ARRANGEMENTS OF A PLANE M-SEXTIC WITH RESPECT TO A LINE

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Abstract. The mutual arrangements of a real algebraic or real pseudoholomorphic plane projective M-sextic and a line up to isotopy are studied. A complete list of the pseudoholomorphic arrangements is obtained. Four of them are proven to be algebraically unrealizable. All the others with two exceptions are algebraically realized.

By a real algebraic curve in $\mathbb{RP}^2$ we mean a complex algebraic curve in $\mathbb{CP}^2$ invariant under the complex conjugation. Given such a curve C, we denote the set of its real points by $\mathbb{R}C$. In this paper we study mutual arrangements in $\mathbb{RP}^2$ of $\mathbb{R}C_6$ and $\mathbb{R}C_1$ where C_6 is a real algebraic M-curve of degree 6 (it has 11 ovals) and C_1 is a real line transverse to C_6. We consider such arrangements up to isotopy in $\mathbb{RP}^2$. We study also the same problem for *real pseudoholomorphic curves* (see [9] and references therein).

In the case when $C_6 \cap C_1$ is contained in a single oval of $\mathbb{R}C_6$, the number of connected components of $\mathbb{R}C_6 \setminus C_1$ is maximal, so, we say in this case that the pair (C_6, C_1) realizes a *maximal arrangement* and $C_6 \setminus C_1$ is an *affine M-sextic*. A complete classification of the maximal arrangements is already done (see [11], [1]): there are exactly 38 pseudoholomorphically realizable arrangements and only 35 of them are algebraically realizable.

An algebraic classification (with a single exception) in the non-maximal case was announced by E. I. Shustin in [12]. However the proofs are not given there (only the method is described) and, moreover, the proofs of the algebraic unrealizability of at least four arrangements are certainly erroneous because these arrangements are pseudoholomorphically realizable whereas the techniques used in the proofs cannot distinguish between the algebraic and pseudoholomorphic realizability.

In the present paper we give a complete list of the pseudoholomorphically realizable non-maximal arrangements. We prove that four of them are algebraically unrealizable; the algebraic realizability of two more of them is open, and all the others are realizable by real algebraic curves.

For plane projective M-sextics, the algebraic and pseudoholomorphic isotopy classifications coincide. There are three isotopy types: $9 \sqcup 1\langle 1\rangle$, $5 \sqcup 1\langle 5\rangle$, and $1 \sqcup 1\langle 9\rangle$ in Viro's notation [13]. Any pseudoholomorphically realizable non-maximal arrangement of an M-sextic and a line belongs to one of the series shown in Figure 1 where the numbers a, b, and c are the numbers of unnested ovals in the corresponding regions. This fact easily follows from Bézout Theorem applied to C_6 and an auxiliary line through some two ovals (and an auxiliary conic for the series E). The notation for the series in Figure 1 is similar to that in [4], [12]. Arrangements in

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Figure 1 (providing one of the three isotopy types of M-sextics after removal of the line) will be called *admissible arrangements.*

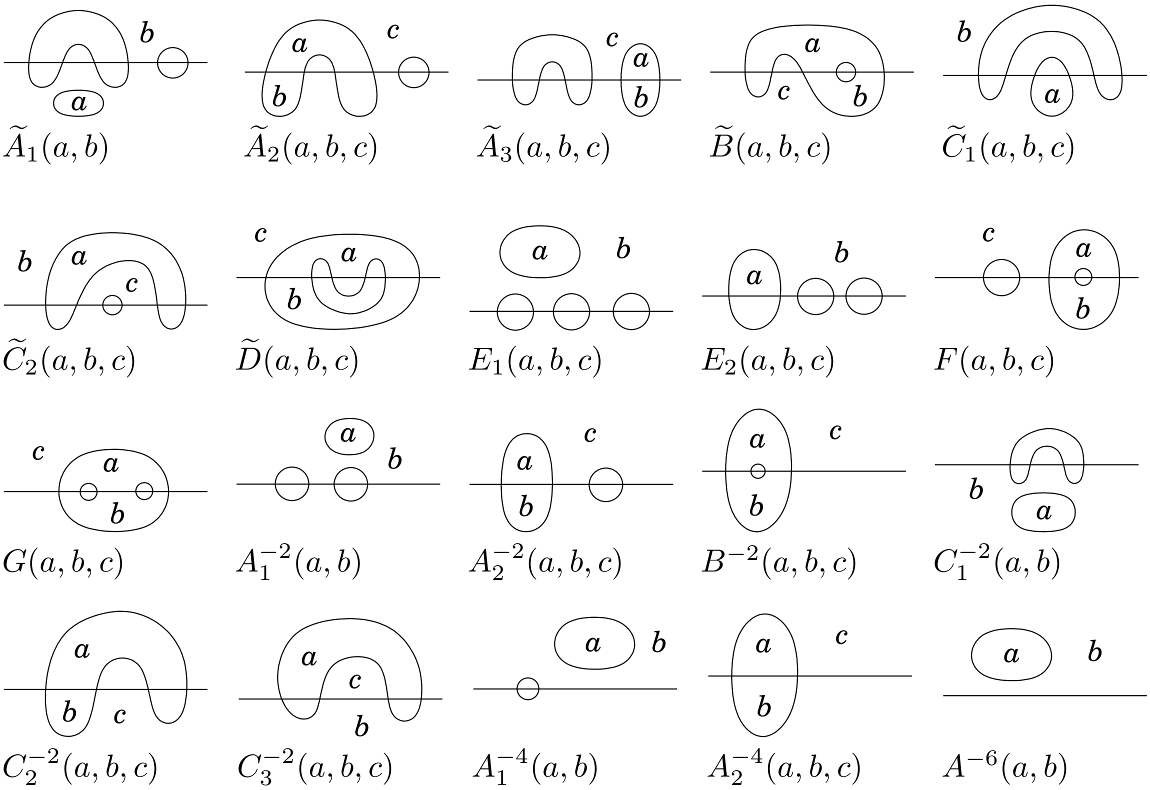

FIGURE 1. Admissible arrangements.

Theorem 1.1. *The following arrangements are realizable as* $\mathbb{R}C_6 \cup \mathbb{R}C_1$ *where* C_6 *is a real algebraic sextic curve and* C_1 *a real line in* $\mathbb{P}^2$*:*

$$
\begin{array}{ll}
\widetilde{A}_2(a,b,c), & (a,b,c) = (2,7,0),(8,1,0),(0,5,4),(2,3,4),(4,1,4),(5,0,4), \\
 & \qquad\qquad (0,1,8),(1,0,8), \\
\widetilde{A}_3(a,b,c), & (a,b,c) = (0,5,4),(0,1,8), \\
\widetilde{C}_2(a,b,c), & (a,b,c) = (1,5,3),(1,6,2),(1,7,1),(1,8,0),(5,1,3),(5,2,2), \\
 & \qquad\qquad (5,3,1),(5,4,0), \\
\widetilde{D}(a,b,c), & (a,b,c) = (1,7,1),(8,0,1),(0,4,5),(1,3,5),(4,0,5),(0,0,9), \\
C_3^{-2}(a,b,c), & (a,b,c) = (1,5,4),(1,6,3),(1,7,2),(1,8,1),(1,9,0),(5,1,4), \\
 & \qquad\qquad (5,2,3),(5,3,2),(5,4,1),(5,5,0),(9,1,0),
\end{array}
$$

and all the admissible arrangements of the other series, i.e., of the series

$$\widetilde{A}_1, \widetilde{B}, \widetilde{C}_1, E, F, G, A_1^{-2}, A_2^{-2}, B^{-2}, C_1^{-2}, C_2^{-2}, A^{-4}, A^{-6}.$$

Theorem 1.2. *Let J be a tame* conj*-anti-invariant almost complex structure on $\mathbb{CP}^2$ and let C_6 and C_1 be smooth real J-holomorphic curves of degree 6 and 1 respectively. Suppose that C_6 is an M-sextic (i.e., $\mathbb{R}C_6$ has 11 ovals), $\mathbb{R}C_1$ is transverse to $\mathbb{R}C_6$, and the arrangement (C_6, C_1) is not maximal, i.e., $C_6 \cap C_1$ is not contained in a single oval of $\mathbb{R}C_6$.*

Then the arrangement of $\mathbb{R}C_6$ with respect to $\mathbb{R}C_1$ is either as in Theorem 1.1 or one of

$$\widetilde{A}_3(1,8,0),\ \widetilde{A}_3(1,4,4), \tag{1}$$

$$\widetilde{C}_2(1,3,5),\ \widetilde{C}_2(1,4,4),\ C_3^{-2}(1,3,6),\ C_3^{-2}(1,4,5). \tag{2}$$

All these arrangements are pseudoholomorphically realizable.

Theorem 1.3. *The arrangements (2) are algebraically unrealizable.*

The algebraic realizability of (1) is open. In Table 1 we present the distribution of realizable arrangements among the series.

Table 1.

	$\widetilde{A}$	$\widetilde{B}$	$\widetilde{C}$	$\widetilde{D}$	E	F	G	A^{-2}	B^{-2}	C^{-2}	A^{-4}	A^{-6}	total
Admissible	29	15	18	15	4	9	6	11	9	36	12	3	167
Pseudohol.	14	15	13	6	4	9	6	11	9	31	12	3	133
Algebraic	12?	15	11	6	4	9	6	11	9	29	12	3	127?

Remark 1.4. It is stated in [12] that all the arrangements listed in Theorem 1.1 and also $\widetilde{A}_3(1,4,4)$ were constructed by A .B. Korchagin, G. M. Polotovskii, and E. I. Shustin, and the constructions are similar to those in [4], but they are not presented in [12]. I indeed found constructions as in [4] for the arrangements of Theorem 1.1 (see §2) but not for $\widetilde{A}_3(1,4,4)$. I doubt that the latter arrangement is algebraically realizable.

Remark 1.5. The algebraic realizability of the affine M-sextic $A_2(8,1,1)$ (see Figure 8 below) is stated in [4, Thm. 2] but its construction is forgotten to be included to that paper. In fact, the construction (that I learned from G. M. Polotovskii) is very simple and I present it in §3.3.

In §§2–4 we prove Theorems 1.1 and 1.2 using the same methods as in [4–8], [13]. In §5 we prove Theorem 1.3. The general strategy of the proof is more or less the same as in [11]: by Hilbert – Rohn – Gudkov method we reduce the problem to the algebraic unrealizability of a certain quadrigonal curve (which is pseudoholomorphically realizable) and then we exclude it using cubic resolvents. However, we evoke some new arguments in the cubic resolvent step (see §5.1).

Remark 1.6. Given an affine M-sextic of the isotopy type $C_2(1,3,6)$ (see Figure 8 in §3), by moving the line one obtains the arrangements (2). Thus Theorem 1.3 gives a new proof of the algebraic unrealizability of $C_2(1,3,6)$ whose Hilbert – Rohn – Gudkov stage is considerably simplified.

We use the approach from [6–8] for the pseudoholomorphic classification. The paper [7] can be used as a general introduction. For the reader's convenience, let us recall some terminology, notation, and main ideas. Given a fibration $\pi : E \to B$, a *fiberwise arrangement* of $X \subset E$ is the equivalence class of X with respect to isotopies $\{H_t\}$ of E such that $\pi \circ H_t = h_t \circ \pi$ for some isotopy $\{h_t\}$ of B. Given a point $p \in \mathbb{RP}^2$ and a subset X of $\mathbb{RP}^2 \setminus \{p\}$, the *$\mathcal{L}_p$-scheme* of X is a fiberwise arrangement for the linear projection $\mathbb{RP}^2 \to \mathbb{RP}^1$. When $X = \mathbb{R}C$ for a real algebraic or real pseudoholomorphic nodal curve C in general position, we encode the $\mathcal{L}_p$-scheme by a word in letters $\subset_k$, $\supset_k$, $\times_k$ which correspond to consecutive non-generic fibers. The subscript k indicates the height of the point with vertical tangent (for $\subset_k$ or $\supset_k$) or the double point (for $\times_k$) similarly to the braid notation. A subword $\subset_k\supset_k$ is abbreviated to o_k (an oval). For example, the upper $\mathcal{L}_p$ scheme in Figure 3(a) is encoded by $[\supset_2 o_2^3 \subset_2 \supset_1 o_2 \subset_1]$. An $\mathcal{L}_p$-scheme determines a braid (or a family of braids). An $\mathcal{L}_p$-scheme is pseudoholomorphically realizable if and only if the braid (at least one braid of the family) is *quasipositive*, i.e., is a product of conjugates of the standard generators of the braid group (see details in [6, 7]).

2. Construction of algebraic curves

In this section we prove Theorem 1.1. The cases A^{-4} and A^{-6} are evident, and below we present constructions for all the other series of arrangements.

2.1. The series $\widetilde{A}_1$, $\widetilde{A}_3$, E, A_1^{-2}. There exist arrangements of a quintic curve C_5 with respect to a line C_1 shown in Figure 2(left); see [3] for $a = 1$ and [2; §7.6] for $a = 5$. By perturbing $C_5 \cup C_1$ in different ways, one obtains $\widetilde{A}_3(0,5,4)$, $\widetilde{A}_3(0,1,8)$, and all the eight admissible arrangements of the series $\widetilde{A}_1$, E_1, E_2, and A_1^{-2} (see Figure 2).

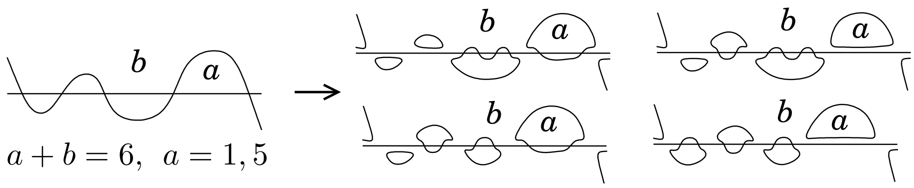

FIGURE 2. Construction of $\widetilde{A}_3(0,5,4)$, $\widetilde{A}_3(0,1,8)$, $\widetilde{A}_1(a,b)$, $E_k(a,b)$.

2.2. The series F, G, A_2^{-2}, B^{-2}. Let C_6 be an M-sextic and p be a point in one of its exterior empty ovals. Then, choosing a line C_1 passing through p, one obtains all the admissible arrangements of the series F and A_2^{-2}.

By Lemma 4.1(a), the interior ovals of C_6 lie in a convex position (if one chooses a point on each oval, the chosen points are the vertices of a convex polygon in some affine chart). Therefore, by different choices of a line C_1 passing through any given point p in an interior oval, one obtains all the admissible arrangements of the series G and B^{-2}.

2.3. The series $\widetilde{A}_2$, $\widetilde{B}$, C_2^{-2}. By [13, §4.2] there exists polynomials $F_k(x,y)$, $k = 1,2$, with Newton polygon $[(0,0),(0,3),(6,0)]$ which define affine curves arranged with respect to the vertical lines as in Figure 3(a). To these curves and their symmetric images we apply the procedure shown in Figure 3(b). Namely, we choose two translates of the same sufficiently narrow parabola and then we apply the transformation $(x,y) \mapsto (x, y + \lambda x^2)$ where λ is chosen so that the parabolas are transformed to lines. The projective closures of the obtained curves have a point of simple tangency of three smooth local branches. By perturbing this singularity (see again [13, §4.2]) we can obtain the arrangements of an M-sextic with respect to a pencil of lines as in Figure 3(c) (for example, the leftmost arrangement in Figure 3(c) corresponds the the case shown in Figure 3(b)). By choosing different lines from this pencil, we obtain all the arrangements of the series $\widetilde{A}_2$, $\widetilde{B}$, C_2^{-2} listed in Theorem 1.1.

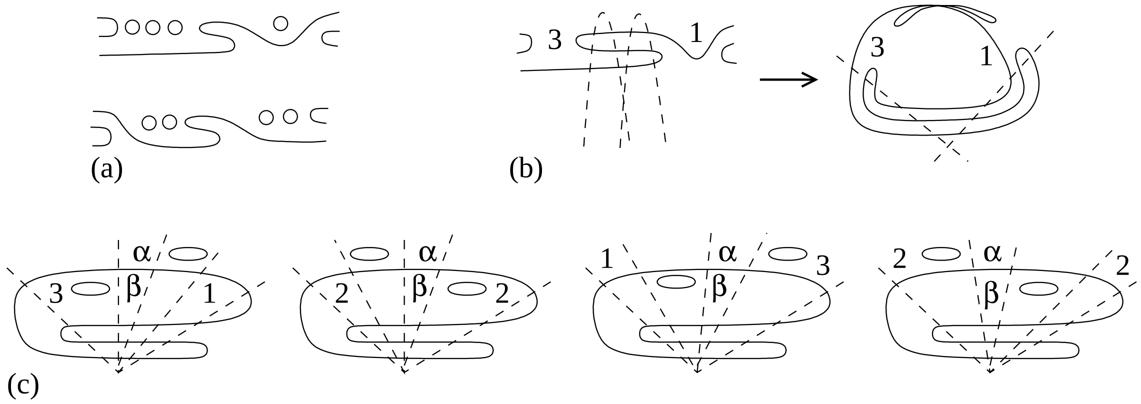

FIGURE 3. Construction of $\widetilde{A}_2$, $\widetilde{B}$, C_2^{-2}; $(\alpha,\beta) \in \{(0,4),(4,0)\}$.

2.4. The series $\widetilde{C}_1$, $\widetilde{D}(a,0,c)$, C_1^{-2}. A construction similar to that in §2.3 yields the required arrangements (in Figure 3(b) we choose a narrow parabola passing through one of the ovals).

2.5. The series $\widetilde{C}_2$ and C_3^{-2}. The curve $(y^2-xz)(y^2-2xz)(y^2-3xz)+x^5z=0$ in projective coordinates $(x:y:z)$ is arranged with respect to the coordinate axes as in Figure 4. It has the singularity E_8 at $(0:0:1)$ and a tangency point of three local branches at $(0:1:0)$. We choose a line L and a point p as in Figure 4. Then, by perturbing the singularities (see [13, §4]) and by rotating L around p (see Figure 4), we obtain all the arrangements $\widetilde{C}_2$ and C_3^{-2} listed in Theorem 1.1.

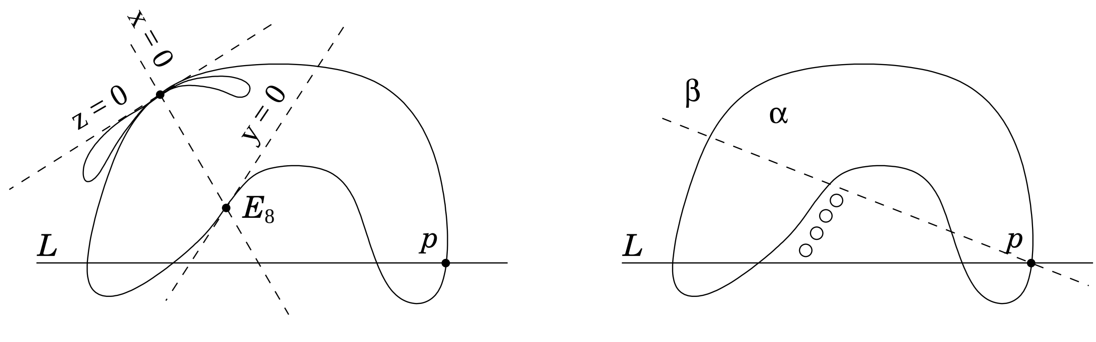

FIGURE 4. Construction of $\widetilde{C}_2$, C_3^{-2}; $(\alpha,\beta) \in \{(1,5),(5,1)\}$.

2.6. The arrangements $\widetilde{D}(a,b,c)$ with $b \neq 0$.

A construction of $\widetilde{D}(1,7,1)$ and $\widetilde{D}(1,3,5)$ is presented in Figure 5. Namely, using the Viro's patchwork shown on the left picture, we obtain the arrangement of a singular quintic curve with respect to coordinate axes shown in the middle picture (the signs of the vertices are represented by colors (black or white) and the tiling is assumed to be subdivided up to a primitive triangulation). By a perturbation of the singular point we obtain the required arrangements. The existence of such perturbations is proven e.g. in [4] or in [13, §4.4].

A construction of $\widetilde{D}(0,4,5)$ is shown in Figure 6. We start with a cuspidal cubic C_3 and three lines in Figure 6(a). Then we perturb L_2 to a line L cutting C_3 at three

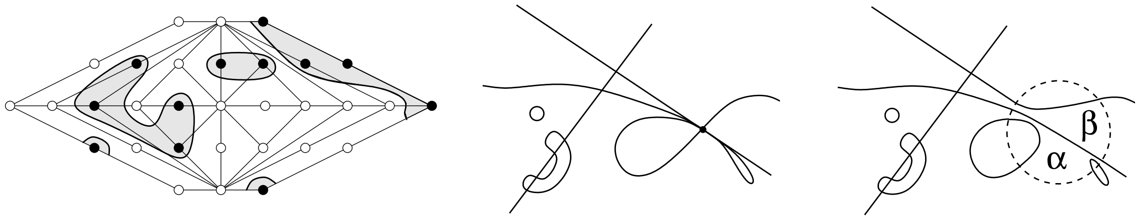

FIGURE 5. Construction of $\widetilde{D}(1,7,1)$, $\widetilde{D}(1,3,5)$; $(\alpha,\beta)\in\{(1,5),(5,1)\}$.

real point, and define a conic C_2 as $L_0^2+\varepsilon L_1 L$, $|\varepsilon|\ll 1$. The resulting arrangement is shown in Figure 6(b). By perturbing the cusp we obtain Figure 6(c). Next we choose the coordinates $(x:y:z)$ as in Figure 6(c) and apply the *hyperbolism* (see [13, §4.5]) $h:(x:y:z)\mapsto(\hat{x}:\hat{y}:\hat{z})=(xy:x^2:yz)$. Then C_3 and C_2 are transformed into a quintic curve and a line respectively which are arranges with respect to the axis $\hat{y}=0$ as in Figure 6(d). By perturbing the singularities as in Figure 6(e), we obtain $\widetilde{D}(0,4,5)$.

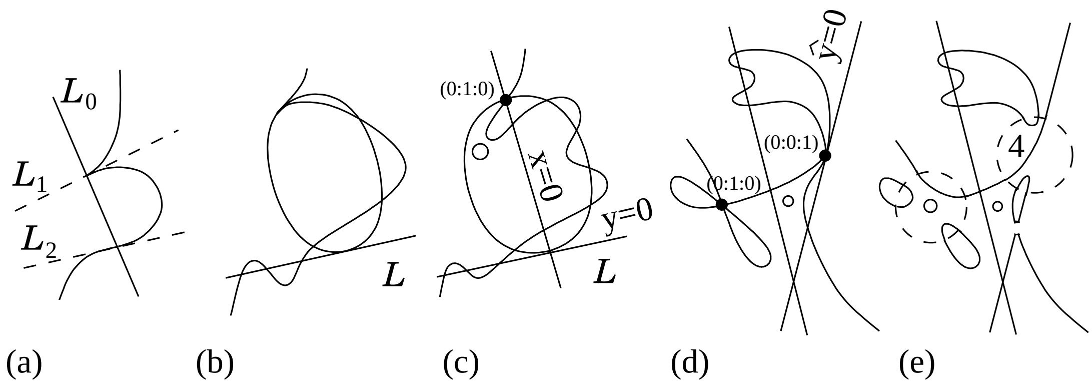

FIGURE 6. Construction of $\widetilde{D}(0,4,5)$.

In Figure 7, for the reader's convenience, we show how the hyperbolism transforming Figure 6(c) into Figure 6(d) decomposes into three blowups and three blowdowns. In Figure 7(a) we depict the arrangement of Figure 6(c) where $\mathbb{RP}^2$ is represented by a disk with opposite boundary points identified, and the boundary of this disk represents the line L. Figure 7(b) is obtained by blowing up q and its infinitely near point on L followed by blowing down the strict transform of L. The resulting surface is the Hirzebruch surface of the second order whose real locus is a torus represented by a rectangle with the opposite sides identified; the horizontal sides corresponding to the (-2)-curve and the vertical sides to a fiber. In Figure 7(c) we show the same as in Figure 7(b) but the surface is cut along another fiber. Finally, to obtain Figure 7(d), we blow up p and then blow down the two curves represented by the sides of the rectangle (this is the transformation inverse to the one which transforms Figure 7(a) to Figure 7(b)).

3. CONSTRUCTION OF PSEUDOHOLOMORPHIC CURVES

In this section we prove the construction part of Theorem 1.2, namely, we pseudoholomorphically realize the six arrangements (1) and (2).

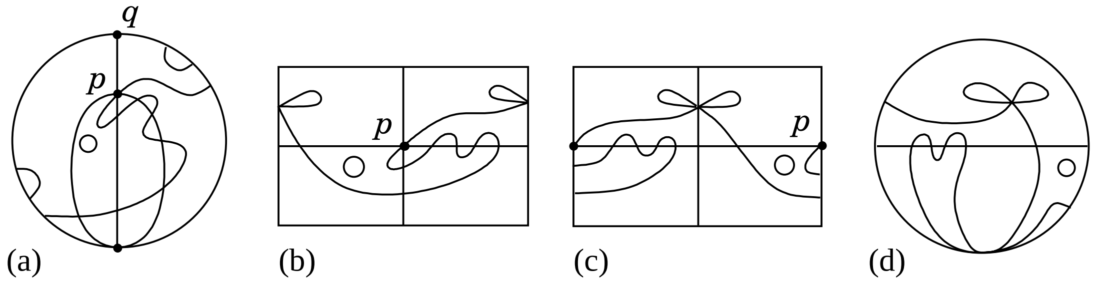

FIGURE 7. The hyperbolism transforming Fig. 6(c) to Fig. 6(d).

3.1. Construction of (2). The affine M-sextic $C_2(1,3,6)$ (see Figure 8) is pseudoholomorphically realized in [6, §7.2]. By rotating the line around the point shown in Figure 8 we obtain all the four arrangements in (2). Indeed, when this line is rotated clockwise, it cannot meet the interior oval while $c>0$ (this follows from Bézout Theorem for an auxiliary line).

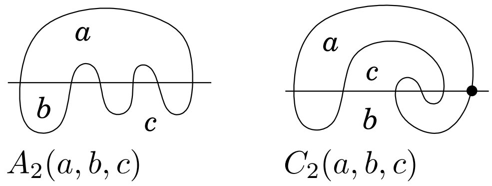

FIGURE 8. Affine M-sextics of the series A_2 and C_2.

3.2. Construction of (1). In [6; §7.2], we show that the braids b_k, $k=1,2$, corresponding to the $\mathcal{L}_p$-schemes $[\times_3\times_4\times_4\times_3\times_2\supset_3 o_3^4 e_8^{(k)}\times_2 o_3\subset_3]$ (in the notation from [6; §3.5]) with $e_8^{(1)}=o_3^3\subset_3\supset_4 o_3$ and $e_8^{(2)}=o_4\subset_3\supset_4 o_4^3$ are quasipositive (note also that $b_1=b_2$ and that these $\mathcal{L}_p$-schemes represent the arrangements $B_2(1,8,1)$ and $B_2(5,4,1)$ respectively, the former one being algebraically realizable [5] but not the latter one [1]). The $\mathcal{L}_p$-schemes $[\times_3\times_4\times_4\times_3\times_2\supset_4\subset_4\supset_3 o_3^4 e_8^{(k)}\times_2\subset_3]$, $k=1,2$, represent the arrangements $\widetilde{A}_3(1,8,0)$ and $\widetilde{A}_3(5,4,0)$. It is easy to see that the corresponding braids b'_k are conjugate to b_k, namely, $b'_k=\sigma_4^{-1}b_k\sigma_4$. Hence, b'_k are also quasipositive whence the pseudoholomorphic realizability of the corresponding arrangements.

3.3. A more geometric construction of (1). In Figure 9, an algebraic realization of $A_2(8,1,1)$ (see Figure 8) is shown (G. M. Polotovskii, a private communication; see Remark 1.5).

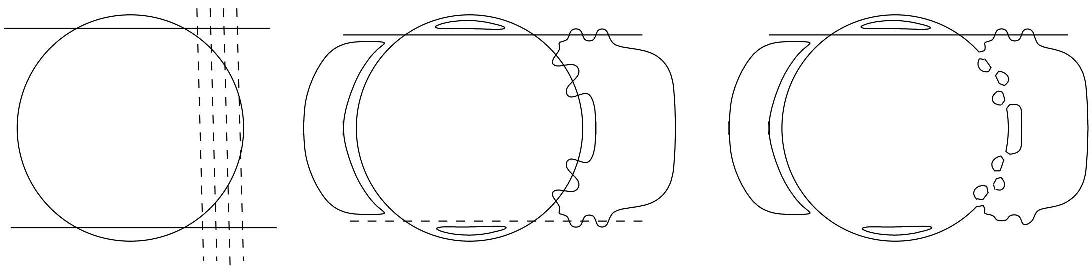

FIGURE 9. Algebraic realization of $A_2(8,1,1)$.

A slight modification of this construction yields a pseudoholomorphic realization of $\widetilde{A}_3(1,8,0)$, see Figure 10(a). Similarly one obtains a real pseudoholomorphic cubic arranged with respect to a conic and two lines as in Figure 10(b). Here the cubic and the conic are tangent to one of the lines at the same point. By perturbing this arrangement one can obtain both $\widetilde{A}_3(1,8,0)$ and $\widetilde{A}_3(1,4,4)$.

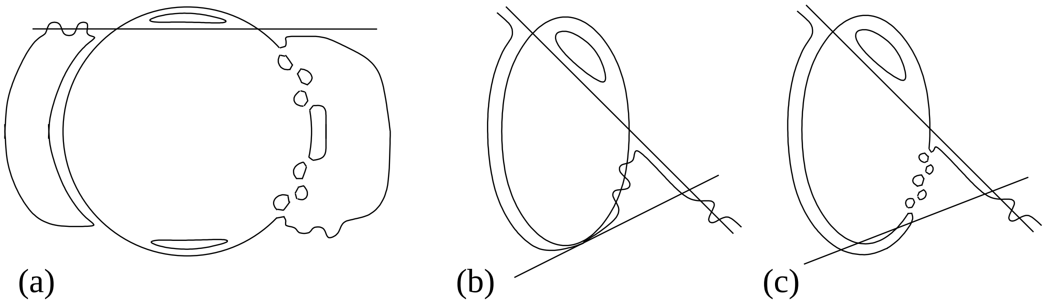

FIGURE 10. Pseudoholomorphic realization of $\widetilde{A}_3(1,8,0)$ and $\widetilde{A}_3(1,4,4)$.

Notice that the arrangement in Figure 10(b) is algebraically unrealizable because if it were, it could be algebraically perturbed into a quintic curve arranged with respect to two lines as in Figure 10(c) which is impossible according to [8, §4.1]. However, for the arrangements (1) and the one in Figure 10(a), it is still unknown whether they are algebraically realizable.

Notice also that using computations similar to those in §4 or in §5.2, Step 2, one can check that any nodal degeneration of the arrangements (1) can be obtained as a perturbation of Figure 10(b). This fact gives a hope to prove the algebraic unrealizability of (1) by some variation of the Hilbert-Rohn-Gudkov method.

4. PROHIBITIONS OF PSEUDOHOLOMORPHIC CURVES

In this section we prove the prohibition part of Theorem 1.2. The following fact is well-known and it immediately follows from the Bézout Theorem applied to an auxiliary conic.

Lemma 4.1. *An M-sextic cannot contain ovals arranged with respect to some lines as in Figure 11 (a), (b), or (c).*

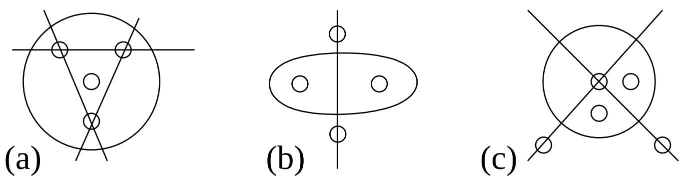

FIGURE 11. Impossible subsets of M-sextics.

4.1. The series $\widetilde{A}_2$. Suppose that (C_6, C_1) is a pseudoholomorphic realization of $\widetilde{A}_2(a,b,c)$. Let v be the oval of C_6 which meets C_1 at two points, and let p be a point on C_1 inside v. All admissible arrangements $\widetilde{A}_2(a,b,c)$ with $c > 0$ and $ab = 0$ are realized in §2, therefore we shall assume that $c = 0$ or $ab \neq 0$. This condition combined with Lemma 4.1(b) implies that the $\mathcal{L}_p$-scheme of C_6 is

$[\supset_3 o_3^a \supset_2 o_2^c \subset_2 o_3^b \subset_2]$. Hence the formula of complex orientations implies that a is even and b is odd. This observation excludes all the arrangements in question except $\widetilde{A}_2(a, 9-a, 0)$ with $a = 0, 4, 6$.

According to [6, 7], the $\mathcal{L}_p$-scheme $[\supset_2 o_3^a \supset_2 \subset_2 o_3^{9-a} \subset_2]$ is pseudoholomorphically realizable if and only if there exists $e \in \mathbb{Z}$ such that the braid

$$\beta_a(e) = \sigma_3^{-a-1}\sigma_2^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_4^{1+e}\sigma_2^{1-e}\sigma_3^{a-9}\sigma_2^{-1}\sigma_3\Delta$$

is quasipositive. By Murasugi-Tristram inequality, a necessary condition for this is the vanishing of $\det \beta_a(e)$. Using the program `ssmW` (see Appendix in [7]) we find that $\pm \det \beta_a(e) = 16(a-2)(a-8)$ (for some mysterious reason it does not depend on e). Hence $\widetilde{A}_2(a, 9-a, 0)$ is unrealizable unless $a \in \{2, 8\}$.

4.2. The series $\widetilde{A}_3$. We have to prohibit $\widetilde{A}_3(2,3,4)$ and the four arrangements $\widetilde{A}_3(a, 9-a, 0)$, $a = 0, 2, 3, 4$. Suppose that (C_6, C_1) is a pseudoholomorphic realization of $\widetilde{A}_3(a,b,c)$. Let v be the oval of C_6 which meets C_1 at four points. We choose a point p on a segment of $\mathbb{R}C_1$ which is exterior to v, has its endpoints on v, and does not have other intersections with C_6.

Then the $\mathcal{L}_p$-scheme of C_6 is $[\supset_1 o_2^a \supset_1 o_1^c \subset_1 o_2^b \subset_1]$. A priori an $\mathcal{L}_p$-scheme realizing $\widetilde{A}(a,b,c)$ could contain some o_1 or o_3 occurring between the ovals of the group o_2^a or o_2^b. However, this is impossible by Lemma 4.1(b). By Bézout Theorem for an auxiliary line, it is also impossible that some part of o_1^c were replaced by $\subset_1 o_2 \dots o_3 \supset_2$ or by $\subset_2 o_3 \dots o_2 \supset_1$ though this replacement does not change the isotopy type.

Thus $\widetilde{A}_3(a, 9-a, 0)$ is pseudoholomorphically realizable if and only if there exists $e \in \mathbb{Z}$ such that the braid

$$\beta_a(e) = \sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2^{-a}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{1+e}\sigma_1^{1-e}\sigma_2^{a-9}\sigma_1^{-1}\sigma_2\Delta$$

is quasipositive. By Murasugi-Tristram inequality, a necessary condition for this is the vanishing of $\det \beta_a(e)$. As in §4.1, using the program `ssmW`, we find that

$$\pm \det \beta_a(e) = 4(36 - 36a + 4a^2 - 9e + 2ae - 2e^2).$$

One easily checks that this polynomial in e does not have integer roots when $a = 0, 2, 3,$ or 4.

Similarly, $\widetilde{A}_3(2,3,4)$ is pseudoholomorphically realizable if and only if there exists $e = (e_1, \dots, e_5) \in \mathbb{Z}^5$ such that the braid

$$\gamma(e) = \sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2^{-a}\sigma_1^{-1}\sigma_3^{-1}\Big(\prod_{j=1}^{5}\sigma_2^{-1}\sigma_3^{e_j}\sigma_1^{-e_j}\Big)\sigma_1\sigma_3\sigma_2^{a-9}\sigma_1^{-1}\sigma_2\Delta$$

is quasipositive. A computation with the help of `ssmW` shows that $|\det \gamma(e)|$ is a polynomial of degree 2 which is positive on $\mathbb{R}^5$ (the quadratic form is positive definite and the value at the minimum is positive).

4.3. The series $\widetilde{C}_2$ and C_3^{-2}. We need to prove the pseudoholomorphic unrealizability of

$$\widetilde{C}_2(1,2,6),\ C_3^{-2}(1,2,7),\ \widetilde{C}_2(1,1,7),\ C_3^{-2}(1,1,8),\ \widetilde{C}_2(1,0,8),\ C_3^{-2}(1,0,9), \quad (3)$$

$$\widetilde{C}_2(5,0,4),\ C_3^{-2}(5,0,5), \tag{4}$$

$$\widetilde{C}_2(9,0,0),\ C_3^{-2}(9,0,1). \tag{5}$$

Notice that if any of (3) (resp. (4), (5)) is realizable, then $\widetilde{C}_2(1,2,6)$ (resp. $\widetilde{C}_2(5,0,4)$, $\widetilde{C}_2(9,0,0)$) is obtained from it by moving the line. Thus it is enough to prohibit only $\widetilde{C}_2(1,2,6)$, $\widetilde{C}_2(5,0,4)$ and $\widetilde{C}_2(9,0,0)$.

To this end we shall use the approach from [7, §3.3]. Suppose that (C_6, C_1) is a pseudoholomorphic realization of $\widetilde{C}_2(a,b,c)$. Let us choose a point p on the line C_1 inside that oval which meets C_1 at two points. Then the $\mathcal{L}_p$-scheme of our curve is $[\supset_2 o_{i_1} \dots o_{i_9} \subset_2]$ where $a = \#\{j \mid i_j = 3\}$, $b = \#\{j \mid i_j = 2\}$, and $c = \#\{j \mid i_j = 4\}$. By Lemma 4.1(b) the a ovals "o_3" are consecutive, i.e., $i_j = \dots = i_{j+a-1} = 3$ for some j. Moreover, by the formula of complex orientations, j is even (it follows already that $\widetilde{C}(9,0,0)$ is unrealizable). By symmetry we may also assume that $j = 2$ for $\widetilde{C}_2(5,0,4)$ and that $j = 2$ or 4 for $\widetilde{C}_2(1,2,6)$.

So we have to consider only one case $[\supset_2 o_4 o_3^5 o_4^3 \subset_2]$ for $\widetilde{C}_2(5,0,4)$ and $2\times\binom{8}{2} = 56$ cases for $\widetilde{C}_2(1,2,6)$. In each case we compute the Alexander polynomial of the corresponding braid and obtain a contradiction with the generalized Fox-Milnor theorem (see [7, §3.3] for details).

4.4. The series $\widetilde{D}$. Suppose that (C_6, C_1) is a pseudoholomorphic realization of $\widetilde{D}(a,b,c)$. Let v be the oval of C_6 which meets C_1 at four points. We choose a point p on a segment of $\mathbb{R}C_1$ which is exterior to v, has its endpoints on v, and does not have other intersections with C_6. Then the $\mathcal{L}_p$-scheme of C_6 is $[\supset_1 o_{i_1} \dots o_{i_9} \subset_1]$ where $a = \#\{j \mid i_j = 1\}$, $b = \#\{j \mid i_j = 3\}$, and $c = \#\{j \mid i_j = 2\}$. By Lemma 4.1(a), the sequence $(i_1 \dots i_9)$ cannot contain $1\dots3\dots1$ or $3\dots1\dots3$. By Lemma 4.1(c), it cannot contain $j\dots2\dots k\dots2$ or $2\dots j\dots2\dots k$ with $j,k \neq 2$. Thus, up to symmetry, the $\mathcal{L}_p$-scheme is one of

(i) $[\supset_1 o_1^{a_1} o_3^{b_1} o_2^{c} o_1^{a_2} o_3^{b_2} \subset_1]$ with $a_1 + a_2 = a$, $b_1 + b_2 = b$, and $b_1 a_2 = 0$;
(ii) $[\supset_1 o_2^{c_1} o_1^{a} o_3^{b} o_2^{c_2} \subset_1]$ with $c_1 + c_2 = c$.

By the generalized Fox–Milnor Theorem [7, §3.3], the determinant $\det\beta$ of the associated braid is (up to sign) a square of an integer number. Using the program `ssmW` (see Appendix in [7]) we find that $\pm\det\beta = 4c(1 + a + 5b - 4ab)$ in Case (i) and $\pm\det\beta = 4c(1 + a + 5b - 4ab) - 16(a+b)c_1c_2$ in Case (ii). Hence $|\det\beta|$ may be a square only in the cases listed in Theorem 1.1.

5. Prohibitions of algebraic curves

In this section we prove Theorem 1.3. As in §4.3, it is enough to prove the algebraic unrealizability of $\widetilde{C}_2(1,4,4)$ because this arrangement can be obtained from any of (2) by moving the line. The rest of this section is devoted to the proof that $\widetilde{C}_2(1,4,4)$ is algebraically unrealizable.

5.1. Self-linking number of a 4-valued function. In this section we refine the observations from [11, Lemma 3.7] and [8, §4.1] concerning the cubic resolvent of a real polynomial of degree 4 in y whose coefficients depend on x. In fact, in [11, Lemma 3.7] we used only one coefficient of the polynomial in y. Therefore, we do not speak here of the cubic resolvent but only of this coefficient.

Lemma 5.1. *Let y_1, y_2, y_3, y_4 be the roots of a polynomial $P(y) = y^4 + a_2y^2 + a_1y + a_0$ with complex coefficients. Then $a_1 = 0$ if and only if y_1, y_2, y_3, y_4 are at the vertices of a parallelogram (maybe degenerated), i.e., $y_1 + y_2 = y_3 + y_4 = 0$ up to permutation of y_1, y_2, y_3, y_4. This condition is also equivalent to the fact that 0 is root of the cubic resolvent of P.*

Proof. Since the coefficient of y^3 is zero, we have $y_4 = -y_1 - y_2 - y_3$. Plugging this into $a_1 = -(y_1y_2y_3 + \dots)$ we obtain $a_1 = (y_1 + y_2)(y_1 + y_3)(y_2 + y_3)$. □

Lemma 5.2. *Let y_1, y_2, y_3, y_4 be the roots of a polynomial $y^4 + a_2y^2 + a_1y + a_0$ with real coefficients. Then the sign of a_1 depends on the mutual position of the roots as shown in Figure 12. More precisely:*

a). If $y_1, \dots, y_4$ are real and $y_1 \le y_2 \le y_3 \le y_4$, then

$$\operatorname{sign} a_1 = \operatorname{sign}\big((y_2 + y_3) - (y_1 + y_4)\big).$$

b). If y_1 and y_2 are real, and $y_3 = \bar y_4$, then

$$\operatorname{sign} a_1 = \operatorname{sign}\big((y_3 + y_4) - (y_1 + y_2)\big) = \operatorname{sign}\big((2\operatorname{Re} y_3) - (y_1 + y_2)\big).$$

c). If $y_1 = \bar y_2$, $y_3 = \bar y_4$, $\operatorname{Im} y_1 \ge 0$, and $\operatorname{Im} y_3 \ge 0$, then

$$\operatorname{sign} a_1 = \operatorname{sign}\big(\operatorname{Re}(y_3 - y_1) \cdot \operatorname{Im}(y_3 - y_1)\big).$$

$a_1 > 0$:

$a_1 < 0$:

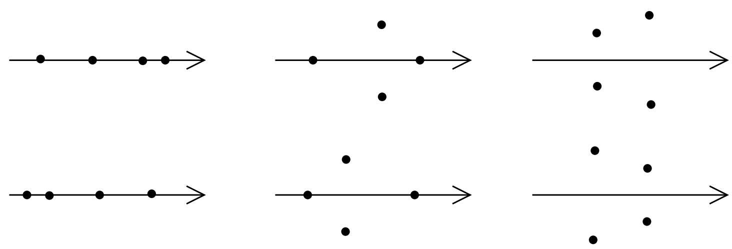

FIGURE 12. Dependence of $\operatorname{sign} a_1$ on the roots.

Proof. In [8, Lemma 4.2] we gave a proof of (a) based on some elementary computations. Of course, (b) and (c) can be proven similarly, but we shall give another proof for all the statements (a)–(c) which does not require any computation.

By Lemma 5.1, in each case (a), (b), (c), the right hand side of the equality vanishes if and only if $a_1 = 0$. Let us consider then the case when the right hand sides are positive. Let A (resp. B, C) be the set of polynomials such that the right hand side of the equality (a) (resp. (b), (c)) is positive. It is easy to check that these sets are connected and the intersections $A \cap B$ and $B \cap C$ are non-empty. Indeed, the polynomial with roots $(-2, 0, 1, 1)$ belongs to $A \cap B$ and the polynomial with roots $(-1, -1, 1 \pm i)$ belongs to $B \cap C$. Since a_1 does not vanish on $A \cup B \cup C$, its sign is constant on this set. So, it is enough to look at the sign of a_1 for any element of A, for example, for $(x - 1)^2 x(x + 2) = x^4 - 3x^2 + 2x$. □

Corollary 5.3. *In the setting of Lemma 5.2(a), if $y_1 < y_2 \le y_3 = y_4$, then $a_1 > 0$, and if $y_1 = y_2 \le y_3 < y_4$, then $a_1 < 0$.* □

Definition 5.4. (Cf. [11, Def. 3.6].) Let $y = f(x)$ be a real 4-valued algebraic function which has no poles on a segment $[x_1, x_2]$. Suppose that

(1) At each of the points x_1 and x_2, two of the analytic branches of f have a simple branching and the other two branches are non-singular;
(2) the values of f are distinct and non-real at any $x \in]x_1, x_2[$.

Let $x_0 = (x_1+x_2)/2$ and let $f_j^{\rm sing}$ and $f_j^{\rm reg}$, $j = 1, 2$, be the branches of f on $[x_j, x_0]$ whose imaginary parts are positive and such that $f_j^{\rm sing}$ is branched at x_j but $f_j^{\rm reg}$ is not. Let $V = \mathbb{R} \times \mathbb{C} = \{(x, y) \mid \operatorname{Im} x = 0\}$ and let $S_+ \subset V$ be the union of the graphs of $f_j^{\rm sing}$ and $f_j^{\rm reg}$, $j = 1, 2$, with the four segments

$$[\,(x'_j, 0), (x_j, f_j^{\rm reg}(x_j))\,], \quad [\,(x_j, 0), (x_j, f_j^{\rm sing}(x_j))\,], \qquad j = 1, 2,$$

where $x'_1 < x_1 < x_2 < x'_2$. Let $S = S_+ \cup r(S_+)$ where r is the rotation of V by 180° around the axis $y = 0$. We endow S^+ with the orientation induced by the projection onto the segment $[x'_1, x'_2]$, and we extend this orientation to the whole S. Then S is the union of two disjoint oriented closed curves. Their linking number is called the *self-linking number of f on $[x_1, x_2]$*.

Lemma 5.5. (cf. [11, Lemma 3.7]). *Let $f(x)$ be a 4-valued algebraic function implicitly defined by the equation $y^4 + a_2(x)y^2 + a_1(x)y + a_0(x) = 0$ where a_0, a_1, a_2 are polynomials with real coefficients. Suppose that $[x_1, x_2]$ satisfies the conditions (1)–(2) of Definition 5.4 and let k be the self-linking number of f on $[x_1, x_2]$. Suppose that $k \neq 0$. Let $x'_1 < x_1 < x_2 < x'_2$ and $a_1(x'_j) \neq 0$, $j = 1, 2$. Then a_1 has at least $|2k + (\varepsilon_1 - \varepsilon_2)/2|$ real roots on the segment $[x'_1, x'_2]$ where $\varepsilon_j = \operatorname{sign} a_1(x'_j)$.*

Proof. We consider only the case when $k > 0$ and the image of S^+ under the plane projection $\pi : \mathbb{R} \times \mathbb{C} \to \mathbb{R}^2$, $(x, y) \mapsto (x, \operatorname{Im} y)$, has exactly k self-crossings. Other cases can be easily reduced to this one.

By Lemma 5.2(c), in this case we have k roots of $a_1(x)$ at the x-coordinates of the self-crossings of $\pi(S^+)$ and $k - 1$ roots between each pair of the consecutive self-crossings.

Let x''_1 (resp. x''_2) be the x-coordinate of the first (resp. the last) self-crossing. Then, for $0 < \delta \ll 1$, we have $a_1(x''_1 - \delta) < 0$ and $a_1(x''_2 + \delta) > 0$ (see Lemma 5.2(c) and Figure 13). Therefore we have at least $(1 + \varepsilon_1)/2$ roots of a_1 on $[x'_1, x''_1 - \delta]$ and at least $(1 - \varepsilon_2)/2$ roots of a_1 on $[x''_2 + \delta, x'_2]$, thus at least

$$k + (k - 1) + (1 + \varepsilon_1)/2 + (1 - \varepsilon_2)/2 = 2k + (\varepsilon_1 - \varepsilon_2)/2$$

roots on $[x'_1, x'_2]$. $\square$

5.2. Algebraic unrealizability of $\widetilde{C}_2(1, 4, 4)$. The proof is similar to the proofs of the algebraic unrealizability of the maximal arrangements (affine M-sextics) $A_4(1, 5, 4)$ and $C_2(1, 3, 6)$ in [10,11].

Step 1. Arrangements of the pseudo-holomorphic curves with respect to an auxiliary pencil of lines. Suppose that C_6 is a pseudo-holomorphic M-sextic and C_1 a line which have the mutual arrangement $\widetilde{C}_2(1, 4, 4)$. Let us choose p as in §4.3. Computing, also like in §4.3, the Alexander polynomials for all a priori possible

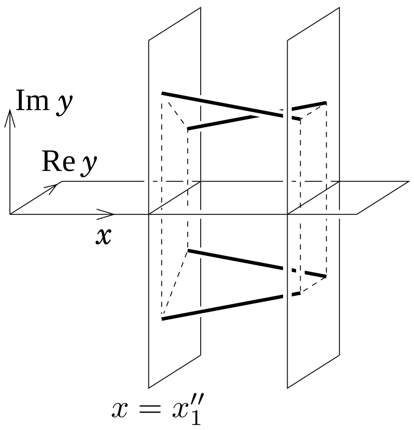

FIGURE 13. $a_1(x_1'') < 0$ for $k > 0$.

sequences $(i_1, \dots, i_9)$ (there are 140 of them), we obtain that only two $\mathcal{L}_p$-schemes do not contradict the generalized Fox-Milnor theorem:

$$[\supset_2 o_2^3 o_3 o_4^4 o_2 \subset_2] \tag{6}$$

$$[\supset_2 o_2 o_3 o_4^4 o_2^3 \subset_2] \tag{7}$$

The $\mathcal{L}_p$-scheme (6) is realized by the pseudo-holomorphic curves constructed in §3.1. I do not know whether (7) is pseudoholomorphically realizable.

Step 2. Essential nodal degenerations of $\mathcal{L}_p$-schemes (6) and (7). Let us say that a nodal degeneration of an $\mathcal{L}_p$-scheme is *essential* if it changes the corresponding braid, i.e. it is not a degeneration of the form $\supset_i \subset_i \to \times_i$ or a contraction of an empty oval to a solitary node but a symbol $\times_i$ is inserted somewhere into the encoding word.

Suppose that C_6' is a pseudo-holomorphic curve whose $\mathcal{L}_p$-scheme is obtained from (6) or (7) by an essential nodal degeneration. By Murasugi-Tristram inequality, the Alexander polynomial of the corresponding braid must identically vanish. The only essential nodal degenerations which satisfy this condition, are

$$[\supset_2 o_2^3 o_3 o_4^4 o_2 \subset_2] \to [\supset_2 o_2^3 o_3 o_4^3 \subset_4 \times_5 \supset_4 o_2 \subset_2] \text{ or } [\supset_2 o_2^3 o_3 o_4^5 \subset_2 \times_1 \supset_2 \subset_2]$$

for (6). In particular, (7) does not admit any essential nodal degeneration.

In fact, in all the cases except four (two cases for each of (6) and (7)) it is not necessary to compute the Alexander polynomial because already the determinant of the braid does not vanish.

Step 3. Application of Hilbert-Rohn-Gudkov method. Now, let us suppose that C_6 and C_1 are real algebraic. Let us choose an equation $f_6 = 0$ of C_6 so that $f_6 < 0$ in the non-orientable component of $\mathbb{RP}^2 \setminus \mathbb{R}C_6$.

Let us deform C_6 in the pencil $f_6 + tg_3^2 = 0$ for a generic cubic polynomial g_3. Then (see details in [10]) C_6 must degenerate into a nodal curve. Choosing another pencil of this form but with a generic g_3 vanishing at the node, we further degenerate the sextic and obtain one more node. Continuing this process, we obtain a sextic C_6' with 10 nodes. By the result of Step 2, the arrangement of C_6' and C_1 is the one depicted in Figures 14. By the genus formula, C_6' is rational.

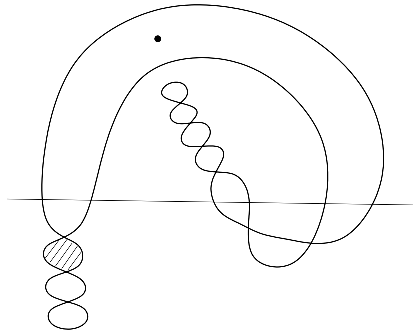

FIGURE 14

Now, let us consider the equisingular deformation of C_6' such that all the nodes are fixed except those that are adjacent to the shadowed digon. One can compute that this is a one-parameter family which is smooth at C_6'. Moving in this family in the direction such that the shadowed digon shrinks, we degenerate C_6' into a curve C_6'' which has a singular point q of the type A_3 (a point of tangency of two smooth branches) instead of the shadowed digon. Rotating C_1 around one of the two middle intersection points p, we obtain a line C_1'' through q such that the right four intersection points of C_1'' and C_6'' are still arranged as in Figure 14.

Step 4. Reduction to a 4*-valued function.* Let T be the tangent to C_6'' at q. In the same way as in [11; Lemmas 3.11 and 3.13(b)], we can prove that C_6'' is arranged with respect to T and C_1'' as in Figure 15 up to isotopy (the rectangular pattern may be replaced as it is shown).

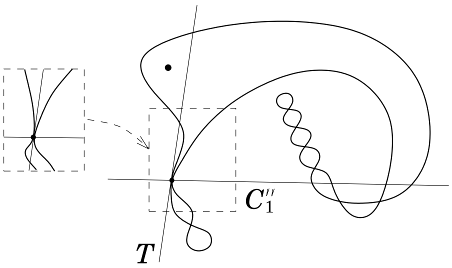

FIGURE 15

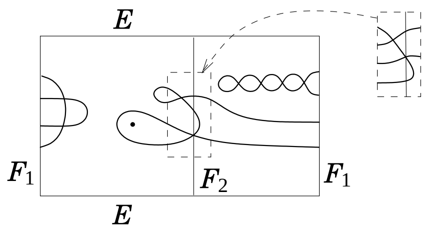

FIGURE 16

Let us blow up twice the point q and then blow down the proper transform of T. Let us denote the exceptional curves of the blowups by E_1 and E_2 (E_2 is the transform of the intersection point of E_1 with the proper transform of C_6'').

We obtain a curve C_4 of bidegree $(4, 8)$ on the Hirzebruch surface $\mathcal{F}_2$ (the quadratic cone blown up at the vertex). Let us denote the proper transforms on $\mathcal{F}_2$ of the curves C_1'', E_1, and E_2 by F_1, E, and F_2 respectively. Then F_1 and F_2 are fibers and E is the exceptional section of the fibration $\pi : \mathcal{F}_2 \to \mathbb{P}^1$. In a standard coordinate system of $\mathcal{F}_2$, the curve C_4 is defined by an equation $y^4 + a_3(x)y^3 + a_2(x)y^2 + a_1(x)y + a_0(x)$ where a_m is a polynomial in x and

$$\deg a_m(x) \le 2(4 - m), \qquad m = 0, 1, 2, 3. \tag{8}$$

By the standard trick, we may kill the coefficient $a_3(x)$.

The fact that $C_6'' \cup C_1''$ is as in Figure 14, implies that the arrangement of C_4 with respect to F_1, F_2, and E is as in Figure 16 where $\mathbb{R}\mathcal{F}_2$ is depicted as a rectangle whose opposite sides are identified (note that $\mathbb{R}\mathcal{F}_2$ is a torus). Hence, the arrangement of C_4 with respect to the fibers of π is $[w_1\subset_1 o_2\subset_3\times_3 w_2\subset_3\times_3^4]$ where w_2 is one of $[\times_1\supset_2]$ or $[\times_2\supset_1]$, and possible values of w_1 are listed in Table 2.

Step 5. Application of the cubic resolvent or counting the roots of $a_1(x)$. For each w_1 we compute the self-linking number k (see Definition 5.4) as it is done in [11, Lemma 3.13(a)]. The results are given in Table 2. The column d contains the lower bound for $\deg a_1(x)$ provided by Lemma 5.5 and Corollary 5.3. We see that in all the cases, this bound contradicts (8).

Table 2.

no.	w_1	k	d	no.	w_1	k	d
1	$\times_3\times_3\supset_2\supset_1$	-4	10	5	$\times_1\times_3\supset_2\supset_1$	-3	10
2	$\times_3\times_2\supset_3\supset_1$	-4	10	6	$\times_1\times_2\supset_3\supset_1$	-3	10
3	$\times_3\times_2\supset_1\supset_1$	-3	10	7	$\times_1\times_2\supset_1\supset_1$	-2	8
4	$\times_3\times_1\supset_2\supset_1$	-3	10	8	$\times_1\times_1\supset_2\supset_1$	-2	8

We shall show in details how the bound $\deg a_1(x) \geq 8$ is obtained for the last line of Table 2. The other cases are similar. Let us choose the points $x_1', x_1, x_2, x_2', x_3, x_4$ on $\mathbb{RP}^1$ as in Figures 17.1 (for $w_2 = \times_1\supset_2$) or in Figure 17.2 (for $w_2 = \times_2\supset_1$). In the both cases we have $a_1(x_1') < 0$, $a_1(x_2') > 0$, $a_1(x_3) < 0$, and $a_1(x_4) > 0$ by Corollary 5.3. Hence a_1 has at least one root on each of the intervals $[x_2', x_3]$, $[x_3, x_4]$, $[x_4, x_1']$, and by Lemma 5.5, it has at least $|2\cdot(-2)+(-1-1)/2| = 5$ roots on $[x_1', x_2']$. Thus, $\deg a_1 \geq 1+1+1+5 = 8$.

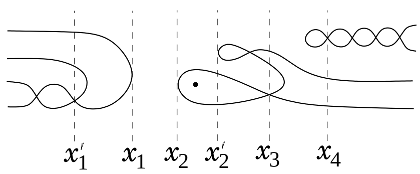

FIGURE 17.1

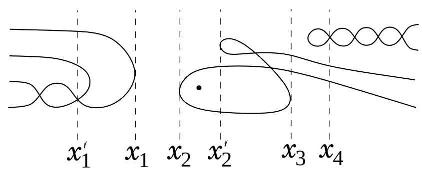

FIGURE 17.2

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