A LA RECHERCHE DE LA TOPOLOGIE PROJECTIVE. DU CÔTÉ DE CHEZ ARNOLD

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Say that a hypersurface H in $\mathbb{RP}^n$ is (k,l)-*convex* or *quasiconvex* if the second fundamental form has the constant signature (k,l), $k+l=n-1$. Arnold [1] discovered that the quasiconvexity imposes very strong restrictions on the topology of H. He conjectured the following properties of such H (proven in [1] for $k=0$):

1 (Standardness). The set of all (k,l)-convex hyperserfaces is connected.

2 (Covering). $H = S^k \times S^l/_{(x,y)\sim(-x,-y)}$, i.e., H is diffeomorphic to a quadric.

3 (Quasistarlikeness). (i). H separates suitable planes $L^+ = \mathbb{RP}^k$ and $L^- = \mathbb{RP}^l$ and (ii) any straight line segment $[a,b]$, $a \in L^+$, $b \in L^-$, transversally meets H at a single point.

4 (Divisors). For any hyperplane section D of H, the pair (H,D) is diffeomorphic to $(\bar{H},\bar{D})$ for a quadric $\bar{H}$ and its suitable hyperplane section $\bar{D}$.

Hypersurfaces with $3(i)$ are called *weakly quasistarlike*. The first aim of my talk is to show that none of these conjectures is true for all (k,l). The second aim is to formulate a similar conjecture which is plausible for any (k,l).

It is amasing that the simplest counter-example (constructed by E. Cartan in 1938) is closely related to the title of [1]. Namely, let $f : \mathbb{CP}^2 \to S^4$ be the double covering constructed in [1] and let $p : S^4 \to \mathbb{RP}^4$ be the standard projection. Let H be a tube of constant raduis over $p(f(\mathbb{RP}^2))$. Then H is $(1,2)$-convex, but it is not homeomorphic to any quadric. It is not weakly quasistarlike neither.

The Cartan's hypersurface H is *isoparametric*, i.e., its principal curvatures are constant. Any isoparametric hypersurface in $\mathbb{RP}^n$ is quasiconvex. Isoparametric hypersurfaces in S^n (hence in $\mathbb{RP}^n$ also) are rather well studied and their classification is almost completed. For $l=k\ge 2$ and for $l=2k$, $k=1,2,4,8$, there exist isoparametric (k,l)-convex hypersurfaces in $\mathbb{RP}^n$ which are not homeomorphic to quadrics (which are not weakly quasistarlike neither). For other values of (k,l), any (k,l)-convex hypersurface in $\mathbb{RP}^n$ is a quadric.

It seams plausible that any $(1,1)$-convex surface $H \subset \mathbb{RP}^3$ has properties 1, 2, $3(i)$, and 4. However, it is easy to construct an example where $3(ii)$ does not hold. Indeed, let $S^3 \to \mathbb{RP}^3 \xrightarrow{h} \mathbb{CP}^1$ be the Hopf fibration. Set $H = h^{-1}(\gamma)$ where γ is a simple closed path which contains a sufficiently long segment of a spiral.

Conjecture. (Self-Duality). $((\mathbb{RP}^n)^*, H^*)$ is diffeomorphic to $(\mathbb{RP}^n, H)$ where H^* is the projectively dual hypersurface of an embedded quasiconvex $H \subset \mathbb{RP}^n$.

References

1. V.I. Arnold, *Ramified covering $CP^2 \to S^4$, hyperbolicity, and projective topology*, Siberian Math. J. **29:5** (1988), 36-47.