ARRANGEMENTS OF A PLANE 
M-SEXTIC WITH RESPECT TO A LINE

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Abstract. The mutual arrangements of a real algebraic or real pseudoholomorphic plane projective M-sextic and a line up to isotopy are studied. A complete list of the pseudoholomorphic arrangements is obtained. Four of them are proven to be algebraically unrealizable. All the others with two exceptions are algebraically realized.

By a real algebraic curve in \( \mathbb{RP}^2 \) we mean a complex algebraic curve in \( \mathbb{CP}^2 \) invariant under the complex conjugation. Given such a curve \( C \), we denote the set of its real points by \( \mathbb{R}C \). In this paper we study mutual arrangements in \( \mathbb{RP}^2 \) of \( \mathbb{R}C_6 \) and \( \mathbb{R}C_1 \) where \( C_6 \) is a real algebraic \( M \)-curve of degree 6 (it has 11 ovals) and \( C_1 \) is a real line transverse to \( C_6 \). We consider such arrangements up to isotopy in \( \mathbb{RP}^2 \). We study also the same problem for real pseudoholomorphic curves (see [9] and references therein).

In the case when \( C_6 \cap C_1 \) is contained in a single oval of \( \mathbb{R}C_6 \), the number of connected components of \( \mathbb{R}C_6 \setminus C_1 \) is maximal, so, we say in this case that the pair \((C_6, C_1)\) realizes a maximal arrangement and \( C_6 \setminus C_1 \) is an affine \( M \)-sextic. A complete classification of the maximal arrangements is already done (see [11], [1]): there are exactly 38 pseudoholomorphically realizable arrangements and only 35 of them are algebraically realizable.

An algebraic classification (with a single exception) in the non-maximal case was announced by E. I. Shustin in [12]. However the proofs are not given there (only the method is described) and, moreover, the proofs of the algebraic unrealizability of at least four arrangements are certainly erroneous because these arrangements are pseudoholomorphically realizable whereas the techniques used in the proofs cannot distinguish between the algebraic and pseudoholomorphic realizability.

In the present paper we give a complete list of the pseudoholomorphically realizable non-maximal arrangements. We prove that four of them are algebraically unrealizable; the algebraic realizability of two more of them is open, and all the others are realizable by real algebraic curves.

For plane projective \( M \)-sextics, the algebraic and pseudoholomorphic isotopy classifications coincide. There are three isotopy types: \( 9 \sqcup 1(1) \), \( 5 \sqcup 1(5) \), and \( 1 \sqcup 1(9) \) in Viro’s notation [13]. Any pseudoholomorphically realizable non-maximal arrangement of an \( M \)-sextic and a line belongs to one of the series shown in Figure 1 where the numbers \( a \), \( b \), and \( c \) are the numbers of unnested ovals in the corresponding regions. This fact easily follows from Bézout Theorem applied to \( C_6 \) and an auxiliary line through some two ovals (and an auxiliary conic for the series \( E \)). The notation for the series in Figure 1 is similar to that in [4], [12]. Arrangements in
Figure 1 (providing one of the three isotopy types of $M$-sextics after removal of the line) will be called admissible arrangements.

\[
\begin{align*}
\tilde{A}_1(a, b) & \quad \tilde{A}_2(a, b, c) & \quad \tilde{A}_3(a, b, c) & \quad \tilde{B}(a, b, c) & \quad \tilde{C}_1(a, b, c) \\
\tilde{C}_2(a, b, c) & \quad \tilde{D}(a, b, c) & \quad E_1(a, b, c) & \quad E_2(a, b, c) & \quad F(a, b, c) \\
G(a, b, c) & \quad A_1^{-2}(a, b) & \quad A_2^{-2}(a, b, c) & \quad B^{-2}(a, b, c) & \quad C^{-2}_1(a, b) \\
C_2^{-2}(a, b, c) & \quad C_3^{-2}(a, b, c) & \quad A_1^{-4}(a, b) & \quad A_2^{-4}(a, b, c) & \quad A^{-6}(a, b)
\end{align*}
\]

**Figure 1.** Admissible arrangements.

**Theorem 1.1.** The following arrangements are realizable as $\mathbb{R}C_6 \cup \mathbb{R}C_1$ where $C_6$ is a real algebraic sextic curve and $C_1$ a real line in $\mathbb{P}^2$:

- $\tilde{A}_2(a, b, c), \quad (a, b, c) = (2, 7, 0), (8, 1, 0), (0, 5, 4), (2, 3, 4), (4, 1, 4), (5, 0, 4), (0, 1, 8), (1, 0, 8)$,
- $\tilde{A}_3(a, b, c), \quad (a, b, c) = (0, 5, 4), (0, 1, 8)$,
- $\tilde{C}_2(a, b, c), \quad (a, b, c) = (1, 5, 3), (1, 6, 2), (1, 7, 1), (1, 8, 0), (5, 1, 3), (5, 2, 2), (5, 3, 1), (5, 4, 0)$,
- $\tilde{D}(a, b, c), \quad (a, b, c) = (1, 7, 1), (8, 0, 1), (0, 4, 5), (1, 3, 5), (4, 0, 5), (0, 0, 9)$,
- $C_3^{-2}(a, b, c), \quad (a, b, c) = (1, 5, 4), (1, 6, 3), (1, 7, 2), (1, 8, 1), (1, 9, 0), (5, 1, 4), (5, 2, 3), (5, 3, 2), (5, 4, 1), (5, 5, 0), (9, 1, 0)$,

and all the admissible arrangements of the other series, i.e., of the series $\tilde{A}_1, \tilde{B}, \tilde{C}_1, E, F, G, A_1^{-2}, A_2^{-2}, B^{-2}, C_1^{-2}, C_2^{-2}, A^{-4}, A^{-6}$.
Theorem 1.2. Let $J$ be a tame conj anti-invariant almost complex structure on $\mathbb{CP}^2$ and let $C_6$ and $C_1$ be smooth real $J$-holomorphic curves of degree 6 and 1 respectively. Suppose that $C_6$ is an $M$-sextic (i.e., $\mathcal{R}C_6$ has 11 ovals), $\mathcal{R}C_1$ is transverse to $\mathcal{R}C_6$, and the arrangement $(C_6,C_1)$ is not maximal, i.e., $C_6 \cap C_1$ is not contained in a single oval of $\mathcal{R}C_6$.

Then the arrangement of $\mathcal{R}C_6$ with respect to $\mathcal{R}C_1$ is either as in Theorem 1.1 or one of:

\begin{align*}
\tilde{A}_3(1,8,0), \ \tilde{A}_3(1,4,4), \\
\tilde{C}_2(1,3,5), \ \tilde{C}_2(1,4,4), \ C_3^{-2}(1,3,6), \ C_3^{-2}(1,4,5).
\end{align*}

All these arrangements are pseudoholomorphically realizable.

Theorem 1.3. The arrangements (2) are algebraically unrealizable.

The algebraic realizability of (1) is open. In Table 1 we present the distribution of realizable arrangements among the series.

<table>
<thead>
<tr>
<th>Admissible</th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$D$</th>
<th>$E$</th>
<th>$F$</th>
<th>$G$</th>
<th>$A^{-2}$</th>
<th>$B^{-2}$</th>
<th>$C^{-2}$</th>
<th>$A^{-4}$</th>
<th>$A^{-6}$</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pseudohol.</td>
<td>14</td>
<td>15</td>
<td>13</td>
<td>6</td>
<td>4</td>
<td>9</td>
<td>6</td>
<td>11</td>
<td>9</td>
<td>31</td>
<td>12</td>
<td>3</td>
<td>133</td>
</tr>
<tr>
<td>Algebraic</td>
<td>12</td>
<td>15</td>
<td>11</td>
<td>6</td>
<td>4</td>
<td>9</td>
<td>6</td>
<td>11</td>
<td>9</td>
<td>29</td>
<td>12</td>
<td>3</td>
<td>127</td>
</tr>
</tbody>
</table>

Table 1.

Remark 1.4. It is stated in [12] that all the arrangements listed in Theorem 1.1 and also $\tilde{A}_3(1,4,4)$ were constructed by A. B. Korchagin, G. M. Polotovskii, and E. I. Shustin, and the constructions are similar to those in [4], but they are not presented in [12]. I indeed found constructions as in [4] for the arrangements of Theorem 1.1 (see §2) but not for $\tilde{A}_3(1,4,4)$. I doubt that the latter arrangement is algebraically realizable.

Remark 1.5. The algebraic realizability of the affine $M$-sextic $A_2(8,1,1)$ (see Figure 8 below) is stated in [4, Thm. 2] but its construction is forgotten to be included to that paper. In fact, the construction (that I learned from G. M. Polotovskii) is very simple and I present it in §3.3.

In §§2–4 we prove Theorems 1.1 and 1.2 using the same methods as in [4–8], [13]. In §5 we prove Theorem 1.3. The general strategy of the proof is more or less the same as in [11]: by Hilbert–Rohn–Gudkov method we reduce the problem to the algebraic unrealizability of a certain quadrigonal curve (which is pseudoholomorphically realizable) and then we exclude it using cubic resolvents. However, we evoke some new arguments in the cubic resolvent step (see §5.1).

Remark 1.6. Given an affine $M$-sextic of the isotopy type $C_2(1,3,6)$ (see Figure 8 in §3), by moving the line one obtains the arrangements (2). Thus Theorem 1.3 gives a new proof of the algebraic unrealizability of $C_2(1,3,6)$ whose Hilbert–Rohn–Gudkov stage is considerably simplified.

We use the approach from [6–8] for the pseudoholomorphic classification. The paper [7] can be used as a general introduction. For the reader’s convenience, let us recall some terminology, notation, and main ideas. Given a fibration $\pi : E \to B$, a fiberwise arrangement of $X \subset E$ is the equivalence class of $X$ with respect to isotopies $\{H_t\}$ of $E$ such that $\pi \circ H_t = h_t \circ \pi$ for some isotopy $\{h_t\}$ of $B$. Given
a point \( p \in \mathbb{RP}^2 \) and a subset \( X \) of \( \mathbb{RP}^2 \setminus \{p\} \), the \( \mathcal{L}_p \)-scheme of \( X \) is a fiberwise arrangement for the linear projection \( \mathbb{RP}^2 \to \mathbb{P}^1 \). When \( X = \mathbb{R}C \) for a real algebraic or real pseudoholomorphic nodal curve \( C \) in general position, we encode the \( \mathcal{L}_p \)-scheme by a word in letters \( \subset_k, \supset_k, \times_k \) which correspond to consecutive non-generic fibers. The subscript \( k \) indicates the height of the point with vertical tangent (for \( \subset_k \) or \( \supset_k \)) or the double point (for \( \times_k \)) similarly to the braid notation. A subword \( \subset_k \supset_k \) is abbreviated to \( o_k \) (an oval). For example, the upper \( \mathcal{L}_p \) scheme in Figure 3(a) is encoded by \( \supset_2 \subset_1 \). An \( \mathcal{L}_p \)-scheme determines a braid (or a family of braids). An \( \mathcal{L}_p \)-scheme is pseudoholomorphically realizable if and only if the braid (at least one braid of the family) is quasipositive, i.e., is a product of conjugates of the standard generators of the braid group (see details in [6, 7]).

2. Construction of algebraic curves

In this section we prove Theorem 1.1. The cases \( A^{-4} \) and \( A^{-6} \) are evident, and below we present constructions for all the other series of arrangements.

2.1. The series \( \tilde{A}_1, \tilde{A}_3, E, A_1^{-2} \). There exist arrangements of a quintic curve \( C_5 \) with respect to a line \( C_1 \) shown in Figure 2(left); see [3] for \( a = 1 \) and [2; §7.6] for \( a = 5 \). By perturbing \( C_5 \cup C_1 \) in different ways, one obtains \( \tilde{A}_3(0, 5, 4), \tilde{A}_3(0, 1, 8), \) and all the eight admissible arrangements of the series \( \tilde{A}_1, E, E_2, \) and \( A_1^{-2} \) (see Figure 2).

![Figure 2: Construction of \( \tilde{A}_3(0, 5, 4), \tilde{A}_3(0, 1, 8), \tilde{A}_1(a, b), E_k(a, b) \).](image)

2.2. The series \( F, G, A_2^{-2}, B^{-2} \). Let \( C_6 \) be an \( M \)-sextic and \( p \) be a point in one of its exterior empty ovals. Then, choosing a line \( C_1 \) passing through \( p \), one obtains all the admissible arrangements of the series \( F \) and \( A_2^{-2}. \)

By Lemma 4.1(a), the interior ovals of \( C_6 \) lie in a convex position (if one chooses a point on each oval, the chosen points are the vertices of a convex polygon in some affine chart). Therefore, by different choices of a line \( C_1 \) passing through any given point \( p \) in an interior oval, one obtains all the admissible arrangements of the series \( G \) and \( B^{-2}. \)

2.3. The series \( \tilde{A}_2, \tilde{B}, C_2^{-2} \). By [13, §4.2] there exists polynomials \( F_k(x, y), k = 1, 2, \) with Newton polygon \([[(0, 0), (0, 3), (6, 0)]\) which define affine curves arranged with respect to the vertical lines as in Figure 3(a). To these curves and their symmetric images we apply the procedure shown in Figure 3(b). Namely, we choose two translates of the same sufficiently narrow parabola and then we apply the transformation \((x, y) \mapsto (x, y + \lambda x^2)\) where \( \lambda \) is chosen so that the parabolas are transformed to lines. The projective closures of the obtained curves have a point of simple tangency of three smooth local branches. By perturbing this singularity (see again [13, §4.2]) we can obtain the arrangements of an \( M \)-sextic with respect to a pencil of lines as in Figure 3(c) (for example, the leftmost arrangement in Figure 3(c) corresponds the the case shown in Figure 3(b)). By choosing different
lines from this pencil, we obtain all the arrangements of the series \( \tilde{A}_2, \tilde{B}, C_2^{-2} \) listed in Theorem 1.1.

2.4. The series \( \tilde{C}_1, \tilde{D}(a, 0, c), C_1^{-2} \). A construction similar to that in §2.3 yields the required arrangements (in Figure 3(b) we choose a narrow parabola passing through one of the ovals).

2.5. The series \( \tilde{C}_2 \) and \( C_3^{-2} \). The curve \((y^2 - xz)(y^2 - 2xz)(y^2 - 3xz) + x^5z = 0\) in projective coordinates \((x : y : z)\) is arranged with respect to the coordinate axes as in Figure 4. It has the singularity \( E_8 \) at \((0 : 0 : 1)\) and a tangency point of three local branches at \((0 : 1 : 0)\). We choose a line \( L \) and a point \( p \) as in Figure 4. Then, by perturbing the singularities (see [13, §4]) and by rotating \( L \) around \( p \) (see Figure 4), we obtain all the arrangements \( \tilde{C}_2 \) and \( C_3^{-2} \) listed in Theorem 1.1.

2.6. The arrangements \( \tilde{D}(a, b, c) \) with \( b \neq 0 \).

A construction of \( \tilde{D}(1, 7, 1) \) and \( \tilde{D}(1, 3, 5) \) is presented in Figure 5. Namely, using the Viro’s patchwork shown on the left picture, we obtain the arrangement of a singular quintic curve with respect to coordinate axes shown in the middle picture (the signs of the vertices are represented by colors (black or white) and the tiling is assumed to be subdivided up to a primitive triangulation). By a perturbation of the singular point we obtain the required arrangements. The existence of such perturbations is proven e.g. in [4] or in [13, §4.4].

A construction of \( \tilde{D}(0, 4, 5) \) is shown in Figure 6. We start with a cuspidal cubic \( C_3 \) and three lines in Figure 6(a). Then we perturb \( L_2 \) to a line \( L \) cutting \( C_3 \) at three
real point, and define a conic \( C_2 \) as \( L_0^2 + \varepsilon L_1 L \), \( |\varepsilon| \ll 1 \). The resulting arrangement is shown in Figure 6(b). By perturbing the cusp we obtain Figure 6(c). Next we choose the coordinates \((x : y : z)\) as in Figure 6(c) and apply the hyperbolism (see [13, §4.5]) \( h : (x : y : z) \mapsto (\hat{x} : \hat{y} : \hat{z}) = (xy : x^2 : yz) \). Then \( C_3 \) and \( C_2 \) are transformed into a quintic curve and a line respectively which are arranged with respect to the axis \( \hat{y} = 0 \) as in Figure 6(d). By perturbing the singularities as in Figure 6(e), we obtain \( \tilde{D}(0, 4, 5) \).

In Figure 7, for the reader’s convenience, we show how the hyperbolism transforming Figure 6(c) into Figure 6(d) decomposes into three blowups and three blowdowns. In Figure 7(a) we depict the arrangement of Figure 6(c) where \( \mathbb{RP}^2 \) is represented by a disk with opposite boundary points identified, and the boundary of this disk represents the line \( L \). Figure 7(b) is obtained by blowing up \( q \) and its infinitely near point on \( L \) followed by blowing down the strict transform of \( L \). The resulting surface is the Hirzebruch surface of the second order whose real locus is a torus represented by a rectangle with the opposite sides identified; the horizontal sides corresponding to the \((-2)\)-curve and the vertical sides to a fiber. In Figure 7(c) we show the same as in Figure 7(b) but the surface is cut along another fiber. Finally, to obtain Figure 7(d), we blow up \( p \) and then blow down the two curves represented by the sides of the rectangle (this is the transformation inverse to the one which transforms Figure 7(a) to Figure 7(b)).

3. Construction of pseudoholomorphic curves

In this section we prove the construction part of Theorem 1.2, namely, we pseudoholomorphically realize the six arrangements (1) and (2).
3.1. **Construction of (2).** The affine $M$-sextic $C_2(1, 3, 6)$ (see Figure 8) is pseudoholomorphically realized in [6, §7.2]. By rotating the line around the point shown in Figure 8 we obtain all the four arrangements in (2). Indeed, when this line is rotated clockwise, it cannot meet the interior oval while $c > 0$ (this follows from Bézout Theorem for an auxiliary line).

3.2. **Construction of (1).** In [6; §7.2], we show that the braids $b_k$, $k = 1, 2$, corresponding to the $L_p$-schemes $[x_3 x_4 x_3 x_2 \supset 3 \alpha_8^{(k)} \times 2 \alpha_3 \subset 3]$ (in the notation from [6; §3.5]) with $e_8^{(1)} = \alpha_3^2 \subset 3 \supset 4 \alpha_3$ and $e_8^{(2)} = \alpha_4 \subset 3 \supset 4 \alpha_4^2$ are quasipositive (note also that $b_1 = b_2$ and that these $L_p$-schemes represent the arrangements $B_2(1, 8, 1)$ and $B_2(5, 4, 1)$ respectively, the former one being algebraically realizable [5] but not the latter one [1]). The $L_p$-schemes $[x_3 x_4 x_3 x_2 \supset 3 \supset 4 \alpha_3 \subset 3 \alpha_8^{(k)} \times 2 \subset 3]$, $k = 1, 2$, represent the arrangements $\tilde{A}_3(1, 8, 0)$ and $\tilde{A}_3(5, 4, 0)$. It is easy to see that the corresponding braids $b_k'$ are conjugate to $b_k$, namely, $b_k' = \sigma_4^{-1} b_k \sigma_4$. Hence, $b_k'$ are also quasipositive whence the pseudoholomorphic realizability of the corresponding arrangements.

3.3. **A more geometric construction of (1).** In Figure 9, an algebraic realization of $A_2(8, 1, 1)$ (see Figure 8) is shown (G. M. Polotovskii, a private communication; see Remark 1.5).
A slight modification of this construction yields a pseudoholomorphic realization of $\tilde{A}_3(1,8,0)$, see Figure 10(a). Similarly one obtains a real pseudoholomorphic cubic arranged with respect to a conic and two lines as in Figure 10(b). Here the cubic and the conic are tangent to one of the lines at the same point. By perturbing this arrangement one can obtain both $\tilde{A}_3(1,8,0)$ and $A_3(1,4,4)$.

![Figure 10. Pseudoholomorphic realization of $\tilde{A}_3(1,8,0)$ and $A_3(1,4,4)$](image)

Notice that the arrangement in Figure 10(b) is algebraically unrealizable because if it were, it could be algebraically perturbed into a quintic curve arranged with respect to two lines as in Figure 10(c) which is impossible according to [8, §4.1]. However, for the arrangements (1) and the one in Figure 10(a), it is still unknown whether they are algebraically realizable.

Notice also that using computations similar to those in §4 or in §5.2, Step 2, one can check that any nodal degeneration of the arrangements (1) can be obtained as a perturbation of Figure 10(b). This fact gives a hope to prove the algebraic unrealizability of (1) by some variation of the Hilbert-Rohn-Gudkov method.

## 4. Prohibitions of pseudoholomorphic curves

In this section we prove the prohibition part of Theorem 1.2. The following fact is well-known and it immediately follows from the Bézout Theorem applied to an auxiliary conic.

**Lemma 4.1.** An $M$-sextic cannot contain ovals arranged with respect to some lines as in Figure 11 (a), (b), or (c).

![Figure 11. Impossible subsets of $M$-sextics](image)

### 4.1. The series $\tilde{A}_2$

Suppose that $(C_6,C_1)$ is a pseudoholomorphic realization of $\tilde{A}_2(a,b,c)$. Let $v$ be the oval of $C_6$ which meets $C_1$ at two points, and let $p$ be a point on $C_1$ inside $v$. All admissible arrangements $\tilde{A}_2(a,b,c)$ with $c > 0$ and $ab = 0$ are realized in §2, therefore we shall assume that $c = 0$ or $ab \neq 0$. This condition combined with Lemma 4.1(b) implies that the $L_p$-scheme of $C_6$ is
\[ [\Omega^3 \subset \Omega^2 \subset \Omega] \]. Hence the formula of complex orientations implies that \( a \) is even and \( b \) is odd. This observation excludes all the arrangements in question except \( \tilde{A}_2(a, 9 - a, 0) \) with \( a = 0, 4, 6 \).

According to [6, 7], the \( L_p \)-scheme \( [\Omega^3 \subset \Omega^2 \subset \Omega^9] \) is pseudoholomorphically realizable if and only if there exists \( e \in \mathbb{Z} \) such that the braid

\[
\beta_a(e) = \sigma^{-a} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \Delta 
\]

is quasipositive. By Murasugi-Tristram inequality, a necessary condition for this is the vanishing of \( \det \beta_a(e) \). Using the program \( \text{ssmW} \) (see Appendix in [7]) we find that \( \pm \det \beta_a(e) = 16(a - 2)(a - 8) \) (for some mysterious reason it does not depend on \( e \)). Hence \( \tilde{A}_2(a, 9 - a, 0) \) is unrealizable unless \( a \in \{2, 8\} \).

**4.2. The series \( \tilde{A}_3 \).** We have to prohibit \( \tilde{A}_3(2, 3, 4) \) and the four arrangements \( \tilde{A}_3(a, 9 - a, 0), a = 0, 2, 3, 4 \). Suppose that \((C_6, C_1)\) is a pseudoholomorphic realization of \( \tilde{A}_3(a, b, c) \). Let \( v \) be the oval of \( C_6 \) which meets \( C_1 \) at four points. We choose a point \( p \) on a segment of \( RC_1 \) which is exterior to \( v \), has its endpoints on \( v \), and does not have other intersections with \( C_6 \).

Then the \( L_p \)-scheme of \( C_6 \) is \([\Omega^3 \subset \Omega^2 \subset \Omega^9] \). A priori an \( L_p \)-scheme realizing \( \tilde{A}(a, b, c) \) could contain some \( o_1 \) or \( o_2 \) occurring between the ovals of the group \( \Omega^2 \) or \( \Omega^9 \).

However, this is impossible by Lemma 4.1(b). By Bézout Theorem for an auxiliary line, it is also impossible that some part of \( \Omega^2 \) or \( \Omega^9 \) is replaced by \( \subset \Omega^2 \... \Omega^9 \) or by \( \subset \Omega^2 \... \Omega^9 \subset \Omega \) though this replacement does not change the isotopy type.

Thus \( \tilde{A}_3(a, 9 - a, 0) \) is pseudoholomorphically realizable if and only if there exists \( e \in \mathbb{Z} \) such that the braid

\[
\beta_a(e) = \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \Delta 
\]

is quasipositive. By Murasugi-Tristram inequality, a necessary condition for this is the vanishing of \( \det \beta_a(e) \). As in §4.1, using the program \( \text{ssmW} \), we find that

\[ \pm \det \beta_a(e) = 4(36 - 36a + 4a^2 - 9e + 2ae - 2e^2). \]

One easily checks that this polynomial in \( e \) does not have integer roots when \( a = 0, 2, 3, 4 \).

Similarly, \( \tilde{A}_3(2, 3, 4) \) is pseudoholomorphically realizable if and only if there exists \( e = (e_1, \ldots, e_5) \in \mathbb{Z}^5 \) such that the braid

\[ \gamma(e) = \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \left( \prod_{j=1}^5 \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \sigma^{-1} \Delta \right) \]

is quasipositive. A computation with the help of \( \text{ssmW} \) shows that \( |\det \gamma(e)| \) is a polynomial of degree 2 which is positive on \( \mathbb{R}^5 \) (the quadratic form is positive definite and the value at the minimum is positive).

**4.3. The series \( \tilde{C}_2 \) and \( C^{-2}_3 \).** We need to prove the pseudoholomorphic unrealizability of

\[ \tilde{C}_2(1, 2, 6), C^{-2}_3(1, 2, 7), \tilde{C}_2(1, 1, 7), C^{-2}_3(1, 1, 8), \tilde{C}_2(1, 0, 8), C^{-2}_3(1, 0, 9) \quad (3) \]
\[ \widetilde{C}_2(5, 0, 4), \ C_3^{-2}(5, 0, 5), \]  
\[ \widetilde{C}_2(9, 0, 0), \ C_3^{-2}(9, 0, 1). \]  

Notice that if any of (3) (resp. (4), (5)) is realizable, then \( \widetilde{C}_2(1, 2, 6) \) (resp. \( \widetilde{C}_2(5, 0, 4), \widetilde{C}_2(9, 0, 0) \)) is obtained from it by moving the line. Thus it is enough to prohibit only \( \widetilde{C}_2(1, 2, 6), \widetilde{C}_2(5, 0, 4) \) and \( \widetilde{C}_2(9, 0, 0) \).

To this end we shall use the approach from [7, §3.3]. Suppose that \((C_6, C_1)\) is a pseudoholomorphic realization of \( \widetilde{C}_2(a, b, c) \). Let us choose a point \( v \) on the line \( L \) inside that oval which meets \( C_1 \) at two points. Then the \( \mathcal{L}_p \)-scheme of our curve is \([\triangleright o_1 \ldots o_9 \subset 2]\) where \( a = \# \{ j \mid i_j = 3 \} \), \( b = \# \{ j \mid i_j = 2 \} \), and \( c = \# \{ j \mid i_j = 4 \} \). By Lemma 4.1(b) the \( a \) ovals “\( o_3 \)” are consecutive, i.e., \( i_j = \cdots = i_{j+a-1} = 3 \) for some \( j \). Moreover, by the formula of complex orientations, \( j \) is even (it follows already that \( \widetilde{C}(9, 0, 0) \) is unrealizable). By symmetry we may also assume that \( j = 2 \) for \( \widetilde{C}_2(5, 0, 4) \) and that \( j = 2 \) or \( 4 \) for \( \widetilde{C}_2(1, 2, 6) \).

So we have to consider only one case \([\triangleright o_1 o_3 o_2^3 o_2^2 \subset 2]\) for \( \widetilde{C}_2(5, 0, 4) \) and \( 2 \times \binom{8}{2} = 56 \) cases for \( \widetilde{C}_2(1, 2, 6) \). In each case we compute the Alexander polynomial of the corresponding braid and obtain a contradiction with the generalized Fox-Milnor theorem (see [7, §3.3] for details).

**4.4. The series \( \widetilde{D} \).** Suppose that \((C_6, C_1)\) is a pseudoholomorphic realization of \( \widetilde{D}(a, b, c) \). Let \( v \) be the oval of \( C_6 \) which meets \( C_1 \) at four points. We choose a point \( p \) on a segment of \( RC_1 \) which is exterior to \( v \), has its endpoints on \( v \), and does not have other intersections with \( C_6 \). Then the \( \mathcal{L}_p \)-scheme of \( C_6 \) is \([\triangleright o_1 \ldots o_9 \subset 1]\) where \( a = \# \{ j \mid i_j = 1 \} \), \( b = \# \{ j \mid i_j = 3 \} \), and \( c = \# \{ j \mid i_j = 2 \} \). By Lemma 4.1(a), the sequence \( (i_1 \ldots i_9) \) cannot contain \( 1 \ldots 3 \ldots 1 \) or \( 3 \ldots 1 \ldots 3 \). By Lemma 4.1(c), it cannot contain \( j \ldots 2 \ldots k \) or \( 2 \ldots j \ldots 2 \ldots k \) with \( j, k \neq 2 \). Thus, up to symmetry, the \( \mathcal{L}_p \)-scheme is one of

1. \([\triangleright o_1^a o_3^b o_5^c o_2^a o_3^b \subset 1]\) with \( a_1 + a_2 = a \), \( b_1 + b_2 = b \), and \( b_1 a_2 = 0 \);
2. \([\triangleright o_2^a o_3^b o_5^c o_2^c \subset 1]\) with \( c_1 + c_2 = c \).

By the generalized Fox–Milnor Theorem [7, §3.3], the determinant \( \det \beta \) of the associated braid is (up to sign) a square of an integer number. Using the program \texttt{ssmW} (see Appendix in [7]) we find that \( \pm \det \beta = 4c(1 + a + 5b - 4ab) \) in Case (i) and \( \pm \det \beta = 4c(1 + a + 5b - 4ab) - 16(a + b)c_1c_2 \) in Case (ii). Hence \( \vert \det \beta \vert \) may be a square only in the cases listed in Theorem 1.1.

**5. Prohibitions of algebraic curves**

In this section we prove Theorem 1.3. As in §4.3, it is enough to prove the algebraic unrealizability of \( \widetilde{C}_2(1, 4, 4) \) because this arrangement can be obtained from any of (2) by moving the line. The rest of this section is devoted to the proof that \( \widetilde{C}_2(1, 4, 4) \) is algebraically unrealizable.

**5.1. Self-linking number of a 4-valued function.** In this section we refine the observations from [11, Lemma 3.7] and [8, §4.1] concerning the cubic resolvent of a real polynomial of degree 4 in \( y \) whose coefficients depend on \( x \). In fact, in [11, Lemma 3.7] we used only one coefficient of the polynomial in \( y \). Therefore, we do not speak here of the cubic resolvent but only of this coefficient.
Lemma 5.1. Let $y_1, y_2, y_3, y_4$ be the roots of a polynomial $P(y) = y^4 + a_2 y^2 + a_1 y + a_0$ with complex coefficients. Then $a_1 = 0$ if and only if $y_1, y_2, y_3, y_4$ are at the vertices of a parallelogram (maybe degenerated), i.e., $y_1 + y_2 = y_3 + y_4 = 0$ up to permutation of $y_1, y_2, y_3, y_4$. This condition is also equivalent to the fact that 0 is root of the cubic resolvent of $P$.

Proof. Since the coefficient of $y^3$ is zero, we have $y_4 = -y_1 - y_2 - y_3$. Plugging this into $a_1 = -(y_1 y_2 y_3 + \ldots)$ we obtain $a_1 = (y_1 + y_2)(y_1 + y_3)(y_2 + y_3)$. □

Lemma 5.2. Let $y_1, y_2, y_3, y_4$ be the roots of a polynomial $y^4 + a_2 y^2 + a_1 y + a_0$ with real coefficients. Then the sign of $a_1$ depends on the mutual position of the roots as shown in Figure 12. More precisely:

a). If $y_1, \ldots, y_4$ are real and $y_1 \leq y_2 \leq y_3 \leq y_4$, then
\[ \text{sign } a_1 = \text{sign } ((y_2 + y_3) - (y_1 + y_4)). \]

b). If $y_1$ and $y_2$ are real, and $y_3 = \bar{y}_4$, then
\[ \text{sign } a_1 = \text{sign } ((y_3 + y_4) - (y_1 + y_2)) = \text{sign } (2 \text{Re } y_3) - (y_1 + y_2)). \]

c). If $y_1 = \bar{y}_2$, $y_3 = \bar{y}_4$, Im $y_1 \geq 0$, and Im $y_3 \geq 0$, then
\[ \text{sign } a_1 = \text{sign } (\text{Re}(y_3 - y_1) \cdot \text{Im}(y_3 - y_1)). \]

\[ \text{Figure 12. Dependence of sign } a_1 \text{ on the roots.} \]

Proof. In [8, Lemma 4.2] we gave a proof of (a) based on some elementary computations. Of course, (b) and (c) can be proven similarly, but we shall give another proof for all the statements (a)–(c) which does not require any computation.

By Lemma 5.1, in each case (a), (b), (c), the right hand side of the equality vanishes if and only if $a_1 = 0$. Let us consider then the case when the right hand sides are positive. Let $A$ (resp. $B, C$) be the set of polynomials such that the right hand side of the equality (a) (resp. (b), (c)) is positive. It is easy to check that these sets are connected and the intersections $A \cap B$ and $B \cap C$ are non-empty. Indeed, the polynomial with roots $(-2, 0, 1, 1)$ belongs to $A \cap B$ and the polynomial with roots $(-1, -1, 1 \pm i)$ belongs to $B \cap C$. Since $a_1$ does not vanish on $A \cup B \cup C$, its sign is constant on this set. So, it is enough to look at the sign of $a_1$ for any element of $A$, for example, for $(x - 1)^2 x(x + 2) = x^4 - 3x^2 + 2x$. □

Corollary 5.3. In the setting of Lemma 5.2(a), if $y_1 < y_2 \leq y_3 = y_4$, then $a_1 > 0$, and if $y_1 = y_2 \leq y_3 < y_4$, then $a_1 < 0$. □
**Definition 5.4.** (Cf. [11, Def. 3.6].) Let \( y = f(x) \) be a real 4-valued algebraic function which has no poles on a segment \([x_1, x_2]\). Suppose that

1. At each of the points \( x_1 \) and \( x_2 \), two of the analytic branches of \( f \) have a simple branching and the other two branches are non-singular;
2. The values of \( f \) are distinct and non-real at any \( x \in ]x_1, x_2[ \).

Let \( x_0 = (x_1 + x_2)/2 \) and let \( f_j^{\text{sing}} \) and \( f_j^{\text{reg}} \), \( j = 1, 2 \), be the branches of \( f \) on \([x_j, x_0]\) whose imaginary parts are positive and such that \( f_j^{\text{sing}} \) is branched at \( x_j \) but \( f_j^{\text{reg}} \) is not. Let \( V = \mathbb{R} \times \mathbb{C} = \{(x, y) \mid \text{Im} x = 0 \} \) and let \( S_+ \subset V \) be the union of the graphs of \( f_j^{\text{sing}} \) and \( f_j^{\text{reg}} \), \( j = 1, 2 \), with the four segments

\[ [(x'_j, 0), (x_j, f_j^{\text{reg}}(x_j))], \quad [(x_j, 0), (x_j, f_j^{\text{sing}}(x_j))], \quad j = 1, 2, \]

where \( x'_1 < x_1 < x_2 < x'_2 \). Let \( S = S_+ \cup r(S_+) \) where \( r \) is the rotation of \( V \) by \( 180^\circ \) around the axis \( y = 0 \). We endow \( S^+ \) with the orientation induced by the projection onto the segment \([x'_1, x'_2]\), and we extend this orientation to the whole \( S \). Then \( S \) is the union of two disjoint oriented closed curves. Their linking number is called the self-linking number of \( f \) on \([x_1, x_2]\).

**Lemma 5.5.** (cf. [11, Lemma 3.7]). Let \( f(x) \) be a 4-valued algebraic function implicitly defined by the equation \( y^4 + a_2(x)y^2 + a_1(x)y + a_0(x) = 0 \) where \( a_0, a_1, a_2 \) are polynomials with real coefficients. Suppose that \([x_1, x_2]\) satisfies the conditions (1)–(2) of Definition 5.4 and let \( k \) be the self-linking number of \( f \) on \([x_1, x_2]\). Suppose that \( k \neq 0 \). Let \( x'_1 < x_1 < x_2 < x'_2 \) and \( a_1(x'_j) \neq 0 \), \( j = 1, 2 \). Then \( a_1 \) has at least \( |2k + (\varepsilon_1 - \varepsilon_2)/2| \) real roots on the segment \([x'_1, x'_2]\) where \( \varepsilon_j = \text{sign} a_1(x'_j) \).

**Proof.** We consider only the case when \( k > 0 \) and the image of \( S^+ \) under the plane projection \( \pi : \mathbb{R} \times \mathbb{C} \to \mathbb{R}^2 \), \((x, y) \mapsto (x, \text{Im} y)\), has exactly \( k \) self-crossings. Other cases can be easily reduced to this one.

By Lemma 5.2(c), in this case we have \( k \) roots of \( a_1(x) \) at the \( x \)-coordinates of the self-crossings of \( \pi(S^+) \) and \( k - 1 \) roots between each pair of the consecutive self-crossings.

Let \( x''_j \) (resp. \( x''_j \)) be the \( x \)-coordinate of the first (resp. the last) self-crossing. Then, for \( 0 < \delta \ll 1 \), we have \( a_1(x''_1 - \delta) < 0 \) and \( a_1(x''_2 + \delta) > 0 \) (see Lemma 5.2(c) and Figure 13). Therefore we have at least \((1 + \varepsilon_1)/2\) roots of \( a_1 \) on \([x'_1, x''_1 - \delta]\) and at least \((1 - \varepsilon_2)/2\) roots of \( a_1 \) on \([x''_2 + \delta, x'_2]\), thus at least

\[ k + (k - 1) + (1 + \varepsilon_1)/2 + (1 - \varepsilon_2)/2 = 2k + (\varepsilon_1 - \varepsilon_2)/2 \]

roots on \([x'_1, x'_2] \). \( \Box \)

### 5.2. Algebraic unrealizability of \( \tilde{C}_2(1, 4, 4) \).

The proof is similar to the proofs of the algebraic unrealizability of the maximal arrangements (affine \( M \)-sextics) \( A_4(1, 5, 4) \) and \( C_2(1, 3, 6) \) in [10,11].

**Step 1.** Arrangements of the pseudo-holomorphic curves with respect to an auxiliary pencil of lines. Suppose that \( C_6 \) is a pseudo-holomorphic \( M \)-sextic and \( C_1 \) a line which have the mutual arrangement \( \tilde{C}_2(1, 4, 4) \). Let us choose \( p \) as in §4.3. Computing, also like in §4.3, the Alexander polynomials for all a priori possible
sequences \((i_1, \ldots, i_9)\) (there are 140 of them), we obtain that only two \(\mathcal{L}_p\)-schemes do not contradict the generalized Fox-Milnor theorem:

\[
\left[ \varnothing_2 o_3 o_4^4 o_2 \subset 2 \right] \tag{6}
\]

\[
\left[ \varnothing_2 o_2 o_3 o_4^4 o_2 \subset 2 \right] \tag{7}
\]

The \(\mathcal{L}_p\)-scheme (6) is realized by the pseudo-holomorphic curves constructed in §3.1. I do not know whether (7) is pseudoholomorphically realizable.

**Step 2.** Essential nodal degenerations of \(\mathcal{L}_p\)-schemes (6) and (7). Let us say that a nodal degeneration of an \(\mathcal{L}_p\)-scheme is **essential** if it changes the corresponding braid, i.e. it is not a degeneration of the form \(\varnothing_i \subset i \rightarrow \times_i\) or a contraction of an empty oval to a solitary node but a symbol \(\times_i\) is inserted somewhere into the encoding word.

Suppose that \(C_6'\) is a pseudo-holomorphic curve whose \(\mathcal{L}_p\)-scheme is obtained from (6) or (7) by an essential nodal degeneration. By Murasugi-Tristram inequality, the Alexander polynomial of the corresponding braid must identically vanish. The only essential nodal degenerations which satisfy this condition, are

\[
\left[ \varnothing_2 o_3 o_4^4 o_2 \subset 2 \right] \rightarrow \left[ \varnothing_2 o_3 o_4^3 o_4 \subset 4 \times 5 \varnothing_4 o_2 \subset 2 \right] \text{ or } \left[ \varnothing_2 o_3 o_4^3 o_4 \subset 2 \times 1 \varnothing_2 \subset 2 \right]
\]

for (6). In particular, (7) does not admit any essential nodal degeneration.

In fact, in all the cases except four (two cases for each of (6) and (7)) it is not necessary to compute the Alexander polynomial because already the determinant of the braid does not vanish.

**Step 3.** **Application of Hilbert-Rohn-Gudkov method.** Now, let us suppose that \(C_6\) and \(C_1\) are real algebraic. Let us choose an equation \(f_6 = 0\) of \(C_6\) so that \(f_6 < 0\) in the non-orientable component of \(\mathbb{RP}^2 \setminus \mathbb{R}C_6\).

Let us deform \(C_6\) in the pencil \(f_6 + t g_3^2 = 0\) for a generic cubic polynomial \(g_3\). Then (see details in [10]) \(C_6\) must degenerate into a nodal curve. Choosing another pencil of this form but with a generic \(g_3\) vanishing at the node, we further degenerate the sextic and obtain one more node. Continuing this process, we obtain a sextic \(C_6'\) with 10 nodes. By the result of Step 2, the arrangement of \(C_6'\) and \(C_1\) is the one depicted in Figures 14. By the genus formula, \(C_6'\) is rational.
Now, let us consider the equisingular deformation of $C'_6$ such that all the nodes are fixed except those that are adjacent to the shadowed digon. One can compute that this is a one-parameter family which is smooth at $C'_6$. Moving in this family in the direction such that the shadowed digon shrinks, we degenerately $C'_6$ into a curve $C''_6$ which has a singular point $q$ of the type $A_3$ (a point of tangency of two smooth branches) instead of the shadowed digon. Rotating $C_1$ around one of the two middle intersection points $p$, we obtain a line $C''_1$ through $q$ such that the right four intersection points of $C''_1$ and $C''_6$ are still arranged as in Figure 14.

**Step 4. Reduction to a 4-valued function.** Let $T$ be the tangent to $C''_6$ at $q$. In the same way as in [11; Lemmas 3.11 and 3.13(b)], we can prove that $C''_6$ is arranged with respect to $T$ and $C''_1$ as in Figure 15 up to isotopy (the rectangular pattern may be replaced as it is shown).

Let us blow up twice the point $q$ and then blow down the proper transform of $T$. Let us denote the exceptional curves of the blowups by $E_1$ and $E_2$ ($E_2$ is the transform of the intersection point of $E_1$ with the proper transform of $C''_6$).

We obtain a curve $C_4$ of bidegree $(4, 8)$ on the Hirzebruch surface $F_2$ (the quadratic cone blown up at the vertex). Let us denote the proper transforms on $F_2$ of the curves $C''_1$, $E_1$, and $E_2$ by $F_1$, $E$, and $F_2$ respectively. Then $F_1$ and $F_2$ are fibers and $E$ is the exceptional section of the fibration $\pi : F_2 \to \mathbb{P}^1$.

In a standard coordinate system of $F_2$, the curve $C_4$ is defined by an equation

$$y^4 + a_3(x)y^3 + a_2(x)y^2 + a_1(x)y + a_0(x)$$

where $a_m$ is a polynomial in $x$ and

$$\text{deg } a_m(x) \leq 2(4 - m), \quad m = 0, 1, 2, 3.$$  \hspace{1cm} (8)

By the standard trick, we may kill the coefficient $a_3(x)$.

The fact that $C''_6 \cup C''_1$ is as in Figure 14, implies that the arrangement of $C_4$ with respect to $F_1$, $F_2$, and $E$ is as in Figure 16 where $\mathbb{R}F_2$ is depicted as a
rectangle whose opposite sides are identified (note that $\mathbb{R}\mathcal{F}_2$ is a torus). Hence, the arrangement of $C_4$ with respect to the fibers of $\pi$ is $[w_1 \subset_1 = 1 2 \subset_3 \times_3 w_2 \subset_3 \times_4 \subset_4 4]$ where $w_2$ is one of $[x_1 \supset_2]$ or $[x_2 \supset_1]$, and possible values of $w_1$ are listed in Table 2.

**Step 5. Application of the cubic resolvent or counting the roots of $a_1(x)$**. For each $w_1$ we compute the self-linking number $k$ (see Definition 5.4) as it is done in [11, Lemma 3.13(a)]. The results are given in Table 2. The column $d$ contains the lower bound for $\deg a_1(x)$ provided by Lemma 5.5 and Corollary 5.3. We see that in all the cases, this bound contradicts (8).

<table>
<thead>
<tr>
<th>no.</th>
<th>$w_1$</th>
<th>$k$</th>
<th>$d$</th>
<th>no.</th>
<th>$w_1$</th>
<th>$k$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x_3 \times_3 \supset_1 \supset_1$</td>
<td>-4</td>
<td>10</td>
<td>5</td>
<td>$x_1 \times_3 \supset_2 \supset_1$</td>
<td>-3</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>$x_3 \times_2 \supset_3 \supset_1$</td>
<td>-4</td>
<td>10</td>
<td>6</td>
<td>$x_1 \times_2 \supset_3 \supset_1$</td>
<td>-3</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>$x_3 \times_2 \supset_1 \supset_1$</td>
<td>-3</td>
<td>10</td>
<td>7</td>
<td>$x_1 \times_2 \supset_1 \supset_1$</td>
<td>-2</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>$x_3 \times_1 \supset_2 \supset_1$</td>
<td>-3</td>
<td>10</td>
<td>8</td>
<td>$x_1 \times_1 \supset_2 \supset_1$</td>
<td>-2</td>
<td>8</td>
</tr>
</tbody>
</table>

We shall show in details how the bound $\deg a_1(x) \geq 8$ is obtained for the last line of Table 2. The other cases are similar. Let us choose the points $x_1', x_1, x_2, x_2', x_3, x_4$ on $\mathbb{R}P^1$ as in Figures 17.1 (for $w_2 = x_1 \supset_2$) or in Figure 17.2 (for $w_2 = x_2 \supset_1$). In the both cases we have $a_1(x_1') < 0$, $a_1(x_2') > 0$, $a_1(x_3) < 0$, and $a_1(x_4) > 0$ by Corollary 5.3. Hence $a_1$ has at least one root on each of the intervals $[x_2', x_3]$, $[x_3, x_4]$, $[x_4, x_1']$, and by Lemma 5.5, it has at least $|2 \cdot (-2) + (-1 - 1)/2| = 5$ roots on $[x_1', x_2']$. Thus, $\deg a_1 \geq 1 + 1 + 1 + 5 = 8$.

**Figure 17.1**

**Figure 17.2**

### References


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