A NEW AFFINE M-SEXTIC

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We shall call an *affine M-curve* an affine real algebraic curve C which has the maximal possible number of connected components $(m^2-m+2)/2$ where m is the degree of C. This is equivalent to the fact that the projective closure $\bar{C}$ of C is a projective M-curve, i.e. it has the maximal possible number of connected components $1+(m-1)(m-2)/2$ and it cuts the infinite line L transversally at m distinct real points which all lie on the same connected component of $\bar{C}$. This definition differs from that, given in [1, 3] but it seems to be more natural.

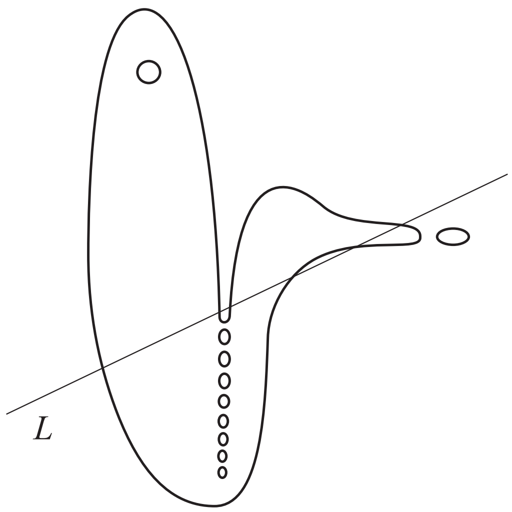

Fig. 1.

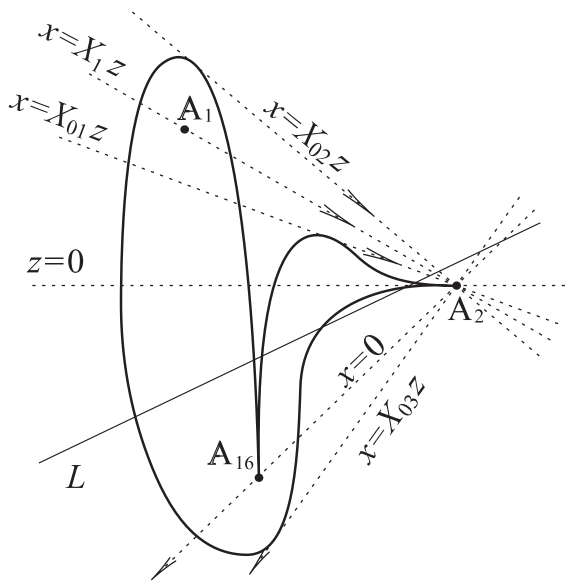

Fig. 2.

33 isotopy types of affine M-curves of degree 6 are constructed in [1]. Other constructions (exposed with more details) of these 33 curves are presented in [2]. It is announced also in [1, 3] that all the other isotopy types but 9 are not realizable, however, the proofs of at least three of these prohibitions are wrong, because the corresponding isotopy types are realizable by smooth surfaces in CP^2 possessing all the properties of algebraic curves used in the proofs. Recently, the author [4] managed to prohibit all the isotopy types except the 33 ones constructed in [1, 2], and except $A_3(0,5,5)^*$, $A_4(1,4,5)^*$, $B_2(1,8,1)$, $B_2(1,4,5)$, $C_2(1,3,6)^*$ in the notation of [1, 2] (the above cases whose prohibition proofs fail in [1, 3], are marked by *).

The present note is devoted to a construction of a curve realizing $B_2(1,8,1)$ (see Fig. 1). We construct it by a perturbation of a suitable singular rational curve using Shustin's lemma [5] on independent smoothing of singularities.

Construction of the line and the sextic shown on Fig. 1. First, we construct an irreducible real sextic C which has singularities A_1, A_2, A_{16} (recall that A_n is the singularity of the form $y^2 \pm x^{n+1} = 0$). The genus formula implies that such a curve is rational and it has no other singular points. Chose coordinates $(x : y : z)$ on $\mathbf{RP}^2$ so that A_{16} and A_2 be at (0:0:1) and (0:1:0), with tangents $y = 0$ and $z = 0$. A parametrization $\mathbf{CP}^1 \to C$, $0 \mapsto (0:0:1)$, $\infty \mapsto (0:1:0)$ has form

$$x(t) = a_2 t^2 + a_3 t^3 + a_4 t^4, \quad y(t) = b_4 t^4 + b_5 t^5 + b_6 t^6, \quad z(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3. \quad (1)$$

By diagonal changes of coordinates in $\mathbf{CP}^1$ and $\mathbf{CP}^2$ one can make

$$a_2 = a_3 = b_4 = c_0 = 1. \quad (2)$$

The condition that one has the singularity A_{16} at (0:0:1), is equivalent to existing of numbers $\gamma_2, \dots, \gamma_7$ such that

$$\operatorname{ord}_{t=0}\Big(y(t)z(t)^6 - \sum_{k=2}^{7} \gamma_k x(t)^k z(t)^{7-k}\Big) = 16 \quad (3)$$

To see this, it is enough to blow up 7 times the singular point. (3) yields a system of simultaneous equations and inequalities for the indeterminates a_4, b_5, b_6, c_1, c_2, c_3, $\gamma_2, \dots, \gamma_7$. Resolving linear (with respect to the corresponding indeterminates) equations, we find successively γ_2, c_1, γ_3, c_2, γ_4, c_3, γ_5, γ_6, γ_7 and obtain 3 non-linear equations for a_4, b_5, b_6. Using resultants we eliminate a_4, b_6 and obtain that b_5 (denote it by β) satisfies the equation

$$311\beta^3 - 293\beta^2 + 85\beta - 7 = 0. \quad (4)$$

The other coefficients in (1) are expressed in terms of β as

$$a_4 = (-317\beta^2 + 221\beta - 24)/56, \quad b_6 = (13\beta^2 - 5\beta)/8, \quad c_1 = 2 - \beta,$$

$$c_2 = (-398431\beta^2 + 312615\beta - 58304)/3624, \quad c_3 = 256(-58843\beta^2 + 46797\beta - 9236)/140883.$$

The equation (4) has a single real root $\beta = 0.1395037384...$ Thus, there exists a unique up to a projective change of coordinates real curve C with the required set of singularities. Let $F(X, Y) = 0$ be its equation in the affine coordinates $X = x/z$, $Y = y/z$. Using the formula $F(X, Y) = \operatorname{Res}_t\big(x(t) - z(t)X, y(t) - z(t)Y\big)$, we express the coefficients of F via β and then compute the polynomial $R(X) = \operatorname{Discr}_Y(F)$. Its factorization over $\mathbf{Q}(\beta)$ has the form $X^{17}(X - X_1)^2 R_0(X)$ where

$$X_1 = (1438630331\beta^2 - 801094822\beta + 83747003)/72828 \approx -0.15259$$

and R_0 is an irreducible over $\mathbf{Q}(\beta)$ polynomial of degree 5 which has 3 real roots $X_{01} \approx -0.15409$, $X_{02} \approx -0.15085$, $X_{03} \approx -0.13551$. The fact that $\operatorname{ord}_{X=0} R(X) = 17$ provides another way to verify that the type of the singularity at $(0:0:1)$ is A_{16}. Computing the multiple root $Y = Y_1$ of $F(X_1, Y)$, we find the ordinate of the singular point of the type A_1:

$$Y_1 = (160515886061\beta^2 - 86960685268\beta + 9007482215)/23409 \approx -0.50314$$

Computing the Hessian at this point

$$F''_{XX}F''_{YY} - (F''_{XY})^2 = (-916243921165066029358781108715 52\beta^2 + 502389472549219212408440681922 56\beta - 522539181096755108975690835558 4)/716297742965892772133 7 \approx 1.6694...\cdot 10^{-9} > 0,$$

we see that (X_1, Y_1) is an isolated double point.

For each real root of R we substitute its approximate value to F and find all the real roots of the obtained polynomial in Y. The results of these calculations are presented in the following table

$X = X_{01}$	$X = X_1$	$X = X_{02}$	$X = X_{03}$	$X = 0$
-1.85807	-1.48933	-0.791026^*	0.691718^*	-4832.11
-0.35177	-0.50314^*	——	——	0.00000^*
-0.30441^*	-0.43017	——	——	27.7307

where multiple roots are marked by $*$.

Finding the number of real roots of polynomials $F(X,\cdot)$ for intermediate values of X and calculating the signs of coefficients responsible for the behavior of the curve at $t \to \infty$ ($a_4 = 0.011... > 0$, $b_6 = -0.055... < 0$, $c_3 = -7.001... < 0$), we see that the curve C looks as it is shown on Fig. 2 (the arrows point to the direction of the increasing of Y). Chose a line L close to the axis $x = 0$ (see Fig. 2). ¿From the result due to Shustin [5, Lemma] we derive that the curve C can be perturbed so that A_{16} gives 8 ovals and each of A_1, A_2 gives one.

Remarks. **1.** Easy to check that $\beta = (97 - 12\alpha - 14\alpha^2)/311$ where $\alpha^3 - \alpha^2 + \alpha = 3$. However, the formulas for γ_j, F and R are rather messy independently of either one use α or β. It seems that the coordinate system fixed by means of (2) is chosen not in the best way.

2. Approximate computations were performed with the accuracy 10^{-1000}.

References

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