ON QUASICONVEX HYPERSURFACES IN REAL PROJECTIVE SPACES

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Abstract. A smooth hypersurface in $\mathbb{RP}^n$ is called *quasiconvex of signature* (k,l), $k+l=n-1$, if the second quadratic form for the spherical metric is non-degenerate at any point and its signature is (k,l). Vladimir Arnold conjectured that such surfaces have some common properties with smooth quadrics. We disprove some of his conjectures and propose improved versions of them. We also discuss some other aspects of surfaces in $\mathbb{RP}^3$ of signature $(1,1)$, in particular, their piecewise linear analogues.

1. Arnold's conjectures

Following [1], we say that a smooth hypersurface H in $\mathbb{RP}^n$ is *quasiconvex of signature* (k,l) (for short, (k,l)-*convex*) at a point $P \in H$ if $k+l=n-1$ and H can be defined by an equation

$$z = x_1^2 + \cdots + x_k^2 - y_1^2 - \cdots - y_l^2 + \text{higher order terms}$$

in some affine chart centered at P. A hypersurface in $\mathbb{RP}^n$ is (k,l)-*convex* if it is (k,l)-convex at each point.

Vladimir Arnold [1] discovered that the quasiconvexity imposes rather strong restrictions on the topology of a real projective hypersurface. He conjectured the following properties of connected (k,l)-convex hypersurfaces H in $\mathbb{RP}^n$ (proven in [1] for $(k,l)=(n-1,0)$):

1° (Standartness). The space of all connected (k,l)-convex hypersufraces is connected, i.e., H is isotopic (through (k,l)-convex hypersurfaces) to a quadric.

2° (Covering). H is homeomorphic to $(S^k \times S^l)/((x,y) \sim (-x,-y))$.

3° (Quasistarlikeness). (i) The complement $\mathbb{RP}^n \setminus H$ has two connected components H_+ and H_- and there exist projective subspaces $L_\pm \subset H_\pm$ of dimensions $\dim L_+ = k$ and $\dim L_- = l$; (ii) any straight line segment $[a,b]$ with $a \in L_+$ and $b \in L_-$ crosses H at a single point.

4° (Divisors). For any hyperplane section D of H, the pair (H,D) is diffeomorphic to $(\bar H, \bar D)$ for a quadric $\bar H$ and its suitable hyperplane section $\bar D$.

Say that a surface satisfying Condition 3°(i) is *weakly quasistarlike*.

None of these conjectures is true for arbitrary (k,l). The first aim of this paper is to give some counter-examples. However, the quasiconvexity still seems to impose strong topological restrictions on H, and the second aim is to improve the conjectures.

This note is an extended version of my talk [10]. We added to it the references [4, 5], a proof of Proposition 1 and everything which is after it.

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2. The general case

The first counter-example to all the above conjectures had been constructed by Elie Cartan [2] in 1938. It is closely related with the main subject of Arnold's paper [1]. Namely, let $f : \mathbb{CP}^2 \to S^4$ be the double covering ramified at $\mathbb{RP}^2$ which is constructed in [1] (the identification of $\mathbb{CP}^2/\operatorname{conj}$ with S^4) and let $p : S^4 \to \mathbb{RP}^4$ be the standard projection. Let H be a tube of constant radius over $p(f(\mathbb{RP}^2))$. Then H is quasiconvex of signature (1,2), but it is not homeomorphic to a quadric. It is not weakly quasistarlike neither.

The Cartan's hypersurface H is *isoparametric*, i.e., its principal curvatures (for a spherical metric on $\mathbb{RP}^4$) are constant. Any isoparametric hypersurface H in $\mathbb{RP}^n$ is, evidently, (k,l)-convex. It follows from [3] and [9] that if H is not a quadric, then either $k = l \geq 2$ or $2k = l$ with $k = 1, 2, 4$, or 8. In the latter case H is either as described above or is obtained by a generalization of the same construction with $\mathbb{R}$ replaced by $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$; see [3]. Note that the classification of isoparametric hypersurfaces in the spheres is recently completed in [4]; see also the survey [5].

A simple example of a connected (k,k)-convex isoparametric hypersurface H in $\mathbb{RP}^n$, $n = 2k+1$, $k \geq 2$, which is not isotopic to a quadric can be described as follows. Let

$$h : \mathbb{RP}^{2k+1} \to \mathbb{CP}^k, \qquad (x \mod \mathbb{R}^\times) \mapsto (x \mod \mathbb{C}^\times), \quad x \in \mathbb{C}^{k+1} \setminus 0, \qquad (1)$$

be the Hopf fibration and let H be the boundary of $H_+ = h^{-1}(U)$ where U is a tubular neighborhood of $\mathbb{RP}^k$ in $\mathbb{CP}^k$. Then H is not isotopic to a quadric because H_+ is not homotopy equivalent to $\mathbb{RP}^k$. The latter fact is seen from the long exact sequence (starting from $\pi_2(S^1)$) of the fibration $H_+ \to U$ with fiber S^1.

Conjecture 1. (Connected components) Any connected component of the set of connected (k,l)-convex hypersurfaces in $\mathbb{RP}^n$ contains an isoparametric hypersurface.

Conjecture 2. (Self-duality) Let H be a connected (k,l)-convex hypersurface in $\mathbb{RP}^n$. Then the pair $(\mathbb{RP}^n, H)$ is diffeomorphic to the pair $(\widehat{\mathbb{RP}}^n, \widehat{H})$, where $\widehat{\mathbb{RP}}^n$ is the dual projective space and $\widehat{H}$ is the dual hypersurface of H. A weaker form of this conjecture is that $\widehat{H}$ does not have self-intersections.

3. The case $n = 3$

3.1. Non quasistarlike surfaces in $\mathbb{RP}^3$. We will call (1,1)-convex surfaces in $\mathbb{RP}^3$ *hyperbolic*. It seems very plausible that any hyperbolic surface $H \subset \mathbb{RP}^3$ has properties 1°, 2°, 3°(i), and 4°. Note that the property 3°(i) is proven in [6–8] under various additional conditions.

However, it is easy to construct an example where 3°(ii) does not hold. Indeed, let $h : \mathbb{RP}^3 \to \mathbb{CP}^1$ be the Hopf fibration (1). Set $H = h^{-1}(\gamma)$, where γ is a smooth simple closed path containing a sufficiently long arc of a spiral (see Figure 1).

Proposition 1. *The surface H is hyperbolic but it is not quasistarlike.*

Proof. The hyperbolicity of $h^{-1}(\gamma)$ follows from the fact that $h^{-1}(S)$ is a hyperboloid for any osculating circle S of γ. Let us show that H is not quasistarlike.

Let U_+ and U_- be the components of $\mathbb{CP}^1 \setminus \gamma$ which are gray and white respectively in Figure 1. Suppose H is quasistarlike. Then there are lines $L_\pm$ in $h^{-1}(U_\pm)$

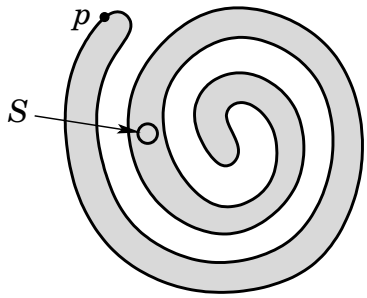


FIGURE 1. U_+ is gray, $S = h(L_+)$, $p = h(P)$.

such that any line relating a point of L_+ to a point of L_- cuts H at two points only. The image of any line by h is a circle in $\mathbb{CP}^1$. Let $S = h(L_+)$. It is a circle in U_+. For any $P \in \mathbb{RP}^3$ there is a unique line L_P that passes through P and has a common point with each of L_+ and L_-. Choose a point $p \in \gamma$ on "the farthest" from S end of U_+ (see Figure 1) and $P \in h^{-1}(p)$. Then $h(L_P)$ is a circle passing through p and cutting S. It is clear that it crosses γ at more than two points. □

3.2. Monotone flows. One might think that a natural way to prove Property 3°(i) for $n = 3$ is to find a monotone flow of hyperbolic surfaces which transforms a given surface to a hyperboloid. By a monotone flow we mean an isotopy $\{H_t\}_{t\geq 0}$ such that, for any t, all the surfaces H_s with $s > t$ are contained in the same component of $\mathbb{RP}^3 \setminus H_t$. However, the following example shows that this is hardly possible to do for an arbitrary hyperbolic surface.

By modifying the construction from Section 3.1 (compare also with the example in [8, §5]), it is possible to construct a hyperbolic torus H in $\mathbb{RP}^3$ such that the set of lines contained in H_+ is disconnected. Let $H_\varepsilon = h^{-1}(\partial I_\varepsilon)$ where I_ε is a very thin neighborhood of a segment $I \subset \mathbb{CP}^1$ (thin of order ε). For some 3-ball $B \subset \mathbb{RP}^3$, the intersection $B \cap H_\varepsilon$ consists of two smooth disks which are ε-close to each other, i.e., $B \cap H_\varepsilon$ (more rigorously, an open subset of it) is represented in some affine chart by the union of graphs of two functions f_1 and f_2 defined in a domain $V \subset \mathbb{R}^2$, such that $0 < f_2 - f_1 < \varepsilon$. Let φ be a smooth function on V which vanishes near the boundary of V but does not vanish on the whole V. Let H be the surface obtained from H_ε by replacing the graphs of f_1 and f_2 with the graphs of $f_1 + \delta\varphi$ and $f_2 + \delta\varphi$, where $0 < \varepsilon \ll \delta \ll 1$. Let H_+ be the "thin" part of $\mathbb{RP}^3 \setminus H$, i.e., the deformation of $h^{-1}(I_\varepsilon)$. Then the set of lines in H_+ is disconnected.

This example can be easily modified so that the set of lines in H_+ and the set of lines in H_- are both disconnected and have any number of connected components.

3.3. Failure of the weak quasistarlikeness for the 3-sphere. Consider the surface in $\mathbb{C}^2$ parametrized by $(s,t) \mapsto (e^{2is}, e^{is} + re^{it})$, $0 < r < 1$. Let H be its radial projection to S^3. A computation shows that H is hyperbolic for $r > 3/4$. However, one of the components of $S^3 \setminus H$ does not contain any geodesic circle, i.e. the S^3-analogue of 3°(i) does not hold for H.

It is easy to check that the projection of H to $\mathbb{RP}^3$ is not an embedding. A more involved computation shows that there is no hyperbolic surface in $\mathbb{RP}^3$ defined as the radial projection of a surface in $\mathbb{C}^2$ parameterized by $(s,t) \mapsto (e^{msi}, e^{nsi}\gamma(t))$, where $m, n \in \mathbb{Z}$, $m \neq \pm n$, and γ is a simple closed curve contractible in $\mathbb{C} \setminus \{0\}$.

4. On hyperbolic disks in $\mathbb{RP}^3$ with boundary on a plane

In this section we discuss another question about hyperbolic surfaces in $\mathbb{RP}^3$.

Let Π be a plane in $\mathbb{RP}^3$ and Δ an embedded 2-disk. Denote the boundary of Δ by $\partial\Delta$. Suppose that

(i) $\partial\Delta$ is a simple closed curve in Π;
(ii) $\Pi \cap \Delta = \partial\Delta$.

Is it possible that Δ is smooth hyperbolic?

This question is somewhat related to the weak form of Conjecture 2 in §2. Indeed, if Π is a bitangent plane for a hyperbolic surface H (i.e., a point on the self-crossing locus of the dual surface), then such disks should exist.

Note that if H is a hyperboloid and Π is a plane tangent to H, then $H \setminus \Pi$ is an embedded open disk. However, its boundary is not a simple closed curve and it cannot be transformed into a simple closed curve on Π by a small perturbation of the embedding.

Gaiane Panina [12] constructed a piecewise linear embedded disk Δ, which is a *saddle surface* (i.e. no plane locally intersects Δ at a single interior point) satisfying Conditions (i) and (ii). Her construction of Δ can be described as follows.

Choose an affine chart in $\mathbb{RP}^3$ with coordinates (x, y, z). Let Π be the plane $z = 0$. Then $\partial\Delta$ is the polygon $ABCDEFE'D'C'B'$ on Π which is depicted in Figure 3 on the left. The points A and F are on the infinite line, the segments AB and DE are collinear, and the polygonal line $AB'C'D'E'F$ is centrally symmetric to $ABCDEF$ with respect to the origin O.

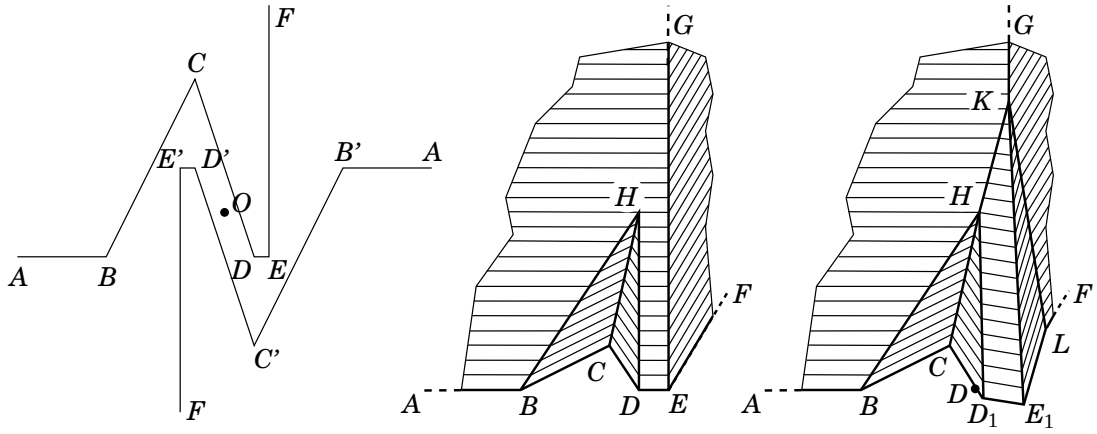


FIGURE 2. $\partial\Delta$ (on the left), a half of Δ (in the middle), and the deformation of the half of Δ described in §5.3 (on the right).

Let G be the point at infinity on the axis Oz and let H be a point in the triangle BEG. The disk Δ is represented by two polyhedral surfaces, which are centrally symmetric to each other (in $\mathbb{R}^3$) with respect to O. One of them is depicted in Figure 3 in the middle. It has four faces: $ABHDEG$, EFG, BCH, and CDH.

One can easily see that Δ is a saddle disk and it satisfies the conditions (i) and (ii). *Can it be approximated by smooth hyperbolic disks?* We discuss this question in the next section.

Remark 1. The disk Δ can be modified near G (see [12]) so that it becomes *simple*, i.e., each vertex has exactly three adjacent edges. Under some additional

hypothesis (which seems easy to get rid of) it is proved in [11, §7] that a simple polyhedral saddle surface can be approximated by a smooth surface such that no plane locally intersects it at a single point. We propose to call such surfaces *weakly hyperbolic*. They can be equivalently characterized by the condition that the principal curvatures κ_1, κ_2 at any point satisfy the inequalities $\kappa_1 \le 0 \le \kappa_2$. However, the weak hyperbolicity is indeed much weaker than the hyperbolicity.

5. A piecewise linear analogue of hyperbolic surfaces in $\mathbb{RP}^3$

In this section we define a class of polyhedral surfaces in $\mathbb{RP}^3$ which are similar to smooth hyperbolic surfaces. In particular, this class is invariant under a properly defined projective duality. Conjecturally, surfaces of this class can be approximated by smooth hyperbolic surfaces.

5.1. 4-biregular saddle surfaces. Let Q be a closed (i.e., without boundary) polyhedral surface in $\mathbb{RP}^3$. We say that Q is *4-biregular* if each face is a convex quadrangle and each vertex is four-valent, i.e. has exactly four adjacent edges. In this case the Euler characteristic of Q is zero, hence it is homeomorphic to a torus (since the Klein bottle is not embeddable in $\mathbb{RP}^3$) and we fix a coorientation on it. Say that an edge e of Q is *convex* if Q is convex (for the chosen coorientation) at interior points of e. Otherwise e is *concave*.

If we assume in addition that Q is a saddle surface (see §4), then its faces and vertices satisfy the following conditions (F) (for faces) and (V) (for vertices):

(F) two opposite edges of the face are convex and the other two edges of this face are concave;

(V) when going along Q around the vertex, convex and concave edges alternate.

For example, if we consider $\mathbb{RP}^3$ as $(\mathbb{C}^2 \setminus 0)/\mathbb{R}^\times$, the hyperboloid $|z|^2 = |w|^2$ can be approximated by saddle 4-regular surfaces ("moka-pot surfaces") with vertices $p_{k,l} = (\omega^k : \omega^l)$, $\omega = e^{\pi i/n}$, and faces $[p_{k,l}\,,\ p_{k+1,l}\,,\ p_{k+1,l+1}\,,\ p_{k,l+1}]$.

Given a polyhedral surface S in $\mathbb{RP}^3$ with boundary, we say that its face or vertex is *interior* if it is disjoint from ∂S. We denote the union of all interior faces of S by S°. By abusing the language, we say that Q is a *4-biregular saddle surface with boundary* if $Q \subset S^\circ$, where S is a polyhedral surface (probably immersed) all whose interior faces are quadrangles satisfying Condition (F) and all interior vertices are 4-valent and satisfy Condition (V).

5.2. Projective duality. Let Q be a 4-biregular saddle surface in $\mathbb{RP}^3$. We define the dual polyhedral surface $\widehat{Q}$ in the dual projective space $\widehat{\mathbb{RP}^3}$ (whose elements are planes in $\mathbb{RP}^3$) as follows. Vertices of $\widehat{Q}$ are planes containing the faces of Q. The face $\widehat{v}$ of $\widehat{Q}$ dual to a vertex v of Q is the set of planes which pass through v and have non-empty intersection with $f \setminus \{v\}$ for each face f containing v.

It is straightforward to check that $\widehat{Q}$ is also a 4-biregular saddle surface and the dual of $\widehat{Q}$ is Q under the natural identification of $\mathbb{RP}^3$ with the dual of $\widehat{\mathbb{RP}^3}$.

5.3. Deformation of Panina's saddle disk to a 4-biregular saddle surface. Choose a point K on the edge EG so that the pentagon $ABHKG$ is convex, and choose any point L on the edge EF. Choose also points D_1 and E_1 on Π close to D and E respectively so that (see Figure 2 on the right):

- D_1 is placed on the line CD so that D is between C and D_1;
- E_1 is on the plane KHD_1, on the right (in Figure 2) of the plane EFG.

Let $\widetilde{\Delta}$ be the piecewise linear saddle disk one of whose halves in depicted in Figure 2 on the right and the other half is centrally symmetric to it with respect to O. Then $\widetilde{\Delta}$ is a 4-biregular saddle surface with boundary. The edges BH, D_1H, E_1K, KG, and their centrally symmetric images are convex. The other edges are concave.

Conjecture 3. Any 4-biregular saddle surface (with or without boundary) can be approximated by smooth hyperbolic surfaces.

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