A NEW AFFINE M-SEXTIC. II

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An affine real algebraic curve is called *affine M-curve* if it has the maximal possible number of connected component allowed by the Harnack inequality, in particular, an affine M-sextic has 16 connected component, 10 of which are ovals and the other 6 are obtained from the 11-th oval of the projectivization by removing the infinite points. In this note we continue the isotopy classification of affine M-sextics started in [1–5] and give the details of the realization of the isotopy type $A_3(0,5,5)$ (see Fig. 1, where L is the infinite line). It was erroneously declared in [3] as prohibited. The result and the idea of the proof were announced in [5, §2].

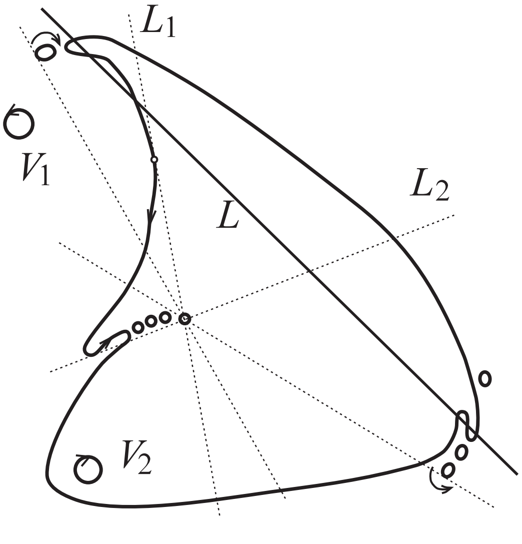

Fig. 1.

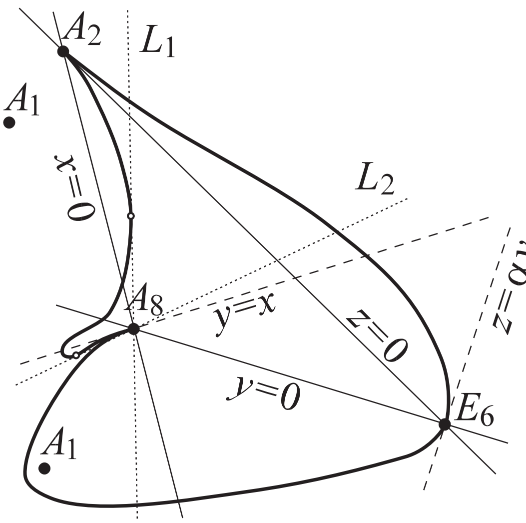

Fig. 2.

The construction is very similar to the realization of $B_2(1,8,1)$ in [4]: we perturb a rational curve obtained by a direct resolving of the simultaneous equations for the coefficients imposed by a given set of singularity types. However, we use here certain geometric arguments to reduce the computations. Similar arguments can be used to simplify the proof in [4].

Let us construct a plane rational sextic C with the singularities E_6, A_8, A_2, A_1, A_1 such that the line through E_6 and A_2 (denote it by L) is tangent to C at A_2. Choose the coordinates $(x:y:z)$ as in Fig. 2 and consider a parametrization $t \mapsto (x:y:z)$ of C such that $0 \mapsto A_8$, $1 \mapsto A_2$, $\infty \mapsto E_6$. Then

$$x(t) = t^2(bt^2+ct+1)(t-1)^2, \qquad y(t) = t^2(at+1), \qquad z(t) = (t-1)^3. \tag{1}$$

Partially supported by Grants RFFI-96-01-01218 and DGICYT SAB95-0502

The singularities E_6 and A_2 are provided by (1). The condition for A_8 has the form

$$\exists\ \gamma_2, \gamma_3 \quad \text{such that} \qquad \mathrm{ord}_{t=0}(vz^2 - \gamma_2 u^2 z - \gamma_3 u^3) = 8 \tag{2}$$

where $u = x+y$, $v = x-y$. Expanding $vz^2-\gamma_2u^2z-\gamma_3u^3$ in the form $e_3t^3+e_4t^4+\dots$ where e_j are polynomial expressions in the variables $a, b, c, \gamma_2, \gamma_3$ we see that (2) gives a system of simultaneous equations and inequalities $e_3 = \dots = e_7 = 0$, $e_8 \neq 0$. This system has 4 solutions two of which are real. We choose the solution

$$a = \alpha, \qquad b = -(6\alpha^2 + 11\alpha + 3)/11, \qquad c = \alpha + 2,$$
$$\gamma_2 = (6\alpha^2 + 33\alpha + 36)/44, \qquad \gamma_3 = -(111\alpha^2 + 374\alpha + 347)/968$$

where $\alpha = -5.1046...$ is the single real root of the equation

$$3\alpha^3 + 24\alpha^2 + 51\alpha + 34 = 0 \tag{3}$$

Lemma. *C is arranged with respect the coordinate axes as in Fig. 2.*

Corollary. (see Lemma in [6].) *C can be smoothed as it is depicted in Fig. 1.*

The Lemma can be proven by the method proposed in [4], but this way requires a lot of calculations which hardly can be performed without a computer. Here we give a geometrical proof where all the computations can be checked manually.

Denote by C_0 the principal connected component of C — the image of $\mathbf{RP}^1$ (other components are isolated double points). The roots (with multiplicities) of the polynomials $x(t)$, $y(t)$, $z(t)$ are respectively:

$$\begin{array}{rcccccc} x(t): & -0.53... & 0 & 0 & 0.20... & 1 & 1 \\ y(t): & 0 & 0 & 1.19... & \infty & \infty & \infty \\ z(t): & 1 & 1 & 1 & \infty & \infty & \infty \end{array}$$

Hence, C_0 intersects the coordinate axes in the order shown in the Fig. 2. Since $\gamma_2 \approx 0.543 > 0$, the branch of C at A_8 is located with respect to the tangent $x = y$ as in Fig. 2. The genus formula implies that C has two more ordinary double points. Let us show that they are real. Indeed, otherwise we would obtain a contradiction with the Fiedler's orientations interchanging rule [7] on the segment of the pencil of lines $[L_1L_2]$ (see Fig. 2).

Let us show that each of the points A_1 provides an empty oval[1] by a suitable smoothing. The coordinate axes cut C_0 into 6 arcs. Three of them can not intersect the others because they are separated by the coordinate axes. We have $y(t) - \alpha z(t) = (3\alpha+1)t^2 - 3\alpha t + \alpha$. The discriminant of this poynomial in t equals $-3\alpha^2 - 4\alpha \approx -57.7 < 0$, hence, the tangent $y = \alpha z$ at E_6 separates all the other pairs of the arcs except the two arcs starting at A_8. If they have two intersection points then an M-smoothing of C with 6 interior ovals would exist which is impossible. If they have one intersection point then three interior ovals of an M-smoothing would be negative which contradicts Rokhlin's formula for complex orientations [8]. Thus, each of A_1 is either an isolated point or a self-intersection point of one of the arcs (it

[1] An oval is called empty if there is no other ovals inside it.

will be seen below that the latter is impossible). In any case there is a smoothing providing an empty oval.

As we have shown above, one of the points A_1 must lie between L_1 and L_2. Denote by V_2 the oval obtained by its smoothing. The same arguments applied to the segment $-\infty < \lambda < 0$ of the pencil of lines $y = \lambda x$ (the perturbations of the lines $x = 0$ and $y = 0$, and some complex orientations are depicted in Fig. 1) show that the other point A_1 (denote the corresponding oval by V_1) must be in the half-plane $H = \{xy > 0\}$. Suppose that V_1 is interior. Then it can not be in the left sector of H (by Bézout theorem), but if it were in the right sector then it would be oriented clockwise by the orientation interchanging rule which again contradicts to the formula for complex orientations.

Thus, V_1 is exterior, hence V_2 is interior, because the number of interior ovals of an M-sextic can not be 4. It follows from the formula for complex orientations that V_2 is oriented clockwise, hence, V_2 is in the left lower sector of $\mathbf{RP}^2 \setminus (L_1 \cup L_2)$ by the orienration interchanging rule. One can see from the complex orientations (see Fig. 1) that the both points A_1 are isolated. The Lemma is proven.

Remarks. **1.** The existence of an independent smoothing of singularities is presented in [6] only in the case of an irreducible projective curve. However, an analoguouse, needed for us result for reducible curves can be proven the same way (see Remark in the end of the paper [6]). Our construction can be reduced to the irreducible case. To this end, it suffices to write down explicitly a deformation C_ε of C putting $z(t) = (t-1)^3(1-\varepsilon t)$ into (1) and preserving the condition (2). A computation shows that the set of real points of C_ε for $\varepsilon = 1/8$ is obtained from Fig. 2 by a rotation of the cusp A_2 counter-clockwise.

2. Since C has a point E_6 of multiplicity 3, one can reduce its degree by birational transformations. However, the coordinate axes will be transformed into conics or cubics and, in our opinion, the problem will not be simplified.

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