ON ARRANGEMENTS OF PLANE REAL QUARTICS WITH RESPECT TO THREE LINES

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Abstract. We complete the classification of mutual arrangements of a smooth real algebraic or real pseudoholomorphic quartic curve and three lines under condition that each oval of the quartic intersects the union of the lines. This classification was started in a recent preprint by Maletto. There is one arrangement which is realizable pseudoholomorphically but not algebraically. It can be constructed in different ways, in particular, by a combinatorial patchworking on an irregular triangulation. This is the first example of a combinatorial patchworking which produces a PL curve in $\mathbb{RP}^2$ whose arrangement relative to the coordinate axes is algebraically unrealizable.

§1. Introduction

A systematic study of mutual arrangements of transversally intersecting non-singular real algebraic curves on $\mathbb{RP}^2$ up to isotopy (an analog of Hilbert's 16th problem for reducible curves) was started by Polotovskiy in [11], [12], [13] and then continued by other authors; see [2], [9], [10] and references in [2].

When the degree increases, in contrary to the case of non-singular curves, the number of *a priori* possible (and even of realizable) arrangements of reducible curves grows much faster than the difficulty of deciding the realizability of any particular arrangement. Therefore this problem for the degree greater than 5 was usually considered under the additional condition that all the crossing points are real (the *maximal intersection condition*) and under other additional conditions for higher degrees. Another difference between the reducible and irreducible cases is that a visual identification of arrangements for reducible curves is more difficult.

An attempt of classification of reducible curves without the maximal intersection condition is undertaken in a recent preprint [4]. Both realizability and non-realizability results (as well as the identification of arrangements constructed in different ways) were obtained using specially written computer programs. In particular, it is proven in [4] that:

- $234 \le N_{411} \le 273$, where N_{411} is the number of arrangements (C_4, L_x, L_z) of a smooth real algebraic quartic curve and two lines in $\mathbb{RP}^2$;
- $1834 \le N_{4111}^{\text{fless}} \le 1883$, where N_{4111}^{fless} is the number of floatless arrangements (C_4, L_x, L_y, L_z) of a smooth real algebraic quartic curve and three lines in $\mathbb{RP}^2$ (*floatless*[1] means that $L_x \cup L_y \cup L_z$ intersects each oval of C_4).

[1] Ovals of C_4 disjoint from $L_x \cup L_y \cup L_z$ are called in [4] *floating*. Usually they are called *free*.

In this count we do not distinguish between arrangements obtained from each other by permuting the lines L_x, L_y, L_z. The realized arrangements are listed in [5] in a machine-readable form. All of them are constructed by Viro's combinatorial patchworking combined with translations defined in [4, §6.3]. The non-realizability proofs in [4] use only Harnack's inequality and Bezout theorem for auxiliary lines, hence these results automatically extend to real pseudoholomorphic curves (we refer to [7, §2], [8, §§1,2] for an introduction to the subject).

The 49 floatless arrangements (C_4, L_x, L_y, L_z) and 39 arrangements (C_4, L_x, L_z) whose realizability is unknown are presented in [4, Figures 20 and 21]. In the present paper we prove that all of them except one are not realizable by subsets of real pseudoholomorphic (in particular, real algebraic) curves. The remaining arrangement (as well as an arrangement obtained from it by adding a free oval; see Figure 2 below) is realizable by real pseudoholomorphic curves but unrealizable by real algebraic curves. The latter fact is proven by a simplest variant of the Hilbert–Rohn–Gudkov method.

Thus the lists in [5] provide a complete classification of algebraically realizable arrangements (C_4, L_x, L_z) and floatless arrangements (C_4, L_x, L_y, L_z). A pseudoholomorphic analogue of these classifications is obtained by adding one more floatless arrangement (C_4, L_x, L_y, L_z).

In our opinion, the most interesting result in the present paper is an observation that the algebraically unrealizable arrangements in Figure 2 are realizable by a patchworking along the unique up to symmetry irregular primitive triangulation of the triangle $[(0,0),(4,0),(0,4)]$. As far as we know, this is the first example of a combinatorial patchworking which produces PL curves whose arrangement relative to the coordinate axes is not realizable by a real algebraic curve of the same degree (we say that a patchworking is *combinatorial* if the only input is a triangulation and a sign distribution). Some examples of non-algebraic patchworking (not combinatorial) were found in [1]. Notice also that an irregular patchworking produces pseudoholomorphic curves under rather general assumptions [3].

It seems plausible that the classification (started in [4]) of all arrangements (not only floatless) (C_4, L_x, L_y, L_z) can be considerably advanced by the methods of the present paper. However, it remains as many as 8198 open cases, thus this should be done on a computer (maybe, by upgrading [5]).

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§2. Classification of (C_4, L_x, L_z)

Throughout the paper the notation (C_4, L_x, L_z) or (C_4, L_x, L_y, L_z) refers to a real algebraic or real pseudoholomorphic (in the same almost complex structure) smooth curve of degree 4 and two or three lines transverse to each other. We say that an arrangement of one-dimensional submanifolds of $\mathbb{RP}^2$ is algebraically or pseudoholomorphically realizable by (C_4, L_x, L_z) or (C_4, L_x, L_y, L_z) if it is realizable by the real loci of these algebraic or pseudoholomorphic curves.

Theorem 1. *The arrangements in [4, Figure 21] are not pseudoholomorphically realizable by* (C_4, L_x, L_z)*. In particular, they are not algebraically realizable by* (C_4, L_x, L_z)*.*

Proof. Any mutual arrangement of a nonhypebolic (i.e. without nested ovals) quartic and a conic which have 8 common points is either one of those shown in [10, Figures 5–7] or is obtained from them by removing some free ovals. In the algebraic case this fact is proved in [13]. A proof in the pseudoholomorphic case is given in [10, Proposition 1]. For each arrangement in [4, Figure 21], after a perturbation of the two lines into a conic, we obtain a non-realizable arrangement of a quartic and a conic. In the following table we give references to Figures 5–7 in [10] for each arrangement from [4, Figure 21]:

5.10	5.1	7.2	5.14	5.14	5.13	5.5
5.4	5.9	5.5	7.1	5.12	5.9	5.14
5.14	5.14	5.5	5.9	5.13	7.1	5.10
7.1	5.4	5.2	7.1	5.5	7.1	7.1
5.4	5.4	5.2	5.14	5.4	5.10	5.9
5.10	5.9	5.4	5.2			

(1)

For example, "5.14" at the 2nd position of the 3rd row of this table means that we exclude the 2nd arrangement in the 3rd row in [4, Figure 21] (it is also shown in Figure 1 on the left) as follows. By perturbing the two lines into a conic we obtain the arrangement in Figure 1 in the middle. It contradicts the fact that any realizable arrangement with such non-free ovals is obtained by removing some free ovals from the 14th item in [10, Figure 5], which is also reproduced in Figure 1 on the right.

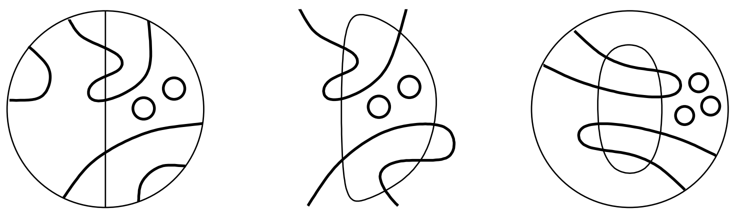

FIGURE 1. Proof for the 2nd arrangement in the 3rd row in [4, Fig. 21].

§2. CLASSIFICATION OF PSEUDOHOLOMORPHIC FLOATLESS ARRANGEMENTS (C_4, L_x, L_y, L_z)

In this section we prove Part (b) of the following theorem.

Theorem 2. *(a). The arrangements in [4, Figure 20] are not algebraically realizable by the floatless part of* (C_4, L_x, L_y, L_z).

(b). The arrangements (C_4, L_x, L_y, L_z) *in Figure 2 are pseudoholomorphically realizable. These are the only pseudoholomorphically realizable arrangements of* (C_4, L_x, L_y, L_z) *whose floatless part is as in [4, Figure 20]. Note that the left arrangement in Figure 2 is the 44th arrangement in [4, Figure 20].*

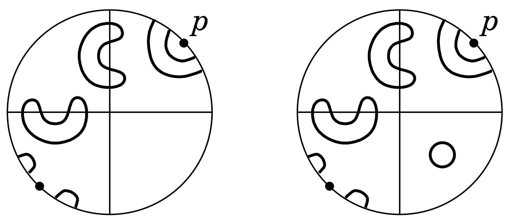

FIGURE 2. Algebraically non-realizable pseudoholomorphic curves

Proof of Theorem 2(b). Notice first that if an arrangement is pseudoholomorphically realizable by (C_4, L_x, L_y, L_z), then so is its floatless part. Therefore it is enough to prove the pseudoholomorphic non-realizability for the floatless arrangements only.

The following table as well as [4, Figure 20] has 7 rows and 7 columns. Its entries refer to the proof for the corresponding arrangements.

Lk(z)	7.1(y)	MT(z)	7.1(z)	5.5(z)	Lk(y)	Lk(z)
7.1(z)	Fig. 3	7.3(z)	Fig. 3	MT(z)	Fig. 3	7.1(x)
7.1(y)	Fig. 3	7.1(y)	7.1(y)	Fig. 3	Fig. 3	5.2(x)
7.2(z)	Fig. 3	7.1(x)	MT(z)	7.1(y)	7.1(y)	Fig. 3
7.2(z)	5.5(z)	MT(z)	7.1(y)	Fig. 3	MT(y)	Lk(y)
7.1(z)	MT(z)	Lk(z)	MT(z)	Lk(x)	Lk(y)	Fig. 3
7.1(x)	$\exists$ps-h(z)	7.1(y)	MT(z)	7.1(x)	7.3(z)	MT(z)

(2)

The entries 7.1(y), 7.1(z), ... refer to arguments as in the proof of Theorem 1 applied after removal of one of the three lines. The letters in the parentheses indicate which line should be removed: L_x, L_y, or L_z (these lines are represented in [4] by vertical segments, horizontal segments, and boundary circles respectively).

An entry "Fig. 3" refers to an unrealizability proof similar to [10, §2.1], which uses an auxiliary conic. Namely, the conic passing through the five points shown in Figure 3 has at least two intersection points with each oval, and hence exactly two by Bezout theorem. We see in Figure 3 that it is impossible to trace the conic so that it crosses each line at two points.

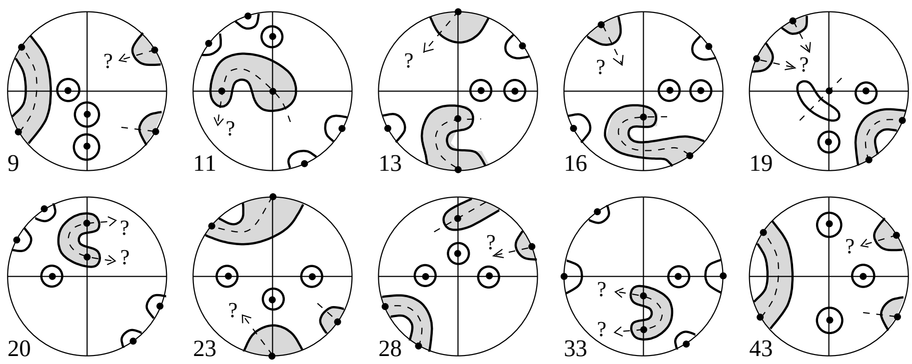

FIGURE 3. Auxiliary conics.

We study the other arrangements using braids. We refer to [6] and [7, §2] for a detailed description of this method. For each arrangement corresponding to the entries MT, Lk, and $\exists$ps-h we choose a line $L \in \{L_x, L_y, L_z\}$ (the letter x, y, or z in the table) so that there is an oval O which crosses L at four points. Let $\{L_1, L_2\}$ be $\{L_x, L_y, L_z\} \setminus \{L\}$. We choose a point p on L which is placed with respect to O as in Figure 2. Let $\mathcal{L}_p$ be the pencil of lines through p. Then we consider the arrangements of $C_4 \cup L_1 \cup L_2$ on the affine plane $\mathbb{RP}^2 \setminus L$ with respect to $\mathcal{L}_p$ (called $\mathcal{L}_p$-schemes in [6], [7] or fiberwise arrangements in [9]). Up to reductions and equivalences from [7, Corollary 2.3], there are three $\mathcal{L}_p$-schemes for the arrangements nos. 1, 7, 34, two for 6, 25, 44, and one for the other arrangements marked by MT or Lk in (2) (the numbering corresponds to [4, Table 21]). For each

$\mathcal{L}_p$-scheme we compute a braid with 6 strands as explained in [7, Algorithm 2.1]. An arrangement is pseudoholomorphically realizable if and only if all the corresponding braids are quasipositive.

All $\mathcal{L}_p$-schemes for the arrangements marked by Lk are excluded using the linking numbers as in [7, §4.5]. Those for MT are excluded by Murasugi-Tristram inequality for braids [7, Corollary 3.2] with the usual signature, i.e. with $\zeta = -1$ in the notation of [7, §3.1].

The arrangement no. 44 is shown in Figure 2 on the left (the corresponding entry in (2) is ∃ps-h). It is pseudoholomorphically realizable. We consider this case in more detail (which also illustrates some underlying computations for the arrangements marked by MT and Lk). By [7, Corollary 2.3] it is enough to consider the two $\mathcal{L}_p$-schemes shown in Figure 4, where $\mathcal{L}_p$ is supposed to be the pencil of vertical lines.

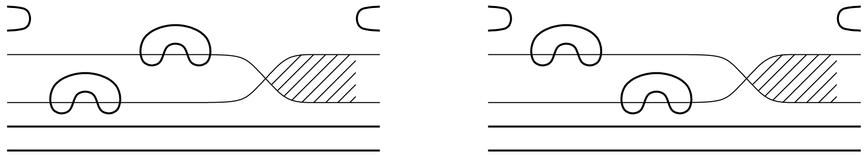

FIGURE 4. $\mathcal{L}_p$-schemes for Figure 2 (no. 44 in [4, Fig. 20]).

The corresponding 6-braids are, respectively,

$$b_1 = \bar\sigma_5\bar\sigma_4\sigma_5\bar\sigma_3^4\bar\sigma_4\bar\sigma_5\bar\sigma_4^3\bar\sigma_5\bar\sigma_3\Delta \qquad \text{and} \qquad b_2 = \bar\sigma_5\bar\sigma_4^3\bar\sigma_5\bar\sigma_4\bar\sigma_3^4\bar\sigma_4\bar\sigma_5\sigma_4\bar\sigma_3\Delta, \tag{3}$$

where $\bar\sigma_i = \sigma_i^{-1}$ and Δ is the Garside's half-twist. Both braids are quasipositive:

$$b_1 = \sigma_4^{\sigma_3^2\sigma_4\sigma_5\sigma_4}\sigma_2^{\sigma_3\sigma_4\sigma_5^2}\sigma_1^{\sigma_2\sigma_3\sigma_4\sigma_5} \qquad \text{and} \qquad b_2 = \sigma_2^{\sigma_3\sigma_4\sigma_5^2}\sigma_1^{\sigma_2^2}\sigma_3^{\sigma_2\sigma_4\sigma_5},$$

where b^a denotes $a^{-1}ba$. Hence Figure 2(left) is pseudoholomorphically realizable.

Finally, let us show that Figure 2(right) is a unique pseudoholomorphically realizable arrangement obtained by adding an oval to Figure 2(left). Proceeding as in the proof of Theorem 1, we easily conclude that a free oval cannot appear anywhere else, i.e. it might appear in the shaded zone in Figure 4 only. The corresponding braids b_1^+ and b_2^+ are obtained by inserting $\bar\sigma_4\sigma_5\bar\sigma_4\bar\sigma_5\sigma_4$ before Δ in b_1 and in b_2 respectively. Then $b_2^+ = \sigma_2^{a\sigma_5}\sigma_1^{\sigma_2 a}$ $(a = \sigma_3\sigma_4\sigma_5)$ is quasipositive whereas a computation of the linking numbers shows that b_1^+ is not.

A more geometric realization of Figure 2 is possible. By perturbing three lines we may obtain a pseudoholomorphic cubic curve arranged as in Figure 5 (cf. [8, before Remark 1.4]). A further perturbation of its union with the fourth line (which is represented by the boundary circle) yields the arrangements in Figure 2.

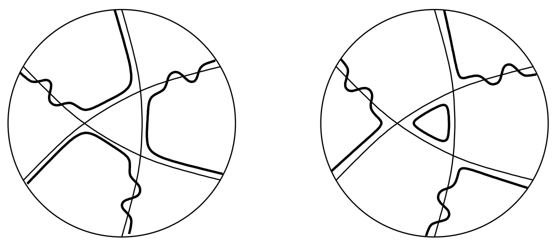

FIGURE 5. Pseudoholomorphic realization of Figure 2.

§4. Algebraic non-realizability of the arrangements in Figure 2

In this section we prove Theorem 2(a) using a simplest version of Hilbert–Rohn–Gudkov method.

Suppose that there exists a real algebraic quartic curve C arranged with respect to three lines L_x, L_y, L_z as in Figure 2. On each non-free oval let us choose a point of intersection with the corresponding line. By Bezout theorem these points are not collinear. Hence there exists a line L_0 such that the chosen points are placed in $\mathbb{RP}^2$ with respect to $L = L_x \cup L_y \cup L_z \cup L_0$ in one of the two ways shown in Figure 6, where L_0 is represented by the boundary circle. It is easy to show that L_0 can be moved off from C, thus $C \cup L$ realizes one of the two arrangements in Figure 6, where the dashed free oval is or is not included.

Note that the free oval cannot appear in the left arrangement in Figure 6 because it cannot appear in the left arrangement in Figure 4; see the end of the proof of Theorem 2(b).

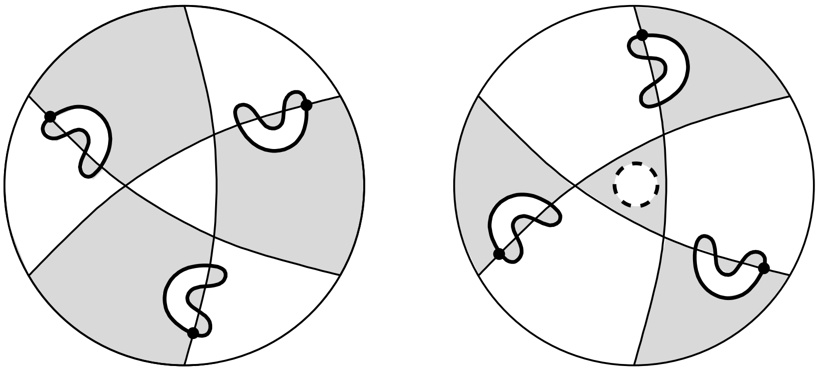

Figure 6. Algebraic non-realizability of Figure 2.

Let $f(x, y, z) = 0$ and $l(x, y, z) = 0$ be the equations of C_4 and L respectively. Consider the pencil of quartics $\{C_t\}$ given by the equations $f + tl = 0$. Up to changing the sign of l we may assume that the gray region in Figure 6 grows and the white region shrinks when t varies from $t = 0$ to $t > 0$. It is clear that C_t cannot tend to L if C_t remains isotopic to C for all $t > 0$. Hence there exists t_0 such that $C_t \cup L$ is isotopic to $C \cup L$ for $0 \le t < t_0$ and C_{t_0} is singular. Let t_1 be such that $0 < t_1 - t_0 \ll 1$. It is easy to see that C_{t_0} cannot be reducible: it is enough to look at the evolution of the intersection of C_t with some auxiliary lines. Hence, by the genus formula, C_{t_0} has a single node.

A priori there are only three possibilities:

(i) a node with real local branches appears in the white part of the disk bounded by a non-free oval O of C when two arcs collide;
(ii) a node with imaginary local branches appears in a white region;
(iii) the free oval of C shrinks into a node with imaginary local branches.

Case (i) is impossible because then O splits into two ovals of C_{t_1}. This gives the arrangement no. 29 in [4, Figure 20], which is already excluded. In Case (ii), the curve C_{t_1} gets a free oval in a white region. This is impossible as we explained above. The proof for the floatless arrangements is completed. Case (iii) is impossible because in this case C_{t_1} realizes a floatless arrangement. Theorem 2 is proven.

Corollary. *The arrangements of a cubic curve and four lines shown in Figure 5 are realizable pseudoholomorphically but not algebraically.*

Proof. A perturbation of these arrangements yields the arrangements in Figure 2. □

Note that this corollary admits a much simpler proof. It can be derived from Abel's theorem on holomorphic 1-forms on Riemann surfaces; cf. [8, Remark 1.4].

§5. Patchworking and rigid isotopies

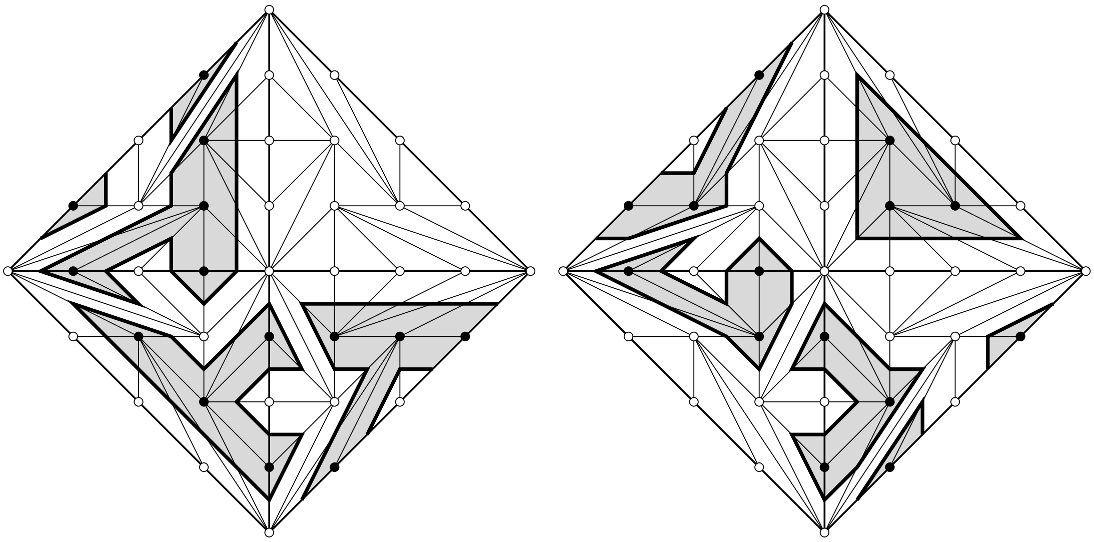

Figure 7. Patchworking of the arrangements in Figure 2.

The patchworking is shown in Figure 7 where the signs of vertices are represented by colors (black and white). Due to [3] it provides one more pseudoholomorphic realization of the arrangements in Figure 2.

Proposition 1. *For each arrangement in Figure 2 there is only one (up to symmetries) combinatorial patchworking which realizes it as a quartic curve and the three coordinate lines.*

Remark. It seems plausible that the patchworks in Figure 7 are the only non-algebraic combinatorial patchworks of degree 4 in $\mathbb{RP}^2$.

Proof. Since Figure 2 is algebraically unrealizable, the underlying triangulation must be irregular. It is well-known that any irregular lattice triangulations of the triangle $T = [(0,0),(4,0),(0,4)]$ is the triangulation in Figure 7 and those obtained from it by removing some edges adjacent to points of ∂T which are not vertices of T. The quartic curve has four intersections with each coordinate line, hence no edge can be removed. The quartic is disjoint from one of the triangles bounded by the coordinate lines, hence it is enough to consider only distributions of signs which are constant on $\partial T \cap \mathbb{Z}^2$. Up to symmetries, there are only four such distributions. Two of them are shown in Figure 7. It is straightforward to check that the two others do not give an arrangement from Figure 2. □

Let $\mathbf{c} : \mathbb{CP}^2 \to \mathbb{CP}^2$, $(x : y : z) \mapsto (\bar{x} : \bar{y} : \bar{z})$, be the involution of complex conjugation. Let A_0 and A_1 be two real pseudoholomorphic curves in $\mathbb{CP}^2$, A_k being

J_k-holomorphic for a tame $\mathbf{c}$-invariant almost complex structure J_k. We say that A_0 and A_1 are *rigidly isotopic* is there exists a $\mathbf{c}$-equivariant isotopy $\{A_t\}_{t\in[0,1]}$ and a continuous family of $\mathbf{c}$-anti-invariant tame almost complex structures $\{J_t\}_{t\in[0,1]}$ such that A_t is J_t-holomorphic for any t.

Let $A_{\rm left}$ and $A_{\rm right}$ be reducible real pseudoholomorphic curves of degree 7 realizing the floatless arrangement in Figure 2 which correspond to the left and right $\mathcal{L}_p$-schemes in Figure 4 respectively. Let $A^{\rm pw}_{\rm left}$ and $A^{\rm pw}_{\rm right}$ be reducible real pseudoholomorphic curves of degree 7 obtained according to [3] from the respective patchworks in Figure 7. Let also $A^{\rm pw-}_{\rm right}$ be the real pseudoholomorphic curve obtained from $A^{\rm pw}_{\rm right}$ by removing the free oval (by a standard procedure well-known in the symplectic topology). Note that we do not know whether all these curves are defined uniquely up to rigid isotopy.

Proposition 2. *$A_{\rm left}$ and $A_{\rm right}$ are not rigidly isotopic.*

Proof. For any continuous deformation of the left $\mathcal{L}_p$-scheme in Figure 4 into the right one, there is a moment when some vertical line passes through all the three ovals, which contradicts the Bezout theorem. □

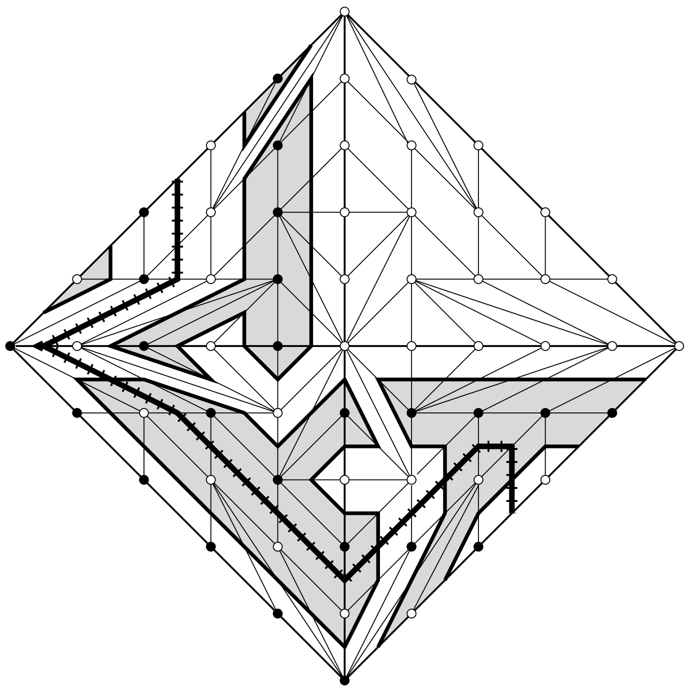

FIGURE 8. Patchworking with an additional line.

In Figure 8 we show a patchworking of the union of a quartic curve and a line; cf. [14]. It gives the left arrangement in Figure 4 with a vertical line passing through one of the two ovals (and, of course, with the line at infinity). This observation naturally leads to the conjecture that $A^{\rm pw}_{\rm left}$ is not rigidly isotopic to $A_{\rm right}$, and hence, to $A^{\rm pw-}_{\rm right}$.

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