ON CURVES OF DEGREE 10 WITH 12 TRIPLE POINTS

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Abstract. We construct an irreducible rational curve of degree 10 in $\mathbb{CP}^2$ which has 12 triple points, and a union of three rational quartics with 19 triple points. This gives counter-examples to a conjecture by Dimca, Harbourne, and Sticlaru. We also prove that there exists an analytic family C_u of curves of degree 10 with 12 triple points which tends as $u \to 0$ to the union of the dual Hesse arrangement of lines (9 lines with 12 triple points) with an additional line. We hope that our approach to the proof of the latter fact could be of independent interest.

Recently Joaquim Roé (private communication) asked if there exists an irreducible curve of degree 10 in $\mathbb{CP}^2$ which has 12 ordinary triple points. If such a curve exists, it should be rational by the genus formula. We give an affirmative answer to that question. This is a counterexample to [1, Conjecture 1.6], which states that *there is no reduced plane curve of degree > 9 whose components are all rational and whose singularities (including infinitely near points) all have multiplicity 3* (note that this conjecture is presented in [1] as "a more accessible target for a counterexample"). Besides an irreducible curve of degree 10 with 12 triple points, we also construct a union of three rational quartics with 19 triple points. Thus the bound 9 in the conjecture should be raised at least to 12.

Proposition 1. *(a). The degree* 10 *curve parameterized by* $t \mapsto (x(t) : y(t) : z(t))$,

$$x = (t^3+2)(t^6+3t^3+3), \quad y = t(t^3+1)(t^3+2)(t^3+3), \quad z = t^9+3t^6-3,$$

has 12 *ordinary triple points. Its Cartesian equation is* $F(x,y,z)=0$ *where* F *is the homogeneous polynomial such that*

$$F(x,y,1) = 9(x-1)y^9 - 3(6x^4+8x^3-3x^2-6x+1)y^6 \\ + (9x^7+24x^6+13x^5-8x^4-11x^3-x^2+2x-1)y^3 - (x+2)x^3(x^2-1)^3.$$

(b). The degree 12 *curve* $f(x,y,z)f(\omega x,y,z)f(\omega^2 x,y,z)=0$, *where*

$$f(x,y,z) = (x-y)(x+y)^3 + (2x+3y)z^3, \qquad \omega = e^{2\pi i/3},$$

has 19 *ordinary triple points. The curve* $f(x,y,z)=0$ *admits parametrization* $t \mapsto (t^4-3t : t^4+2t : 2t^3-1)$.

In §1 we explain how the examples in Proposition 1 were constructed.

In §2 (which is independent of §1) we obtain an irreducible curve of degree 10 with 12 triple points as a perturbation of the dual Hesse arrangement (9 lines with 12 triple points) with an additional line. Namely, we prove the following fact.

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Proposition 2. *There exists an analytic family C_u, $|u| < 1$, of curves of degree 10 in $\mathbb{CP}^2$ such that C_u for $u \neq 0$ is irreducible and has 12 ordinary triple points, and C_0 is given by the equation*

$$(2x + 2y - z)(x^3 - y^3)(y^3 - z^3)(z^3 - x^3) = 0. \tag{1}$$

We do not know if there exists an equisingular deformation of the curve in Proposition 1(a) to those in Proposition 2.

Remark 1. Proposition 2 holds for a generic line instead of the line $2x+2y = z$ in (1), i.e., for any element of some Zariski open subset of $\check{\mathbb{P}}^2$ (see Remark 4 below).

Remark 2. There are results on the existence of perturbations with prescribed singularities in rather general settings (see e.g. [2], [3], [5]). Those which deal with Cartesian equations are not applicable to our problem. We construct the perturbation in a parametric form. We do not know if non-equigeneric perturbations (i.e. decreasing the number of irreducible components and/or increasing the sum of their genera) were studied somewhere in a parametric form. It is interesting to see if our approach could be used for such perturbations in more general settings.

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§1. Constructions based on a symmetry of order 3

1.1. Reduction by a birational transformation. Let us show how the example from Proposition 1(a) was found. We look for a homogeneous polynomial $f(x, y, z)$ of degree 10 of the form $f(x, y, z) = f_1(x, y^3, z)$ such that the curve $C = \{f = 0\} \subset \mathbb{P}^2$ is rational and has 12 triple points. Then the Newton polygon of $f_1(x, y, 1)$ is contained in the trapezoid $[(0, 0), (10, 0), (1, 3), (0, 3)]$, so, it is natural to consider the curve C_1 given by the equation $f_1(x, y, z) = 0$ on the weighted projective plane $\mathbb{P}^2_{1,3,1}$ or on the Hirzebruch surface Σ_3, which is $\mathbb{P}^2_{1,3,1}$ blown up at $(0 : 1 : 0)$. The curve C is the preimage of C_1 under the three-fold branched covering $\rho : \mathbb{P}^2_{1,3,1} \to \mathbb{P}^2$, $(x : y : z) \mapsto (x : y^3 : z)$.

It is easy to see that, under the above assumptions, three triple points of C are placed on the line $L = \{y = 0\}$ so that their images on Σ_3 are points where C_1 has a cubic tangency with the line $L_1 = \rho(L)$. The remaining nine triple points of C are preimages of three triple points of C_1. The curves L_1 and C_1 (considered as curves on Σ_3) belong to the linear systems $|G|$ and $|3G + F|$, where F is a fiber and G a general section of the projection $\Sigma_3 \to \mathbb{P}^1$. Let us blow up the three triple points of C_1 and then blow down the strict transforms of the fibers passing through them. The resulting surface is $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$. The strict transforms of L_1 and C_1 are smooth curves on it. We denote them by L_2 and C_2. They belong to the linear systems $|A + 3B|$ and $|3A + B|$ respectively, where A is the strict transform of the (-3)-section of Σ_3 and B is the image of F.

1.2. The construction. Thus, in order to construct the desired curve of degree 10 in $\mathbb{P}^2$, it is enough to find smooth curves $L_2 \in |A + 3B|$ and $C_2 \in |3A + B|$ on Σ_0 such that $(L_2 \cdot C_3)_{p_k} = 3$, $k = 1, 2, 3$, and to apply the above transformations backward. We observe that L_2 and C_2 are expressed in A and B symmetrically. So, it is natural to search these curves in the form $C_2 = \{f_2(x, y) = 0\}$ and

$L_2 = \{f_2(y,x) = 0\}$ where $\deg_x f_2 = 1$ and $\deg_y f_2 = 3$ (here (x, y) corresponds to $\big((x : 1), (y : 1)\big) \in \Sigma_0$). We also observe that if C_2 is tangent to a line $x + y = 2a$ at a diagonal point $p = (a, a)$, then $(C_2 \cdot L_2)_p = 3$.

So, we set $f_2(x, y) = p(y)x + q(y)$ where $p = y^3 - 3y^2 + 1$ and $q = y^3 - y^2 + y$. Then C_2 is tangent to the lines $x + y = 2a$ for $a = 0, \pm 1$ at the diagonal points $(0, 0)$, $(1, 1)$, $(-1, -1)$, thus it has a cubic tangency with L_2 at these points. The passage from Σ_0 to Σ_3 should transform the equation $p(x)y + q(x) = 0$ of L_2 into the equation $y = 0$ of L_1. This means that it transforms any equation $g(x, y) = 0$, $\deg_y g = n$, into

$$g\Big(\frac{y - q(x)}{p(x)}\Big)p(x)^n = 0,$$

and we obtain that the required curve C in $\mathbb{P}^2$ is given by $f(x, y, z) = 0$, where

$$f(x, y, 1) = \bigg(xp\Big(\frac{y^3 - q(x)}{p(x)}\Big) + q\Big(\frac{y^3 - q(x)}{p(x)}\Big)\bigg)\, p(x)^3.$$

In Figure 1 the arrangement of $\mathbb{R}C_k$ and $\mathbb{R}L_k$ on $\mathbb{R}\Sigma_0$ and $\mathbb{R}\Sigma_3$ are shown (we denote the real locus of a complex variety X by $\mathbb{R}X$). The imaginary local branches of C_1 at triple points are schematically depicted by dashed lines; $\mathbb{R}\Sigma_k$ are represented by squares whose opposite sides are glued according to the arrows.

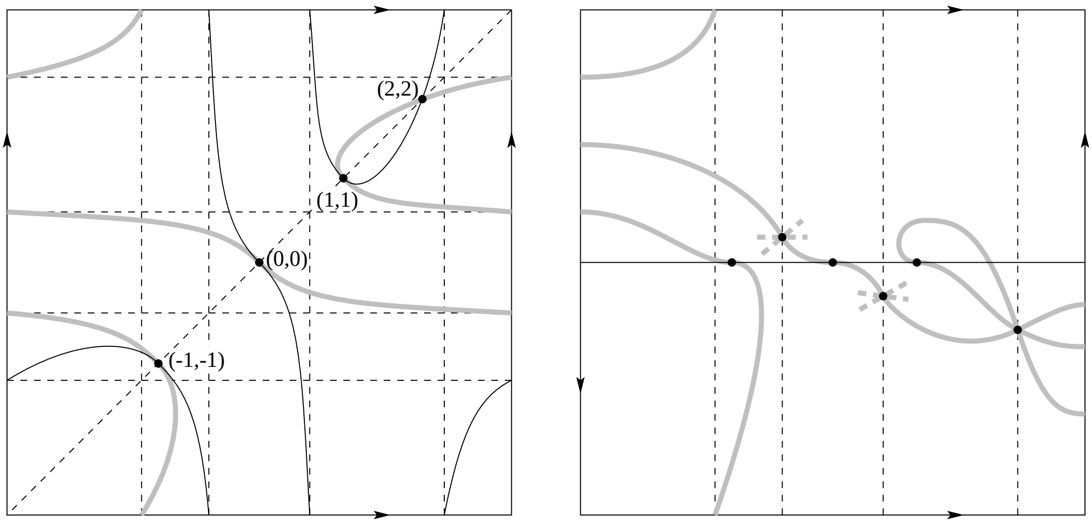

Figure 1. $\mathbb{R}L_2, \mathbb{R}C_2$ on $\mathbb{R}\Sigma_0$ and $\mathbb{R}L_1, \mathbb{R}C_1$ on $\mathbb{R}\Sigma_3$.

A parametrization can be computed as follows. It is clear that C_2 is parametrized by $y = s$, $x = -q(s)/p(s)$. Then C_1 is parametrized by $s \mapsto (q(s) : r(s) : -p(s)) \in \mathbb{P}^2_{1,3,1}$, where $r(s) = 3(s-2)(s^2-1)^3 s^3$. The parameter change $s = t^3 + 2$ yields a parametrization $t \mapsto \big(q(s) : r(s)^{1/3} : -p(s)\big)\big|_{s=t^3+2}$ of C. The formulas in Proposition 1 are obtained by further "cosmetic" rescalings $(x, y, z) \to (x, 3^{1/3}y, -z)$.

1.3. What about curves of degree 17 with 20 quadruple points. This construction would be generalized to irreducible rational curves of degree $n^2 + 1$ with $n^2 + n$ ordinary n-fold points as soon as there exist curves in the linear systems $|A + nB|$ and $|nA + B|$ on Σ_0 which have n tangency points of order n. If we assume in addition that the curves are symmetric to each other with respect to the diagonal (as in §1.2), then the tangency points cannot be placed on the diagonal when n is even. We studied the symmetric case for $n = 4$. In this case the coordinates can be chosen so that the tangency points are at $(c, 0)$, $(1, \infty)$ and at their symmetric images. Then, in the notation of §1.2,

$$f_2(x, y) = (y - 1)(y^3 + a_2y^2 + a_1y + a_0)x - (y - c)(y^3 + b_2y^2 + b_1y + a_0).$$

The tangency conditions give a system of 6 polynomial equations with 6 indeterminates. Using Gröbner bases, we checked that there are no irreducible solutions. In the general (i.e., asymmetric) case for $n = 4$, the system of equations is twice bigger and we did not succeed to solve it. We did not try $n \geq 5$.

1.4. Three quartics with 19 triple points. The construction of the curve in Proposition 1(b) is similar to the previous one. We look for a curve of the form $F(x^3, y^3, z^3) = 0$ which has a triple point at $p_1 = (0 : 0 : 1)$, three more triple points on each coordinate axis, and 9 triple points somewhere else. Then the quartic curve $F(x, y, z) = 0$ passes through p_1, has a cubic tangency with each axis, and has a triple point p_2 somewhere else. Let us apply the Cremona quadratic transformation with the base points p_1, p_2, p_3, where p_3 is the point of cubic tangency with the axis $y = 0$; see Figure 2 on the left. We obtain a cuspidal cubic C_3, a conic C_2, and four lines arranged as shown in Figure 2 on the right. If we forget the two lines passing through the point q (see Figure 2), the remaining configuration is unique up to automorphism of $\mathbb{P}^2$, thus the whole picture is determined by a choice of q on C_2. If the coordinates are chosen as shown in Figure 2, then $C_3 = \{x^3 = y^2z\}$, $C_2 = \{2xy + xz = y^2 + 2yz\}$, and the equation in the statement of Proposition 1(b) corresponds to $q = (3 : 2 : 8)$.

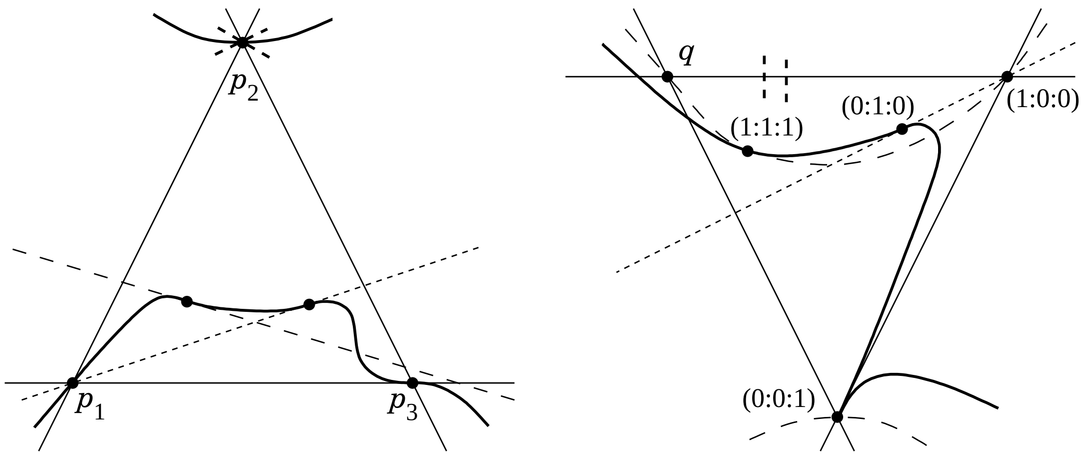

FIGURE 2. Cremona transformation in §1.4.

The parametrizations of the irreducible components of $F(x^3, y^3, z^3) = 0$ can be found using the following observation: if $t \mapsto \big(tx(t)^n : ty(t)^n : z(t)^n\big)$ is a parametrization of a curve $h(x, y, z) = 0$, then the curve $h(x^n, y^n, z^n) = 0$ has n irreducible components parametrized by

$$t \mapsto \big(\zeta^k tx(t^n) : ty(t^n) : z(t^n)\big), \qquad \zeta = e^{2\pi i/n}, \qquad k = 1, \dots, n.$$

§2. Perturbation of the union of the dual Hesse arrangement with a generic line

2.1. The arrangement to perturb. Denote the components of the dual Hesse line arrangement $(x^3-y^3)(y^3-z^3)(z^3-x^3)=0$ by

$$\begin{aligned} &L_1=\{y=z\}, && L_4=\{z=x\}, && L_7=\{x=y\},\\ &L_2=\{y=\omega z\}, && L_5=\{z=\omega x\}, && L_8=\{x=\omega y\},\\ &L_3=\{y=\overline{\omega} z\}, && L_6=\{z=\overline{\omega} x\}, && L_9=\{x=\overline{\omega} y\}, \qquad \omega=e^{2\pi i/3}. \end{aligned}$$

It has twelve triple points $p_{ijk}=L_i\cap L_j\cap L_k$:

$$\begin{aligned} & p_{123}=(1:0:0), && p_{456}=(0:1:0), && p_{789}=(0:0:1),\\ p_1=&p_{147}=(1:1:1), & p_4=&p_{159}=(\overline{\omega}:1:1), & p_7=&p_{249}=(1:\omega:1),\\ p_2=&p_{258}=(1:\overline{\omega}:\omega), & p_5=&p_{267}=(1:1:\overline{\omega}), & p_8=&p_{168}=(\omega:1:1),\\ p_3=&p_{369}=(1:\omega:\overline{\omega}), & p_6=&p_{348}=(1:\overline{\omega}:1), & p_9=&p_{357}=(1:1:\omega). \end{aligned}$$

Let us denote $\mathcal{L}_1=\{L_1,L_2,L_3\}$, $\mathcal{L}_2=\{L_4,L_5,L_6\}$, $\mathcal{L}_3=\{L_7,L_8,L_9\}$.

Let $L_0=\{2x+2y=z\}$. We parametrize it by $t\mapsto f_0(t)=(t:1-t:2)$. Note that L_0 does not pass through the 12 triple points, nor through the intersections of the L_i's with the coordinate lines. Let $t_1,\dots,t_9$ be such that $f_0(t_i)=L_0\cap L_i$, $i=1,\dots,9$, i.e.,

$$(t_1,\dots,t_9)=(-1,1-2\omega,1-2\overline{\omega},\quad 2,2\overline{\omega},2\omega,\quad 1/2,-\overline{\omega},-\omega). \tag{2}$$

We are going to perturb $L_0\cup L_1\cup\dots\cup L_9$ so that the intersections of L_0 with the other lines are smoothed while all the triple points are preserved.

2.2. A preliminary irreducible perturbation with double points. Notice first that if A is a rational curve in $\mathbb{P}^n$ and L a line that crosses A at a point p without tangency, then a smoothing of the intersection point can be described as follows (cf. [4, Lemma 6.1]). Let us choose homogeneous coordinates $(x_0:\dots:x_n)$ in $\mathbb{P}^n$ so that L passes through $(1:0:\dots:0)$. Let $t\mapsto f(t)=(x_0(t):\dots:x_n(t))$ be a polynomial parametrization of A and let t_0 be such that $f(t_0)=p$. Let C be the curve parametrized by $t\mapsto(\hat{x}_0(t):\dots:\hat{x}_n(t))$ where $\hat{x}_0(t)=(t-t_0-u)x_0(t)$ and $\hat{x}_i(t)=(t-t_0)x_i(t)$ for $i\ge 1$. Then C tends to $A\cup L$ when $u\to 0$. The degeneration of C into $A\cup L$ and its analog for pseudoholomorphic curves is often called *bubbling.* This construction can be easily generalized to the case when several lines cross A under condition that each line passes through a base point of $\mathbb{P}^n$. Fortunately, this condition is satisfied by the dual Hesse arrangement.

So, we consider a curve $C=C_{u,\mathbf{a}}$ whose parametrization

$$t\mapsto f(t)=f_{u,\mathbf{a}}(t)=\big(tf_1(t):(1-t)f_2(t):2f_3(t)\big),$$

depends on $u\in\mathbb{C}$ and $\mathbf{a}=(a_1,\dots,a_9;b_1,\dots,b_9)\in\mathbb{C}^9\times(\mathbb{C}\setminus\{0\})^9$ as follows:

$$f_\nu(t)=\prod_{i=1}^{9}\big(t-t_i-(a_i+b_{i,\nu})u\big),\quad b_{i,\nu}=\begin{cases} b_i & \text{if } L_i\in\mathcal{L}_\nu,\\ 0 & \text{otherwise.}\end{cases}$$

If $\mathbf{a}$ is fixed and u tends to zero, then C tends to $L_0\cup L_1\cup\dots\cup L_9$. By construction, C has triple points at $(1:0:0)$, $(0:1:0)$, and $(0:0:1)$. However, each of the other nine triple points in general splits into three nodes. Our aim is to find an analytic germ $u\mapsto\mathbf{a}(u)$ such that all the 12 triple points are preserved in $C_{u,\mathbf{a}(u)}$.

2.3. A perturbation with the triple points preserved up to $O(u^2)$. From now on we pass to the affine chart $z \neq 0$, which we identify with $\mathbb{C}^2$. So, we denote a point $(x : y : z)$ by $(x/z, y/z)$ and say that C is parametrized by

$$t \mapsto f(t) = \big(X(t), Y(t)\big), \qquad X(t) = \frac{tf_1(t)}{2f_3(t)}, \qquad Y(t) = \frac{(1-t)f_2(t)}{2f_3(t)},$$

where $f_\nu(t)$ are as above. Then

$$X = \frac{t}{2}\frac{A_1A_2A_3\,B_7B_8B_9}{B_1B_2B_3\,A_7A_8A_9}, \quad Y = \frac{(1-t)}{2}\frac{A_1A_2A_3\,B_4B_5B_6}{B_1B_2B_3\,A_4A_5A_6}, \quad \begin{cases} A_i = t - t_i - a_iu, \\ B_i = A_i - b_iu. \end{cases}$$

If $p_k \in L_i$ and $L_i \in \mathcal{L}_j$, we set $i_j(k) = i$. Let $\tau_{k,j} = \tau_{k,j}(\mathbf{a})$ be found from the condition

$$\lim_{u\to 0} f(t_i + \tau_{k,j}u) = p_k, \qquad i = i_j(k).$$

Each $\tau_{k,j}$ is of the form $a_i + \lambda_{k,j}b_i$, $i = i_j(k)$, $\lambda_{k,i} \in \mathbb{Q}(\omega)$. For example, we have $p_1 = (1,1)$, $i_2(1) = 4$, $t_4 = 1/2$ (see (2)), and

$$\lim_{u\to 0} Y(t_4 + \tau u) = \frac{(1-t_4)(\tau - a_4 - b_4)}{2(\tau - a_4)},$$

hence $\tau_{1,2} = a_4 + \frac{1}{3}b_4$. Let us set

$$p_{k,j}(u) = \begin{cases} f(t_i + \tau_{k,j}u), & u \neq 0,\ i = i_j(k), \\ p_k, & u = 0. \end{cases}$$

Both components of the derivatives $p'_{k,j}(0)$ are linear combinations of $a_1, \dots, a_9$, $b_1, \dots, b_9$ with coefficients in $\mathbb{Q}(\omega)$, for example, the X-component of $6p'_{1,2}(0)$ is

$$-2b_1 + (2+4\omega)(b_2 - b_3) + 3a_4 + b_4 + 4b_7 + (4+2\omega)b_8 + (2-2\omega)b_9.$$

Let $k \in \{1, \dots, 9\}$. We are going to find a condition on $\mathbf{a}$ to assure that the triple point at p_k is preserved in $C_{u,\mathbf{a}}$ up to $O(u^2)$ (i.e., the distances between the double points of $C_{u,\mathbf{a}}$ appearing near p_k are of that order). To this end, in a neighborhood of p_k, we approximate $C_{u,\mathbf{a}}$ by $L'_1 \cup L'_2 \cup L'_3$, where L'_j is the line parallel to $L_{i_j(k)}$ passing through the point $p_k + up'_{k,j}(0)$. These lines are concurrent if and only if there exist ξ_1, ξ_2, ξ_3 such that

$$v_1 + \xi_1 w_1 = v_2 + \xi_2 w_2 = v_3 + \xi_3 w_3,$$

where $v_j = p'_{k,j}(0)$ and $w_1 = (1,0)$, $w_2 = (0,1)$, $w_3 = p_k$ (the vector w_j is parallel to $L_{i_j(k)}$). This is a system of four linear non-homogeneous equations for three unknowns, thus it has a solution if and only if $e_k = 0$, where

$$e_k = \det\begin{pmatrix} w_1 & -w_2 & 0 & v_2 - v_1 \\ w_1 & 0 & -w_3 & v_3 - v_1 \end{pmatrix} \tag{3}$$

(here the v_j and w_j are interpreted as columns). Since the w_j are constant, e_k is a linear combination of $a_1, \dots, a_9, b_1, \dots, b_9$ with coefficients in $\mathbb{Q}(\omega)$, for example, we have

$$e_1 = \tfrac{1}{2}(a_1 + b_1 + a_4 + b_4) - 4(a_7 + b_7) + b_8 + b_9 + \tfrac{2\omega-1}{7}(b_2 + b_6) + \tfrac{2\overline{\omega}-1}{7}(b_3 + b_5).$$

Thus we obtain a system of homogeneous linear equations $e_1 = \cdots = e_9 = 0$. Each solution with all b_i's non-zero yields a desired perturbation with all the triple points preserved up to $O(u^2)$. For such a solution we can choose

$$\widehat{\mathbf{a}} = \left(-\tfrac{1}{3}, \alpha, \bar{\alpha}, -\tfrac{1}{3}, \bar{\alpha}, \alpha, \tfrac{37}{42}, 0, 0;\ \ 1,1,1,1,1,1,-\tfrac{1}{2},1,1\right), \quad \alpha = \frac{38\omega - 5}{21}. \tag{4}$$

Remark 3. If L_0 is generic, then the existence of a solution with all b_i's non-zero can be derived from the symmetry of the Hesse configuration. Indeed, suppose that for any solution there exists i such that $b_i = 0$ in this solution. Then there exists i such that $b_i = 0$ in any solution. Since the configuration is symmetric and L_0 is generic, it follows that $b_1 = \cdots = b_9 = 0$ in any solution. This fact would imply that the coefficient of each a_i in each e_k is zero, which is not the case.

However, we do not know how to do the next step (in §2.4) without the above computations for a concrete line (for which we have chosen $2x + 2y = z$ though this particular choice is not at all important).

2.4. A perturbation with all triple points preserved. Roughly speaking, we just blow up the space of parameters at $(0, \widehat{\mathbf{a}})$ (see (4)) and apply the implicit function theorem. Let us explain it in more detail.

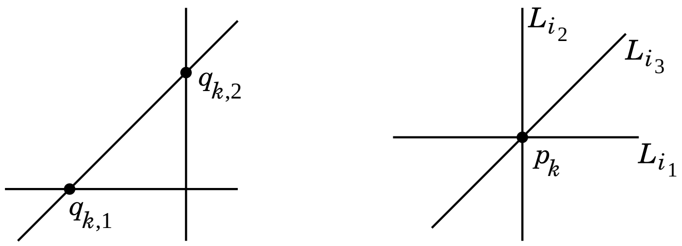

FIGURE 3. The double points of $C_{u,\mathbf{a}}$ that converge to p_k.

For a fixed $k \in \{1, \ldots, 9\}$, let $q_{k,j}(u, \mathbf{a})$, $j = 1, 2$, be the double point of $C_{u,\mathbf{a}}$ whose local branches converge to the germs of $L_{i_j(k)}$ and $L_{i_3(k)}$ at p_k as $u \to 0$ (see Figure 3). Let $s_{k,1}(u, \mathbf{a})$ and $s_{k,2}(u, \mathbf{a})$ be such that

$$f_{u,\mathbf{a}}\big(t_{i_3(k)} + s_{k,j}(u, \mathbf{a})u\big) = q_{k,j}(u, \mathbf{a}), \qquad j = 1, 2.$$

i.e. $t_{i_3(k)} + s_{k,j}(u, \mathbf{a})u$ is the parameter of the center of the local branch of $C_{u,\mathbf{a}}$ at $q_{k,j}(u, \mathbf{a})$ that converges to $(L_{i_3(k)}, p_k)$. The condition that the triple point p_k is preserved under the perturbation is equivalent to the condition that

$$s_{k,1}(u, \mathbf{a}) = s_{k,2}(u, \mathbf{a}). \tag{5}$$

Let $\widehat{\mathbf{a}}$ be as in (4). The fact that it is a solution of the equations $e_1 = \cdots = e_9 = 0$ (see §2.3) implies that (5) holds up to $O(u^2)$ when $\mathbf{a} = \widehat{\mathbf{a}} + \widetilde{\mathbf{a}}u$, $\widetilde{\mathbf{a}} \in \mathbb{C}^{18}$, i.e.,

$$s_{k,2}(u, \widehat{\mathbf{a}} + \widetilde{\mathbf{a}}u) - s_{k,1}(u, \widehat{\mathbf{a}} + \widetilde{\mathbf{a}}u) = u^2 \Phi_k(u, \widetilde{\mathbf{a}}),$$

where Φ_k is analytic in a neighborhood of $\{(0, \widetilde{\mathbf{a}}) \mid \widetilde{\mathbf{a}} \in \mathbb{C}^{18}\}$ in $\mathbb{C} \times \mathbb{C}^{18}$.

One can compute that $\Phi_k(0, \widetilde{\mathbf{a}})$ are affine linear functions of $\widetilde{\mathbf{a}}$ with coefficients in $\mathbb{Q}(\omega)$ (see more details in the recursion step for $n = 1$ in §2.5), for example, we have $\Phi_k(0,\widetilde{\mathbf{a}}) = c_k e_k(\widetilde{\mathbf{a}}) + d_k$ with $c_k, d_k \in \mathbb{Q}(\omega)$ for $k = 1,5,9$ though the linear part of $\Phi_k(0,\widetilde{\mathbf{a}})$ is not proportional to $e_k(\widetilde{\mathbf{a}})$ for other values of k (recall that e_k is defined in (3)). Moreover $\Phi|_{u=0}$ is surjective for

$$\Phi = (\Phi_1,\dots,\Phi_9) : \mathbb{C}\times\mathbb{C}^{18} \to \mathbb{C}^9,$$

in particular, $\det\big(\Phi|_{0\times E}\big) = 64/(3^{12}\times 49)$, where

$$E = \{\widetilde{\mathbf{a}} \in \mathbb{C}^{18} \mid \tilde a_1 = \tilde a_4,\ \ \tilde a_7 = \tilde b_1 = \tilde b_4 = \tilde b_5 = \tilde b_6 = \tilde b_7 = \tilde b_8 = \tilde b_9 = 0\}. \qquad (6)$$

Hence, by the implicit function theorem, for a neighborhood U of 0 in $\mathbb{C}$ there exists an analytic mapping $U \to E$, $u \mapsto \widetilde{\mathbf{a}}(u)$, such that $\Phi(u,\widetilde{\mathbf{a}}(u)) = 0$. Then $t \mapsto f_{u,\widehat{\mathbf{a}}+u\widetilde{\mathbf{a}}(u)}(t)$ is the required perturbation of $L_0 \cup L_1 \cup \dots \cup L_9$ into a rational curve with 12 triple points.

Remark 4. (Cf. Remarks 1 and 3.) Proposition 2 remains true for a generic line L_0 because the set of lines such that $\operatorname{rank}(\Phi|_{u=0}) = 9$ is Zariski open and non-empty.

Remark 5. One might think that the same arguments would yield an irreducible perturbation of a curve $(ax+by+cz)(x^n-y^n)(y^n-z^n)(z^n-x^n)$ with n^2 triple points and three n-fold points for $n = 4$ and 5. The results of §2.3 indeed extend to these cases. However, the analog of $\Phi|_{u=0}$ is no longer surjective: it is a mapping $\mathbb{C}^{24} \to \mathbb{C}^{16}$ of rank 15 for $n = 4$, and $\mathbb{C}^{30} \to \mathbb{C}^{25}$ of rank 20 for $n = 5$.

2.5. Power series expansions. Let $\widetilde{\mathbf{a}}(u)$ be as in §2.4. Denote $\mathbf{a}(u) = \widehat{\mathbf{a}} + u\widetilde{\mathbf{a}}(u)$, $C_u = C_{u,\mathbf{a}(u)}$ and $f_u = f_{u,\mathbf{a}(u)}$. Let $p_k(u)$ be the triple point of C_u which tends to p_k as $u \to 0$, and let $T_{k,j}(u) \in f_u^{-1}(p_k(u))$ be such that the germ of f_u at $T_{k,j}(u)$ defines the local branch of C_u at $p_k(u)$ that tends to $(L_{i_j(k)}, p_k)$ as $u \to 0$. By the definition of $\tau_{k,j}$ (see §2.3) there exist analytic functions $S_{k,j}(u)$ such that

$$T_{k,j}(u) = t_{i_j(k)} + u\tau_{k,j}\big(\mathbf{a}(u)\big) + u^2 S_{k,j}(u).$$

In this subsection we explain how to recursively compute the power series expansion of $\mathbf{a}(u)$ simultaneously with the expansions of the $S_{k,j}(u)$. Let us set:

$$\mathbf{a}(u) = \sum_{n=0}^{\infty} \mathbf{a}_n u^n, \quad \mathbf{a}^{[n]}(u) = \sum_{m=0}^{n} \mathbf{a}_m u^m, \quad \mathbf{a}_n = (a_{1,n},\dots,a_{9,n};\, b_{1,n},\dots,b_{9,n}),$$

$$S_{k,j}(u) = \sum_{n=0}^{\infty} S_{k,j,n} u^n, \qquad S_{k,j}^{[n]}(u) = \sum_{m=0}^{n} S_{k,j,m} u^m.$$

Then $C_{u,\mathbf{a}^{[n]}(u)}$ is a perturbation with all triple points preserved up to $O(u^{n+2})$.

The recursive computation of $\mathbf{a}^{[n]}$ and $S_{k,j}^{[n]}$ goes as follows. Note that all the equations below are linear (because we expand $S_{k,j}$ instead of $T_{k,j}$). Moreover, the homogeneous part of (7) does not depend on n.

1°. *Initialization of the recursion.* For each $k = 1,\dots,9$, find s_1, s_2, s_3 from the system of equations

$$f_{u,\widehat{\mathbf{a}}}\big(t_{i_j(k)} + u\tau_{k,j}(\widehat{\mathbf{a}}) + u^2 s_j\big) = f_{u,\widehat{\mathbf{a}}}\big(t_{i_3(k)} + u\tau_{k,j}(\widehat{\mathbf{a}}) + u^2 s_3\big) + O(u^2), \quad j = 1,2,$$

(these are four linear equations for three unknowns, but a solution exists due to the definition of $\widehat{\mathbf{a}}$). Then set $S_{k,j}^{[0]} = s_j$, $j = 1, 2, 3$, and $\mathbf{a}^{[0]} = \widehat{\mathbf{a}}$.

2°. *The recursion step.* Assuming $\mathbf{a}^{[n-1]}(u)$ and all the $S_{k,j}^{[n-1]}(u)$ known, compute $\mathbf{a}^{[n]}(u)$ and all the $S_{k,j}^{[n]}(u)$ as follows.

- For each $k = 1, \dots, 9$ and $j = 1, 2$, find $s_{k,j} = s_{k,j}(\mathbf{a}_n)$ and $s^*_{k,j} = s^*_{k,j}(\mathbf{a}_n)$ from the equation

$$\phi_{k,3}^{[n]}(0, s_{k,j}) = \phi_{k,j}^{[n]}(0, s^*_{k,j}),$$

  where $\phi_{k,j}^{[n]}$ is the analytic function such that

$$f_{u,\mathbf{a}^{[n]}(u)}\Big(t_{i_j(k)} + u\tau_{k,j}\big(\mathbf{a}^{[n]}(u)\big) + u^2\big(S_{k,j}^{[n-1]}(u) + su^n\big)\Big) = u^{n+1}\phi_{k,j}^{[n]}(u, s).$$

- Find $\mathbf{a}_n \in E$ (see (6)) from the system of equations

$$s_{k,1}(\mathbf{a}_n) = s_{k,2}(\mathbf{a}_n), \qquad k = 1, \dots, 9. \tag{7}$$

- Set $S_{k,j,n} = s^*_{k,j}(\mathbf{a}_n)$, $j = 1, 2$, and $S_{k,3,n} = s_{k,1}(\mathbf{a}_n)$.

The result of computation of $\mathbf{a}^{[7]}$ and $S_{j,k}^{[7]}$, as well as the Maple code with comments, is available at `https://www.math.univ-toulouse.fr/~orevkov/12tp.html`.

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