY  */
          if( (h=(1-C)*B/(BC-1)) > hmax ) hmax=h;
   }}}}}}}
   printf( "hmax=%lf", hmax );
}
```

Fig. 2.

§9. Proof of Theorem 1′

Let things be like in §6.

Lemma 9.1. *Suppose that $r \geq 17$. Then:*
(a). $h \geq 6.5$.
(b). If $h < 6.8$ then $r = 17$, and up to a permutation, $(d_1, ..., d_{17})$ is either $(4, 2, ..., 2)$ or $(3, 2, ..., 2)$.

Proof. (a). See Lemma 6.1.

(b). If $h < 6.8$ then $r = 17$ by Lemma 6.1. Without loss of generality we may assume that $d_1 \geq d_2 \geq ... \geq d_{17}$. If $d_2 \geq 3$, we would have $h = 17 - 2 - 1/d_1 - ... - 1/d_{17} \geq 15 - 1/3 - 1/3 - 1/2 - ... - 1/2 = 6\,5/6 > 6.8$. Thus $d_2 = ... = d_{17} = 2$ and $1/d_1 = 17 - 2 - 1/2 - ... - 1/2 - h > 7 - h > 1/5$. $\square$

Lemma 9.2. *Suppose that $r \geq 17$ and $h \geq 6.8$. Then (up to a permutation) one of the following possibilities holds:*
(1). (T_1, T_2) is one of the three pairs listed in 8.3(a) and either
(1.1). $r = 18$ and $d_3 = ... = d_{18} = 2$, or
(1.2). $r = 17$ and $(d_3, ..., d_{17})$ is one of $(6, 3, 2, ..., 2)$, $(4, 4, 2, ..., 2)$, $(3, 3, 3, 2, ..., 2)$.
(2). $r = 17$, $d_2 = ... = d_{17} = 2$ and T_1 is the twig depicted in 8.3(b).

Proof. By 6.3, X is rational. Hence, there exists a smooth rational (-1)-curve C. It does not coincide with one of $D_1, ..., D_n$ by the minimality, and $C \neq D_0$ by Lemma 6.2. Thus, it follows from 6.5 and 4.2 that $CD = 2$.

Introduce the notation like in §7, §8. If the both b_i and b_k were ≥ 2, then by Corollary 8.9 we would have $h < 6.8$. Thus, one of them, say, b_k is equal to 1 and by Lemma 7.2 we have $Q_{ik} = 0$.

Case 1. (Like in 8.1) $b_k = 1$, $k \neq 0$.

Since D_i and D_k do not belong to the same twig, without loss of generality we may assume that $D_i \subset T_1$, $D_k \subset T_2$ (i.e. $d_1 = a_i$, $d_2 = a_k$) and that $d_3 \leq d_4 \leq \dots$. Then

$$h^+ = r - 2 - 1/d_3 - 1/d_4 - \dots - 1/d_r \geq r - 2 - {}^1\!/_2 - \dots - {}^1\!/_2 = (r-2)/2. \tag{23}$$

Since $r \geq 17$, it follows that $h^+ \geq 7\,{}^1\!/_2$. Hence, by 8.1 we have $h^+ = 8$.

Subcase 1.1. $r \geq 18$. Then (23) turns out into $8 = \dots \geq (r-2)/2 \geq 8$. Hence, all the "$\geq$" can be replaced with "=", and we have $r = 18$ and $d_3 = \dots = d_{18} = 2$.

Subcase 1.2. $r = 17$. If $d_6 \geq 3$, then like in (23) we would have $8 \geq 15 - ({}^1\!/_3 - {}^1\!/_3 - {}^1\!/_3 - {}^1\!/_3) - {}^1\!/_2 - \dots = 8\,{}^1\!/_6$. Thus, $d_6 = \dots = d_{17} = 2$ and $1/d_3 + 1/d_4 + 1/d_5 = 15 - h^+ - {}^1\!/_2 - \dots = 1$.

Case 2. (Like in 8.2) $k = 0$.

Subcase 2.1. Without loss of generality assume that $D_i \in T_1$, i.e. $d_1 = a_i$. Then $r \geq 17$ implies like in (23) that $h^+ = h + 1 + 1/d_1 \geq r - 1 - {}^1\!/_2 - \dots - {}^1\!/_2 = (r-1)/2 \geq 8$, and by 8.2 we have $h^+ = 8$. Hence, all the "$\geq$" can be replaced with "=" and we obtain $r = 17$ and $d_2 = \dots = d_{17} = 2$. □

Lemma 9.3. *Let X be a smooth rational projective surface. Then $K^2 + b = 10$ where $K = K_X$ is the canonical class and $b = b_2(X)$ is the second Betti number.*

Proof. Since X is rational, it is obtained from $\mathbf{P}^2$ by successive blow-ups and -downs. Clearly that $K^2 + b = 10$ for $\mathbf{P}^2$ and that $K^2 + b$ is invariant under blow-ups. □

Corollary 9.4. (See e.g. [**FZ**; 1.3]) *Let notation be like in 9.3. Suppose that D is an SNC-curve such that $X - D$ is $\mathbf{Q}$-acyclic. Then*

$$(K + D)^2 = 8 - s - 3b \tag{24}$$

where s denotes the sum of all the weights of Γ_D.

Proof. Let $D_1, \dots, D_b$ be the irreducible components of D. Write $(K+D)^2 = K^2 + 2KD + D^2$ and compute each summand in the right hand side:

$K^2 = 10 - b$ by Lemma 9.3;

$KD = \sum D_i(K + D_i) - \sum D_i^2 = -2b - s$ by adjunction formula;

$D^2 = \sum D_i^2 + \sum_{i \neq k} D_i D_k = \sum D_i^2 + 2(\text{number of edges of } \Gamma_D) = s + 2(b-1)$. □

Now let (X, D) be again as in §6. Introduce the following notation. For a twig T denote $s(T) = \sum (w_\nu + 3)$, where w_ν are the weights and the summation is over all the vertices. Recall that $e(T)$ denotes the inductance of a twig T (cf. §2). Let $e'(T) = e(T')$ where T' is the twig obtained from a twig T by reversing the order of the vertices. Denote $e(T) + e'(T) - s(T)$ by $\varphi(T)$, and put: $e_j = e(T_j)$, $e'_j = e'(T_j)$, $s_j = s(T_j)$ and $\varphi_j = \varphi(T_j)$.

Lemma 9.5. $\sum \varphi_j \geq 2h - 5$.

Proof. By Lemma 1.2 and (2) we have $-\Delta = p\cdot(D_0^2 + \sum e'_j)$. Hence, $D_0^2 = -\Delta/p - \sum e'_j = h/\varepsilon - \sum e'_j$. Further, by 4.1(a) and 2.4(i) we have $(K+D)^2 = H^2 + N^2 = \varepsilon h - \sum e_j$. Putting these expressions for D_0^2 and $(K+D)^2$ into (24) (where, in our notation, $s + 3b = D_0^2 + 3 + \sum s_j$), we obtain $5 + \sum \varphi_j = h(\varepsilon + 1/\varepsilon) \geq 2h$. $\square$

Now let us complete the proof of Theorem $1'$. Suppose that $r \geq 17$. Then by 9.1(a) we have $h \geq 6.5$, hence, 9.5 implies $\sum \varphi_i \geq 13 - 5 = 8$. However, each φ_j depends only on the twig, and by 9.1 and 9.2 only few types of twigs can appear. The values of $\varphi(T)$ for these twigs are as follows:

Table 3.

$d(T)$	T	$\varphi(T)$	$d(T)$	T	$\varphi(T)$
2	[2]	0	5	[5]	2.4
3	[3]	$2/3$		[3, 2]	0
	[2, 2]	$-2/3$		[2, 2, 2, 2]	-2.4
4	[4]	1.5	6	[6]	$3\,1/3$
	[2, 2, 2]	-1.5		[2, 2, 2, 2, 2]	$-3\,1/3$

Here the twig with the weights $w_1, w_2, \ldots$ is denoted by $[-w_1, -w_2, \ldots]$. In Table 3 we listed all the twigs with discriminants ≤ 6. The values $\varphi(T)$ for those twigs which appear in 8.3, are

$$\varphi([2,5]) = 1\,{}^7\!/_9, \quad \varphi([3,5]) = 2\,{}^4\!/_7, \quad \varphi([4,5]) = 3\,{}^9\!/_{19}, \quad \varphi([4,2,2,2]) = -{}^{12}\!/_{13}.$$

It is easy to check that in all the cases allowed by 9.1 and 9.2 we can not have $\sum \varphi_j \geq 8$. Theorem $1'$ is proven.

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