ARRANGEMENTS OF AN M-QUINTIC WITH RESPECT TO A CONIC WHICH MAXIMALLY INTERSECTS ITS ODD BRANCH

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Introduction

0.1. Statement of main results.

Connected components of the set of real points of a plane projective real curve are called *branches*. A branch is called *even* (or an *oval*), if it is zero-homologous in $\mathbb{RP}^2$. Otherwise it is called *odd* (or a *pseudoline*).

Theorem 1. *a). Let $\mathcal{J}$ be a tame almost complex structure in $\mathbb{CP}^2$ which is invariant under the complex conjugation, and let C_5 and C_2 be nonsingular real $\mathcal{J}$-holomorphic M-curves in $\mathbb{RP}^2$ of degrees 5 and 2 respectively. Let J_5 be the odd branch of C_5. Suppose that J_5 intersects C_2 at ten distinct real points. Then the arrangement of $C_5 \cup C_2$ in $\mathbb{RP}^2$ is one of those listed in Sect. 0.5 up to isotopy. All these arrangements are realizable.*

b). All the arrangements except the six of them labeled by "$\nexists^*$ alg." or "$\nexists$ alg." are realizable by real algebraic curves of degrees 5 and 2.

c). The two arrangements labeled by "$\nexists$ alg." are unrealizable by real algebraic curves of degrees 5 and 2.

Theorem 2. *a). Let $\mathcal{J}$ be a tame almost complex structure in $\mathbb{CP}^2$ which is invariant under the complex conjugation, and let C_5, L_1, and L_2 be nonsingular real $\mathcal{J}$-holomorphic M-curves in $\mathbb{RP}^2$ of degrees 5, 1, and 1 respectively. Suppose that the odd branch J_5 of C_5 intersects each of the lines L_1 and L_2 at five distinct real points. Then either the arrangement of $C_5 \cup L_1 \cup L_2$ in $\mathbb{RP}^2$ is one of those listed in Sections 0.6, 0.7, and in Figures 16.1–16.22, or $C_5 \cup L_1 \cup L_2$ realizes one of the sixteen arrangements such that $L_1 = \{x = 0\}$, $L_2 = \{x + \varepsilon y = 0\}$ where L_1 is a line intersecting J_5 at five points and $0 < \varepsilon \ll 1$.*

All these arrangements are realizable.

b). Among them, algebraically realizable are:

— The arrangements listed in Sections 0.6 and 0.8;

— The arrangements listed in Sect. 0.7 except the five ones labeled by "$\nexists^*$ alg." or "$\nexists$ alg.";

— The arrangements in Figures 16.1 – 16.11 and 16.13 – 16.22.

c). The three arrangements in Sect. 0.7 labeled by "$\nexists$ alg." and the one in Fig. 16.12 are unrealizable by real algebraic curves.

Theorems 1 and 2 are proved in §§ 2–7. A general scheme of the proof is given in Sections 0.2 and 0.4.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Remark 1. I know a proof of the algebraic unrealizability of the four arrangements in Sect. 0.5 and of the two ones in Sect. 0.7, which are labeled by "$\nexists^*$ alg.", but this paper is so long that I decided not to include it. Maybe, I shall write it somewhere else.

Remark 2. A big part of the classification described by Theorem 2 was obtained earlier in [9, 10] (see in more detail at the end of Sect. 0.4). However, it happens that the most part of unrealizability statements (restrictions) proven there, can be obtained "for free" as corollaries of other results of the present paper (see Sect. 7.1). So, for the reader's convenience, I added Sect. 7.2 (only half a page long) where I give another proof of the remaining restrictions from [9, 10], making the restriction part of Theorems 1 and 2 self-contained. In contrary, I give here proofs of the realizability (constructions) only when they are not given in [9, 10].

A particular case of Theorem 1 admits the following generalization for an arbitrary degree. This statement is used essentially in the proof of Theorems 1 and 2 (see Sect. 6.5). Let O be an oval in $\mathbb{RP}^2$. It divides $\mathbb{RP}^2$ into a disk D and a Möbius band M. Let us say that n pairwise disjoint smoothly embedded segments $I_1, \dots, I_n$ with the endpoints on O form a *nest of depth n inside O* (respectively, *outside O*), if there exist embedded disks $D_1 \subset D_2 \subset \dots \subset D_n \subset \mathbb{RP}^2$ such that $I_j = D \cap (\partial D_j)$ (respectively, $I_j = M \cap (\partial D_j)$).

Theorem 3. *Let C_2 and C_n be real pseudoholomorphic (for example, real algebraic) nonsingular M-curves in $\mathbb{RP}^2$ of degrees 2 and n respectively. Suppose that C_2 intersects a branch B_n of C_n at $2n$ distinct real points and that all other branches of C_n are contained in the exterior of C_2. Suppose that B_n has the parity of n (i.e., either n is even, or n is odd and then B_n is the odd branch of C_n). Then there exists a nest of depth n inside C_2 formed by arcs of B_n.*

This result is proven in §1. The proof is based on the complex orientations formula for curves on a 2-sphere and on a generalization for quadrics of Fiedler's theorem [3] on symmetric M-curves. It is independent of the rest of the introduction.

Remark 3. The following question suggests itself from comparing Theorem 3 with the classification of arrangements of an M-quintic with respect to a conic which maximally intersects one of its ovals [14]:

Suppose that B_n and n have different parities (i.e., n is odd and B_n is an oval of C_n) but all the other conditions of Theorem 3 are satisfied. Does it imply that there exists a nest of depth $n-1$ inside C_2 formed by arcs of B_n?

The proof of Theorem 3 given below does not extend to this case by the following reason. The lift of the odd branch of C_n onto the double covering branched along C_2 is not connected, hence the lift of C_n is an $(M-1)$- rather than an M-curve on the hyperboloid, and the rest of the proof fails. On the other hand, this is an additional informal argument for the plausibility of the affirmative answer, because many properties of $(M-1)$-curves "differ by one" (whatever it means) from corresponding properties of M-curves.

I am grateful to G.M. Polotovskii for useful discussions and the informing me about main results of the paper [10] before finishing its writing.

0.2. Arrangements which are pseudoholomorphically realizable, but algebraically unrealizable.

As is seen in the statements of Theorems 1 and 2, some arrangements (those labeled by "$\nexists$ alg." in Sections 0.5, 0.7 and the one in Fig. 16.12) contribute to the collection of known examples of isotopy types which are realizable by real pseudoholomorphic curves but unrealizable by real algebraic curves.

The algebraic unrealizability of two of them (in Figures 21.1 and 16.12) is proven in Sections 4.1 and 4.4 respectively. The remaining cases are reduced to these two. Namely, the unrealizability of Fig. 21.2 and Fig. 21.3 is reduced to that of Fig. 16.12 and of Fig. 21.1 in Sect. 4.2 and in Sect. 4.3 respectively. The unrealizability of the two arrangements of a quintic and a conic labeled by "$\nexists$ alg." is reduced to the corresponding arrangements of a quintic and a pair of lines by Corollary 5.2.

The proof of the algebraic unrealizability of Fig. 21.1 repeats almost word-by-word the end of the proof of non-existence of a real algebraic affine sextic of the type $C_2(1,3,6)$ in [22; §3.4]. However, the proof for Fig. 16.12 in Sect. 4.4 involves a new approach, more precisely, a new combination of two old ones: 1) consideration of the cubic resolvent R of a curve of bidegree $(4,n)$ on a ruled surface and its arrangement with respect to the core L (this idea was already used by the author in [18; §6] and in [22; §3]); and 2) application of Burau representation of the groupoid of colored braids to the braid corresponding to $R\cup L$ (this idea was used in [15]). In Remark 4.7, a more detailed (with respect to [15]) description of the Burau representation of the groupoid of colored braids is given. In my opinion, it is more convenient for practical computations.

In Appendix A, this approach is applied to obtain a new "and a more reliable" (see the discussion after Proposition A.1) proof of one of main results of [21].

In Appendix C, we apply the method of cubic resolvents to prove the algebraic unrealizability of some (non-fiberwise!) isotopy types on ruled surface $\mathcal{F}_4$. Moreover, these isotopy types are pseudoholomorphically realizable in a tame almost complex structure with the exceptional curve.

0.3. Zigzag removal.

All the pseudoholomorphic unrealizability statements not provided by Theorem 3, are proven in this paper by using the method proposed in [13, 16], i.e., by reducing the question of the realizability of a curve to the question of the quasipositivity of some braids (but the collection of quasipositivity criterions is enriched here by a result of Florens [5], see Theorem 6.6).

The braids are determined by the arrangements of the curve with respect to a pencil of lines (fiberwise arrangements). To exclude a given isotopy type, one should consider all corresponding fiberwise arrangements. The number of cases can be considerably reduced using the fact that the unrealizability of some fiberwise arrangements follows from the unrealizability of the arrangements obtained from them by certain elementary moves (for example, by rectifying S-like zigzags).

In all previous papers where this method was used by myself or by other authors, the reduction of this sort of all fiberwise arrangements to some restricted subset of them was supposed evident, and the details (consisting in routine case-by-case considerations) were omitted. Usually, it was really evident, but in the case of a quintic and a conic (as well as of a quartic and a cubic) the number of cases to be considered becomes too large.

One very particular arrangement of a quintic and a conic was excluded in [13;

Theorem 1.2A] as an example of application of the method of braids. However, this statement is wrong and the mistake appeared because I missed one possibility for the fiberwise arrangement, the one which is realizable (this mistake is discussed in [9]). Some other similar restrictions are formulated in [7], but most of them are erroneous by the same reason. To avoid repeating of this kind of errors, some rigorous statements about reduction of fiberwise arrangements to each others are formulated and proven in Sections 6.3 and 6.4.

Note, that the statement on removing of S-like zigzags holds only for pseudo-holomorphic curves. An example given in Appendix B shows that it in general it is wrong for algebraic curves.

0.4. Types and series. General scheme of proofs of Theorems 1 and 2. Following Polotovskii, let us divide all arrangements mentioned in Theorems 1 and 2 into eight types ($8 = 3 + 5$) as follows.

Let C_5, J_5, and C_2 be as in Theorem 1. Let Γ_∞ be a pseudoline (i.e., an embedded circle which is not contractible in $\mathbb{RP}^2$) disjoint from C_2 and intersecting J_5 at the minimal possible number of points n. Then the intersection points are called *passages through infinity* and n is called the *type* of the arrangement of C_5 with respect to C_2. It is clear that n can take only the values 1, 3, or 5.

Let C_5, J_5, L_1, and L_2 be as in Theorem 2. The lines L_1 and L_2 divide $\mathbb{RP}^2$ into two domains. Let us denote them by D_1 and D_2. The lines cut J_5 into ten arcs. Let d_j, $j = 1, 2$, be the number of the arcs lying in D_j and connecting one line to the other one. The unordered pair (d_1, d_2) is called the *type* of the arrangement of C_5 with respect to L_1 and L_2. It is clear that each of d_1, d_2 can take only the values 1, 3, and 5, the type (5,5) being impossible. Thus, all the arrangements are naturally divided into five types: (1,1), (1,3), (3,3), (1,5), and (3,5).

We shall subdivide each type into *series* according to the isotopy types of the arrangements of J_5 with respect to C_2 or $L_1 \cup L_2$. In the lists in Sections 0.5 – 0.7 and throughout the paper, the series are referred by numbers typed in the bold face.

It is clear that the objects described by Theorems 1 and 2 are very close to each other. Indeed, when a pair of lines is perturbed into a nonsingular conic, a curve $C_5 \cup L_1 \cup L_2$ satisfying the hypothesis of Theorem 2 is transformed into a curve $C_5 \cup C_2$ satisfying the hypothesis of Theorem 1. Vice versa, in many cases (see §5) a curve $C_5 \cup C_2$ satisfying the hypothesis of Theorem 1 can be degenerated into a curve satisfying the hypothesis of Theorem 2. By this reason, many cases of the classification of curves of the form $C_5 \cup C_2$ may be reduced to the classification of $C_5 \cup L_1 \cup L_2$ and vise versa. This concerns equally algebraic and pseudoholomorphic curves.

In constructions we use only perturbations $L_1 \cup L_2 \to C_2$. However, in restrictions we use such reductions in the both directions. Therefore, to eliminate any possible reader's suspecting in a vicious circle, let us present the main steps of the proof of the restrictions in the pseudoholomorphic case (the algebraic case was discussed in Sect. 0.2).

Step 1. In the following cases, both algebraic and pseudoholomorphic classifications are reduced to the classification of arrangements of $C_5 \cup C_1$ — maximally intersecting quintic and single line:

(1.1) arrangements of $C_5 \cup L_1 \cup L_2$ of the types (1,5) and (3,5);
(1.2) arrangements of $C_5 \cup C_2$ with five passages through infinity;

(1.3) arrangements of $C_5 \cup C_2$ with one or three passages through infinity which have five nested arcs inside C_2 (see the definition in Sect 0.1 before the formulation of Theorem 3).

The classification of $C_5 \cup C_1$ was originally obtained by Polotovskii [23] using methods working only for algebraic curves. However, it can also be derived from Kharlamov-Viro congruence [8] (which has been found partially due to this classification). This congruence holds for pseudoholomorphic curves as well.

Step 2. (§3). Complete classification of the arrangements of $C_5 \cup L_1 \cup L_2$ of the type (3,3).

Step 3. (§6). Restrictions for the arrangements of $C_5 \cup C_2$ with one passage through infinity of the series listed in Sect 6.1.

Step 4. (Sect. 7.1). Restrictions for the arrangements of $C_5 \cup L_1 \cup L_2$ of the types (1,1) and (1,3) which are reduced to the restrictions for $C_5 \cup C_2$ proven in Steps 1.2, 1.3, and 3.

Step 5. (Sect. 7.2 or a reference to [9, 10]). Restrictions for the arrangements of $C_5 \cup L_1 \cup L_2$ of the types (1,1) and (1,3) which were not proven in Step 4. This completes the proof of Theorem 2.

Step 6. By Corollary 5.2, all the remaining restrictions for the arrangements of $C_5 \cup C_2$ (i.e., for the series with one passage through infinity which were not considered in Step 3 and for all arrangements with three passages through infinity) are reduced to Theorem 2.

In the following table, we present the distribution of the realizable arrangements over the types (according to Remark 1 in Sect. 0.1, the last two lines may be summed up).

	$C_5 \cup C_2$			$C_5 \cup L_1 \cup L_2$				
Type of arrangement	1	3	5	(1,1)	(1,3)	(3,3)	(1,5)	(3,5)
number of series	33	15	2	8	19	7	3	1
number of ps-hol. arr.	124	48	4	20	56	22	12	4
number of "$\nexists$ alg."	–	2	–	–	3	1	–	–
number of "$\nexists^*$ alg."	3	1	–	–	2	–	–	–

A complete classification of the arrangements of $C_5 \cup L_1 \cup L_2$ for the type (1,1) and an almost complete (except about 15 cases) for the type (1,3) is obtained in [9, 10]. Almost all the proofs of restrictions in [9, 10] extend automatically to the case of pseudoholomorphic curves (taking into account the remark in Step 1 about the classification of $C_5 \cup C_1$). The only exception is the arrangement denoted in [9] by $D4$. Its unrealizability is derived in [9] from the algebraic unrealizability of affine sextics of the isotopy type $B_2(1,4,5)$ proven in [4]. This argument cannot work in the pseudoholomorphic case by the simple reason that such an affine pseudoholomorphic sextic is constructed in [13] (see also [4]). All proofs in [9] become valid in for pseudoholomorphic curves after adding the following line into [9; Table 2]:

$D4$	—"—	$X_2X_3X_4 \supset_3 \zeta_3\varepsilon_4X_3X_2X_2X_2X_2X_2\varepsilon_3 \subset_4 X_5X_5$	6	2; 4

0.5. The list of all pseudoholomorphic arrangements of an M-quintic and a nonsingular conic maximally intersecting the odd branch.

0.5.1. Encoding of the series. The list is organized by the same rules as the list in [18, §5]. To identify a series (isotopy type of a mutual arrangement of the intersecting branches), we shall use the encoding proposed by Polotovskii. Namely, let us number the points of $J_5 \cap C_2$ by $0, 1, \ldots, 9$ in their order along C_2, so that the point 0 is the endpoint of a component of $J_5 \setminus C_2$ crossing Γ_∞ and the point 1 is not (Γ_∞ means the same as in the beginning of Sect. 0.4). The isotopy type of $J_5 \cup C_2$ will be encoded by a word $i_0 i_1 \ldots i_9$ composed of the digits $0, \ldots, 9$ appearing in the same order as the corresponding points appear on J_5. Moreover, the arc (i_9, i_0) of the quintic must pass through infinity (i.e., it must intersect Γ_∞). Among all words encoding the same isotopy type (if there are no symmetries, the number of them is twice the number of the passages through infinity), we shall always choose the minimal one with respect to the lexicographical order. For the reader's convenience, we shall denote the passages through infinity by the slash "/".

First, we list the arrangements with one passage through infinity, then those with three passages, and then those with five passages. Series of the same type are ordered lexicographically (ignoring the symbol "/"). Arrangements of the same series are placed in a random order. The points $0, \ldots, 9$ are not labeled in the pictures, but they are supposed to be placed on C_2 clockwise, the point "0" being always the uppermost. The numbers written in the pictures denote the quantities of the ovals of the quintic in the corresponding regions.

0.5.2. Encoding of the constructions. Under each arrangement, a reference to its construction(s) is given. This is either the number of a figure with a singular curve to be perturbed (an expression of the form 16.2,7 being an abbreviation for 16.2, 16.7), or one of the following notation.

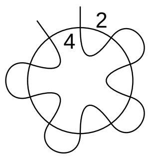
$5+1$. Means that $C_2 = \{c_1^2 = \varepsilon f\}$ where $\{c_1 = 0\}$ is a line intersecting J_5 at five distinct points and $0 < \varepsilon \ll 1$.

m_n. Means that C_2 is a perturbation of $L_1 \cup L_2$ in the n-th arrangement of $C_5 \cup L_1 \cup L_2$ from the m-th series of the type (1,3) in Sect. 0.7.

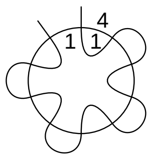
A1, B5, etc. mean that C_2 is a perturbation of $L_1 \cup L_2$ in an arrangement of $C_5 \cup L_1 \cup L_2$ of the type (1,1) constructed in [9] (see also Sect. 0.6).

0.5.3. Arrangements of $C_5 \cup C_2$ with one passage through infinity.

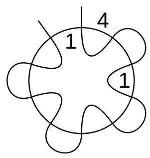
1. 0123456789



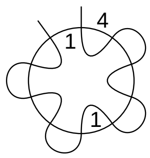
A15, 5.2, 9.3



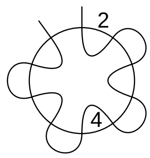
A12, 2.1



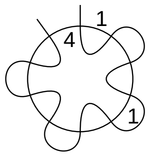
A11, 9.3



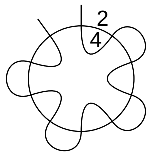
A13



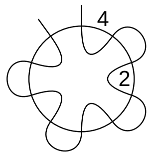
A10



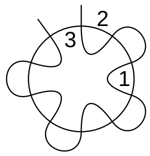
A14



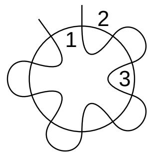
5.1



5.3, 9.1

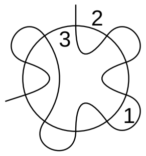


2.2, 9.1, 9.2

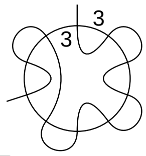


9.2

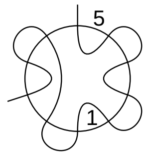
2. 0123456987



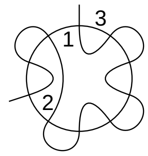
D11



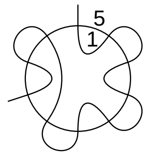
D13, 5.2, 5.3, 9.3



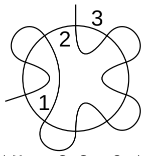
D14



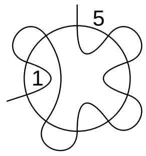
D3, 9.3



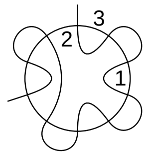
D7, 2.1, 7.2



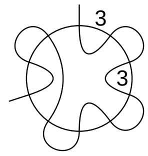
15_1, 2.2, 9.1, 9.2



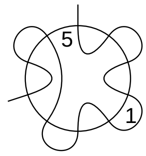
$15_{2,3}$



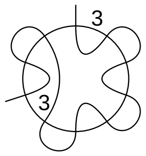
15_4, 9.5



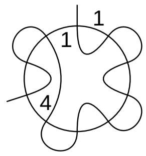
15_5, 9.5



15_6

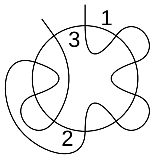


9.1

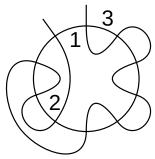


9.2

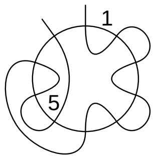
3. 0123458769



B1

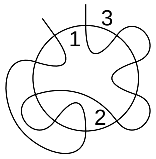


B1, 2.1

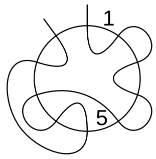


5.1

4. 0123476589

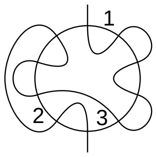


C1

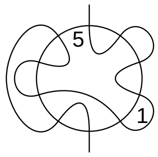


C2

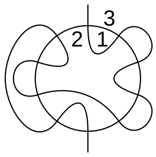
5. 0123478965



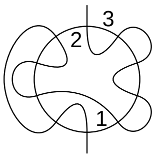
D3, 13_1



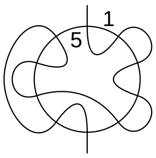
D7, 13_3



D11, 13_4

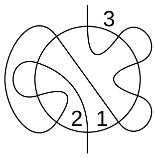


D13, 13_2

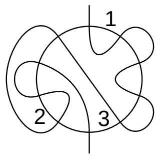


D14, 5.2

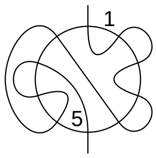
6. 0123496785



B1, 11_3

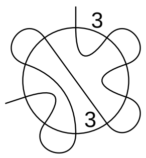


B2, 11_2

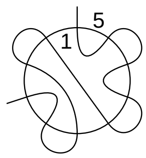


11_1 ($\nexists^*$ alg.)

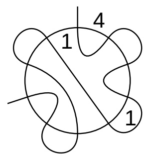
7. 0123498567



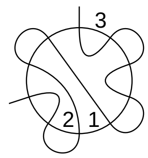
C2, 5.3



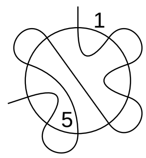
C5



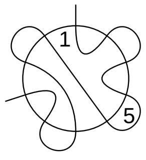
4_1



4_2, 9.8

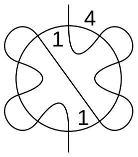


4_3, 9.8

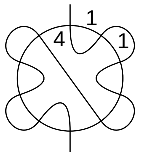


4_4

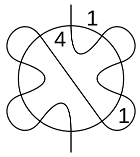
8. 0123498765



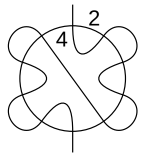
A10, A15, 18_3



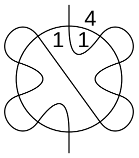
A11, 18_1



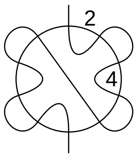
A12, 18_2



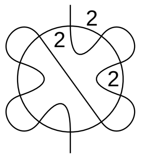
A13, 5.2



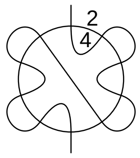
A14, 18_4



19_2, 9.8

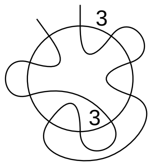


19_1, 9.8

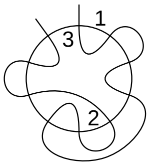


18_5 ($\nexists^*$ alg.)

9. 0123654789

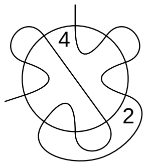


5.3, 9.4

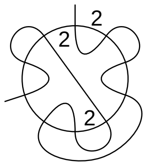


2.2, 9.4

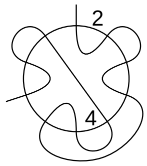
10. 0123654987



17_1

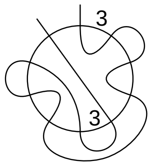


17_2, 2.2, 9.4

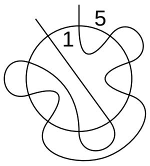


5.2, 9.4

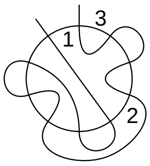
11. 0123678549



5.1

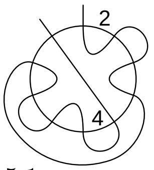


2.1

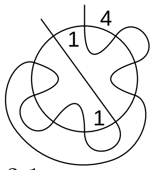


11.4

12. 0123876549

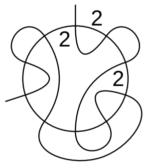


5.1



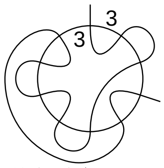
2.1

13. 0125436987

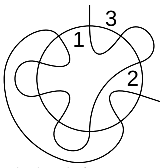


16_1

14. 0125678943

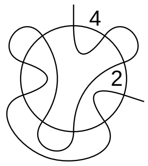


5.2, 9.7

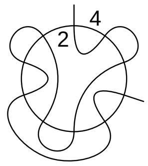


2.2, 9.7

15. 0125698743

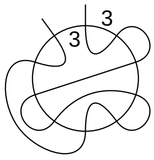


7_1, 2.2

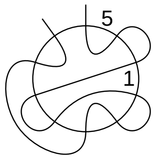


7_2, 5.2

16. 0127634589

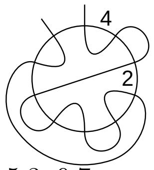


1_1, 2.2, 9.6

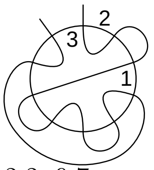


1_2, 5.3, 9.6

17. 0127654389

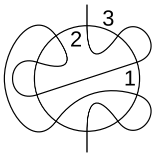


5.3, 9.7

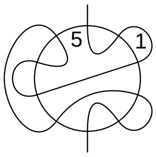


2.2, 9.7

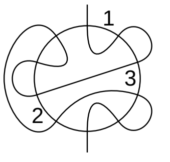
18. 0127896345



2_1,3_1

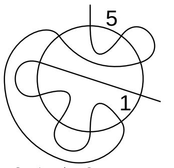


2_2,3_2



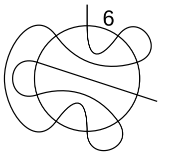
2_3,3_3

19. 0129456783

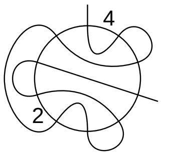


2.1, 7.2

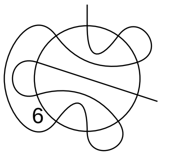
20. 0129654783



$5 + 1$

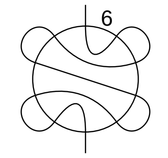


$5 + 1$

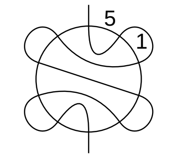


$5 + 1$

21. 0129834765



$5 + 1$



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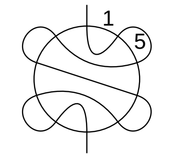
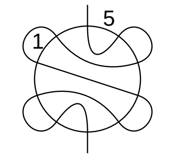
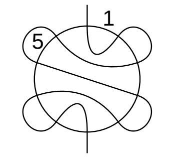


$5 + 1$

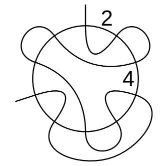


$5 + 1$



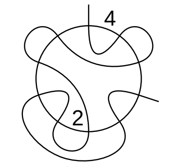
$5 + 1$

22. 0129854367



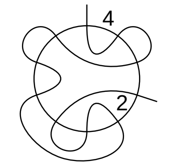
F2, 5.3

23. 0129856743

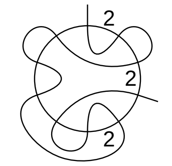


F2

24. 0129874563

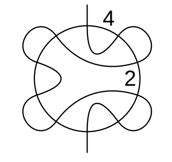


E2, 2.1, 7.2

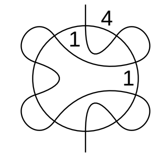


E4

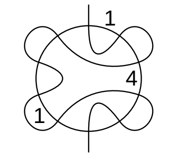
25. 0129876345



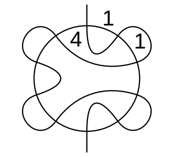
G5, 5.3



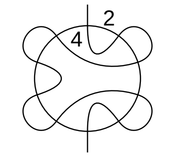
$6_{1,4}$, 9.5



$6_{2,3}$

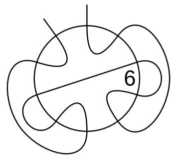


$6_{5,7}$



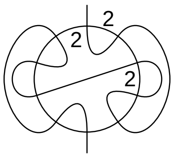
6_6, 9.5

26. 0143276589



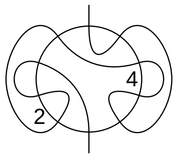
H5

27. 0143278965

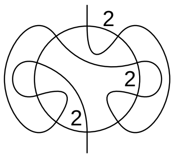


G5, 14_1

28. 0143296785

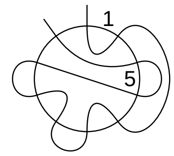


E2, 12_1

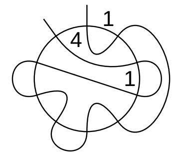


E4, 12_2

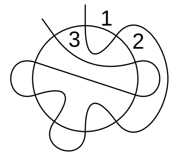
29. 0145678329



8_1, 5.1, 7.1

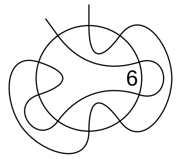


8_2, 7.1

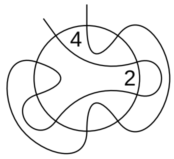


8_3 ($\nexists^*$ alg.)

30. 0145876329

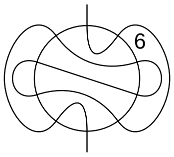


9_1, 5.1, 7.1

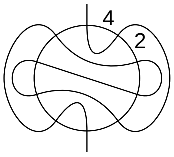


9_2, 7.1

31. 0147832965

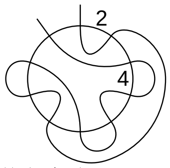


$5 + 1$

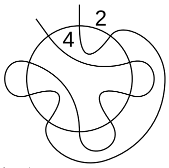


$5 + 1$

32. 0167854329

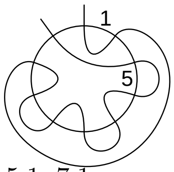


5.1, 7.1

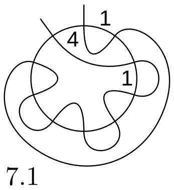


7.1

33. 0187654329



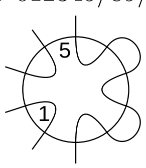
5.1, 7.1



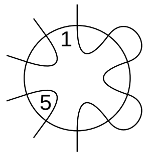
7.1

0.5.4. Arrangements of $C_5 \cup C_2$ with three passages through infinity.

34. 012345/89/67

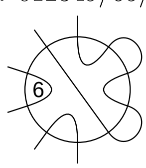


5_2

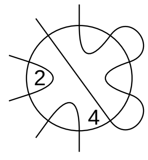


5_1

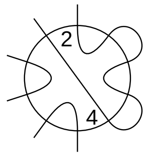
35. 012349/65/87



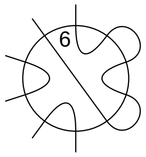
16.19



16.20

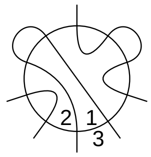


16.21

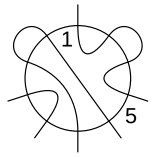


16.22

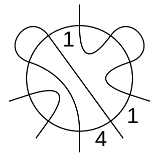
36. 0123/67/4985



16.12, 8_3 ($\nexists$ alg.)

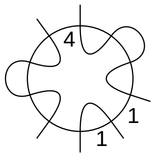


8_1, 16.13

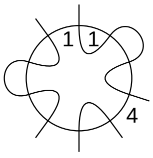


8_2, 16.14

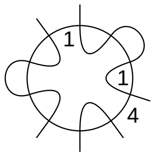
37. 0123/6789/45



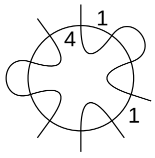
6_1,18_4



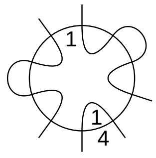
$6_{2,6}$,18_1



$6_{3,5}$,18_2

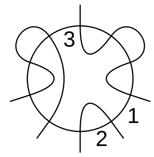


6_4,18_3

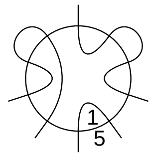


18_5 ($\nexists^*$ alg.)

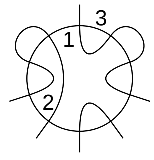
38. 0123/6987/45



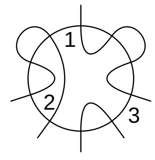
3_1,11_3



3_2,11_1 ($\nexists$ alg.)

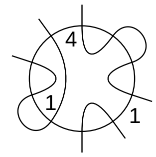


3_3

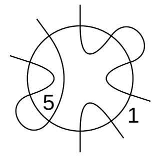


11_2

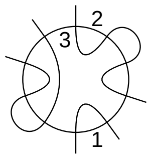
39. 0123/8769/45



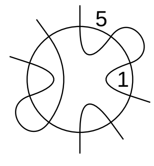
15_2, 16.1,6



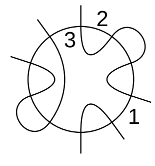
15_3, 16.2,7



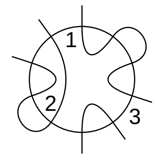
13_4,15_1, 16.3,5



2_2,13_3,15_6, 16.4

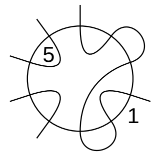


2_1,13_2,15_4

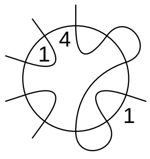


2_3,13_1,15_5

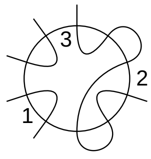
40. 012543/89/67



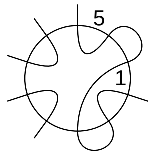
4_4, 16.15



4_1, 16.16

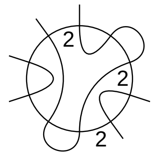


4_2, 16.17

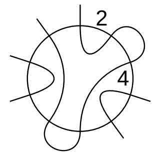


4_3, 16.18

41. 012569/43/87

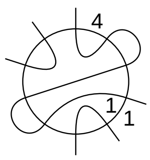


7_1, 16.9,10

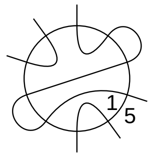


7_2, 16.8,11

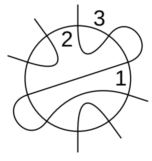
42. 012763/89/45



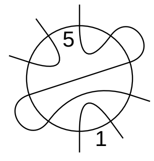
16.1,16



16.2,15

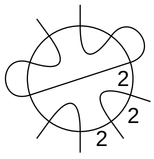


1_1, 16.3,17

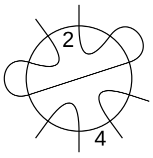


1_2, 16.4,18

43. 012789/43/65

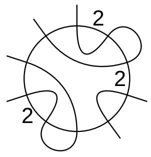


19_1, 16.10



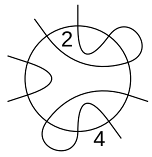
19_2, 16.11

44. 0129/43/8567

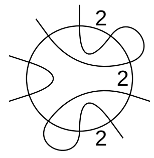


14_1,16_1

45. 0129/4563/87

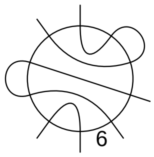


12_1

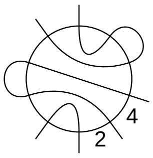


12_2

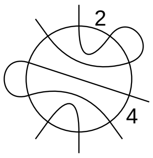
46. 0129/4783/65



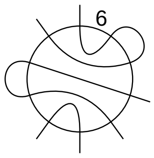
16.19



16.20

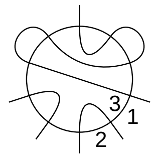


9_2, 16.21

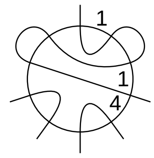


9_1, 16.22

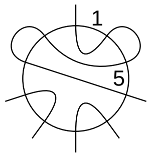
47. 012983/67/45



16.5,12

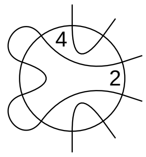


16.6,14

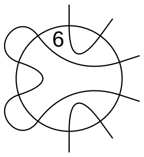


16.7,13

48. 01/45/298763



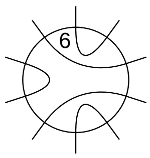
10_1



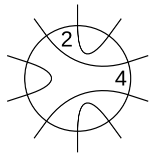
10_2

0.5.5. Arrangements of $C_5 \cup C_2$ with five passages through infinity.

49. 01/63/87/29/45

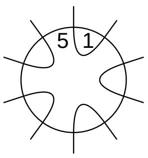


$5+1$

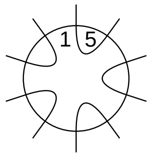


$5+1$

50. 01/67/23/89/45



$5+1$

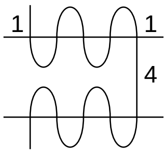


$5+1$

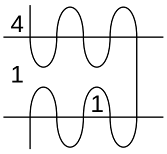
0.6. Pseudoholomorphically realizable arrangements of an M-quintic and two lines of the type (1,1). Here we reproduce from [9] the list of realizable arrangements of $C_5 \cup L_1 \cup L_2$ of the type (1,1). The list is organized more or less as in Sect. 0.5. The encoding of arrangements of the intersecting branches is as follows (as in Sect. 0.5, it was introduced by Polotovskii).

Let us denote the points of $L_1 \cap J_5$ by 0,1,2,3,4, and the points of $L_2 \cap J_5$ by 5,6,7,8,9, placed in the order in which they appear on $L_1 \setminus \{p\}$ and on $L_2 \setminus \{p\}$ respectively (here $\{p\} = L_1 \cap L_2$). Let D_2 be that connected component of $\mathbb{RP}^2 \setminus (L_1 \cup L_2)$ on whose boundary the points $0, 1, \dots, 9$ appear in this order, and let D_1 be the other component. A series is encoded by a word $i_0 \dots i_9$, composed of the digits $0, \dots, 9$, ordered in the same way as the corresponding points lye on J_5, and so that the points i_0 and i_9 are on different lines and the arc (i_0, i_9) of J_5 is contained in D_1. As in Sect. 0.5, among all words we choose the lexicographically minimal one. The lines L_1 and L_2 are depicted horizontally (p being at the infinity), and the points $0 \dots 9$ are placed in the order $\begin{smallmatrix} 0 & 1 & 2 & 3 & 4 \\ 9 & 8 & 7 & 6 & 5 \end{smallmatrix}$ (this implies that D_2 is represented in the pictures by a horizontal band). A notation from [9] is written under each arrangement.

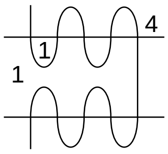
1. 0123456789



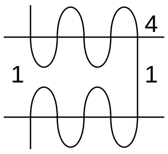
A10



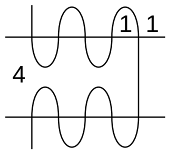
A11



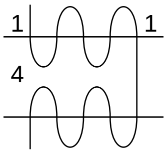
A12



A13



A14



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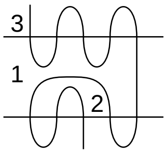

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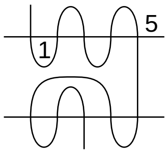

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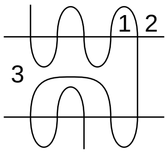

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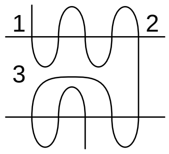

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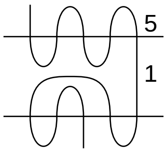

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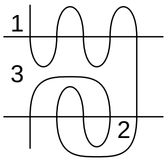

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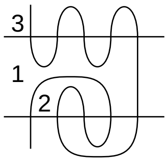

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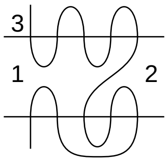

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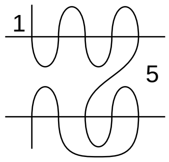

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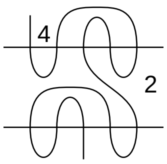

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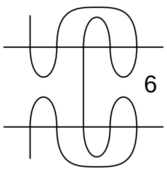

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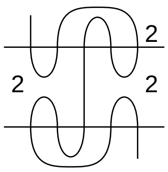

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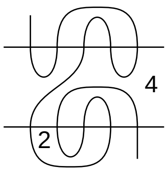

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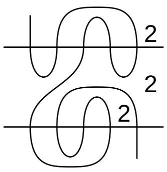

E4

0.7. Pseudoholomorphic arrangements of $C_5 \cup L_1 \cup L_2$ of the type (1,3). The list is organized as in Sect. 0.6, but we assume in the encoding that $d_1 = 1$, $d_2 = 3$ (d_1 and d_2 are as in Sect. 0.4). Under some arrangements we refer to its construction (singular curve to perturb). The others are constructed in [10].

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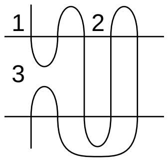

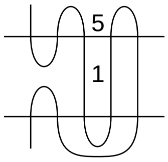

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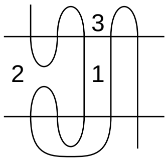

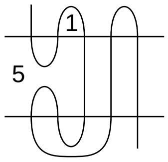

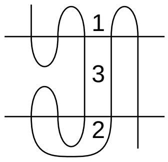

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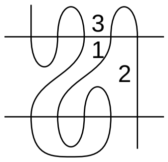
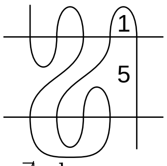
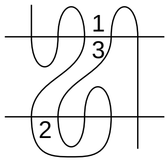




∄ alg.



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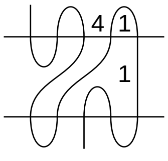

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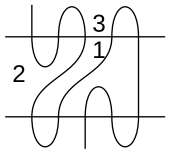

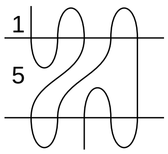

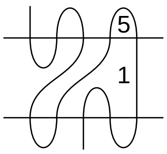

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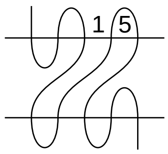

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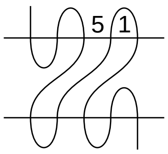

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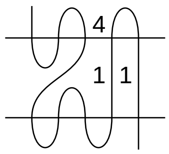

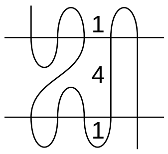

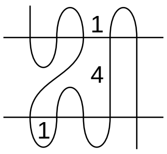

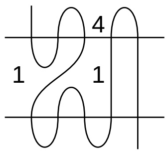

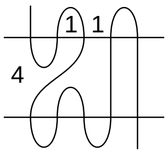

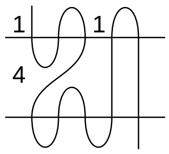

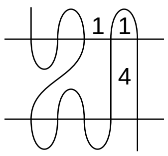

∄* alg.

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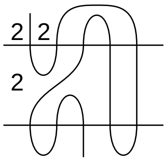

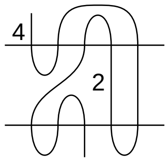

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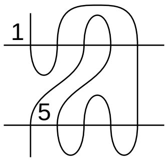

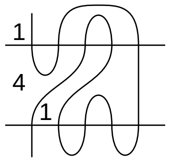

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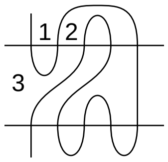

∄ alg.

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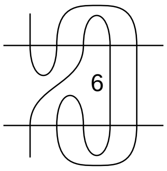

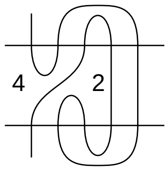

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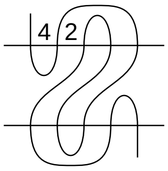

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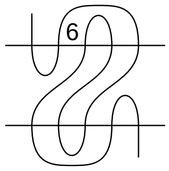

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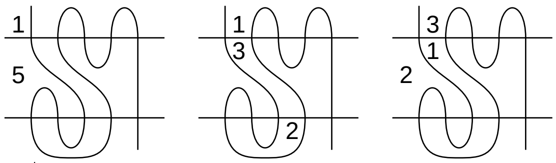

∄ alg.

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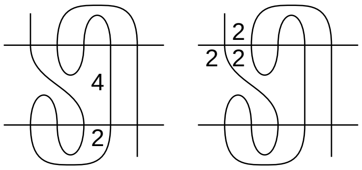

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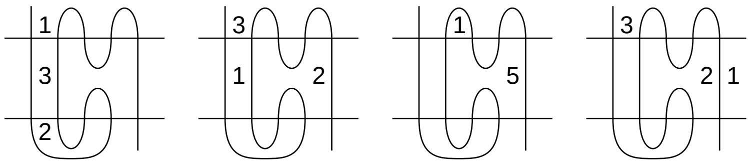

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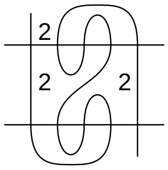

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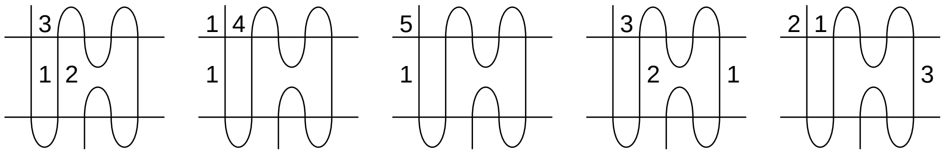
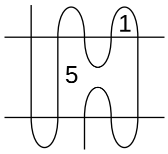


15.1 – 15.4



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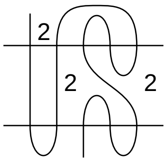

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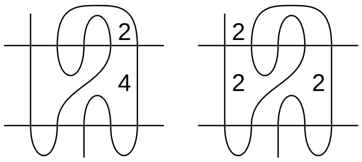

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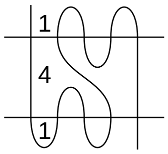
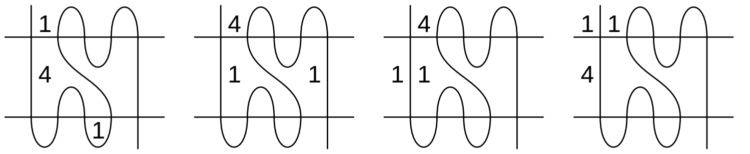
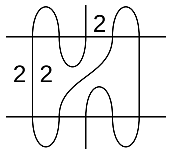
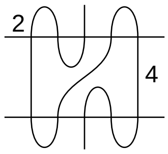




∄* alg.

19. 2109834567





§1. Curves on quadrics: formula of complex orientations, a generalization of Fiedler's theorem on symmetric M-curves, and the proof of Theorem 3

1.1. Rohlin's formula of complex orientations and a characterization of hyperbolic curves on $\mathbb{RP}^2$.

Let A be a nonsingular real algebraic curve on $\mathbb{RP}^2$ (the set of its complex points) and let $\mathbb{R}A$ be the set of its real points. Suppose that the curve A is *dividing*, i.e., $A \setminus \mathbb{R}A$ has two connected components whose closures we denote by A_+ and A_-. A *complex orientation* of $\mathbb{R}A$ is the boundary orientation coming from one of the halves A_+ or A_-.

Let Π_+ (respectively, Π_-) be the number of pairs of ovals (O, o) of $\mathbb{R}A$ such that o is inside O and $[o] = -[O]$ (respectively, $[O] = [o]$) in the first homology group of the annulus bounded in $\mathbb{RP}^2$ by these ovals. Let l be the number of ovals of $\mathbb{R}A$. Suppose that the degree of A is even and is equal to $2k$. Then the following Rohlin's formula of complex orientations holds

$$2(\Pi_+ - \Pi_-) = l - k^2. \tag{1}$$

Now, let us suppose that the degree of A is odd and is equal to $2k+1$. Let J be the odd branch of A. Let Λ_+ (respectively, Λ_-) be the number of ovals O such that $[O] = -2[J]$ (respectively, $[O] = 2[J]$) in the first homology group of the exterior of O. Then the following Rohlin-Mishachev's formula of complex orientations holds

$$(\Lambda_+ - \Lambda_-) + 2(\Pi_+ - \Pi_-) = l - k^2 - k. \tag{2}$$

One of corollaries of (1) and (2) is the following fact, also discovered by Rohlin. Let us recall that a real algebraic curve on $\mathbb{RP}^2$ is called *hyperbolic*, if there exists a point $p \in \mathbb{RP}^2$ such that any real line through p intersects A at d real points where d is the degree of A. This condition is equivalent to the fact that $\mathbb{R}A$ consists of $[d/2]$ nested ovals, and also of the odd branch when d is odd.

Proposition 1.1. *Let A be a dividing real algebraic curve on $\mathbb{RP}^2$ of degree $2k$ or $2k+1$ which has l ovals. Then $l \geq k$ and if $l = k$, then A is hyperbolic.*

Proof. We give a proof for the case of even degree. When the degree is odd, the proof is similar. It is clear that $(l^2 - l)/2 =$ (the number of pairs of ovals) $\geq$ (the number of nested pairs of ovals) $= \Pi_- + \Pi_+ \geq \Pi_- - \Pi_+$ and the latter expression is equal to $(k^2 - l)/2$ by (1). This implies $l^2 \geq k^2$ and hence, $l \geq k$. Moreover, if $l = k$, then we have the equality sign in all the inequalities, hence each pair of ovals is nested. $\square$

1.2. Zvonilov's formula of complex orientations and a characterization of hyperbolic curves on an ellipsoid.

Here we present Zvonilov's [25] formula of complex orientations for curves on an ellipsoid and rewrite it in a more invariant form, i.e. so that its ingredients do not depend on a choice of an auxiliary point. In the same way as for curves on $\mathbb{RP}^2$, we derive from it a characterization of hyperbolic curves on an ellipsoid as dividing curves which have the minimal possible number of ovals.

Let X be a complex quadric given in $\mathbb{CP}^3$ by $x_1^2 + x_2^2 + x_3^2 = x_0^2$, and let $\mathbb{R}X$ be the set of its real points, i.e., the ellipsoid given in $\mathbb{RP}^3$ by the same equation. Let

A be a nonsingular dividing real algebraic curve on X of degree $2k$ (in other words, of bidegree (k,k)). Let us fix a complex orientation on $\mathbb{R}A$.

Let us choose any point $p \in \mathbb{R}X \setminus \mathbb{R}A$. Let $O_1, \dots, O_l$ be the ovals of A. Each O_i divides $\mathbb{R}X$ into two disks. Let D_i be that of them which does not contain p. Let us endow these disks by the orientation induced from $\mathbb{R}X$. Let us set $\varepsilon_i = 1$ and call an oval O_i *positive*, if its complex orientation coincides with the boundary orientation induced from D_i. Otherwise we say that O_i is *negative* and set $\varepsilon_i = -1$ (these definitions depend on a choice of p).

Let Λ^p_+ and Λ^p_- be the number of positive and negative ovals respectively. Let $\Pi^p_\pm$ be the number of pairs (i,j) such that $i < j$, $D_i \cap D_j \ne \varnothing$, and $\varepsilon_i\varepsilon_j = \mp 1$. Then the following formula of complex orientations holds [25] (see also [2]):

$$2l - (\Lambda^p_- - \Lambda^p_+)^2 + 4(\Pi^p_- - \Pi^p_+) = k^2. \tag{3}$$

Now, let us express the left hand side of (3) in terms of quantities independent of a choice of p. Any pair of disjoint embedded circles divides a 2-sphere into three parts, two of whom are homeomorphic to a disk and the third one is homeomorphic to an annulus. Let us call a pair of ovals of $\mathbb{R}A$ *negative*, if they realize the same first homology class of the annulus bounded by them, and *positive* otherwise. Let Π_+ and Π_- be the number of positive and negative pairs of ovals of $\mathbb{R}A$.

Proposition 1.2. *The formula (1) holds for any dividing curve of bidegree (k,k) on an ellipsoid, which has l ovals.*

Proof. Let $\tilde\Pi^p_\pm$ be the number of pairs (i,j) such that $i < j$, $D_i \cap D_j = \varnothing$, and $\varepsilon_i\varepsilon_j = \pm 1$. It is clear that $\Lambda^p_+ - \Lambda^p_- = \varepsilon_1 + \dots + \varepsilon_l$. Hence

$$\begin{aligned}(\Lambda^p_- - \Lambda^p_+)^2 &= \Big(\sum_{i=1}^{l} \varepsilon_i^2\Big) + 2\Big(\sum_{D_i\cap D_j \ne \varnothing} \varepsilon_i\varepsilon_j\Big) + 2\Big(\sum_{D_i\cap D_j = \varnothing} \varepsilon_i\varepsilon_j\Big)\\ &= l + 2\big(\Pi^p_- - \Pi^p_+\big) - 2\big(\tilde\Pi^p_- - \tilde\Pi^p_+\big).\end{aligned}$$

Substituting $\tilde\Pi^p_\pm = \Pi_\pm - \Pi^p_\pm$ and adding (3), we obtain (1). $\square$

Let us say that pairwise disjoint smoothly embedded circles $O_1, \dots, O_r$ form a nest if there exist disks $D_1, \dots, D_r$ such that $\partial D_i = O_i$ and $D_1 \subset D_2 \subset \dots \subset D_r$. In this case r is called the *depth* of the nest. For example, two ovals always form a nest, but three ovals not always. A curve of bidegree (k,k) on an ellipsoid is called *hyperbolic*, if it contains a nest of depth k (and hence, by Bezout's theorem it cannot contain other ovals).

Proposition 1.3. *Let A be a dividing real algebraic curve of bidegree (k,k) on an ellipsoid, which has l ovals. Then $l \ge k$ and if $l = k$, then the curve A is hyperbolic.*

Proof. As in the proof of Proposition 1.1, formula (1) implies that $l \ge k$ and $\Pi_+ = 0$ for $l = k$. It remains to apply the following two evident facts:

1). If, among three ovals, any two of them form a negative pair, then all the three ovals form a nest.

2). If, among k ovals, any three of them form a nest, then all the k ovals also form a nest. $\square$

1.3. Fiedler's theorem on symmetric curves in $\mathbb{RP}^2$.

Let $(x_0 : x_1 : x_2)$ be homogeneous coordinates on $\mathbb{RP}^2 \subset \mathbb{CP}^2$. Let $s : \mathbb{CP}^2 \to \mathbb{CP}^2$ be the holomorphic involution $(x_0 : x_1 : x_2) \mapsto (-x_0 : x_1 : x_2) = (x_0 : -x_1 : -x_2)$. Let F be the set of fixed points of this involution. It is the union of the line $F_1 = \{x_0 = 0\}$ and the point $F_0 = (1 : 0 : 0)$. Let $c : \mathbb{CP}^2 \to \mathbb{CP}^2$ be the involution of the complex conjugation. We have $s \circ c = c \circ s$ (let us denote this involution by $\tilde{c}$). Any real (i.e., commuting with c) holomorphic involution of $\mathbb{CP}^2$ can be written in this form in suitable coordinates. Let $\widetilde{\mathbb{RP}}^2$ be the set of fixed points of $\tilde{c}$.

Let A be a real algebraic curve on $\mathbb{RP}^2$ which is symmetric with respect to s. Set $\mathbb{R}A = A \cap \mathbb{RP}^2$ and $\widetilde{\mathbb{R}A} = A \cap \widetilde{\mathbb{RP}}^2$. Let us call $\widetilde{\mathbb{R}A}$ the *complementary curve* of $\mathbb{R}A$.

In coordinates, the symmetricity of A means that it is defined by an equation $f(x_0^2, x_1, x_2) = 0$. After the coordinate change $\tilde{x}_0 = ix_0$, $\tilde{x}_1 = x_1$, $\tilde{x}_2 = x_2$, we have $\widetilde{\mathbb{RP}}^2 = \{(\tilde{x}_0 : \tilde{x}_1 : \tilde{x}_2) \,|\, \tilde{x}_0, \tilde{x}_1, \tilde{x}_2 \in \mathbb{R}\}$ and $\widetilde{\mathbb{R}A} = \{(\tilde{x}_0 : \tilde{x}_1 : \tilde{x}_2) \in \widetilde{\mathbb{RP}}^2 \,|\, f(-\tilde{x}_0^2, \tilde{x}_1, \tilde{x}_2) = 0\}$.

Theorem 1.4. (Fiedler [3]). *Suppose that $\mathbb{R}A$ is an M-curve of degree d on $\mathbb{RP}^2$. Then:*

a). If $\mathbb{R}A \cap F \neq \varnothing$, then $\widetilde{\mathbb{R}A}$ is a hyperbolic curve on $\widetilde{\mathbb{RP}}^2$.

b). If $d \notin \{2, 4\}$, then $\mathbb{R}A \cap F \neq \varnothing$ (and hence, part (a) implies that $\widetilde{\mathbb{R}A}$ is a hyperbolic curve on $\widetilde{\mathbb{RP}}^2$).

Remark. A perturbation of the union of the conics $x_0^2 = x_1^2 + 2x_2^2$ and $x_0^2 = 2x_1^2 + x_2^2$ (respectively, any of them) provides an example which shows that Part (b) of Theorem 1.4 does not hold for $d = 4$ (respectively, for $d = 2$).

In the following subsection, we shall reproduce the proof of this theorem from [3], extracting intermediate statements which hold for abstract (i.e., nowhere embedded) symmetric real curves, those which hold for curves on arbitrary surfaces, and those which hold for curves on $\mathbb{RP}^2$. As a result, we shall obtain an analogue of Theorem 1.4 for symmetric curves on a real quadric.

1.4. Proof and generalization of Fiedler's theorem on symmetric curves.

1.4.1. Abstract (nowhere embedded) curves.

Let A be a smooth Riemann surface of genus g supplied with an antiholomorphic involution $c : A \to A$ and a holomorphic involution $s : A \to A$ which commutes with c. Set

$$\tilde{c} = s \circ c = c \circ s, \quad \mathbb{R}A = \mathrm{Fix}(c), \quad \widetilde{\mathbb{R}A} = \mathrm{Fix}(\tilde{c}), \quad F = \mathrm{Fix}(s).$$

Lemma 1.5. *If a curve $\mathbb{R}A$ is dividing and $F \cap \mathbb{R}A \neq \varnothing$, then $F = \mathbb{R}A \cap \widetilde{\mathbb{R}A}$.*

Proof. (see [3]). In a neighbourhood of any point of $F \cap \mathbb{R}A$, the involution s looks like $z \mapsto -z$. Hence, s exchanges the halves of $A \setminus \mathbb{R}A$. $\square$

Lemma 1.6. *Suppose that $\mathbb{R}A$ is an M-curve (i.e., the number of its connected components is $g + 1$). Then:*

a). If $F \cap \mathbb{R}A \neq \varnothing$, then the curve $\widetilde{\mathbb{R}A}$ is dividing and the number of its connected components is one half of the cardinality of F.

b). If $F \cap \mathbb{R}A = \varnothing$, then $\mathrm{Card}\, F = 0$, 2, *or* 4.

Proof. (see [3]). Let us fill the holes of one of the halves of $A \setminus \mathbb{R}A$ by disks, extend $\tilde c$ to the obtained sphere, and apply Lemma 1.5. $\square$

Remark. Lemma 1.6, of course, is an immediate corollary of the topological classification of pairs of commuting antiholomorphic involutions of Riemann surfaces obtained in [12].

1.4.2. Curves on arbitrary surfaces.

Now let A be a nonsingular connected curve on a smooth surface X and let $s, c, \tilde c : (X, A) \to (X, A)$ be a holomorphic and two antiholomorphic involutions such that $\tilde c = c \circ s = s \circ c$. Set $F = \mathrm{Fix}(s) = F_0 \sqcup F_1$ where $\dim F_0 = 0$, $\dim F_1 = 1$, and

$$\mathbb{R}Z = \mathrm{Fix}(c|_Z), \quad \widetilde{\mathbb{R}Z} = \mathrm{Fix}(\tilde c|_Z) \qquad \text{for } Z = X, A, F, F_k.$$

It is clear that $\mathbb{R}F_k = \widetilde{\mathbb{R}F}_k$, $k = 0, 1$.

Lemma 1.7. *If $\mathbb{R}A$ is an M-curve and $\mathbb{R}A \cap F \neq \varnothing$, then $\widetilde{\mathbb{R}A}$ is a dividing curve whose number of components $b_0(\widetilde{\mathbb{R}A})$ satisfies the inequalities*

$$0 \le 2\, b_0(\widetilde{\mathbb{R}A}) - (A \cdot F_1) \le \mathrm{Card}(\mathbb{R}F_0).$$

Proof. Immediately follows from Lemma 1.6. $\square$

1.4.3. Curves on $\mathbb{RP}^2$ and on a real quadric.

Proof of Theorem 1.4. It follows from Lemma 1.6(b) that $\mathbb{R}A \cap F \neq \varnothing$. Hence, it follows from 1.7 that the curve $\widetilde{\mathbb{R}A}$ is dividing and $2\, b_0(\widetilde{\mathbb{R}A}) \le \deg A + 1$. It remains to apply Proposition 1.1. $\square$

Now let us proceed to the case when $\mathbb{R}X$ is a hyperboloid defined in $\mathbb{RP}^3$ by the equation $x_0^2 + x_3^2 = x_1^2 + x_2^2$, X is its complexification, and $s : X \to X$ the involution $(x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1 : x_2 : -x_3)$. Let us introduce all the other notation as in Sect. 1.4.2. Let us set $\tilde x_3 = i x_3$ and $\tilde x_j = x_j$ for $j \le 2$. Then

$$\widetilde{\mathbb{R}X} = \{(\tilde x_0 : \cdots : \tilde x_3) \in X \mid \tilde x_j \in \mathbb{R}\} = \{(\tilde x_0 : \cdots : \tilde x_3) \in \widetilde{\mathbb{R}P}^3 \mid \tilde x_0^2 = \tilde x_1^2 + \tilde x_2^2 + \tilde x_3^2\}$$

is an ellipsoid. It is clear that $F_0 = \varnothing$ and F_1 is a curve of bidegree (1,1).

Theorem 1.8. *Let $\mathbb{R}A$ be a nonsingular real M-curve of bidegree (k, k) on the hyperboloid $\mathbb{R}X$, symmetric with respect to s. Then:*

a). If $\mathbb{R}A \cap F \neq \varnothing$, then the complementary curve $\widetilde{\mathbb{R}A}$ is a hyperbolic curve on the ellipsoid $\widetilde{\mathbb{R}X}$.

b). If $k > 2$, then $\mathbb{R}A \cap F \neq \varnothing$ (and hence, Part (a) implies that $\widetilde{\mathbb{R}A}$ is a hyperbolic curve on $\widetilde{\mathbb{R}X}$).

Proof. Use Proposition 1.3 rather than Proposition 1.1 in the proof of 1.4. $\square$

1.5. Proof of Theorem 3.

Let us introduce coordinates $(y_0 : y_1 : y_2)$ on $\mathbb{RP}^2$ so that the conic C_2 is given by $y_0^2 = y_1^2 + y_2^2$. Let X, c, s mean the same as in Sect. 1.4.3. Let $\xi : X \to \mathbb{CP}^2$ be the double covering $(x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1 : x_2)$ branched along C_2. Then $\xi = \xi \circ s$, hence c and $\tilde c$ are two lifts onto X of the involution of the complex conjugation of $\mathbb{CP}^2$.

Let us denote the connected components of $\mathbb{RP}^2\backslash\mathbb{R}C_2$ by D (disk) and M (Möbius band). It is easy to check that $\mathbb{R}X$ is a hyperboloid (topologically, a torus) and ξ maps it onto $\overline{M}$ with a fold along $\mathbb{R}F_1$. Moreover, $\mathbb{R}F_1$ is mapped diffeomorphically onto $\mathbb{R}C_2$ and the restriction of ξ to $\mathbb{R}X \setminus \mathbb{R}F_1$ (which is homeomorphic to an open annulus) is a connected double covering of M. Further, $\widetilde{\mathbb{R}X}$ is a 2-sphere and ξ maps it onto $\overline{D}$ with a fold along $\mathbb{R}F_1$. Moreover, $\widetilde{\mathbb{R}X} \setminus \mathbb{R}F_1$ is a disjoint union of two disks each of which is mapped homeomorphically onto D.

Let us set $A = \xi^{-1}(C_n)$. It is a curve of bidegree (n, n) on X. Hence, its genus is $g(A) = (n-1)^2$ (the number of integer points inside the square $n \times n$). Let us show that $\mathbb{R}A$ is an M-curve on the hyperboloid $\mathbb{R}X$. Indeed, each oval of C_n lying outside of $\mathbb{R}C_2$ (the number of them is $g(C_n)$), being zero homologous in $H_1(M)$, provides two ovals of $\mathbb{R}A$ on $\mathbb{R}X$, and each exterior arc of B_n provides one oval on $\mathbb{R}X$. Thus, we have $2\,g(C_n) + n = (n^2 - 3n + 2) + n = g(A) + 1$ ovals, i.e., $\mathbb{R}A$ is an M-curve.

Therefore, Theorem 1.8 implies that $\widetilde{\mathbb{R}A}$ is a hyperbolic curve on the ellipsoid $\widetilde{\mathbb{R}X}$ and Theorem 3 follows.

§2. Constructions

2.1. Definitions and notation.

Recall that a curve has a *singularity of the type* A_n (respectively, *of the type* E_8) at a point p, if it can be defined by $y^2 = \pm x^{n+1}$ (respectively, by $y^3 = x^5$) in suitable local analytic coordinates centered at p.

Definition 2.1. Suppose that one curve is nonsingular at a point p and another curve has a singularity of the type A_n at p. Let us say that these curves have a *maximal* (respectively, *almost maximal*) *intersection* at p if the local intersection multiplicity is equal to $n+1$ (respectively, to n).

Note, that if one curve is nonsingular at a point p and another curve has a singularity of the type A_{2k} at p, then the intersection is maximal if and only if one of the curve is situated on the both sides from the other one.

Notation 2.2. Let C be a curve in $\mathbb{RP}^2$ and p be a nonsingular point of C which is not a flex point. Choose coordinates $(x:y:z)$ so that the line $z=0$ is tangent to C at p. Choose a real parameter a so that the intersection multiplicity of the conic $yz = ax^2$ with C is ≥ 3 at p. Let $f_{C,p}$ be the birational quadratic transformation $(x:y:z) \mapsto (xz : yz - ax^2 : z^2)$, i.e., $(X, Y) \mapsto (X, Y - aX^2)$ in the affine coordinates $X = x/z$, $Y = y/z$.

Notation 2.3. Let p and q be two points in $\mathbb{RP}^2$ and let L be a line passing through q but not passing through p. Choose coordinates $(x:y:z)$ so that $p = (0:1:0)$, $q = (0:0:1)$, $L = \{y=0\}$. Let $h_{p,q,L}$ be the birational quadratic transformation $(x:y:z) \mapsto (x^2 : xy : yz)$. In the literature on the topology of real algebraic curves, this transformation is usually called a *hyperbolism* (Viro introduced this term referring to Newton).

2.2. Construction of some mutual arrangements of a singular quintic and a nonsingular conic.

Lemma 2.4. *There exist arrangements of an M-quartic C_4 with respect to three lines L, L', and (pq) depicted in Figures 1.1 – 1.2.*

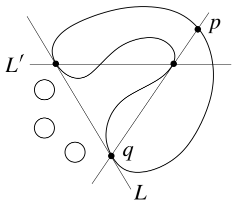


Fig. 1.1

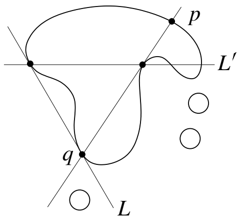


Fig. 1.2

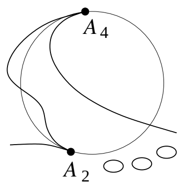


Fig. 2.1

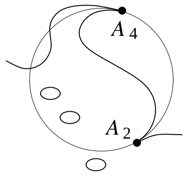


Fig. 2.2

Proof. Fig. 1.1 can be easily constructed starting from an M-quartic which is obtained as a perturbation of the union of two conics. A construction of the arrangement of Fig. 1.2 is shown in Figures 3.1-3.3. □

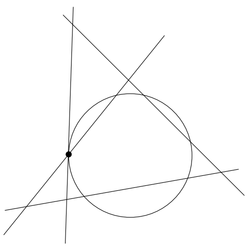


Fig. 3.1

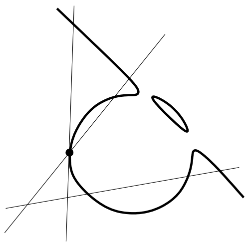


Fig. 3.2

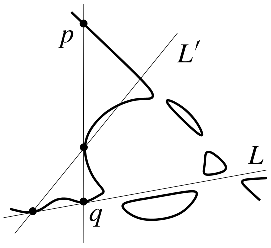


Fig. 3.3

Lemma 2.5. *There exist mutual arrangements depicted in Figures 2.1 – 2.2 of a singular four-component quintic C_5 with singularities A_2 and A_4 and a nonsingular conic C_2, which have maximal intersection at the singular points.*

Proof. Apply $h_{p,q,L}$ to the curves in Lemma 2.4. Then $C_4 \to C_5$, $L' \to C_2$. □

Lemma 2.6. *Let C be a nonsingular M-quartic and let O be one of its ovals. Let L_1 be a line, tangent to O at points p and q. Let L_2 be a line through q which intersects O at four distinct real points. Then C, L_1, and L_2 are arranged on $\mathbb{RP}^2$ as in one of Figures 4.1 – 4.3. All these arrangements are realizable.*

Proof. Easily follows from the classification of maximal mutual arrangements of an M-quartic and two lines. Moreover, Fig. 4.1 and Fig. 4.2 can be obtained from Fig. 1.1 and Fig. 1.2 respectively by forgetting one of the lines and changing the notation. □

Lemma 2.7. *Let C_2 be a nonsingular conic and let C_5 be a singular four-component quintic whose odd branch J_5 has a singularity of the type A_6. Suppose that J_5 and C_2 have a maximal intersection at A_6 and three transversal intersections. Then C_2 and C_5 are arranged on $\mathbb{RP}^2$ as in one of Figures 5.1 – 5.3. All these arrangements are realizable.*

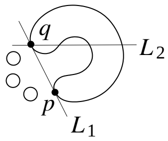


Fig. 4.1

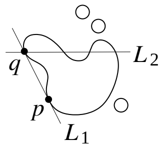


Fig. 4.2

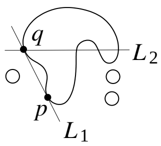


Fig. 4.3

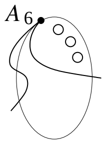


Fig. 5.1

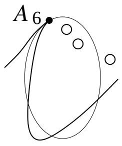


Fig. 5.2

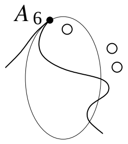


Fig. 5.3

Proof. Apply $f_{C,p}$ to the arrangements from Lemma 2.6. Then $C \to C_5$, $L_2 \to C_2$, $q \mapsto A_6$. $\square$

Lemma 2.8. *There exist mutual arrangements of a cuspidal cubic C_3, a nonsingular conic C'_2, and two lines L and (pq) depicted in Fig. 6.1 – 6.2.*

Proof. It is clear that if one forgets the conic C'_2 then the required arrangement exists (it is the same in the both cases). Let ℓ, ℓ', ℓ'', and ℓ_0 be linear functions defining the lines L, L', (pq), and L_0 respectively (see Fig. 6.3). Let us set $C'_2 = \{\ell'\ell'' = \varepsilon\ell\ell_0\}$, $|\varepsilon| \ll 1$. Then we obtain Fig. 6.1 or Fig. 6.2 depending on the sign of ε. $\square$

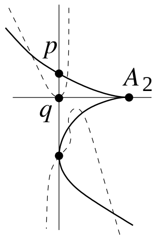


Fig. 6.1

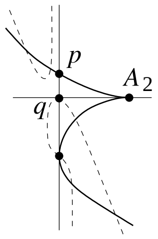


Fig. 6.2

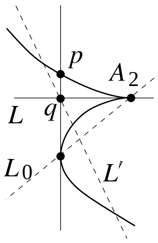


Fig. 6.3

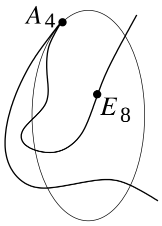


Fig. 7.1

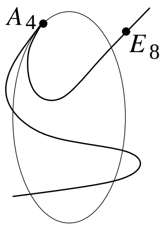


Fig. 7.2

Lemma 2.9. *There exist mutual arrangements depicted in Fig. 7.1 – 7.2 of a rational quintic C_5 with singularities A_4 and E_8 and a nonsingular conic C_2 which have a maximal intersection at A_4.*

Proof. Apply $h_{p,q,L}$ to the curves from Lemma 2.8. Then $C_3 \to C_5$, $C'_2 \to C_2$. $\square$

Definition 2.10. Let $\mathcal{F}_n$ denote a rational ruled surface (Hirzebruch surface) of degree n, i.e., a fiberwise compactification of the line bundle $\mathcal{O}(n)$ over $\mathbb{P}^1$. The surface $\mathcal{F}_n$ can be covered by affine coordinate charts $(x_1, y_1), \dots, (x_4, y_4)$ with the transition functions

$$\begin{array}{llll} x_2 = x_1^{-1} & x_4 = x_3^{-1} & x_3 = x_1 & x_4 = x_2 \\ y_2 = y_1 x_1^{-n} & y_4 = y_3 x_3^{n} & y_3 = y_1^{-1} & y_4 = y_2^{-1} \end{array}$$

these are the toric coordinates corresponding to the fan spanned on the vectors $(1,0)$, $(0,\pm 1)$, $(-1,n)$. In this case, the coordinates (x_1, y_1) are called *standard.* We shall consider only that real structure on $\mathcal{F}_n$ where the standard coordinates are real.

In standard coordinates, the fibration $\mathcal{F}_n \to \mathbb{P}^1$ is the projection onto the x_1-axis. The exceptional section is defined by $y_3 = 0$ or $y_4 = 0$. We shall depict $\mathbb{R}\mathcal{F}_n$ as a rectangle whose opposite sides are identified. The horizontal sides correspond to the exceptional section, the vertical sides correspond to some fiber.

The *bidegree* of a curve C on $\mathcal{F}_n$ is the pair (k, m), where k and m are the intersection numbers of C with $x_1 = 0$ and with $y_1 = 0$ respectively.

Lemma 2.11. *Let C, G, and F be real curves of bidegrees $(2,8)$, $(1,4)$, and $(0,1)$ respectively on $\mathcal{F}_4$ (G is a section and F is a fiber of $\mathcal{F}_4 \to \mathbb{P}^1$). Suppose that C is an M-curve (i.e., it has four ovals) which intersects G at eight real points. Suppose that all the intersection points lye on the same oval of C and that F is tangent to C at one of them.*

Then C, G, and F are arranges on $\mathbb{R}\mathcal{F}_4$ as in one of Figures 8.1 – 8.4. All these arrangements are realizable.

Moreover, any free (i.e., non-intersecting with G) oval can be replaced by a solitary simple double point.

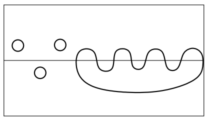


Fig. 8.1

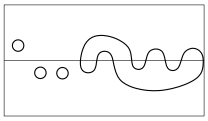


Fig. 8.2

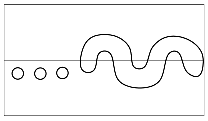


Fig. 8.3

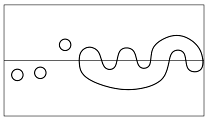


Fig. 8.4

Proof. One can construct the curves in Figures 8.1 – 8.4, for example, by Viro T-construction (patchwork) as it is done in [20; §2.2]. They can also be constructed by applying a general method of construction of trigonal curves on rules surfaces [17].

The nonexistence of other arrangements easily follows, for example from the general algorithm of realizability recognition for real pseudoholomorphic trigonal curves [19].

The fact that any oval of a trigonal curve can be replaced by a solitary double point without changes of fiberwise arrangement of the rest of the curve, is proven in [17; Lemma 2]. □

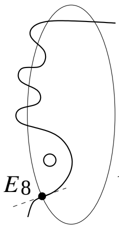


Fig. 9.1

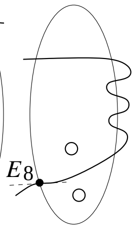


Fig. 9.2

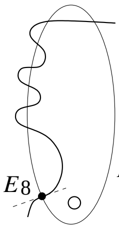


Fig. 9.3

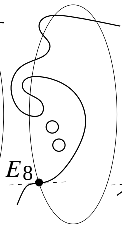


Fig. 9.4

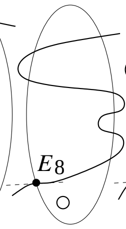


Fig. 9.5

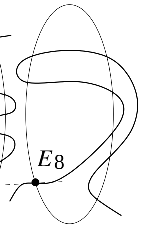


Fig. 9.6

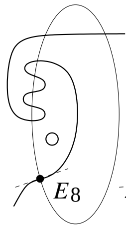


Fig. 9.7

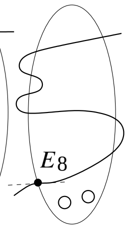


Fig. 9.8

Lemma 2.12. *Let C_2 be a nonsingular conic and C_5 a three-component quintic whose odd branch J_5 has a singularity of the type E_8. Suppose that C_2 passes through E_8 and it also intersects J_5 at seven other points. Then C_2 and C_5 are arranged on $\mathbb{RP}^2$ as in one of Figures 9.1 – 9.8 (we do not depict the ovals of C_5 lying in the component of $\mathbb{RP}^2 \setminus (C_2 \cup J_5)$ whose closure is non-orientable). All these arrangements are realizable.*

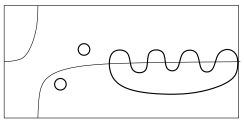


Fig. 10.1

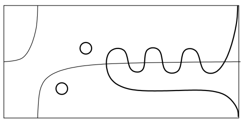


Fig. 10.2

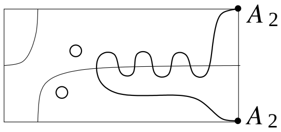


Fig. 10.3

Remark. The dashed line in Figures 9.1 – 9.8 is the tangent line to C_5 at E_8. The local arrangement of C_5 at E_8 with respect to the tangent line is important for perturbations.

Proof. Let us choose one of the arrangements from Lemma 2.11 and contract one of its ovals into a double point. Let us blow up this point and then, blow down the proper transform of the fiber passing through it. This transforms C, G, and F into the curves C', G', and F' on $\mathcal{F}_3$ of bidegrees $(2,7)$, $(1,4)$, and $(0,1)$ respectively. Their arrangement on $\mathcal{F}_3$ is depicted in Fig. 10.1 for the choice of the leftmost oval in Fig. 8.1. Now let us perform such transformation twice at the intersection point of F' and G'. Then the curves C' and G' are transformed into curves C'' and G'' on $\mathcal{F}_1$ of bidegrees $(2,5)$ and $(1,2)$ respectively, which are arranged as in Fig. 10.3 (the result of the intermediate transformation is depicted in Fig. 10.2). The curve C'' has a cusp (a singularity A_2) where C'' has a maximal intersection with the exceptional section. Finally, we blow down the exceptional section and obtain the required arrangement on $\mathbb{RP}^2$. Applying this construction to every arrangement from Lemma 2.11 and to every choice of the free oval, we obtain Figures 9.1 – 9.8. The correspondence between the pictures is the following (here n_k means the choice of the k-th oval from the left in Fig. 8.n):

$$\begin{array}{llll} 1_1 \to 9.1, & 1_2 \to 9.2, & 1_3 \to 9.3, & 2_1 \to 9.4, \\ (2_2, 2_3) \to 9.5, & (3_1, 3_2, 3_3) \to 9.6, & (4_1, 4_2) \to 9.7, & 4_3 \to 9.8. \end{array} \qquad \square$$

Lemma 2.13. *There exists a mutual arrangement depicted in Fig. 11.4 of a singular six-component quintic C_5 with a singularity A_2 and a nonsingular conic C_2 which have a maximal intersection at the singular point.*

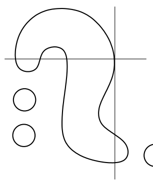


Fig. 11.1

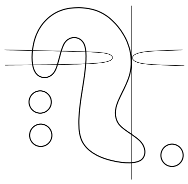


Fig. 11.2

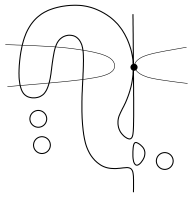


Fig. 11.3

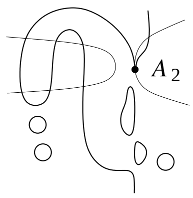


Fig. 11.4

Proof. See Fig. 11.1 – 11.4. $\square$

2.3. Construction of some mutual arrangements of a singular quintic and two lines.

Lemma 2.14. *There exists a quintic with singularities A_1 and A_3 arranged with respect to lines L_1, L_2 as in Fig. 12.1*

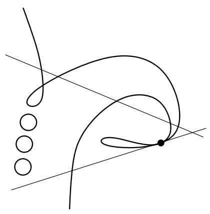


Fig. 12.1

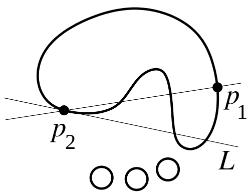


Fig. 12.2

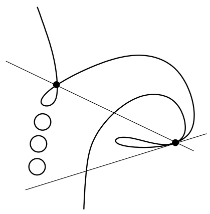


Fig. 12.3

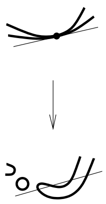


Fig. 12.4

Proof. Consider a quartic arranged with respect two lines as in Fig. 12.2 and apply the hyperbolism $h_{p_1,p_2,L}$ to it. As the result, we obtain the arrangement in Fig. 12.3. By a small shift of the upper line, it can be transformed into Fig. 12.1. □

The second arrangement of the series **9** in the list in Sect. 0.7 is constructed by applying the perturbation shown in Fig. 12.4 to Figure 12.1.

Lemma 2.15. *There exists a quintic with singularities A_2 and E_8 arranged with respect to lines L_1, L_2 as in Fig. 13.1.*

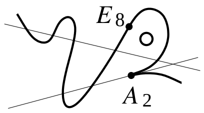


Fig. 13.1

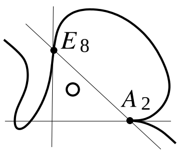


Fig. 13.2

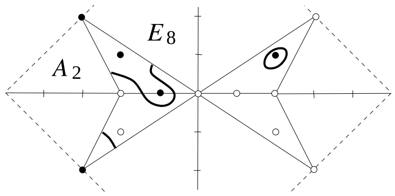


Fig. 13.3

Proof. It is easy construct a curve as in Fig. 13.2 using Viro patchworking (see the corresponding subdivision into charts in Fig. 13.3). This means just that the curve in Fig. 13.2 is given in homogeneous coordinates $(x{:}y{:}z)$ by the equation

$$z^5 + axz^4 + x^2z^3 + x^3y^2 - bx^2yz^2 = 0 \qquad \text{for } 1 \ll a \ll b.$$

Let us choose the axis $y = 0$ as the line L_1 and let L_2 be a line obtained from it by a small rotation clockwise around A_2 (this is the point $(1{:}0{:}0)$) followed by a yet smaller sift up. □

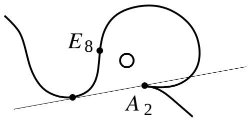


Fig. 14.1

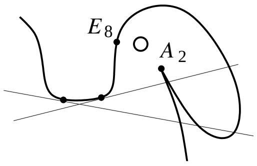


Fig. 14.2

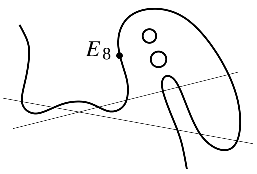


Fig. 14.3

Fig. 14.1 can be obtained from Fig. 13.2 by the rotation of the axis $y = 0$ clockwise around A_2 till the first tangency with the quintic. Fig. 14.2 can be obtained from Fig. 14.1 by replacing the depicted tangent line with two close tangents. Fig. 14.3 is a perturbation of Fig. 14.2.

Construction of the second arrangement of the series 15.

Let us consider an M-quartic obtained by a small perturbation of the union of two conics. Let V be one of its ovals and let q be a flex point on V. Let ℓ be a line through q intersecting V at four points. Let p_1 be that intersection point for which the segment $[qp_1]$ lies inside V. Choose a point p_0 on the concave part of V. Consider the tangent to V at p_0 and denote its intersection points with V by p_2 and p_3 so that p_2 is closer to q than p_3 in the sense that ℓ meets the arc qp_2 of V only at q. It is clear that each of the lines p_1p_0 and p_1p_2 tends to ℓ as $p_0 \to q$. Therefore,

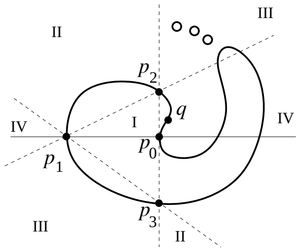


Fig. 15.1

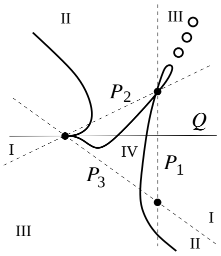


Fig. 15.2

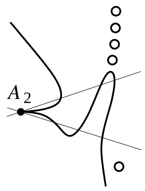


Fig. 15.3

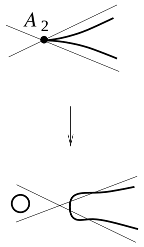


Fig. 15.4

if the point p_0 is chosen sufficiently close to q, then we obtain an arrangement of an M-quartic with respect to four lines as in Fig. 15.1.

Applying the birational quadratic transformation centered at p_1, p_2, p_3, we obtain a singular quintic with singularities A_1, A_1, and A_2, arranged with respect to the lines P_1, P_2, P_3, Q (being the transforms of the points p_1, p_2, p_3 and the line p_1p_0 respectively), as in Fig. 15.2, where the Roman literals I, . . . ,IV indicate the correspondence between the quarters.

Let L_1 and L_2 be two lines through A_2 which are close to Q and placed on different sides of it. Perturbing simple double points, we obtain Fig. 15.3. Finally, let us perturb A_2 as shown in Fig. 15.4.

§3. Classification of pseudoholomorphic arrangements of a quintic and two lines of the type (3,3)

Proposition 3.1. *a). Let C_5 be a real pseudoholomorphic (for example, algebraic) M-quintic in $\mathbb{RP}^2$, and let L_1, L_2 be two lines each of which intersects the odd branch J_5 of C_5 at five distinct points. Let $\mathcal{L}$ be the pencil of lines which contains L_1 and L_2. Suppose that the mutual arrangement of the curves C_5, L_1, L_2 is of the type $(3,3)$. (types of arrangements are defined in §0.4). Then C_5, L_1, and L_2 are arranged as in Figures 16.1–16.22 up to isotopy. Moreover, the fiberwise arrangement of C_5 with respect to $\mathcal{L}$ is as indicated in the capture to the corresponding figure up to insertion of zigzags of the form $\subset_j \supset_{j\pm1}$ (see Remark 3.2).*

b). All the arrangements in Figures 16.1–16.22 are realizable by real pseudoholomorphic curves. All of them except Fig. 16.12 are realizable by real algebraic curves.

c). The arrangement in Fig. 16.12 is algebraically unrealizable.

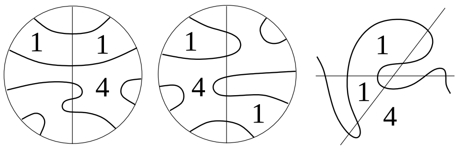


Fig. 16.1. $[\supset_1 o_3 \subset_1 \supset_2 o_3 o_2 o_2 o_2 o_2 \subset_2]$.

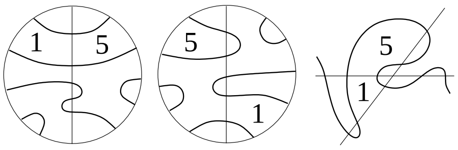


Fig. 16.2. $[\supset_1 o_3 \subset_1 \supset_2 o_3 o_3 o_3 o_3 o_3 \subset_2]$.

Remark 3.2. In Figures 16.1–16.22, we present each arrangement in three forms. In two of them, $\mathbb{RP}^2$ is cut along one of the lines. The third form is somewhat similar to the singular curve whose perturbation provides the given arrangement.

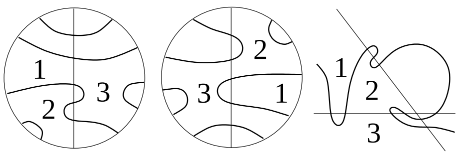


Fig. 16.3. $[\supset_1 o_1o_1o_2 \subset_1 \supset_2 o_2o_2o_2 \subset_2]$.

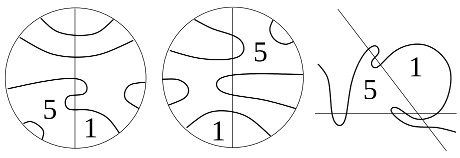


Fig. 16.4. $[\supset_1 o_1o_1o_1o_1o_1 \subset_1 \supset_2 o_1 \subset_2]$.

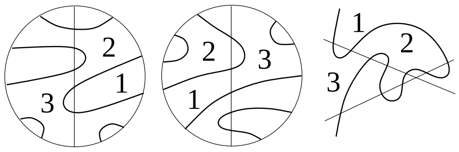


Fig. 16.5. $[\supset_1 o_1o_1o_1 \subset_1 \supset_3 o_3o_3o_2 \subset_1]$.

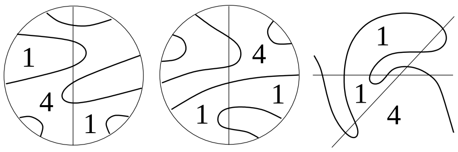


Fig. 16.6. $[\supset_1 o_2o_1o_1o_1o_1 \subset_1 \supset_3 o_1 \subset_1]$.

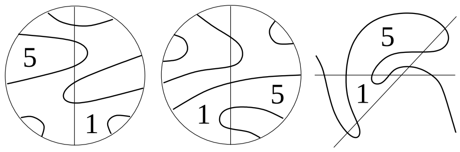


Fig. 16.7. $[\supset_1 o_2o_2o_2o_2o_2 \subset_1 \supset_3 o_1 \subset_1]$.

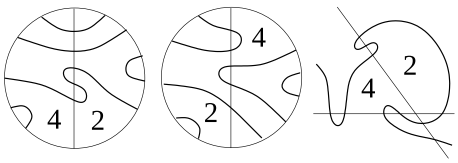


Fig. 16.8. $[\supset_1 o_1o_1o_1o_1 \subset_2 \supset_1 o_1o_1 \subset_2]$.

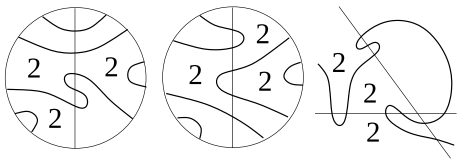


Fig. 16.9. $[\supset_1 o_1o_1o_2o_2 \subset_2 \supset_1 o_2o_2 \subset_2]$.

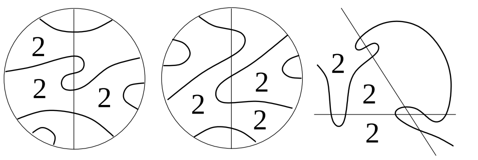


Fig. 16.10. $[\supset_1 o_3o_3o_2o_2 \subset_2 \supset_3 o_2o_2 \subset_2]$.

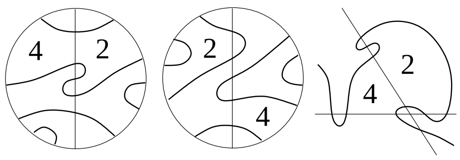
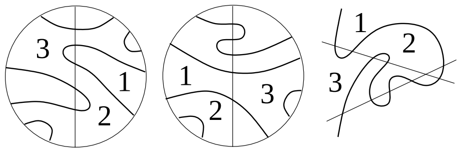


Fig. 16.11. $[\supset_1 o_3o_3o_3o_3 \subset_2 \supset_3 o_3o_3 \subset_2]$.



(Algebraically unrealizable.)
Fig. 16.12. $[\supset_1 o_3o_3o_3 \subset_3 \supset_1 o_1o_1o_2 \subset_3]$.

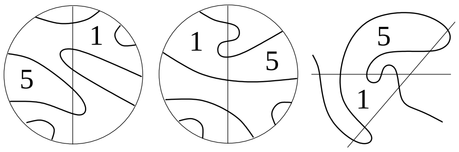


Fig. 16.13. $[\supset_1 o_2o_2o_2o_2o_2 \subset_3 \supset_1 o_3 \subset_3]$.

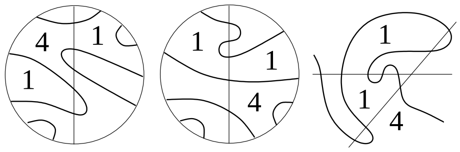


Fig. 16.14. $[\supset_1 o_2o_3o_3o_3o_3 \subset_3 \supset_1 o_3 \subset_3]$.

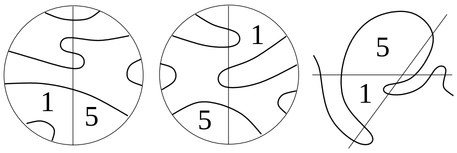


Fig. 16.15. $[\supset_1 o_1 \subset_3 \supset_2 o_1 o_1 o_1 o_1 o_1 \subset_2]$.

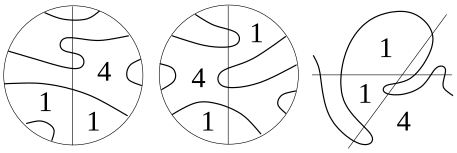


Fig. 16.16. $[\supset_1 o_1 \subset_3 \supset_2 o_1 o_2 o_2 o_2 o_2 \subset_2]$.

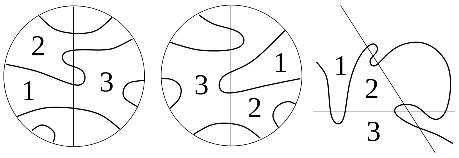


Fig. 16.17. $[\supset_1 o_3 o_3 o_2 \subset_3 \supset_2 o_2 o_2 o_2 \subset_2]$.

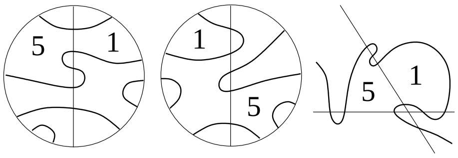


Fig. 16.18. $[\supset_1 o_3 o_3 o_3 o_3 o_3 \subset_3 \supset_2 o_3 \subset_2]$.

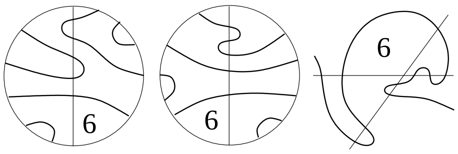


Fig. 16.19. $[\supset_1 \subset_4 \supset_2 o_1 o_1 o_1 o_1 o_1 o_1 \subset_3]$.

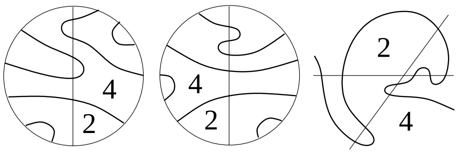


Fig. 16.20. $[\supset_1 \subset_4 \supset_2 o_2 o_2 o_2 o_2 o_1 o_1 \subset_3]$.

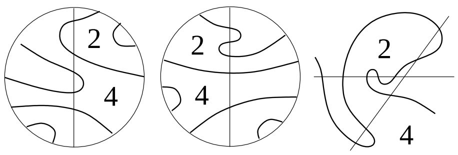


Fig. 16.21. $[\supset_1 \subset_4 \supset_2 o_2 o_2 o_2 o_2 o_3 o_3 \subset_3]$.

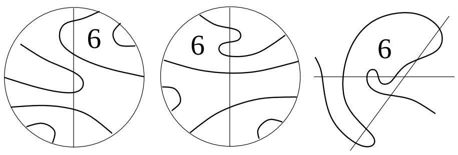


Fig. 16.22. $[\supset_1 \subset_4 \supset_2 o_3 o_3 o_3 o_3 o_3 o_3 \subset_3]$.

In the captures, we give the fiberwise arrangement of C_5 with respect to $\mathcal{L}$ encoded as described in [16; §2] (see also Sect. 6.2 below) assuming that one of the lines L_1, L_2 is chosen as the infinite line. All the encoding words have the form

$$[\supset_a o_{i_1} \dots o_{i_k} \subset_b \supset_c o_{i_{k+1}} \dots o_{j_6} \subset_d]. \tag{4}$$

Depending on the choice of the infinite line and the orientations on it and on the pencil, 8 encoding words are possible. We choose always the one providing that the vector $[a, b, c, d]$ is minimal possible with respect to the lexicographic order. Figures 16.1 – 16.22 are numbered in the ascending order of the vectors $[a, b, c, d, k, i_1, \dots, i_6]$.

Restrictions.

Proof of Part (a) of Proposition 3.1. The vector $[a, b, c, d]$ discussed in Remark 3.2 encodes the mutual arrangement of J_5, L_1, and L_2. It is clear that it must satisfy the conditions

$$1 \le a, b, c, d \le 4, \quad b \ne c, \quad d \ne 5 - a, \quad a + b + c + d \equiv 0 \mod 2. \tag{5}$$

The dihedral group of order 8 acts on the set of such vectors. It is generated by the mappings

$$[a, b, c, d] \mapsto [c, d, 5 - a, 5 - b], \quad \text{and} \quad [a, b, c, d] \mapsto [d, c, b, a]. \tag{6}$$

Vectors from the same orbit define the same mutual arrangement of J_5, L_1, and L_2 up to swapping of the lines and the orientations changes.

Taking into account the restrictions (5) and the symmetries (6), we have 8 vectors to consider:

$$[1,1,2,2],\ [1,1,3,1],\ [1,2,1,2],\ [1,2,3,2],\ [1,3,1,3],\ [1,3,2,2],\ [1,4,2,3], \tag{7}$$

and $[1,2,4,1]$. However, the last one does not provide any connected curve J_5.

The formula of complex orientations (2) implies that an M-quintic has three positive ovals and three negative ones. It is easy to deduce that the word (4) must satisfy the condition

$$k \equiv a+b \mod 2. \tag{8}$$

Finally, it follows from Bezout's theorem for auxiliary lines that

$$\begin{array}{ll} |i_j - i_l| \le 1 & \text{for } 1 \le j < l \le k \text{ or } k < j < l \le 6, \\ |i_j - a| \le 2,\ |i_j - b| \le 2 & \text{for } 1 \le j \le k, \\ |i_j - c| \le 2,\ |i_j - d| \le 2 & \text{for } k < j \le 6. \end{array} \tag{9}$$

For each word of the form (4) where the vector $[a,b,c,d]$ is one of (7) and integers $k; i_1, \dots, i_6$ satisfy (8), (9), and $1 \le i_j \le 4$, we have performed the following computations:

(1) Find the braid. All the links which are the closures of the obtained braids have three components.
(2) Check if all pairwise linking numbers of the link components are zero.
(3) If yes, then using the program from [16; Appendix], check if the Murasugi-Tristram inequality holds.

As the result, we have obtained only those words of the form (4) which occur in the captures to Figures 16.1–16.22.

Constructions.

Proof of Part (b) of Proposition 3.1.

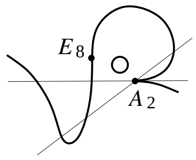


Fig. 17

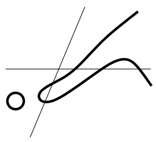


Fig. 18.1

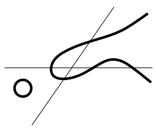


Fig. 18.2

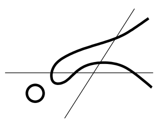


Fig. 18.3

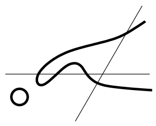


Fig. 18.4

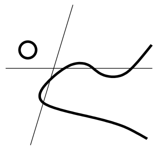


Fig. 18.5

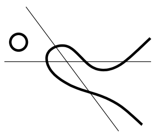


Fig. 18.6

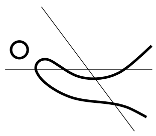


Fig. 18.7

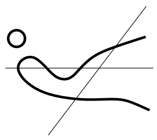


Fig. 18.8

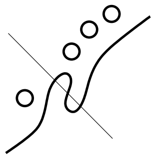


Fig. 19.1

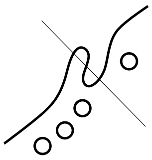


Fig. 19.2

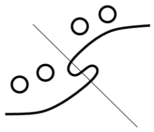


Fig. 19.3

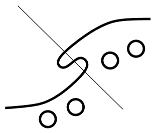


Fig. 19.4

By a small rotation of the horizontal axis around the point A_2 in Fig. 13.2, one obtains Fig. 17. Perturbing the singular point A_2 as shown in Figures 18.1–18.5 and Fig. 18.8 and perturbing each time E_8 in two possible ways, we obtain the arrangements in Fig. 16.n for $n = 1, 2, 6, 7, 13, 14, 15, 16, 19, 20, 21, 22$.

$$18.1 \to \begin{cases} 16.15 \\ 16.16 \end{cases} \quad 18.2 \to \begin{cases} 16.1 \\ 16.2 \end{cases} \quad 18.3 \to \begin{cases} 16.6 \\ 16.7 \end{cases}$$

$$18.4 \to \begin{cases} 16.13 \\ 16.14 \end{cases} \quad 18.5 \to \begin{cases} 16.19 \\ 16.20 \end{cases} \quad 18.8 \to \begin{cases} 16.21 \\ 16.22 \end{cases}$$

Forgetting the vertical line in Fig. 13.1, perturbing A_2 as shown in Fig. 18.6 and 18.7, and perturbing E_8 as shown in Fig. 19.1 – 19.4, we obtain the arrangements in Fig. 16.n for $n = 3, 4, 8, 9, 10, 11, 17, 18$

$$18.6 \to (16.10,\ 16.11,\ 16.17,\ 16.18), \qquad 18.7 \to (16.3,\ 16.4,\ 16.8,\ 16.9).$$

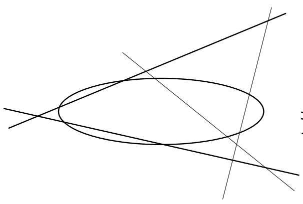


Fig. 20.1

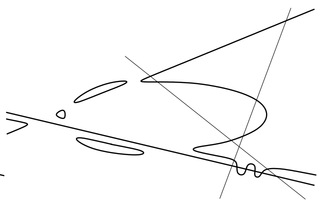


Fig. 20.2

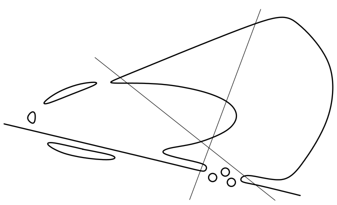


Fig. 20.3

An algebraic realization of Fig. 16.5 is shown in Fig. 20.1 – 20.3.

§4. Pseudoholomorphic arrangements of a quintic and two lines which are algebraically unrealizable

In this section we prove that the arrangements in Figures 21.1 – 21.3 and in Fig. 16.12 are realizable by real pseudoholomorphic curves (see Sect. 4.5), but they are unrealizable by real algebraic curves (see Sections 4.1 – 4.4). Moreover, in Sect. 4.5 we give a pseudoholomorphic realization of the two arrangements which are labeled by "$\nexists^*$ alg." in the list in Sect. 0.7.

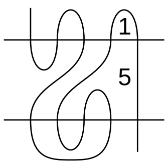


Fig. 21.1. (**3**-2).

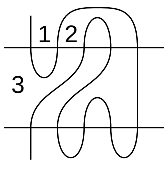


Fig. 21.2. (**8**-3).

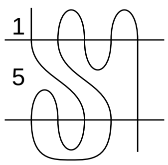


Fig. 21.3. (**11**-1).

4.1. Algebraic realizability of the arrangement in Fig. 21.1.

Rotating L_1 around q as shown in Fig. 22.1 till the first tangency with the quintic, we obtain an arrangement as in Fig. 22.2. Rotating L_2 around q' as shown in Fig. 22.2, we obtain Fig. 22.3 where $\mathbb{RP}^2$ is depicted as a disk with opposite boundary points identified. The boundary of the disk represents L_1. Blowing up p, we obtain Fig. 22.4 where the exceptional divisor is denoted by P. Blowing up p' and then blowing down L_1, we obtain a curve C of bidegree $(4, 8)$ on $\mathcal{F}_2$ (see Definition 2.10) depicted in Fig. 22.5. This curve has a singular point of the type D_4 (a simple triple point) on the line P'. In Fig. 22.5, fibers of $\mathbb{R}\mathcal{F}_2 \to \mathbb{RP}^1$ correspond to vertical lines.

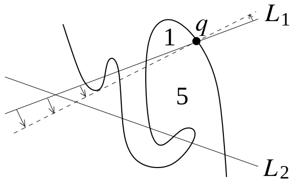


Fig. 22.1

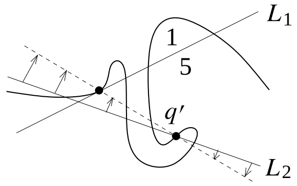


Fig. 22.2

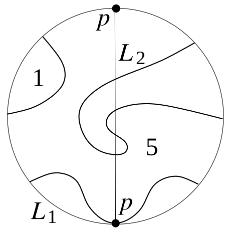


Fig. 22.3

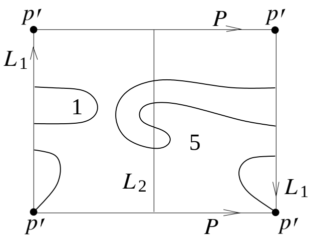


Fig. 22.4

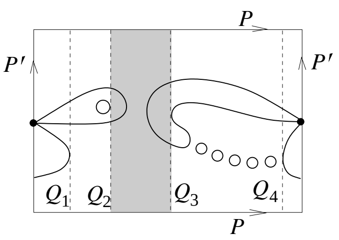


Fig. 22.5

The end of the proof of the algebraic unrealizability of Fig. 21.1 is similar to that in [22, §3]. Let k be the self-linking number of C on the interval of the pencil of vertical lines corresponding to the gray rectangle in Fig. 22.5. Then the braid associated to the curve C respectively to the pencil of vertical lines has the form

$$b = \sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{1+k}\sigma_1^{1-k}\sigma_1^{-6}\,\delta^{-1}\,\Delta^2,$$

where $\delta = \sigma_3\sigma_2\sigma_3$ is the part of the braid word corresponding to the singularity D_4 and $\Delta = \delta\,\sigma_1\sigma_2\sigma_3$ is the Garside element of the group of braids with 4 strings.

Lemma 4.1. *If Fig. 21.1 is pseudoholomorphically realizable, then $k = -3$.*

Proof. The algebraic length of b is zero. Hence, its quasipositivity is equivalent to its triviality. Let us show that b is non-trivial for $k \neq -3$. Indeed, if k is even, then b defines a non-trivial permutation. If k is odd, then the linking number of the second and the third strings is equal to $k + 3$. $\square$

Lemma 4.2. *Suppose that a polynomial $P(y) = y^4 + a_2y^2 + a_3y + a_4$ has four real roots $y_1, \dots, y_4$ such that $y_1 \le \dots \le y_4$. If $y_1 + y_4 < y_2 + y_3$, then $a_3 > 0$. If $y_1 + y_4 > y_2 + y_3$, then $a_3 < 0$.*

Proof. We shall consider only the case when $y_1 + y_4 < y_2 + y_3$. The opposite case is similar. Let us denote $(y_2 + y_3)/2$ by b. Since the coefficient of y^3 is zero, we have $y_1 + \dots + y_4 = 0$, and hence, $(y_1 + y_4)/2 = -b$. Since $y_1 + y_4 < y_2 + y_3$, we have $b > 0$. Let us set $c = (y_3 - y_2)/2$ and $d = (y_4 - y_1)/2$. Then

$$y_1 = -b - d, \quad y_2 = b - c, \quad y_3 = b + c, \quad y_4 = -b + d.$$

Therefore,

$$a_3 = -y_1y_2y_3 - y_1y_2y_4 - y_1y_3y_4 - y_2y_3y_4 = 2b(d^2 - c^2).$$

It remains to note that $d - c = y_4 - y_3 + 2b \ge 2b > 0$, and $c > 0$. $\square$

Let (x, y) be standard coordinates on $\mathcal{F}_2$ (see Definition 2.10). The equation of C in the coordinates (x, y) has the form

$$y^4 + a_2(x)y^2 + a_3(x)y + a_4(x) = 0, \qquad \deg_x a_m(x) = 2m$$

(as usually, we kill the coefficient of y^3 by the variable change $y \to y - a_1(x)/4$). Let $x = x_0, \dots, x = x_4$ be the equations of the lines P', Q_1, Q_2, Q_3, Q_4 respectively (see Fig. 22.5). As shown in [22; Lemma 3.7], $a_3(x)$ has at least $2|k| - 1$ roots in the segment $[x_2, x_3]$. We have $k = -3$ by Lemma 4.1. It follows from Lemma 4.2 that $a_3(x_0) > 0$, $a_3(x_1) < 0$, and $a_3(x_4) > 0$, hence $a_3(x)$ has at least one root at each of the segments $[x_0, x_1]$ and $[x_4, x_0]$. Thus, $a_3(x)$ has $\ge 5 + 2 = 7$ roots. This contradicts to $\deg_x a_3(x) = 6$ (see Remark 4.4 in Sect. 4.4).

4.2. Algebraic unrealizability of the arrangement in Fig. 21.2.
Starting from Fig. 21.2 and rotating the line around p, as shown in Fig. 24, we obtain successively Fig. 16.12 and Fig. 16.5. The algebraic unrealizability of Fig. 16.12 is proven in Sect. 4.4 (note that the arrangement in Fig. 16.5 is algebraically realized at the end of §3).

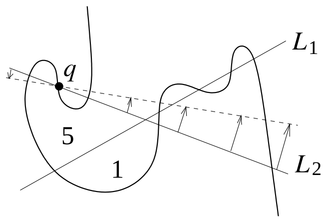


Fig. 23

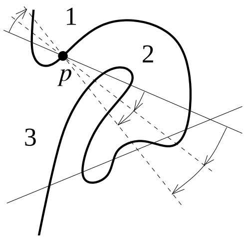


Fig. 24

4.3. Algebraic unrealizability of the arrangement in Fig. 21.3.
Rotating L_1 around q as shown in Fig.23, we transform Fig. 21.3 into Fig. 21.1 whose algebraic unrealizability is proven in Sect. 4.1.

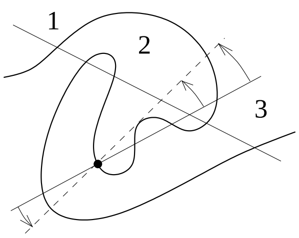


Fig. 25.1

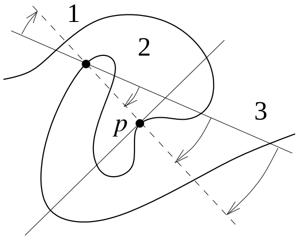


Fig. 25.2

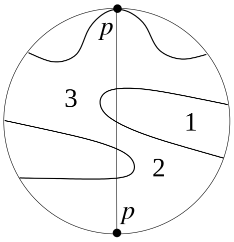


Fig. 25.3

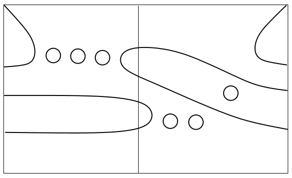


Fig. 25.4

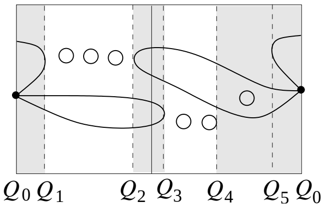


Fig. 25.5

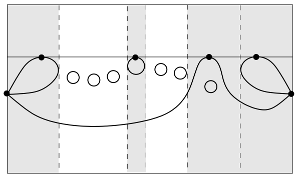


Fig. 25.6

4.4. Algebraic unrealizability of the arrangement in Fig. 16.12.

After preliminary manipulations depicted in Figures 25.1 – 25.5 and similar to those performed in the beginning of Sect. 4.1, we obtain an algebraic curve C of bidegree $(4, 8)$ on the Hirzebruch surface $\mathcal{F}_2$, depicted in Fig. 25.5. This curve has a singularity of the type D_4 (simple triple point). The three rightmost ovals appear in this order with respect to the fibers of $\mathbb{R}\mathcal{F}_2 \to \mathbb{RP}^1$ (in Fig. 25.5, fibers correspond to vertical lines), because otherwise it were too many real intersections of C with an auxiliary line (i.e., with a curve of bidegree $(1, 2)$) passing through the triple point and two of these ovals.

Let R and L be the cubic resolvent and the core of the curve C. These are curves of bidegrees $(3, 12)$ and $(1, 4)$ respectively on $\mathcal{F}_4$. Their definitions and main properties see in [22; §3]. Let us just recall that all the intersections of R and L are tangencies, and their x-coordinates (i.e., the projections on $\mathbb{P}^1$) are the roots of $a_3(x)$ where, as above, $y^4 + a_2(x)y^2 + a_3(x)y + a_4(x)$ is the defining polynomial of C in some standard coordinates on $\mathcal{F}_2$.

It follows from [22; Lemma 3.3] that R is arranged with respect to the fibers as in Fig. 25.6, and that it lies beneath the core L (in the sense of *non-strict* inequalities). Moreover, Lemma 4.2 (see Sect. 4.1) implies that each interval corresponding to a gray rectangle bounded by dashed lines in Fig. 25.5, contains an odd number of roots of $a_3(x)$. Hence, each gray rectangle in Fig. 25.6 contains an odd number of tangency points of the curves R and L. By the reason explained in Remark 4.5, it is sufficient to consider only the case when each gray rectangle contains a single tangency point and there are no other tangencies, i.e., when R is arranged with respect to L as in Fig. 25.6.

Thus, the algebraic unrealizability of Fig. 16.12 follows from the following statement (taking into account Remark 4.5).

Proposition 4.3. *Suppose that R and L are real algebraic curves on $\mathbb{R}\mathcal{F}_4$ of bidegrees* $(3, 12)$ *and* $(1, 4)$ *respectively, such that all* (*including non-real*) *the intersection points of R and L are tangency points and the curve R has a singular point of the* *type D_4. Then the mutual arrangement of R and L cannot be as in Fig. 25.6.*

Proof. The genus of R is equal to 7 (the number if interior integral points in the triangle (0,0)-(9,0)-(0,3)). It follows that the curve R is dividing, in particular, it has complex orientations.

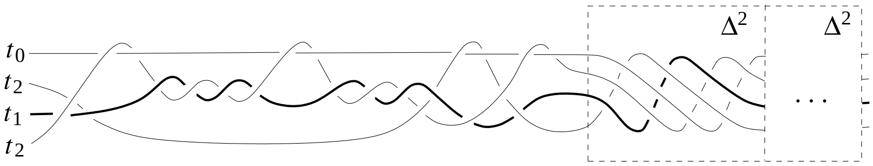


Fig. 26.

Let b be the braid associated to the reducible curve $A = R \cup L$. We have

$$b = \underbrace{(\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1})}_{D_4}\ \underbrace{(\sigma_3^{-2})}_{A_3}\ \sigma_1^{-4}\ \underbrace{(\sigma_3^{-2})}_{A_3}\ \sigma_1^{-3}\ \tau_{2,1}\ \underbrace{(\sigma_3^{-2})}_{A_3}\ \sigma_1^{-1}\ \tau_{1,2}\ \underbrace{(\sigma_3^{-2})}_{A_3}\ \Delta^4, \tag{10}$$

where $\tau_{2,1} = \sigma_1^{-1}\sigma_2 = \tau_{1,2}^{-1}$ and $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$ (see Fig. 26).

Let us denote the standard projection by $\pi : \mathbb{C}\mathcal{F}_4 \to \mathbb{CP}^1$. Let H° be one of the halves of $\mathbb{CP}^1 \setminus \mathbb{RP}^1$, namely, that which induces the orientation of $\mathbb{RP}^1$ corresponding to the direction from the left to the right according to Fig. 25.6. Let $H \subset H^\circ$ be a disk, sufficiently close to H°. Let $N = \pi^{-1}(H) \cap \mathbb{C}A$. Then the closure $\hat{b}$ of b is the boundary of N. Let $N_R = N \cap \mathbb{C}R$ and $N_0 = N \cap \mathbb{C}L$.

Counting the Euler characteristics of the components of N and the linking numbers of the braid components, one can conclude that N_R has two connected components, one of which (let us denote it by N_1) is mapped bijectively onto H and the other one (let us denote it by N_2) is a double covering of H branched at a single point. Moreover, N_1 is disjoint from N_0 and N_2 has a simple tangency with N_0 at a single point. Let us set $L_j = \partial N_j$, $j = 0, 1, 2$.

It follows that b must have the form

$$b = (a_1\sigma_1 a_1^{-1})(a_2\sigma_1^4 a_2^{-1}). \tag{11}$$

Moreover, the following refinement of (11) takes place. Let $\hat{B}_{1,1,2} = \hat{B}_{1,1,2}(t_0, t_1, t_2)$ be the groupoid of colored braids introduced in [15]. Let us color the strings of b in the colors t_0, t_1, t_2 according to the decomposition $\hat{b} = L_0 \sqcup L_1 \sqcup L_2$. Then $b \in G = \mathrm{Aut}(t_2, t_1, t_2, t_0)$ and the above description of the mutual arrangement of the surfaces N_0, N_1, N_2 means that b has the form $b = b_1 b_4$ where the colored braid b_1 is a conjugate of

$$\sigma_1 : (t_2, t_2, t_0, t_1) \to (t_2, t_2, t_0, t_1),$$

(i.e., σ_1 whose first two strings are colored in the color t_2), and b_4 is a conjugate

$$\sigma_1^4 : (t_0, t_1, t_2, t_2) \to (t_0, t_1, t_2, t_2).$$

Let us show that this is impossible for the braid (10). Let ρ be the Burau representation of $\hat{B}_{1,1,2}$ described in [15] with the values of the parameters $t_0 = t_2 = -\tau$, $t_1 = \tau^2$, where $\tau = (-1 + i\sqrt{3})/2 = \exp(2\pi i/3)$. Then the eigenvalues of

$\rho(b_4)$ are $(1,1,t_0^2t_1^2)=(1,1,\tau^6)=(1,1,1)$. Moreover, it is easy to check that $\rho(b_4)$ is the identity matrix. Thus, $b=b_1b_4$ implies $\rho(b)=\rho(b_1)$. The latter equality is impossible because the eigenvalues of $\rho(b_1)$ must be $(1,1,-t_0)=(1,1,\tau)$ whereas it is easy to compute that the eigenvalues of $\rho(b)$ are $(\alpha,\tau\alpha,\tau^2\alpha)$, where $\alpha^3=\tau$. To compute $\rho(b)$ (especially, by hands), it is convenient to use the fact that $\rho(\Delta^2)$ is a scalar matrix for any choice of parameters (see also Remark 4.7 below). $\square$

Remark 4.4. In terms of cubic resolvents, the proof of algebraic unrealizability of Fig. 21.1 given in Sect. 4.1 means the following. Topological properties of the curve in Fig. 22.5 impose that R and L have at least $(2|k|-1)+2=7$ real tangency points. This contradicts to the fact that the intersection number of these divisors on $\mathcal{F}_4$ is equal to 12.

Remark 4.5. Lemma 4.2 allows us to find only the parity of the number of roots of $a_3(x)$ on every interval bounded by vertical tangents to C. However, when writing the braid word, we choose each time the minimal possible values of these numbers which are equal to 0 or 1. What happens if we increase some value by 2? It is clear that in this case, two successive tangency points appear on the corresponding interval. This means that the subword σ_3^{-4} is inserted somewhere into the braid word of b. Thus, this does not affect the subsequent arguments.

Remark 4.7. One can use the formulas given in [15] for the Burau matrices of colored braids. However, the computation becomes easier if one changes the base (see Remark 4.8) as explained below. For example, 3×3-matrices are needed in this case rather than 4×4. Let

$$b=\sigma_{j_1}^{\varepsilon_1}\dots\sigma_{j_n}^{\varepsilon_n},\qquad \varepsilon_\nu\in\{-1,+1\},$$

be a braid with m strings colored in the colors $t_{i_1},\dots,t_{i_m}$ along the left hand side, and hence, in the colors $t_{i_{\pi(1)}},\dots,t_{i_{\pi(m)}}$ along the right hand side, where π is is the permutation defined by b. Then we have

$$\rho(b)=S_{j_1}(t_{k_1})^{\varepsilon_1}\dots S_{j_n}(t_{k_n})^{\varepsilon_n}$$

where t_{k_ν} is the color of the lower string (in the sense of over-/underpasses) at the ν-th crossing, i.e.,

$$k_\nu=\begin{cases}\pi_{\nu-1}^{-1}(j_\nu)=\pi_\nu^{-1}(j_\nu+1), & \varepsilon_\nu=+1,\\ \pi_{\nu-1}^{-1}(j_\nu+1)=\pi_\nu^{-1}(j_\nu), & \varepsilon_\nu=-1,\end{cases}$$

(here π_ν denotes the permutation defined by $\sigma_{j_1}^{\varepsilon_1}\dots\sigma_{j_\nu}^{\varepsilon_\nu}$), and $S_j(t)$ is the $(m-1)\times(m-1)$-matrix obtained by deleting the first and the last rows and columns from the $(m+1)\times(m+1)$-matrix

$$I_{j-1}\oplus\begin{pmatrix}1&0&0\\ t&-t&1\\ 0&0&1\end{pmatrix}\oplus I_{m-j-1},$$

here I_p is the identity $p\times p$-matrix and $A\oplus B$ is the block-diagonal matrix $\left(\begin{smallmatrix}A&0\\0&B\end{smallmatrix}\right)$.

For example, to check the proof of Proposition 4.1, one needs to do the following matrix computation. Let us set $S_j = S_j(t_0) = S_j(t_2) = S_j(-\tau)$ and $T_j = S_j(t_1) = S_j(\tau^2)$, i.e.,

$$S_1 = \begin{pmatrix} \tau & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 & 0 \\ -\tau & \tau & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\tau & \tau \end{pmatrix},$$

$$T_1 = \begin{pmatrix} -\tau^2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 0 \\ \tau^2 & -\tau^2 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \tau^2 & -\tau^2 \end{pmatrix}.$$

Then (see Fig. 26)

$$\rho(b) = T_1^{-1} S_2^{-1} S_1^{-1} S_3^{-2} \, (S_2^{-1} T_2^{-1})^2 \, S_3^{-2} \, (S_2^{-1} T_2^{-1} S_2^{-1}) \, S_1^{-1} S_2 T_1^{-1} S_3^{-2} S_2^{-1} T_1 S_3^{-2} \times$$
$$\times \left(S_1 S_2 S_3 T_1 T_2 T_3 (S_1 S_2 S_3)^2 \right)^2 = \begin{pmatrix} -1 & \tau & 1 \\ \tau & -\tau^2 & -2\tau \\ -\tau^2 & 0 & -\tau \end{pmatrix},$$

$$\rho(b_1) \sim S_1, \qquad \text{and} \quad \rho(b_4) \sim S_1 T_1 S_1 T_1 = I.$$

To compute faster $\rho(b)$, one can use the following identities (recall that $\tau^3 = 1$ and $\tau^2 = -\tau - 1$): $(S_2^{-1} T_2^{-1})^2 = I$ (hence, also $S_2^{-1} T_2^{-1} S_2^{-1} = T_2$) and $\rho(\Delta^2) = S_1 S_2 S_3 T_1 T_2 T_3 (S_1 S_2 S_3)^2 = -\tau^2 I$.

Remark 4.8. When speaking in the previous remark about the change of the base in the Burau representation described in [15], we were not quite rigorous. Recall that a *groupoid* is a category all whose morphisms are invertible and its representation is a functor to the category of modules over some ring. Burau representation is defined in [15] as a functor which associates the same free module V to every object of the groupoid of colored braids (i.e., to every permutation of the colors). The matrices given in [15] are the matrices of the images of morphisms with respect to a fixed base of V. Actually, *there does not exist* any base of V such that the matrices of the images of braids take the form described in Remark 4.7. However, it is possible to associate different copies of V to different object, and to find suitable bases at each copy so that the matrices become as in Remark 4.7.

4.5. Construction of pseudoholomorphic arrangements.

Proposition 4.9. *The arrangements in Figures 21.1 – 21.3, in Fig. 16.12, and in Fig. 27.1 – 27.2 are realizable by a real pseudoholomorphic quintic and two lines.*

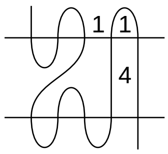


Fig. 27.1. (**6**-7).

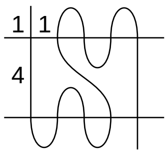


Fig. 27.2. (**18**-5).

Proof. It is easy to check that the braid b defined in Sect. 4.1 is trivial for $k=-3$. Indeed,

$$b^{-1} = \Delta^{-2}\delta\,\sigma_1^2\sigma_3^2\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1^{-1}\sigma_2\sigma_1 = \Delta^{-2}\delta\,\sigma_1\sigma_3\,(\underbrace{\sigma_1\sigma_3\sigma_2\sigma_3\sigma_1\sigma_2}_{\Delta})\,\sigma_2\sigma_1\sigma_2^{-1}$$

$$= (\Delta^{-2}\sigma_2)\,(\sigma_3\sigma_2\sigma_1\sigma_3\,\Delta)\,\sigma_2\sigma_1\sigma_2^{-1} = (\sigma_2\,\Delta^{-2})\,(\Delta\,\underbrace{\sigma_1\sigma_2\sigma_3\sigma_1)\,\sigma_2\sigma_1}_{\Delta}\,\sigma_2^{-1} = 1.$$

It follows that Fig. 21.1 is pseudoholomorphically realizable. The pseudoholomorphic realizability of Fig. 21.3 is equivalent to that of Fig. 21.1 because the braid associated to the pencil of lines centered at q (see Fig. 23) is the same in the both cases.

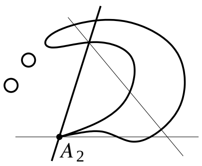


Fig. 28.1

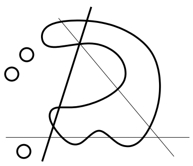


Fig. 28.2

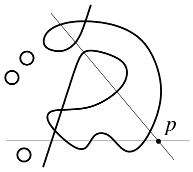


Fig. 28.3

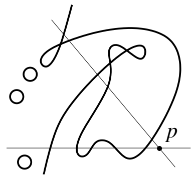


Fig. 28.4

To prove that Fig. 21.2 is pseudoholomorphically realizable, I could just write down a quasipositive decomposition of the corresponding braid, leaving to the reader to check the identity in the braid group. Instead, I give here a geometric proof which does not require any computation. It is depicted in Figures 28.1 – 28.4. By a small perturbation of a double conic, it easy to construct an algebraic cuspidal quartic arranged as in Fig. 28.1 with respect to three lines. Then, by successive perturbations, one can obtain Fig. 28.2 and Fig. 28.3 (still remaining in the class of algebraic curves). Further, a pseudoholomorphic realizability of Fig. 28.3 implies that of Fig. 28.4, because they define the same braid with respect to the pencil of lines centered at p. Finally, a perturbation of double points of the quintic in Fig. 28.4 yields Fig. 21.2.

Pseudoholomorphic realizations of Fig. 16.5 and 16.12 can be obtained from Fig. 21.2 by a rotation of one of the lines as shown in Fig. 24 (see Sect. 4.2).

Finally, to realize pseudoholomorphically Fig. 27.1 – 27.2, note that the above braid b corresponds not only to the curve depicted in Fig. 22.5, but also to the curve obtained from it by removing of one of the five ovals situated between Q_3 and Q_4, and by replacing of the three lower intersection points on Q_3 with a singularity of the type A_2 tangent to Q_3. Perturbing this singularity, we obtain Fig. 27.1. Fig. 27.2 can be obtained from it.

§5. Degeneration of a conic into a pair of lines

Let O be a simple closed curve dividing $\mathbb{RP}^2$ into a disk D and a Möbius band M. Let C be a smooth real curve on $\mathbb{RP}^2$, a and b two points on O not lying on C, and let γ be one of the two arcs into which a and b divide O. Let us say that γ is *minimal with respect to C inside O* (respectively, *outside* O), if any path γ' from a to b contained in D (respectively, contained in M, and homotopic in M to γ), has at least as many intersections with C as γ.

Proposition 5.1. *Let C_2 and C_n be nonsingular real algebraic curves on $\mathbb{RP}^2$ of degrees 2 and n respectively, which have $2n$ real intersections. Suppose that the following condition holds.*

(*) *There exist points a, b, c, d lying on C_2 in this cyclic order, such that the arcs ab and cd are minimal with respect to C_n inside C_2, and the arcs bc and da are minimal with respect to C_n outside C_2.*

Then the mutual arrangement of C_2 and C_5 is isotopic to a mutual arrangement of C_2' and C_5 where C_2' is a smooth perturbation of a union of two lines each of which has n real intersection points with the curve C_n.

The same statement takes place for real pseudoholomorphic curves.

Proof. Consider the pencil of conics through a, b, c, d. □

Let C_2 and C_5 be a conic and an M-quintic which have 10 real intersection points all of them lying on the odd branch J_5 of C_5. Recall that the notion of *passage through infinity* is defined in Sect. 0.4, and a nest of arcs inside (outside) an oval is defined in Sect. 0.1 before the formulation of Theorem 3.

Corollary 5.2. *Suppose that one of the following conditions holds:*

(1) *there are at least three passages through infinity;*
(2) *there is a nest of depth 5 inside C_2 formed by arcs of C_5,*
(3) *there are two disjoint nests outside C_2 formed by arcs of C_5,*
(4) *J_5 and C_2 are arranged as in one of Fig. 29.1 – 29.2 (series* **16, 18** *in Sect. 0.5).*

Then the condition (*) holds, and hence, $C_5 \cup C_2$ can be degenerated into a quintic and two lines.

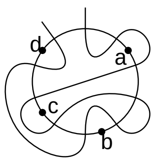


Fig. 29.1.

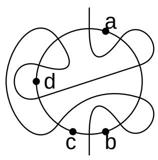


Fig. 29.2.

Proof. In each of the cases (1) – (4), let us describe a choice of a, b, c, d providing (*). Let D and M be the interior and the exterior components of $\mathbb{RP}^2 \setminus C_2$.

(1). There are five arcs of C_5 outside C_2 with the ends on C_2. If all of them pass through infinity (i.e., non-trivial in $H_1(M, C_2)$), then (*) holds for any choice of a, b, c, d such that the arcs ab and cd do not meet C_5. Otherwise, there exist two arcs α_1, α_2 of C_5 which are trivial in $H_1(M, C_2)$. Let D_j be the disk which is cut from M by α_j, and let $\beta_j = C_2 \cap (\partial D_j)$. Choose $a, b \in \beta_1$ and $c, d \in \beta_2$.

(2). Let $D_1 \subset \cdots \subset D_5$ be the disks involved in the definition of nest, i.e., such that $D \cap (\partial D_j) \subset C_5$, $j = 1, \dots, 5$. Choose $b, c \in C_2 \cap D_1$ and $a, d \in C_2 \setminus D_5$.

(3). Let $\alpha_1, \dots, \alpha_4$ be the exterior arcs of C_5 forming the nests, and let D_j be the disk, which is cut from M by α_j. Let us set $\beta_j = C_2 \cap (\partial D_j)$. Let $D_1 \subset D_2$ and $D_3 \subset D_4$. It follows from Bezout's theorem for an auxiliary line that a nest of depth three is impossible, hence, $D_2 \cap D_4 = \varnothing$. Choose $a, b \in \beta_1$ and $c, d \in \beta_3$.

(4). Choose a, b, c, d as shown in Fig. 29.1 – 29.2. $\square$

Thus, if one of conditions (1) – (4) of Corollary 5.2 holds, then those and only those arrangements are realizable by real pseudoholomorphic (respectively, algebraic) curves which can be obtained as a perturbation of the union of a quintic and two lines.

§6. Classification of arrangements of $C_5 \cup C_2$ of series not covered by Corollary 5.2

6.1. Enumeration of arrangements of $J_5 \cup C_2$ to consider.

Now we shall consider those arrangements of $C_2 \cup J_5$ which do not satisfy conditions (1) – (4) of Corollary 5.2, because it is natural to postpone the classification of the others untill the classification of $C_5 \cup L_1 \cup L_2$ is finished. Moreover it is not necessary to consider arrangements which contradict to Bezout's theorem for auxiliary lines, for instance those which have a nest of exterior arcs of J_5 deeper than two. Up to symmetry, there are 24 of such arrangements. These are the series **1–15, 17, 19, 22–25, 29, 32, 33** in the list in Sect. 0.5.

It is not difficult to check it directly, but also it can be seen from the list [7] of all arrangements of $J_5 \cup C_2$ with one passage through infinity which do not contradict to Bezout's theorem. For the reader's convenience, in the following table we give the correspondence between the numbers of series in Sect. 0.5 (**1, 2,...**) and in [7] ($M1, M2, \dots$):

1	$M30$	**12**	$M21$	**23**	$M2$	$M1$	**9**	$M12$	**20**	$M23$	**33**
2	$M33$	**13**	$M3$	**24**	$M20$	$M2$	**23**	$M13$	**21**	$M24$	**6**
3	$M18$	**14**	$M6$	**25**	$M10$	$M3$	**13**	$M14$	**26**	$M25$	**19**
4	$M28$	**15**	$M26$	**26**	$M14$	$M4$	**17**	$M15$	**30**	$M26$	**15**
5	$M32$	**16**	$M27$	**27**	$M16$	$M5$	**28**	$M16$	**27**	$M27$	**16**
6	$M24$	**17**	$M4$	**28**	$M5$	$M6$	**14**	$M17$	**29**	$M28$	**4**
7	$M31$	**18**	$M9$	**29**	$M17$	$M7$	**10**	$M18$	**3**	$M29$	**8**
8	$M29$	**19**	$M25$	**30**	$M15$	$M8$	**22**	$M19$	**11**	$M30$	**1**
9	$M1$	**20**	$M12$	**31**	$M11$	$M9$	**18**	$M20$	**24**	$M31$	**7**
10	$M7$	**21**	$M13$	**32**	$M22$	$M10$	**25**	$M21$	**12**	$M32$	**5**
11	$M19$	**22**	$M8$	**33**	$M23$	$M11$	**31**	$M22$	**32**	$M33$	**2**

6.2. Arrangements with nested exterior arcs.

Let C_2 and C_5 be a conic and an M-quintic, such that the odd branch of J_5 cuts C_2 at ten distinct points. Let D (disk) and M (Möbius band) be the components of $\mathbb{RP}^2 \setminus C$. Suppose that there is a nest outside C_2 formed by arcs of J_5, i.e., there exist arcs α_1 and α_2 which cut from M disks D_1 and D_2, such that $D_1 \subset D_2$. Choose $p \in D_1 \cap C_2$ and denote the pencil of lines through p by $\mathcal{L}_p$. Let $L_\infty \in \mathcal{L}_p$ be the tangent to C_2 at p. Choose affine coordinates (x, y) such that L_∞ is the infinite line, $\mathcal{L}_p$ is the pencil of vertical lines $x = \text{const}$, and C_2 is the parabola $y = x^2$.

We shall use the encoding of the arrangement of $C = C_5 \cup C_2$ with respect to $\mathcal{L}_p$ (fiberwise arrangements) proposed in [13] (and used in [14, 16, 20, 9] and in §3). Namely, we shall encode such an arrangement by a word composed of the symbols $\subset_k$, $\supset_k$, $\times_k$ which denote respectively a point of minimum and maximum of the x-coordinate and a double point on C. In all the three cases, k is the height of the point, i.e., the vertical line through this point has $k-1$ transversal intersections with C, with a smaller y-coordinate. We abbreviate a subword $\subset_k \supset_k$ up to o_k.

The braid corresponding to a fiberwise arrangement of C has the form $b = b_{\mathbb{R}} b_\infty$ where $b_{\mathbb{R}}$ is computed from the encoding word by the rules formulated in [13] for the case $p \notin C$, but $b_\infty = \Delta\sigma_1\sigma_2 \dots \sigma_6$ (rather that $b_\infty = \Delta$ as it is for $p \notin C$).

By the above assumption, we need to consider only the series listed in the first two columns of Table 1. Then, up to zigzag removing (deleting subwords of the form $\subset_j \supset_{j\pm1}$) the arrangement of C with respect to $\mathcal{L}_p$ is encoded by the word from the third column of Table 1 where the variables u_1, u_2, u_3 should be replaced by sequences of the form $o_{i_1} o_{i_2} \dots$ of the total length 6.

If we drop C_2, it remains a quintic whose fiberwise encoding with respect to $\mathcal{L}_p$ either has the form $[\supset_3 o_{i_1} \dots o_{i_6} \subset_4]$, or $[\supset_3 o_{i_1} \dots o_{i_k} \subset_3 \supset_4 o_{i_{k+1}} \dots o_{i_6} \subset_4]$, where the indices $i_1, \dots, i_k$ are computed starting from the encoding word for C using evident rules.

Among all words of this form, only the following can encode a fiberwise arrangement of an M-quintic

$$[\supset_3 o_2^6 \subset_4], \qquad [\supset_3 o_3^4 o_2^2 \subset_4], \qquad [\supset_3 o_3^4 o_4^2 \subset_4], \qquad [\supset_3 o_4^6 \subset_4]$$

and

$$[\supset_3 o_3^4 o_2 \subset_3 \supset_4 o_2 \subset_4], \quad [\supset_3 o_2^5 \subset_3 \supset_4 o_2 \subset_4], \quad [\supset_3 o_3^3 \subset_3 \supset_4 o_3 o_4^2 \subset_4], \quad [\supset_3 o_4 \subset_3 \supset_4 o_4^5 \subset_4].$$

In the former case, this easily follows from the classification of affine quintics and from Bezout's theorem for auxiliary lines. In the latter case, this is proven in §3.

Hence, it is sufficient to consider only those values of the parameters in the words in Table 1 which provide one of these eight words after dropping the conic. Moreover, since $\times_3$ commutes with o_j for $j \neq 4$, in the words containing subwords of the form $u_{\nu-1} \times_3^\alpha u_\nu$, it is sufficient to consider only the cases when u_ν begins with o_4.

For all such words, we checked with a computer if Murasugi-Tristram inequality for the usual signature holds (see details in [13, 16]). All the cases when it does, are listed in the last column of Table 1 where an expression of the form $[i_1 \dots i_k][i_{k+1} \dots i_m] \dots$ means that $u_1 = o_{i_1} \dots o_{i_k}$, $u_2 = o_{i_{k+1}} \dots o_{i_m}, \dots$; moreover, if parameters α and $\beta = 5 - \alpha$, $3 - \alpha$, or $2 - \alpha$ occur in the encoding word,

Table 1.

3.	M18	$[\times_5\times_4^6\supset_3 u_1\times_3 u_2\subset_4\times_5^2]$	[434422][], [433355][], [555][455]
4.	M28	$[\times_5\times_4^2\supset_3 u_1\subset_3\times_4^5\supset_5 u_2\subset_4\times_5^2]$	[334][355], [5][45555]
5.	M32	$[\times_5\times_4^2\supset_3 u_1\times_3^\alpha u_2\times_3^{5-\alpha} u_3\subset_4\times_5^2]$	[344422][][], $[333]_4[455]_1[]$, [343355][][], $[]_1[455555]_4[]$, [55][][4555]
6.	M24	$[\times_5\times_4\times_3^5\supset_4 u_1\times_3 u_2\subset_4\times_5^2]$	[344422][], [343355][], [][455555]
9.	M1	$[\times_5\times_4^4\supset_3 u_1\subset_3\times_4^3\supset_5 u_2\subset_4\times_5^2]$	[444][355], [5][44455]
10.	M7	$[\times_5\times_4^4\supset_3 u_1\times_3^\alpha u_2\times_3^{3-\alpha} u_3\subset_4\times_5^2]$	[444433][][], [443355][][], [55][][4455]
11.	M19	$[\times_5\times_4^4\supset_3 u_1\times_3 u_2\subset_4\times_5\times_4^2\times_5]$	[433322][], [4333][44], [555][444]
12.	M21	$[\times_5\times_4^4\supset_3 u_1\times_3 u_2\subset_4\times_5^4]$	[4333][45], [555][445]
13.	M3	$[\times_5\times_4\times_3^2\times_4\supset_3 u_1\subset_3\times_4^3\supset_5 u_2\subset_4\times_5^2]$	[344][355]
14.	M6	$[\times_5\times_4^2\supset_3 u_1\times_3^\alpha u_2\times_3^{3-\alpha} u_3\times_4\times_5^4]$	[4433][45], [55][4445]
15.	M26	$[\times_5\times_4^2\supset_3 u_1\times_3^\alpha u_2\times_3^{3-\alpha} u_3\subset_4\times_5\times_4^2\times_5]$	[4433][][44], [55][][4444]
17.	M4	$[\times_5\times_4^2\supset_3 u_1\subset_3\times_4^3\supset_5 u_2\subset_4\times_5^4]$	[444][345], [5][44445]
19.	M25	$[\times_5\times_4\times_3^3\supset_4 u_1\times_3 u_2\subset_4\times_5^4]$	[3333][45], [][444445]
22.	M8	$[\times_5\times_4^2\supset_3 u_1\times_3^\alpha u_2\times_3^{2-\alpha}\subset_3\times_4^3\supset_5 u_3\subset_4\times_5^2]$	[5][][44555]
23.	M2	$[\times_5\times_4^2\supset_3 u_1\subset_3\times_4\times_3^2\times_4^2\supset_5 u_2\subset_4\times_5]$	[333][355]
24.	M20	$[\times_5\times_4^3\times_3^2\times_4\supset_3 u_1\times_3 u_2\subset_4\times_5^2]$	[334422][], [333355][], [][444455]
29.	M17	$[\times_5\times_4^2\supset_3 u_1\times_3 u_2\subset_4\times_5\times_4^4\times_5]$	[444322][], [44445][4], [55555][4]
32.	M22	$[\times_5^3\times_4^2\supset_3 u_1\times_3 u_2\subset_4\times_5\times_4^2\times_5]$	[4444][44], [5555][44]
33.	M23	$[\times_5\times_4^2\supset_3 u_1\times_3 u_2\subset_4\times_5^6]$	[4444][45], [5555][45]

then their values, if they are needed, are indicated as lower indices between the brackets.

It remains to check that the parameter values listed in the last column provide only those arrangements which are listed in Sect. 0.5.

6.3. Zigzag removal and fiberwise models.

Let (x, y) be an affine coordinate system on $\mathbb{RP}^2$, and let p be the infinite point on the y-axis. Let $\mathcal{L}_p$ be the pencil of lines centered at p, i.e., the pencil of vertical lines $x = \text{const}$. Let A be a nodal real pseudoholomorphic curve of degree m, not passing through p, and let $\mathbb{R}A$ be the set of its real points. After an arbitrarily small shift of p, we may assume that A is *in general position with respect to* $\mathcal{L}_p$, i.e., any line from $\mathcal{L}_p$ neither is tangent to $\mathbb{R}A$ at a flex point or at a double point, nor passes through two critical points (we call *critical point* either a double point or a point with the vertical tangent).

We call *fiberwise isotopy with respect to the pencil of lines* $\mathcal{L}_p$ (or just *fiberwise isotopy* if it clear what pencil is meant) an isotopy of $\mathbb{RP}^2$ such that the image of any line from $\mathcal{L}_p$ at any moment is a line (maybe, another) from the same pencil. We call a *fiberwise arrangement of a curve with respect to* $\mathcal{L}_p$ (or just *fiberwise arrangement*) the equivalence class of a curve under fiberwise isotopies. As above, we shall encode fiberwise arrangements by symbols $\subset_k$, $\supset_k$, $\times_k$, o_k.

We say that a real nodal curve is *in almost general position* with respect to $\mathcal{L}_p$, if all the above genericity conditions hold with the only exception: one of the branches at a double point may have a vertical tangent, but this branch may not have a flex at the double point. To encode such points, we shall use the symbols $\in_k$ and $\ni_k$ (a perturbation of $\in_k$ yields $\subset_k\times_{k+1}$ or $\subset_{k+1}\times_k$).

A smooth family $\{B_t\}_{t\in[0,1]}$ of immersed curves not passing through p will be called an *admissible isotopy* with respect to $\mathcal{L}_p$, if there exists a finite set $T =$

$T' \cup T_{\text{flex}} \subset]0,1[$, such that:

(i) if $t \notin T \cup \{0,1\}$, then the curve B_t is in general position with respect to $\mathcal{L}_p$;

(ii) if $t = 0$ or 1, then the curve B_t is in almost general position with respect to $\mathcal{L}_p$;

(iii) if $t \in T$, then the genericity conditions hold with the following two exceptions: 1) for $t \in T'$, there exists a unique vertical line passing through exactly two critical points at least one of whom is a double point; 2) for $t \in T_{\text{flex}}$, there exists a unique flex point with vertical tangent, and the number of critical points decreases when t passes through such a value.

A passage through $t \in T_{\text{flex}}$ will be called a *zigzag removal.* It is easy to see that during an admissible isotopy the minimal number of intersections with vertical lines is constant, and the maximal number does not increase.

We define a *fiberwise model* of a curve A, as a set B such that there exists an admissible isotopy $\{B_t\}$ with $B_0 = \mathbb{R}A$ and $B_1 = B$.

Let us say that a union of immersed circles B satisfies the *k-condition with respect to p*, if any line from $\mathcal{L}_p$ meets B at least at k points. We say that B satisfies the *strong k-condition* if the image of B under any diffeomorphism of the pair $(\mathbb{RP}^2, p)$ onto itself satisfies the k-condition. If a curve B satisfies the $(m-2)$-condition and any line of $\mathcal{L}_p$ meets it at $\le m$ points, then this curve uniquely defines an m-braid (which we denote by $\operatorname{br}(B)$) such that $B = \mathbb{R}A$ for some real pseudoholomorphic curve A if and only if the braid $\operatorname{br}(B)$ is *quasipositive*, i.e., can be written in the form $\prod_j a_j \sigma_{i_j} a_j^{-1}$ (see details in [13, 16]).

Using this terminology, one can reformulate the statement on zigzag removal [16; Proposition 2.2] (more precisely, one of its corollaries) as follows.

Proposition 6.1. *Let A be a nodal real pseudoholomorphic curve in $\mathbb{RP}^2$ of degree m, in general position with respect to $p \in \mathbb{RP}^2$, such that $\mathbb{R}A$ satisfies the $(m-2)$-condition. Then for any fiberwise model B of A, the braid $\operatorname{br}(B)$ is quasipositive.*

The following statement or its analogs for other contexts I used implicitly in all my previous papers on application of quasipositive braids in problems of real geometry (including the previous section of this paper).

Proposition 6.2. *Let A be a nodal real pseudoholomorphic curve in $\mathbb{RP}^2$ of degree m, in general position with respect to $p \in \mathbb{RP}^2$, such that $\mathbb{R}A$ satisfies the strong $(m-2)$-condition. Suppose that $\mathbb{R}A$ is a union of embedded pairwise transversal circles. Then there exists a fiberwise model of $\mathbb{R}A$, which can be encoded by a word containing only symbols $\times_k, \mathrel{-\!\!\in}_k, \mathrel{\ni\!\!-}_k$, and o_k (i.e., not containing $\subset_k$ and $\supset_k$).*

Proof. Suppose that the encoding word contains $\subset_k$ (the case of $\supset_k$ is similar) and let us prove that there exists an admissible isotopy such that the number of "$\subset$" decreases and the number of "$\supset$" does not increase. Choosing, if necessary, another line from $\mathcal{L}_p$ as the infinite line (this corresponds to a cyclic permutation of the encoding word followed by an evident change of the indices), we may assume that there exists a subword of the form $\subset_k \times_{i_1} \dots \times_{i_n} u$ where u is either $\supset_l$, or $\mathrel{\ni\!\!-}_l$. If $n > 0$ and $i_1 \notin \{k-1, k, k+1\}$, then replacing $\subset_k \times_{i_1}$ with $\times_{i_1} \subset_k$ for $i_1 < k$ or with $\times_{i_1 - 2} \subset_k$ for $i_1 > k$ one can reduce the length of the subword $\times_{i_1} \dots \times_{i_n}$ (it is clear that such a replacement corresponds to an admissible isotopy). Therefore, we may assume that either $n = 0$, or $k - 1 \le i_1 \le k + 1$.

Case 1. $n = 0$ and $u = \supset_k$. Replace $\subset_k \supset_k$ with o_k.

Case 2. $n = 0$ and $u = \supset_{k\pm1}$. Remove $\subset_k\supset_{k\pm1}$ (zigzag removal).

Case 3. $n = 0$ and $u = \ni_k$ or $\ni_{k-1}$. Self-intersection (i.e., this contradicts to the condition that $\mathbb{R}A$ is a union of embedded circles).

Case 4. $n = 0$ and $u = \ni_{k+1}$ (the case of $u = \ni_{k-2}$ is similar). Replace $\subset_k\ni_{k+1} \to \subset_k\times_{k+2}\supset_{k+1} \to \times_k\subset_k\supset_{k+1} \to \times_k$.

Case 5. $n = 0$, but u is none of the above symbols. Then there exists an isotopy (certainly, non-admissible!), exchanging $\subset_k$ and u. This contradicts to the strong $(m-2)$-condition.

Case 6. $n > 0$, $i_1 = k \pm 1$. Replace $\subset_k\times_{k\pm1}$ with $\in_k$ or $\in_{k-1}$.

Case 7. $n > 0$, $i_1 = k$. Self-intersection. $\square$

6.4. Clusters of double points.

Let B be a union of embedded circles in $\mathbb{RP}^2$, and let $p \in \mathbb{RP}^2 \setminus B$. A subset D of $\mathbb{RP}^2 \setminus \{p\}$ is called a *digon of B with respect to p*, if

1. D is homeomorphic to an open disk,
2. there exists a connected component B' of B, such that D is a connected component of $\mathbb{RP}^2 \setminus (B' \cup \{p\})$,
3. q_1 and q_2 are the only double points of B lying on ∂D.

In this case we say that the points q_1 and q_2 are *connected by the digon D*. A digon D of B is called *empty*, if $D \cap B = \varnothing$.

Let Γ_B be the graph whose vertices are the double point of B and whose edges correspond to digons connecting them to each other. We define a *cluster of double points* of B as the set of vertices of a connected component of Γ_B. A cluster is called *degenerate*, if it contains two points connected by two digons which have a common side. It is clear that the graph of any non-degenerate cluster c is homeomorphic to a circle, a segment, or a point. In these cases we shall call the cluster *cyclic*, *linear*, or *trivial* respectively.

The following Lemma is evident.

Lemma 6.3. *Let B be a union of embedded circles in $RP^2\setminus\{p\}$ which are transversal to each other, and let c be a non-degenerate non-trivial cluster of B with respect to p. Suppose that the fiberwise arrangement of B is encoded by a word which contains only "$\in$", "$\ni$", "$\times$" and "o". Let $\pi_p : \mathbb{RP}^2 \setminus \{p\} \to \mathbb{RP}^1$ be the projection from p. Then there exist two smooth mappings $\gamma_1, \gamma_2 : [0,1] \to B$ such that*

1. $\pi_p \circ \gamma_1 = \pi_p \circ \gamma_2$ *(denote this mapping by f),*
2. $f'(t) \neq 0$ *for* $0 < t < 1$,
3. $c = \gamma_1([0,1]) \cap \gamma_2([0,1])$.

$\square$

Lemma 6.4. *Let A be a real pseudoholomorphic curve in $\mathbb{RP}^2$ of degree m such that $\mathbb{R}A$ is a union of pairwise transversal embedded circles satisfying the strong $(m-2)$-condition with respect to some point p. Let c be a non-degenerate cluster of B with respect to p, and let Γ be a subgraph of Γ_c homeomorphic to a segment, whose edges correspond to empty digons. Then there exists a fiberwise model of B encoded by a word which contains only "o", "$\times$", "$\in$", "$\ni$", such that the set of the vertices of Γ corresponds to a subword of the form $u\times_k^l v$ where $u \in \{\times_k, \in_k, \in_{k-1}\}$ and $v \in \{\times_k, \ni_k, \ni_{k-1}\}$.*

Moreover, an admissible isotopy transforming $\mathbb{R}A$ into B can be chosen so that it is constant on the complement to an arbitrarily small neighbourhood of the union of the digons corresponding to the edges of Γ.

Proof. By Proposition 6.2, there exists a fiberwise model B of $\mathbb{R}A$ encoded by a word in "o", "$\times$", "$\multimap$", "$\multimapinv$". Let γ_1 and γ_2 be as in Lemma 6.3. Let $T = \{t_1, \dots, t_n\}$ be a finite subset of $[0,1]$, such that $\gamma_1(T) = \gamma_2(T) =$ (the set of the vertices of Γ) and $t_1 < \cdots < t_s$. Let us set $N(B) = \text{Card}\left([t_1, t_n] \cap \gamma_1^{-1}(S)\right)$ where S is the union of those lines of $\mathcal{L}_p$ which pass through the critical points of B. Lemma 6.4 can be proved using the induction by $N(B)$, similarly to Lemma 6.3. $\square$

Lemma 6.5. *Let A be a real pseudoholomorphic curve in $\mathbb{RP}^2$ of degree 7. Suppose that $\mathbb{R}A$ is a union of pairwise transversal embedded circles. Let O be an oval of A, i.e., a smooth even branch not passing through double points. Suppose that $\mathbb{R}A$ does not contain degenerate clusters and satisfies the strong 5-condition with respect to some point p lying inside O. Then there exists a fiberwise model of $\mathbb{R}A$ encoded by a word which contains only "o", "$\times$", "$\multimap$", "$\multimapinv$", such that each non-trivial cluster of double points c_i corresponds to a subword of the form $u_i w_i v_i$ where $u_i \in \{\times_{k_i}, \multimap_{k_i}, \multimap_{k_i-1}\}$, $v_i \in \{\times_{k_i}, \multimapinv_{k_i}, \multimapinv_{k_i-1}\}$ for all $i = 1, \dots, n$, $w_i = \times_{k_i}^{l_i}$ for $i > 1$, and $w_1 = \times_{k_1}^{l_0} o_{j_1} \dots o_{j_s} \times_{k_1}^{l_1}$.*

Proof. Since there are no degenerate clusters, the curve $\mathbb{R}A$ does not contain any two digons with a common side. Hence, Bezout's theorem for an auxiliary line implies that $\mathbb{R}A$ has at most one non-empty digon. If all digons are empty, our statement follows from Lemma 6.4. Suppose that there is a non-empty digon D. We may assume that its vertices belong to c_1.

Let $\pi_p : \mathbb{RP}^2 \setminus \{p\} \to \mathbb{RP}^1$ be the projection from p. Lemma 6.4 implies that there exists a fiberwise model B of $\mathbb{R}A$ encoded by a word in "o", "$\times$", "$\multimap$", "$\multimapinv$", such that for any empty digon D', the open band $\pi_p^{-1}(\pi_p(D'))$ does not contain any critical point. The end of the proof is the same as for Lemma 6.4, and we omit it also. $\square$

6.5. Arrangements of $C_2 \cup C_5$ without nested arc.

It remains to consider the arrangements of $J_5 \cup C_2$ of the series **1, 2, 7, 8, 25.**

By Theorem 3, we can always choose the center of the pencil of lines inside an oval of the quintic which lies in its turn inside the conic.

So, for each of the domains into which J_5 divides the interior of the conic, we shall check the hypothesis that this domain contains an oval of the quintic (let us denote it by O_5). In each case, we shall consider all possible fiberwise arrangements of $C = C_2 \cup C_5$ with respect to $\mathcal{L}_p$ where p is chosen inside O_5.

It is clear that if the center of the pencil of lines is inside C_2 and inside O_5, then the arrangement satisfies the strong $(m-2)$-condition (see Sect. 6.3). Therefore, we may consider only fiberwise models of $C = C_2 \cup C_5$ whose existence is proven in Lemma 6.5. It is easy to check that in this case the fiberwise model of $C_2 \cup J_5 \cup O_5$ is uniquely determined by the fiberwise arrangement of the clusters of double points on C_2. Hence, it suffices to try one by one all possible fiberwise arrangements of n points on a conic C_2 where n is the number of clusters, constructing each time the fiberwise model of $C_2 \cup J_5 \cup O_5$, and then, to consider all possible positions for the ovals of the quintic.

Let us number the intersection points of C_2 and J_5 by $0, \dots, 9$ as in Sect. 0.5. For each choice of p, we shall denote the clusters by $c_1, c_2, \dots$ clockwise along C_2 (i.e., in the same order as for the intersection points) starting with the cluster containing the point "0". *Domains*, i.e., the connected components of $\mathbb{RP}^2 \setminus (J_5 \cup C_2)$, will be denoted by $D_{i_1 i_2 \dots}$ where $i_1, i_2, \dots$ are the intersection points belonging to the boundary of the domain.

In Table 2, we present the results of the primary analysis of all possible choices of the domain containing O_5 and for all choices of the fiberwise arrangement of the clusters. We do not consider the cases when a choice of a domain containing O_5 contradicts Bezout's theorem for auxiliary lines. For example, this is the reason why no line of Table 2 corresponds to the choice of D_{01} for the series **7**. Indeed, suppose that $p \in D_{01}$ and consider a line passing through p and a point on the arc (67) of the conic. It cuts O_5 at two points and it cuts also the arcs (01), (49), (58), and (67) of J_5, so it has too many intersections with C_5.

Let p be any point in the interior of C_2 and let $q_1, \dots, q_n$ be points on C_2, pairwise non-collinear with p. Under these conditions, there exist 2^{n-1} fiberwise arrangements of $C_2, q_1, \dots, q_n$ with respect to p. Indeed, let us choose an affine chart such that $\mathcal{L}_p$ is the pencil of vertical lines, and C_2 is the hyperbola $y^2 - x^2 = 1$. Let $q_i = (x_i, y_i)$. Let $(i_1, \dots, i_n)$ be the permutation providing $x_{i_1} < \dots < x_{i_n}$. We may assume that $i_1 = 1$ and $y_1 > 0$. Then the fiberwise arrangement of q_i's is uniquely determined by the sequence $(\operatorname{sign} y_{i_2}, \dots, \operatorname{sign} y_{i_n})$. In Table 2, we denote the corresponding fiberwise arrangement of the clusters $c_1, \dots, c_n$ by a $2\times n$-matrix whose lines correspond to the branches of the hyperbola. For example, ${}^{1}{}_{4}{}^{23}$ corresponds to the case when $x_1 < x_4 < x_2 < x_3$ and $\operatorname{sign} y_1 = \operatorname{sign} y_2 = \operatorname{sign} y_3 = 1 = -\operatorname{sign} y_4$.

By Proposition 6.2, it is sufficient to consider only fiberwise arrangements of $C_2 \cup J_5$, such that all the maxima of the restriction of the x-coordinate onto J_5 are situated on C_2. Such arrangements are uniquely determined by fiberwise arrangements of clusters. Let us illustrate it in the following example. Let us consider the first line of Table 2 (Series **25**, $p \in D_{78}$) and the fiberwise arrangement ${}^{1}{}_{4}{}^{23}$ of the clusters. To simplify pictures, we shall present the hyperbola C_2 as two horizontal lines. Let us place the points $0, \dots, 9$ on C_2 according to the given cluster arrangement (see Fig.30.1) and let us draw the arcs of J_5 one by one starting, for example, with (01). The x-coordinate is monotone on each arc. Therefore, there are two choices for the arc (01): Fig. 30.2 and Fig. 30.3. Since $p \in D_{78}$, this arc must cut the segment which is contained inside C_2 and which relates p to some point q of the arc (01) of the conic (for our choice of the affine coordinates, this is the vertical ray from q disjoint from the other branch of the hyperbola). By this reason, we have to choose Fig. 30.2 rather than Fig. 30.3. Similarly, we construct all other arcs, and we obtain the fiberwise arrangement of $J_5 \cup C_2$ depicted in Fig. 30.4. This arrangement contradicts Bezout's theorem for an auxiliary line through the point "4". In such cases, we write "B." in the corresponding square of Table 2.

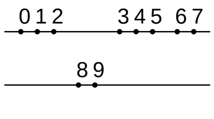


Fig. 30.1

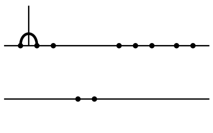


Fig. 30.2

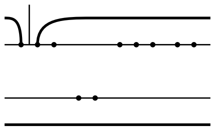


Fig. 30.3

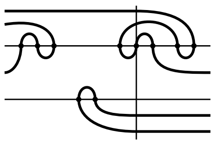


Fig. 30.4

When drawing by hand a fiberwise arrangement of $J_5 \cup C_2$, it is convenient to start with the exterior arc of J_5 passing through infinity and the arcs of J_5 bounding the domain containing O_5. After this, all other arcs will be uniquely determined by the condition that they do not cross the previously constructed arcs.

Table 2.

$C_2 \cup J_5$	p	clusters	^{1234}	${}^{123}{}_4$	${}^{12}{}_4{}^3$	${}^{12}{}_{34}$	${}^1{}_4{}^{23}$	${}^1{}_3{}^2{}_4$	${}^1{}_{34}{}^2$	${}^1{}_{234}$
25. (M10)	D_{78}	012, 345, 67, 89	B.	O.K.	B.	B.	B.	B.	B.	B.

$C_2 \cup J_5$	p	clusters	^{123}	${}^{12}{}_3$	${}^1{}_3{}^2$	${}^1{}_{23}$
2. (M33)	D_{01}	0, 123456, 789	B.	B.	$\leftarrow c_1$	O.K.
	D_{23}	012, 3456, 789	B.	B.	B.	O.K.
	D_{45}	01234, 56, 789	B.	B.	B.	O.K.
	D_{78}	0123456, 7, 89	B.	O.K.	$c_3 \rightarrow$	B.
7. (M31)	D_{012349}	01234, 567, 89	O.K.	B.	$\leftarrow c_3$	B.
	D_{4589}	01234, 567, 89	$c_3 \rightarrow$	$\leftarrow c_3$	O.K.	O.K.
	D_{5678}	01234, 567, 89	B.	O.K.	$c_3 \rightarrow$	B.
8. (M29)	D_{01}	0, 1234, 56789	B.	B.	$c_2 \rightarrow$	O.K.
	D_{23}	012, 34, 56789	B.	B.	B.	O.K.
25. (M10)	D_{236789}	012, 345, 6789	$c_3 \rightarrow$	$\leftarrow c_3$	O.K.	O.K.
	D_{0129}	012, 345, 6789	O.K.	B.	B.	$\leftarrow c_3$

$C_2 \cup J_5$	p	clusters	^{12}	${}^1{}_2$
1. (M30)	D_{01}	0, 123456789	$c_2 \rightarrow$	O.K.
	D_{23}	012, 3456789	O.K.	O.K.
	D_{45}	01234, 56789	O.K.	O.K.
2. (M33)	$D_{01234569}$	0123456, 789	O.K.	$c_1 \rightarrow$
	D_{6789}	0123456, 789	$c_2 \rightarrow$	O.K.
8. (M29)	D_{012349}	01234, 56789	O.K.	$c_1 \rightarrow$

$C_2 \cup J_5$	p	clusters	fiberwise arrangements of clusters
1. (M30)	$D_{01\dots89}$	0123456789	O.K.

An expression of the form "$c_i \rightarrow$" (respectively, "$\leftarrow c_i$") in Table 2 means that the given fiberwise arrangement can be reduced to another one by means of an admissible isotopy (see the definition in Sect. 6.3) such that c_i moves to the right (respectively, to the left) and other clusters are fixed. Consider, for example, the fiberwise cluster arrangement ${}^1{}_3{}^2$ in the second line of Table 2. It corresponds to the arrangement of J_5, O_5, and C_2 depicted in Fig. 31. By Bezout's theorem for vertical lines, the remaining five ovals of C_5 must be contained in the band bounded by the vertical lines through the points 9 and 1. Moreover, they are contained in a smaller band which we depict in Fig. 31 as a gray rectangle. Then there exists an admissible isotopy shown by arrows in Fig. 31 which moves c_1 (consisting of a single point "0") to the left. As a result, we obtain the fiberwise arrangement ${}_{31}{}^2$ which can be transformed into ${}^1{}_{23}$ changing the infinite line.

Finally, we write "O.K." in Table 2 if the corresponding fiberwise arrangement of C_2, J_5, and O_5 should be further studied. Every such an arrangement corresponds to a line of Table 3.

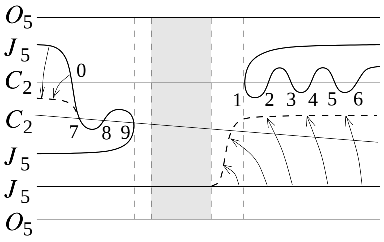


Fig. 31

Table 3.

1.	D_{01}	9	$[\ni_3 u_1 \times_3 u_2 \in_2 \times_3^7]$	$[22323][]$, $[43333][]$, $[4][4444]$	7,2,2
	D_{23}	0	$[\times_4^2 \ni_4 u_1' \in_4 \times_4^6]$	$[44443]$, $[44454]$	3,8
	–”–	9	$[\ni_3 u_1 \in_3 \times_4 \ni_4 u_2 \in_2 \times_3^5]$	$[422][44]$, $[444][33]$	10,9
	D_{45}	0	$[\times_4^4 \ni_4 u_1' \in_4 \times_4^4]$	$[45554]$	5
	–”–	9	$[\ni_3 u_1 \in_3 \times_4^3 \ni_4 u_2 \in_2 \times_3^3]$	$[334][33]$, $[433][44]$	4,4
	$D_{01\dots9}$	0	$[u_1' \times_3^\alpha u_2' \times_3^{10-\alpha}]$	$[]_1[43333]_9$, $[3333]_1[4]_9$, $[32223][]^*$, $[32322][]$, $[32325][]$, $[32355][]$, $[34555][]$, $[]_2[43222]_8^*$, $[]_3[43322]_7$, $[]_3[43325]_7$, $[]_3[43355]_7$, $[33]_3[444]_7$, $[3333]_3[4]_7$, $[]_4[43222]_6$, $[322]_4[45]_6$, $[323]_4[44]_6^*$, $[325]_4[45]_6$, $[355]_4[45]_6$, $[33]_5[433]_5$	2,2,1*,1, 1,1,1,11*, 9,9,9,10, 3,6,6,12*, 6,6,4
2.	D_{01}	9	$[\ni_3 u_1 \times_3 u_2 \in_2 \times_3^5 \times_4^2]$	$[33333][]$, $[3][4444]$, $[][44443]$	5,5,5
	D_{23}	9	$[\ni_3 u_1 \in_3 \times_3 \ni_4 u_2 \in_2 \times_3^3 \times_4^2]$	$[322][44]$, $[344][33]$	9,8
	D_{45}	9	$[\ni_3 u_1 \in_3 \times_3^3 \ni_4 u_2 \in_2 \times_3 \times_4^2]$	$[333][44]$	3
	D_{78}	0	$[\times_3^6 \ni_3 u_1 \times_3 u_2 \in_2 \times_3]$	$[][43333]$, $[][44444]$	7,7
	$D_{0\dots569}$	9	$[\ni_2 u_1 \times_3^\alpha u_2 \times_3^{7-\alpha} u_3 \in_3 x_4]$	$[3322]_4[4]_3[]$, $[33232][][]$, $[333][][42]$, $[3]_3[4332]_4[]$, $[]_4[42222]_3[]$, $[4][][4444]$, $[444][][44]$	1,2,6, 8,10, 12,4
	D_{6789}	1	$[\times_4^5 \ni_4 u_1 \times_2^\alpha u_2 \times_2^{3-\alpha} u_3 \in_3]$	$[3322][][3]$, $[][][34442]$, $[43322][][]$, $[444][][33]$	12,4, 11,6
7.	D_{012349}	9	$[\ni_2 u_1 \times_3^\alpha u_2 \times_3^{5-\alpha} u_3 \in_3 \times_3^2 \times_4]$	$[3333]_4[4]_1[]$, $[43333][][]$, $[]_4[44444]_1[]$, $[44444][][]$	3,2, 6,2
	D_{4589}	0	$[\times_4^4 \ni_4 u_1' \times_2 u_2' \times_2 u_3' \in_4 \times_4^2]$	$[33444][][]$, $[45544][][]$	4,1
	–”–	1	$[\times_4^3 \ni_4 u_1 \in_2 \times_3 \ni_3 u_2 \times_2 u_3 \times_2 u_4 \in_3]$	$[4233][2][][]$	1
	D_{5678}	0	$[\times_3^4 \ni_3 u_1 \times_3 u_2 \times_3^2 u_3 \in_2 \times_3]$	$[2222][][4]$, $[24333][][]$	5,4
8.	D_{01}	9	$[\ni_3 u_1 \times_3 u_2 \in_2 \times_3^3 \times_4^4]$	$[][42223]$, $[34333][]$	8,5
	D_{23}	9	$[\ni_3 u_1 \in_3 \times_4 \ni_4 u_2 \in_2 \times_3 \times_4^4]$	$[222][44]$, $[244][33]$	6,7
	D_{012349}	9	$[\ni_2 u_1 \times_3^\alpha u_2 \times_3^{5-\alpha} u_3 \in_3 \times_4^3]$	$[3222]_4[4]_1[]$, $[32232][][]$, $[333][][43]$, $[3]_1[4333]_4[]$, $[]_2[42223]_3[]$, $[4333][][4]$, $[]_3[44332]_2[]$	3,4,1, 5,2, 1,7
25.	D_{78}	0	$[\times_3^2 \ni_3 u_1 \in_3 \times_3^2 \times_4 \ni_4 u_2 \in_2 \times_3]$	—–	–
	D_{236789}	0	$[\times_4^2 \ni_4 u_1 \times_2^\alpha u_2 \times_2^{4-\alpha} u_3 \in_4 \times_4^2]$	$[]_1[32224]_3[]$, $[4222]_3[3]_1$, $[43444][][]$, $[444][][34]$	3,3, 2,2
	–”–	1	$[\times_4 \ni_4 u_1' \in_2 \times_3 \ni_3 u_2' \times_2^\alpha u_3' \times_2^{4-\alpha} u_4' \in_3]$	$[4433][2][][]$	1
	D_{0129}	9	$[\ni_2 u_1 \times_3^2 u_2 \times_3 u_3 \in_3 \times_3^2 \times_4^3]$	$[3222][4][]$, $[333][][43]$, $[42223][][]$	4,2,5

Table 3 is organized similarly to Table 1 above. In the fourth column, we write the encoding words for fiberwise arrangements of $C = C_5 \cup C_2$. The variables $u_1, u_2, \dots$ or $u'_1, u'_2, \dots$ (the meaning of the primes is explained below) stand for subwords (including the empty one) of the total length 5 and of the form $o_{i_1} o_{i_2} \dots$ where $2 \le i_1, i_2, \dots \le 5$.

It follows from the commuting properties of symbols "$\times$" and "o", that if a word contains a subword of the form $u^{(\prime)}_{i-1} \times^m_k u^{(\prime)}_i$, then it is enough to consider only the cases when $u^{(\prime)}_i$ starts with $o_{k'}$ for $k' \ne k+1$. By the same reason, in the case (**1,** $p \in D_{0\dots9}$), we may assume that either the word u'_1 is empty, or it starts with o_3, or $u'_1 = o_2^5$ and $u'_2 = \varnothing$.

The encoding word depends on the choice of the infinite line. Always when it is possible (i.e., in all cases except the last line for Series **1**), we choose it so that it meets J_5 at three points, and so that the fiberwise arrangement of clusters on the affine chart is as similar as possible to that given in Table 2. In the third column of Table 3, we write the number of the leftmost point.

Now let us explain the difference between u_k and u'_k in the encoding words. Suppose that we study the case when the leftmost point has the number i and O_5 is contained in D_α. If this domain contains more than one ovals, then any of them can be chosen as O_5. Let us consider the tangent to C_2 at the i-th point and let us rotate it (in one direction or another) up to the first tangency with an oval of C_5 contained in D_α, and let us choose this oval to be O_5. The obtained line (let us denote it by L) divides the interior of C_2 into two components but only one of them can contain other ovals from D_α. This means that either o_2 or o_5 does not occur in the encoding word (depending on the direction of the rotation). The absence of prime of a variable u_k means that it may be replaced only by words without o_5.

In the cases when two different fiberwise models correspond to the same line of Table 2, we apply this trick only to one of them for not to bother about the compatibility of the choices of O_5. The fact that this precaution is not in vain, can be illustrated by the arrangement **1** with $p \in D_{45}$. Analyzing the last column of Table 3 (whose meaning is explained below), we see that if we consider only words without o_5, then we would "prove" the unrealizability of the fifth arrangement of Series **1**.

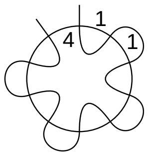
Also, this argument cannot be applied to (**1.** $p \in D_{0\dots9}$), because it is possible in this case that the line L cuts $J_5 \cup O_2$ at three real points only. Hence, ovals of C_5 may pass through L during an admissible isotopy which transforms o_5 into o_2 and vice versa.

For each fiberwise arrangement listed in Table 3 and satisfying the above restrictions, we have checked on a computer if the corresponding braid b satisfies Murasugi-Tristram inequality (see details in [16]), and if the Alexander polynomial $\Delta_b(t)$ satisfies the Fox-Milnor condition

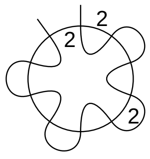
$$\exists f \in \mathbb{Z}[t] \quad : \quad \Delta_b(t) \doteq f(t) f(t^{-1})$$

for $e(b) = 6$ (i.e., in the case **1.** $p \in D_{0\dots9}$), or the condition $\Delta_b(t) = 0$ for $e(b) \le 5$ (i.e., in the remaining cases). Here $e(b)$ denotes the exponent sum of b, i.e., $e(b) = \sum k_j$ for $b = \prod \sigma_{i_j}^{k_j}$, and $\doteq$ means the equality up to multiplication by units of the ring $\mathbb{Z}[t, t^{-1}]$.

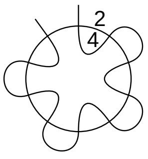
All fiberwise arrangements satisfying these conditions are listed in the fifth column of Table 3 (the data format is the same as in the last column of Table 1). For each of them, in the last column we give the reference to the corresponding (non-fiberwise) arrangement in Sect. 0.5, namely, we write the order number (according to the order in Sect. 0.5) of the corresponding arrangement among all arrangements of the same series. The cases excluded below are marked by the asterisk. The numbers 11 and 12 of Series **1** refer to Fig. 32.1 and Fig. 32.2.



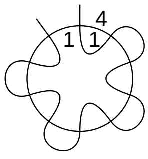
1-11 (unrealizable)
Fig. 32.1



1-12 (unrealizable)
Fig. 32.2



1-7
Fig. 33.1



1-2
Fig. 33.2

For example, the first line of Table 3 means that we have considered all fiberwise arrangements encoded by

$$[\ni_3 o_{i_1} \dots o_{i_k} \times_3 o_{i_{k+1}} \dots o_{i_5} \in_2 \times_3^7],$$

where $0 \le k \le 5$, $2 \le i_j \le 4$ for all j, and if $k < 5$, then $i_{k+1} = 4$. Those among them, which satisfy Murasugi-Tristram inequality and the condition $\Delta_b(t) = 0$ are only

$$[\ni_3 o_2^2 o_3 o_2 o_3 \times_3 \in_2 \times_3^7], \quad [\ni_3 o_4 o_3^4 \times_3 \in_2 \times_3^7] \quad \text{and} \quad [\ni_3 o_4 \times_3 o_4^4 \times_3 \in_2 \times_3^7],$$

moreover, the former case corresponds to the 7th arrangement of Series **1** (i.e., Fig. 33.1), and the two latter cases both correspond to the second arrangement of the same series (Fig. 33.2).

It remains to prove the unrealizability of the arrangements in Fig. 32.1 and Fig. 32.2. We shall apply the following generalization of Fox-Milnor theorem.

Theorem 6.6. (Florens [5]). *Let $F = F_1 \cup \dots \cup F_\mu$ be a smooth compact surface embedded into the 4-ball $\mathbb{B}^4$ such that $L = \partial F \subset \mathbb{S}^3 = \partial\mathbb{B}^4$. Let us set $L_i = \partial F_i$. Let $\Delta_L(t_1, \dots, t_\mu)$ be the multi-variable Alexander polynomial corresponding to the partition $L = L_1 \cup \dots \cup L_\mu$.*

Suppose that the Euler characteristic $\chi(F)$ is equal to one. Then there exists a polynomial $f(t_1, \dots, t_\mu)$ with integer coefficients such that

$$\Delta_L(t_1, \dots, t_\mu) \prod_{i=1}^{\mu} (t_i - 1)^{-\chi(F_i)} \doteq f(t_1, \dots, t_\mu)\, f(t_1^{-1}, \dots, t_\mu^{-1}), \tag{12}$$

where $\doteq$ means the equality up to multiplication by units of the ring $\mathbb{Z}[t_1^{\pm1}, \dots, t_\mu^{\pm1}]$.

Let b be the braid associated to one of the fiberwise arrangements listed in the last line of Series **1** in Table 3. As explained in [13], if this fiberwise arrangement is a fiberwise model of a real pseudoholomorphic curve $C = C_2 \cup C_5$, then the closure L of b is isotopic to the boundary of a surface F, which is the intersection of the complexification of C with some 4-ball embedded into $\mathbb{CP}^2$. Let $L = L_2 \cup L_5$ and $F = F_2 \cup F_5$ be the partitions corresponding to the partition $C = C_2 \cup C_5$. By Riemann-Hurwitz formula, we have $\chi(F) = \deg C - e(b) = 7 - 6 = 1$. Hence, the multi-variable Alexander polynomial $\Delta_L(t_2, t_5)$ associated to the partition $L = L_2 \cup L_5$ must have the form (12). An explicit computation shows that this is not so in the cases marked by the asterisk in Table 3.

Remark 6.7. The arrangement in Fig. 32.1 (respectively, in Fig. 32.2) can be also excluded without Theorem 6.6. One can check that if the center of the pencil of lines is chosen inside an oval contained in the digon (12) (respectively, (34)), then all fiberwise models contradict Murasugi-Tristram inequality. As above, it is sufficient to consider only models corresponding to all possible fiberwise arrangements of the clusters $c_1 = (01)$ and $c_2 = (2 \dots 9)$ (respectively, $c_1 = (0123)$ and $c_2 = (456789)$) on the conic C_2, which can be represented in this case by an ellipse in the affine plane. By Proposition 6.2, it is enough to consider only the cases when the extrema of x-coordinate on C_2 are the leftmost or rightmost points of the clusters.

Remark 6.8. The arrangements in Figures 32.1 – 32.2 are pseudoholomorphically unrealizable, but the corresponding Alexander polynomials in one variable do not contradict to Fox-Milnor theorem. Perhaps, this is just a coincidence. But its probability is rather small (in contrary to a situation when, say, Murasugi-Tristram inequality occasionally holds). So, the following question rises naturally: is this an indication that these arrangements are realizable as the sets of real points of some objects (for instance, flexible curves in the sense of Viro?) which generalize real algebraic curves but in a weaker sense than real pseudoholomorphic curves?

§7. Completing of the classification

7.1. Restriction for $C_5 \cup L_1 \cup L_2$, deduced from already proven restriction for $C_5 \cup C_2$.

It follows from Bezout's theorem for auxiliary lines that if an M-quintic C_5 and two lines L_1, L_2 realize an arrangement of the type (1,1) (respectively, (1,3)), then the odd branch J_5 of C_5 is arranged with respect to $L_1 \cup L_2$ in one of the 8 (respectively, 19) ways corresponding to the series in Sect. 0.6 (respectively, in Sect. 0.7). The question is how the ovals of the quintic are distributed between the components of the complement to $J_5 \cup L_1 \cup L_2$.

For the most of the series, in particular, for all the series of the type (1,1), the answer to this question follows from the partial classification of arrangements of $C_5 \cup C_2$ with one passage through infinity which is already obtained in §6, combined with the classification of the arrangements of $C_5 \cup C_2$ with five passages through infinity or with five nested interior arcs which evidently reduces to Polotivskii's classification of arrangements of an M-quintic and a line.

Example 7.1. Series **8** of the type (1,1), i.e., Series E according to [9]. Let us denote the components of the complement to $J_5 \cup L_1 \cup L_2$ according to Fig. 34.1 (as usually, each of the letters $a, b, \dots$ denotes simultaneously the domain and the number of ovals of the quintic contained in it). Bezout's theorem for auxiliary lines implies that there are no ovals in other domains. Let us perturb $L_1 \cup L_2$ into a nonsingular conic as shown in Fig. 34.2. We obtain the arrangement of $J_5 \cup C_2$ of Series **28**. The classification for this series is not completed yet (in contrary, in §5 we decided to reduce this case to the classification of $C_5 \cup L_1 \cup L_2$). Let us try then to redraw this arrangement as shown in Fig. 35.3 and to perturb it as in Fig. 35.4. This time, we obtain an arrangement of $J_5 \cup C_2$ of Series **24**, whose classification is completed in Sect. 6.2. It follows from this classification that only two cases are possible: (1) $a = d = 0$, $b = 4$, $c = 2$ or (2) $c = 0$, $a = b = d = 2$. Both of them can be found in the list of realized arrangements in Sect. 0.6.

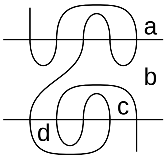


Fig. 34.1

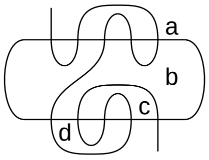


Fig. 34.2

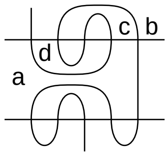


Fig. 34.3

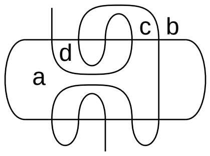


Fig. 34.4

Example 7.2. Series **16** of the type (1,3). Denote the domains as in Fig. 35.1. Since one of two possible perturbations of two lines into a conic provides an arrangement with three passages through infinity, whose classification we did not even started yet (it will be obtained in Sect. 7.3), we shall consider only the other perturbation, the one depicted in Fig. 35.2. This is Series **13** where, according to Sect. 6.2, only one arrangement is realizable and we have $a + b = c = d = 2$ for it. It remains to show that $a = 0$ and $b = 2$. This follows from the fact that after forgetting one of the lines, only these values of a and b provide an arrangement which does not contradicts to Polotovskii's classification of arrangements of a quintic and a line.

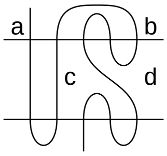


Fig. 35.1

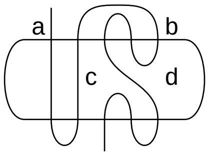


Fig. 35.2

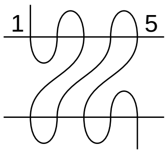


Fig. 36.1

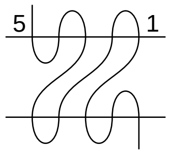


Fig. 36.2

Proceeding as in these examples, it is not difficult to exclude all arrangements absent in Sections 0.6 – 0.7, except the series **1, 2, 3, 9, 10, 12, 14** of the type (1,3) and except two more arrangements of Series **5** of the type (1,3) which are depicted in Fig. 36.1 – 36.2.

7.2. End of proof of Theorem 2.

In all the remaining cases, we apply Murasugi-Tristram inequality. For the arrangements in Fig. 36.1 – 36.2, the choice of the center of the pencil of lines in the solitary oval provides unique fiberwise models $[\ni_4 o_3^5 \times_3 \in_4 \times_4^2 \times_3^2 \times_4^2 \times_3^2]$ and $[\supset_4 o_4^4 \times_2 \times_3^3 \times_2^2 \times_3^2 \times_2^3 \subset_3]$ respectively.

In other cases, we choose the center of the pencil of lines on one of the lines L_1, L_2 (let it be L_1) inside the nest formed by arcs of the quintic. We choose L_1 as the infinite line. The results of computations are presented in Table 4 which is organized similarly to Tables 1 and 3 in Sections 6.2 and 6.4. As in Sect. 6.2, the variables u_i stand only for those subwords of the form $o_{i_1} o_{i_2} \dots$ which provide realizable fiberwise arrangements of C_5 after dropping L_2. If u_i follows $\times_k$, then either it is empty, or it starts with o_{k+1}.

7.3. End of proof of Theorem 1.

By Corollary 5.2, the classification of the series not considered in §6 follows from Theorem 2.

Table 4.

1.	$[\supset_2 u_1 \!-\!\!\in_2 \times_2 \ni\!\!-_1 u_2 \!-\!\!\in_1 \times_2]$	[222][322], [1][22222]	1,2
2.	$[\times_3 \ni\!\!-_2 u_1 \times_1^{\alpha} u_2 \times_1^{3-\alpha} u_3 \subset_1]$	[322244][][], [323311][][], $[]_1[211111]_2[]$	3,1,2
3.	$[\supset_1 u_1 \times_1 u_2 \times_1 u_3 \!-\!\!\in_2 \times_3^2]$	[44][][2223], [1133][][23], [11111][2][]	3,1,2
9.	$[\supset_1 u_1 \ni\!\!-_2 \times_2 \times_1^2 \times_2]$	[112222], [111111]	2,1
10.	$[\supset_2 u_1 \times_1 u_2 \!-\!\!\in_1 \times_2 \times_3^2]$	[444444][], [333344][], [3333][22], [][222222]	1,2,2,1
12.	$[\supset_1 u_1 \!-\!\!\in_2 \times_2 \ni\!\!-_1 u_2 \times_1 u_3 \!-\!\!\in_3]$	[1111][22], [1122][33]	1,2
14.	$[\supset_1 u_1 \times_1 u_2 \!-\!\!\in_1 \times_2 \ni\!\!-_2 u_3 \!-\!\!\in_2]$	[113][][223]	1

Appendix A. A new proof of the algebraic unrealizability of a certain pseudoholomorphic M-curve of bidegree $(4,8)$ on the quadratic cone

Proposition A.1. *There does not exists a real algebraic curve of bidegree* $(4,8)$ *on the Hirzebruch surface* $\mathbb{R}\mathcal{F}_2$ (*i.e., on the blown up quadratic cone*) *which is arranged with respect to the exceptional divisor* E *and one of fibers* F *as depicted in Fig. 37 where* $\mathbb{R}\mathcal{F}_2$ *is represented by a rectangle whose opposite sides are identified. The horizontal sides represent* E *and the vertical sides represent* F.

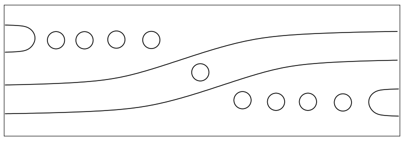


Fig. 37.

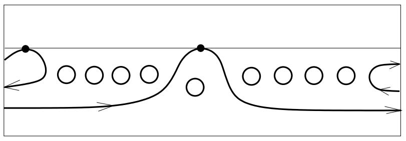


Fig. 38.

This result is proven in [21] using Hilbert-Rohn method. Note that Fig. 37 is realizable by a real pseudoholomorphic curve in an almost complex structure where the self-intersection number of E is equal to -2. In this appendix, I give a new proof of Proposition A.1, similar to the proof of the algebraic unrealizability of Fig. 16.12 given in Sect. 4.4, but here I use also Agnihotri-Woodward inequalities. In my opinion, the new proof is simpler and "more reliable" than the old one (unfortunately, proofs based on Hilbert-Rohn method sometimes have mistakes because some cases of possible degenerations are missed; the first version of §4 of this text nearly contributed to the list of such erroneous proofs).

On the other hand, the result stated in Proposition A.1 is obtained in [21] as a corollary of a stronger result stating that the $(M-1)$-curve obtained from Fig. 37 by removing the central oval is algebraically unrealizable. At the present time, I do not know any proof of this stronger result, other than the proof given in [21] and based on Hilbert-Rohn method.

In this context, it is natural to ask the following question. *Is algebraically realizable the arrangement of an* $(M-1)$*-quintic and two lines on* $\mathbb{RP}^2$ *obtained from Fig. 16.12 by deleting the solitary empty oval* (*i.e., by erasing the digit "1" on the picture*)?

Proof of Proposition A.1. If Fig. 37 were algebraically realizable, then the cubic resolvent R should be arranged with respect to the core L as in Fig. 38 (see Remark 4.5). The braid associated to the curve $A = R \cup L$ has the form

$$b = \sigma_3^{-2}\sigma_2^{-5}\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_3^{-2}\sigma_2^{-1}\sigma_1\sigma_2^{-4}\Delta^4.$$

The curve R is maximal and its complex orientation must be as shown in Fig. 38, because otherwise the linking number of positive and negative strings would not be equal to zero. Let us introduce the notation H, N, N_R, and N_0 as in §4.

It follows from counting of the Euler characteristics and liking numbers (as in §4) that N consists of two connected components N_1, N_2 which cover H once and twice respectively. The projection $N_2 \to H$ has two branching points and each of N_1 and N_2 once touches N_0.

Hence, coloring the strings of b into the colors (t_2, t_1, t_2, t_0), we conclude that it must have the form $b = b_1^{(1)} b_1^{(2)} b_4^{(1)} b_4^{(2)}$ where the colored braid $b_1^{(j)}$, $j = 1, 2$ is a conjugate of $\sigma_1 : (t_2, t_2, t_0, t_1) \to (t_2, t_2, t_0, t_1)$, and the colored braids $b_4^{(j)}$, $j = 1, 2$, are conjugates of $\sigma_1^4 : (t_0, t_j, t_{3-j}, t_2) \to (t_0, t_j, t_{3-j}, t_2)$.

The images of $b_1^{(j)}$ and $b_4^{(j)}$ have the spectra $(-t_2, 1, 1)$ and $(t_0^2 t_j^2, 1, 1)$ respectively.

A computation similar to that in [15] leads to a contradiction with one of Agnihotri-Woodward inequalities for the values of the parameters $t_0 = t_2 = i = \sqrt{-1}$, $t_1 = -\bar{\tau} = (1 + i\sqrt{3})/2$.

Appendix B. Example of an unremovable zigzag on a real algebraic curve

Proposition B.1. *a). There exists a nonsingular real algebraic curve of bidegree* $(3, 6)$ *on* $\mathcal{F}_2$ (*see Definition 2.10*) *whose fiberwise arrangement is encoded by* $w = [\supset_1 o_1 \subset_1 \supset_2 o_2 \subset_2 \supset_1 \subset_2 \supset_1 \subset_2]$.

b). There does not exist any nonsingular real algebraic curve of bidegree $(3, 6)$ *on* $\mathcal{F}_2$ *whose fiberwise arrangement is encoded by* $w' = [\supset_1 o_1 o_2 \subset_2 \supset_1 \subset_2 \supset_1 \subset_2]$.

Proof. a). Such a curve can be easily constructed by Viro's method subdividing the triangle (0,0)-(6,0)-(0,3) by the segment (3,0)-(0,3). The chart gluing corresponds to the subdivision of the word w as $w = u_1 v u_2$ where $u_1 = \supset_1 o_1 \subset_1$, $v = \supset_2 o_2 \subset_2 \supset_1 \subset_2$, $u_2 = \supset_1 \subset_2$. Here v represents the chart in (0,0)-(3,0)-(0,3), and $u_1 u_2$ represents the chart in (3,0)-(6,0)-(0,3).

b). Follows from [18; Lemma 6.6] which states that the encoding word of a curve of bidegree $(3, 3n)$ on $\mathcal{F}_n$ cannot contain more than n subwords of the form $\supset_1 \subset_2$ or $\supset_2 \subset_1$. $\square$

The word w' is obtained from w by deleting the subword $\subset_1 \supset_2$, i.e., by a zigzag removal.

Remark. A generalization of [18; Lemma 6.6] (which was used in the proof of Part (b) of Proposition B.1) for any degree was recently obtained by Brugallé [1].

Appendix C. Smooth pseudoholomorphic curves of bidegree $(4, 16)$ on $\mathcal{F}_4$ whose (non-fiberwise) isotopy type is algebraically unrealizable

Recall that we denote the n-th Hirzebruch surface by $\mathcal{F}_n$ (see Definition 2.10 about the choice of a real structure on $\mathcal{F}_n$). When speaking about real pseudoholomorphic curves on $\mathcal{F}_n$, we consider tame conj-invariant almost complex structures, such that there exists a pseudoholomorphic curve E whose self-intersection number is $-n$ (the exceptional curve). Such a curve is necessarily real.

Welschinger [24] constructed real pseudoholomorphic curves A on $\mathcal{F}_n$, $n \geq 2$, for generic conj-invariant almost complex structures such that the isotopy type of

$\mathbb{R}A$ is algebraically unrealizable. The reason of the algebraic unrealizability of the examples from [24] is that otherwise the number of real intersections of $\mathbb{R}A$ with $\mathbb{R}E$ would be greater than the intersection number $A.E$. In particular, this implies that the examples from [24] are unrealizable for almost complex structures with an exceptional curve.

In the case when the exceptional curve exists, restrictions based on intersection numbers with auxiliary rational curves cannot work. However, it is possible to apply Hilbert-Rohn method as well as the cubic resolvent techniques. Using these methods, algebraic unrealizability of some pseudoholomorphically realizable fiberwise arrangements is proven in [21, 22] (see also Appendix A above). However, the curves in these examples are isotopic (non-fiberwise) to algebraic ones.

In this appendix, using cubic resolvents, algebraic unrealizability of some (non-fiberwise) isotopy types by curves of bidegree $(4,16)$ on $\mathcal{F}_4$ is proven. Moreover, these isotopy types are realizable by pseudoholomorphic curves in a tame conj-invariant almost complex structure with an exceptional curve. I obtained this result when I was preparing the paper [16], but I did not include it there because I was going to write another paper devoted to cubic resolvent method. Now, when it is already exposed in [18, 22] and in the present paper, a separate paper on this subject is not needed. So, I decided to write this appendix.

A circle smoothly embedded into $\mathbb{R}\mathcal{F}_n$, is called an *oval* if it bounds a disk (since $\mathbb{R}\mathcal{F}_n$ is a torus or a Klein bottle, this condition is equivalent to the fact that the circle is zero-homologous). If a curve has only ovals, we shall use Viro's notation for its isotopy class (usage of this notation means that we suppose that the curves have only ovals).

Proposition .1. *Isotopy types* $1 \sqcup 1\langle 1\rangle \sqcup 1\langle 18\rangle$ *and* $1 \sqcup 1\langle 7\rangle \sqcup 1\langle 12\rangle$ *are*

a). unrealizable by real algebraic curves of bidegree $(4,16)$ *on* $\mathbb{R}\mathcal{F}_4$*;*

b). realizable by real $\mathcal{J}$*-holomorphic curves of bidegree* $(4,16)$ *on* $\mathbb{R}\mathcal{F}_4$ *where* $\mathcal{J}$ *is a tame* conj*-invariant almost complex structure with an exceptional* $\mathcal{J}$*-holomorphic curve.*

Proof. a). If one of these isotopy types is realizable by a real pseudoholomorphic curve A, then (up to symmetry and zigzag removals) the fiberwise isotopy type of $\mathbb{R}A$ is either $A(\alpha_1,\alpha_2,\alpha_3) = [\supset_1\subset_1 o_2^{\alpha_1} o_3 o_2^{\alpha_2} \supset_1\subset_2 o_2^{\alpha_3}]$, $\alpha_1+\alpha_2+\alpha_3 = 19$, $\alpha_3 \in \{1,7,12,18\}$, $\alpha_1 \ge \alpha_2$, or $B(\alpha_1,\alpha_2) = [\supset_1\subset_1 o_2^{\alpha_1} \supset_1 o_1 \subset_1 o_2^{\alpha_2}]$, $\alpha_1+\alpha_2 = 19$, $\alpha_3 \in \{1,7\}$. Passing to cubic resolvents, we obtain M-curves of bidegree $(3,24)$ on $\mathcal{F}_8$ of fiberwise isotopy types $A'(\alpha_1,\alpha_2,\alpha_3) = [o_2 o_1^{\alpha_1} o_2 o_1^{\alpha_2} o_2 o_1^{\alpha_3}]$ and $B'(\alpha_1,\alpha_2) = [o_2 o_1^{\alpha_1} o_2^2 o_1^{\alpha_2}]$ respectively.

The algebraic length of the corresponding braids is equal to two. It can be easily derived from the quasipositivity criterium [19], that A' algebraically realizable only for $(\alpha_1,\alpha_2,\alpha_3) \in \{(17,1,1), (11,7,1), (11,1,7), (7,5,7)\}$. and B' is unrealizable for all (α_1,α_2).

Let us denote the braid corresponding to $A(\alpha_1,\alpha_2,\alpha_3)$ by $b = b(\alpha_1,\alpha_2,\alpha_3; e_1, e_2)$. It depends on two unknown integers e_1, e_2 and it has the form

$$b = a\,\sigma_3^{1+e_1}\sigma_1^{1-e_1}\sigma_2^{-\alpha_1}\tau_{2,3}\sigma_3^{-1}\tau_{3,2}\sigma_2^{-\alpha_2}a\,\sigma_3^{1+e_2}\sigma_1^{1-e_2}\sigma_2^{-\alpha_3}\Delta^4,$$

where $a = \sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}$, $\tau_{2,3} = \tau_{3,2}^{-1} = \sigma_3^{-1}\sigma_2$, and $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$.

If $e_1 \not\equiv e_2 \mod 2$, then the image of b in the permutation group S_4 is $(13)(24)$, hence, the closure L of b consists of two components. Their linking number is equal to -3 which is impossible. Therefore, we shall assume that $e_1 \equiv e_2 \mod 2$. The image of b in S_4 is trivial in this case. The computation of the Euler characteristic yields that L bounds a three-component surface in the 4-ball. Computing linking numbers between the link components, one can see that two cases are a priori possible:

(i) $e_1 \equiv e_2 \equiv 0 \mod 2$ and $e_2 - e_1 = 10$;
(ii) $e_1 \equiv e_2 \equiv 1 \mod 2$ and $e_2 - e_1 = 11 - \alpha_3$.

It follows from [22; 3.7], that $(2|e_1| - 1) + (2|e_2| - 1) \le 12$, i.e., $|e_1| + |e_2| \le 7$ (see the end of §4.1). This excludes Case (i) for all $\alpha_3 \in \{1, 7\}$, and also Case (ii) for $\alpha_3 = 1$.

It remains to consider Case (ii) for $\alpha_3 = 7$. Murasugi-Tristram inequality implies in this case that the determinant d of L must vanish. Expressing e_2 via e_1 and computing d using the computer program from [16; Appendix] (or computing it by hands using Göritz matrices as in [21]), we obtain $d = 24(347 - 876e_1 - 223e_1^2)$ for $\alpha_1 = 7$ and $d = 24(31e_1 + 137)(3 - e_1)$ for $\alpha_1 = 11$. In the former case, there are no integral solutions of the equation $d(e_1) = 0$. In the latter case, there is a unique solution $e_1 = 3$, which yields $e_2 = e_1 + 11 - \alpha_3 = 3 + 11 - 7 = 7$, which contradicts the inequality $|e_1| + |e_2| \le 7$ (this solution corresponds to the pseudoholomorphic curve constructed in [16; §5.2]).

b). The isotopy type $1 \sqcup 1\langle 7\rangle \sqcup 1\langle 12\rangle$ is realized in [16; §5.2].

To realize $1 \sqcup 1\langle 1\rangle \sqcup 1\langle 18\rangle$, note that the above braid $b(17, 1, 1; e_1, 10 - e_1)$ does not depend on e_1 and that it is quasipositive. Moreover, if $e_2 = 10 - e_1$, then even the braid

$$a\,\sigma_3^{1+e_1}\sigma_1^{1-e_1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-16}\tau_{2,3}\sigma_3^{-1}\tau_{3,2}\sigma_2^{-1}a\,\sigma_3^{1+e_2}\sigma_1^{1-e_2}\sigma_2^{-1}\Delta^4$$

(corresponding to $[\supset_1\subset_1 o_2\subset_2\times_1\supset_2 o_2^{15} o_3 o_2\supset_1\subset_2 o_2]$) is quasipositive. Since, the algebraic length of the latter braid is equal to one, its quasipositivity is equivalent to the fact that the braid is conjugate to a standard generator.

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