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SECOND ORDER EPI DERIVATIVES AND THE DUPIN INDICATRIX FOR NONSMOOTH FUNCTIONS

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1. INTRODUCTION

Since the presentation of Clarke's generalized derivative in the mid-1970s, it has been a main issue in nonsmooth analysis and optimization to develop notions of generalized first and second derivatives for nonsmooth functions which may serve as a substrate for the classical derivatives. In this paper we discuss two related concepts of generalized second derivatives for nonsmooth functions which have been under an intense discussion over recent years. On the one hand, *first and second epi derivatives* in the sense of Rockafellar have been studied at least since 1987. This concept proves to be useful, e.g. when studying optimization problems for various classes of nonsmooth functions including convex functions, convex-concave saddle functions, lower C^2-functions, or functions of the form $g \circ \phi$, with g convex and ϕ of class C^2. See [1–5] and the references given therein for an overview on these.

Recently, second epi derivatives have been extended to infinite dimensions (see [6 and 7]). In [6], Borwein and Noll have used second Mosco derivatives for convex functions on Hilbert spaces, while in [7], second epi derivatives turned out to be an important tool when describing the second order behavior of not necessarily convex integral functionals on L^2-spaces and Sobolev spaces.

Parallel to this, on the other hand, Hiriart-Urruty's theory of *approximate first and second derivatives* for convex functions has been under discussion since the early 1980s, and has been further developed over recent years. See [8–12] for an overview on this. It seems at first glance that both theories should play, in some sense, a complementary role. Namely, approximate second derivatives in the first place are designed as a one-dimensional notion which reflects the second variation of a function f at a point x along a fixed direction h. As a contrast, second order epi derivatives are sensitive to the second order variation of f along all directions h simultaneously.

Nevertheless, we present a link for both theories here by showing that the generalized Dupin indicatrix for a convex function, which was presented in [11, 12] as part of the theory of approximate derivatives, may be obtained quite naturally in the context of second epi derivatives. This requires a more geometric view of second epi derivatives, and so we continue our line of investigation from [13], where second epi derivatives were looked at from a geometric point of view. We derive some consequences of our new sight of the Dupin indicatrix. For instance, we answer the question about the limiting behavior of the difference quotient

$$\lim_{\varepsilon \to 0^+} \frac{\partial_\varepsilon f(x) - \partial f(x)}{\sqrt{2\varepsilon}} \tag{1.1}$$

where $\partial_\varepsilon f(x)$ denotes the ε-subdifferential of f at x (cf. [9, 12]). Namely, the limit (1.1) exists if and only if f has a second order epi derivative at x. The limit behaviour of (1.1) was already discussed in [12] under an unnecessarily strong side condition.

2. SECOND EPI DERIVATIVES

Let us briefly recall the notion of epi convergence. For an overview see [14–17].

Let f_t, $f\colon \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$, $t > 0$ be extended real-valued lower semicontinuous functions. Then f_t is said to *epi converge* to f as $t \to 0^+$, noted $f_t \xrightarrow{e} f$, if the following conditions are satisfied:

(α) given any sequence $t_n \to 0^+$ and $h \in \mathbb{R}^d$, there exist $h_n \to h$ such that $f_{t_n}(h_n) \to f(h)$ as $n \to \infty$;

(β) given any $t_n \to 0^+$, $h \in \mathbb{R}^d$ and $h_n \to h$, we have $\liminf_{n\to\infty} f_{t_n}(h_n) \geq f(h)$.

We use a slight extension of the concept of epi convergence which was already discussed in [13], and which has a geometric motivation in the context of epi convergence of second order difference quotients to be presented later. Namely, we write $f_t(h) \xrightarrow{e} \theta \in \mathbb{R} \cup \{\infty\}$ if conditions (α) and (β) above are satisfied with $f(h)$ replaced by θ. So $f_t \xrightarrow{e} f$ if and only if $f_t(h) \xrightarrow{e} f(h)$ for every h.

Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz function. The *second order difference quotient* of f at $x \in \mathbb{R}^d$ relative to $y^* \in \partial f(x)$, where $\partial f(x)$ denotes the Clarke generalized subdifferential of f at x (cf. [18]), is defined as

$$\Delta_{f,x,y^*,t}(h) = \frac{f(x+th) - f(x) - t\langle y^*, h\rangle}{t^2}, \qquad h \in \mathbb{R}^d. \tag{2.1}$$

Notice that, for every $t \neq 0$, $\Delta_{f,x,y^*,t}$ is a locally Lipschitz function of h. With these notions at hand we are ready to give the following definition, which the reader should compare with [1-4, 13].

Definition 2.1. The function f is said to have *second epi derivative* q_{x,y^*} at x relative to $y^* \in \partial f(x)$ if the second order difference quotient (2.1) epi converges to q_{x,y^*} as $t \to 0^+$, i.e. $\Delta_{f,x,y^*,t} \xrightarrow{e} q_{x,y^*}$ as $t \to 0^+$.

Remark 1. Notice that q_{x,y^*}, when it exists, is automatically lower semicontinuous and extended real-valued. Owing to the relation

$$\Delta_t(\lambda h) = \lambda^2 \Delta_{\lambda t}(h), \tag{2.2}$$

(where $\Delta_t := \Delta_{f,x,y^*,t}$), we see that q_{x,y^*} is quadratic, i.e. positively homogeneous of degree 2.

Remark 2. If q_{x,y^*} is symmetric, i.e. satisfies $q_{x,y^*}(-h) = q_{x,y^*}(h)$, then the second epi derivative is said to be two-sided. Notice that this implies in particular that $\operatorname{dom}(q_{x,y^*})$ is a linear subspace. Further, if q_{x,y^*} is *purely quadratic*, i.e. has a representation of the form

$$q_{x,y^*}(h) = \tfrac{1}{2}\langle T_{x,y^*}h, h\rangle, \tag{2.3}$$

with T_{x,y^*} a symmetric linear operator defined on $\operatorname{dom}(q_{x,y^*})$, then we use the notation $q_{xy^*} = II_{x,y^*}$, calling this the *generalized second fundamental form* of f at x relative to y^*. The operator T_{x,y^*} is then called the generalized Hessian.

Remark 3. Second derivatives may be understood as one-sided second derivatives in the following sense. Namely, instead of basing our notion of convergence on the epigraphs, we might as well work with hypographs. Now hypo convergence of $\Delta_{f,x,y^*,t}$ to a limit function q having values in $\mathbb{R} \cup \{-\infty\}$ is equivalent to epi convergence of $-\Delta_{f,x,y^*,t} = \Delta_{-f,x,-y^*,t}$ to the limit $-q$. Now suppose $\Delta_{f,x,y^*,t}$ both epi and hypo converges to a limit q, which consequently must have finite values and be continuous. Then $\Delta_{f,x,y^*,t}$ converges to q pointwise and even uniformly on compact sets. The latter is equivalent to saying that f has a second order Taylor's expansion at x, i.e. a representation of the form

$$f(x + h) = f(x) + \langle \nabla f(x), h \rangle + q(h) + o(\|h\|^2) \qquad (h \to 0). \tag{2.4}$$

Conversely, if f has a second order Taylor expansion (2.4) at x, i.e. if $\Delta_{f,x,y^*,t}$ converges uniformly on compact sets to some fully defined quadratic limit q, then $\Delta_{f,x,y^*,t} \xrightarrow{e} q$ and $\Delta_{-f,x,-y^*,t} \xrightarrow{e} -q$, i.e. we have both convergence in the epi and in the hypo sense.

It may happen that, besides its epi limit q_{x,y^*}, the second order difference quotient (2.1) has a limit in the sense of pointwise convergence, i.e.

$$q^{\#}_{x,y^*}(h) = \lim_{t \to 0^+} \Delta_{f,x,y^*,t}(h) \tag{2.5}$$

exists for every $h \in \mathbb{R}^d$. Condition (β) for the epi convergence then automatically implies $q_{x,y^*} \le q^{\#}_{x,y^*}$. Again we use the notation $q^{\#}_{x,y^*} = II^{\#}_{x,y^*}$ when (2.5) is a quadratic form defined on a subspace of $\mathbb{R}^d$. Clearly, if f has a derivative at x in the classical sense, then the usual second fundamental form II_x at x exists and equals $II^{\#}_{x,y^*}$, where $y^* = \nabla f(x)$. However, is it true that $II_x = II_{x,y^*}$ in this case, and if not, what is the reason for introducing an object like II_{x,y^*}?

We shall deal with these and related questions here. As it turns out, both q_{x,y^*} and $q^{\#}_{x,y^*}$ have a geometric interpretation, so there is no question about giving preference to one of them. It may happen even for convex f that both II_{x,y^*} and $II^{\#}_{x,y^*}$ exist but are different, although this situation may be considered as a somewhat pathological case. A class of examples of this type was already presented in [13], and we shall go further into the analysis of these phenomena here. Before doing this, let us quote the following facts from [13] which justify the use of the second epi derivative q_{x,y^*} resp. II_{x,y^*}.

In the convex case, the most obvious reason for using epi convergence is of course its invariance under Young-Fenchel conjugation. More precisely, if $\Delta_{f,x,y^*,t} \xrightarrow{e} q_{f,x,y^*}$, then the conjugate of the second difference quotient, which is $(\Delta_{f,x,y^*,t})^* = \Delta_{f^*,y^*,x,t}$, has epi limit q^*_{f,x,y^*}, i.e. $\Delta_{f^*,y^*,x,t} \xrightarrow{e} q^*_{f,x,y^*} = q_{f^*,y^*,x}$.

Let us now consider another more sophisticated observation, which justifies the use of second epi derivatives, and which was obtained in [13].

Fact 1. For a convex f, $x \in \mathbb{R}^d$, $y^* \in \partial f(x)$, let F be the convex hypersurface in $\mathbb{R}^{d+1}$ represented as the graph of f in a neighbourhood of the point $p = (x, f(x))$. Consider the outer parallel surface F_ε at distance $\varepsilon > 0$. Let n_p be the outer unit normal vector of F corresponding with the subgradient y^* of f at x. Consider the corresponding point $p_\varepsilon = p + \varepsilon n_p$ on the surface F_ε. Then second order epi differentiability of f at x with respect to y^* is equivalent to p_ε being a point of second order smoothness of F_ε in the classical sense (sometimes called a normal point of F_ε). In other terms, if F_ε is represented as the graph of a convex function f_ε in a neighbourhood of y and such that $p_\varepsilon = (y, f_\varepsilon(y))$, then f is second order epi differentiable at x with respect to y^* if and only if f_ε is second order differentiable at y in the classical sense, i.e. admits a second order Taylor expansion at y. See [13] for this.

As a consequence of fact 1, the epi limit q_{x,y^*}, resp. II_{x,y^*}, arises naturally as the limit ($\varepsilon \to 0^+$) of the second fundamental forms of the outer parallel surfaces. In more abstract terms, the usual point of view of convex geometry leads us to consider II_{x,y^*}, since the information given by this form is hidden in the second order behaviour of the outer parallel surfaces (see [13]). On the other hand, the function $q^{\#}_{x,y^*}$, resp. $II^{\#}_{x,y^*}$, cannot be reconstructed in a natural way from the information on the outer parallel surfaces. See example 5.3 for this.

Let us quote a second fact which clarifies the interrelation between the pointwise and epi limits of the second difference quotient and justifies the use of second epi derivatives.

Fact 2. Let f be a locally Lipschitz function, $x \in \mathbb{R}^d$, $y^* \in \partial f(x)$. Let F be the Lipschitzian hypersurface in $\mathbb{R}^{d+1}$ represented as the graph of f in a neighbourhood of $p = (x, f(x))$. Let n_p be the unit normal vector corresponding with y^*, and let τ_p the corresponding generalized tangent hyperplane at p (cf. [18]). Let t_p be any ray emanating from p and contained in τ_p, then t_p and n_p determine a normal section. Let us choose our local coordinates such that $p = (0, 0) = (0, f(0))$ and $0 = y^* \in \partial f(0)$. Then $\tau_p = \mathbb{R}^d \times \{0\}$, n_p is the x_{d+1}-axis, and $t_p = \mathbb{R}h \times \{0\}$ for some $h \in \mathbb{R}^d$. Then the following are true (cf. [13]).

(1) $\Delta_{f,0,0,t}(h) \to \theta_1$, $(t \to 0^+)$ is satisfied if and only if the surface curve κ_h lying in the normal section spanned by t_p and n_p has curvature $2\theta_1$ at p.

(2) Given any other surface curve κ having the same tangent line t_p at p, and such that its projection κ' onto the tangent hyperplane τ_p has a finite curvature at p, has normal curvature $2\theta_1$ at p (theorem of Meusnier).

(3) $\Delta_{f,0,0,t}(h) \xrightarrow{e} \theta_2$ is equivalent to saying that every surface curve κ having tangent t_p at p has upper normal curvature at least $2\theta_2$, and there exists at least one such curve which has exactly normal curvature $2\theta_2$ at p.

Roughly speaking, these statements show that the second order behaviour of f at x, resp. F, at p with respect to a certain direction h may be completely understood by considering all surface curves κ having tangent t_p at p and evaluating the curvature of their projection κ'' onto the normal section spanned by t_p and n_p. However, only those curves κ whose projection κ' onto the tangent hyperplane have a finite curvature at p contribute to the value $q^{\#}_{x,y^*}(h)$, while the remaining curves κ still have an effect on the value $q_{x,y^*}(h)$. As part of the discussion here, we will see that $\theta_2 < \theta_1$ is possible.

Let us now collect some known facts concerning the interrelation between pointwise and epi convergence of the second difference quotient, resp., the interrelation between q_{x,y^*} and $q^{\#}_{x,y^*}$. For convex functions f, the following are known.

(1) Suppose q_{x,y^*} exists and $\mathrm{dom}(q_{x,y^*})$ has nonempty interior. Then the second difference quotient converges pointwise to q_{x,y^*} on $\mathrm{int}(\mathrm{dom}(q_{x,y^*}))$. In particular, if q_{x,y^*} is fully defined, then $q^{\#}_{x,y^*}$ exists and equals q_{x,y^*}, (see [2, 13]). Though q_{x,y^*} and $q^{\#}_{x,y^*}$ may differ on the boundary of $\mathrm{int}(\mathrm{dom}(q_{x,y^*}))$ (see example 5.1).

(2) If $q^{\#}_{x,y^*}$ exists and is fully defined, then q_{x,y^*} exists and they coincide. Convexity is essential for this, as example 5.2 shows.

(3) When $q_{x,y^*} = II_{x,y^*}$ exists with domain a *proper* subspace, then $q^{\#}_{x,y^*}$ may still fail to exist. Even worse, $q^{\#}_{x,y^*} = II^{\#}_{x,y^*}$ may exist, but the pathological phenomenon $II_{x,y^*}(h) < II^{\#}_{x,y^*}(h)$ for some h may occur (see example 5.3).

(4) $q^{\#}_{x,y^*} = II^{\#}_{x,y^*}$ may exist with domain a proper subspace, while q_{x,y^*} fails to exist (see example 6.3 in [13]).

The situation is even worse for nonconvex f, for in this case II_{x,y^*} and $II^{\#}_{x,y^*}$ may both exist, have domain $\mathbb{R}^d$, and yet be different, i.e. $II_{x,y^*}(h) < II^{\#}_{x,y^*}(h)$ for some h (see example 5.2).

3. THE DUPIN INDICATRIX

In classical differential geometry, the Dupin indicatrix of a surface F at a given point p is the hypersurface $II_p(h) = \pm 1$ of the tangent space T_p of F at p. In convexity, we may define an upper and lower Dupin indicatrix as proposed in [19]. On every ray t_p emanating from p and contained in the tangent hyperplane, we mark the upper and lower curvatures of the plane surface curve lying in the normal section spanned by t_p and the normal n_p, thus obtaining a convex domain γ representing the lower Dupin indicatrix, and a star shaped domain Γ representing the upper Dupin indicatrix. If $\gamma = \Gamma$, this set is called the Dupin indicatrix (see [19] for details). It is clear, however, that these constructions reflect the behaviour of the *pointwise* limit $q^{\#}_{x,y^*}$ rather than the *epi* limit of the second difference quotient. We shall, therefore, use the notation $\mathrm{Ind}^{\#}$ for the pointwise indicatrix. Now, Busemann and Feller [19] also propose a third way of constructing a Dupin indicatrix as a set-valued limit of certain domains. This idea also appears in Hiriart-Urruty and Seeger [12], and we use this as our definition of a generalized Dupin indicatrix for a convex function. Clearly all definitions coincide if the function f is twice differentiable at the point x under consideration.

Let f be a convex function on $\mathbb{R}^d$, $x \in \mathbb{R}^d$, $y^* \in \partial f(x)$. For $t > 0$ let

$$\Gamma_t = \{h \in \mathbb{R}^d : \Delta_{f,x,y^*,t}(h) \le \tfrac{1}{2}\},$$

then

$$\begin{aligned} \liminf_{t\to 0^+} \Gamma_t &= \{h \in \mathbb{R}^d : \forall\, t_n \to 0^+\ \exists\, h_n \to h,\ h_n \in \Gamma_{t_n}\} \quad \text{resp.} \\ \limsup_{t\to 0^+} \Gamma_t &= \{h \in \mathbb{R}^d : \exists\, t_n \to 0^+\ \exists\, h_n \to h,\ h_n \in \Gamma_{t_n}\} \end{aligned} \tag{3.1}$$

are called the lower, resp. upper, Dupin indicatrices of f at x with respect to y^*, noted $\underline{\mathrm{Ind}}_{f,x,y^*}$, resp. $\overline{\mathrm{Ind}}_{f,x,y^*}$. If the lower and upper Dupin indicatrices coincide, we call this the Dupin indicatrix, using the notation Ind_{f,x,y^*}. The limit inferior and limit superior (3.1) are usually referred to as limits in the sense of Kuratowski, so the Dupin indicatrix exists when the level sets Γ_t are convergent in the Kuratowski sense (as $t \to 0^+$).

In [12], the authors give sufficient conditions on when the Dupin indicatrix (3.1) exists. Here we have the following theorem.

THEOREM 3.1. Let f be convex. Then the Dupin indicatrix Ind_{f,x,y^*} of f at x relative to $y^* \in \partial f(x)$ exists if and only if f has second epi derivative q_{x,y^*} at x. Moreover, in this case we have the representation

$$\mathrm{Ind}_{f,x,y^*} = \{h \in \mathbb{R}^d : q_{x,y^*}(h) \le \tfrac{1}{2}\}. \tag{3.2}$$

Proof. (1) Suppose the Dupin indicatrix Ind_{f,x,y^*} exists. By (3.1) this means that the level sets $(\Delta_t \le \frac{1}{2})$ are convergent in the Kuratowski sense (as $t \to 0^+$). As a consequence of formula (2.2), we deduce that the level sets $(\Delta_t \le \alpha)$ are convergent for any fixed positive α. We show that $\mathrm{epi}(\Delta_t)$ converges in the Kuratowski sense (as $t \to 0^+$). Clearly, we have to show that $\limsup \mathrm{epi}(\Delta_t) \subset \liminf \mathrm{epi}(\Delta_t)$. Let (x, μ) be an element of the left-hand side. Fix a sequence $t_n \to 0^+$. Using the definition (3.1) of the limit superior, we find $s_n \to 0^+$, $x_n \to x$ and

$\mu_u \to \mu$ such that $\Delta_{s_n}(x_n) \le \mu_n$. Fixing $\varepsilon > 0$, we find $\mu_n \le \mu + \varepsilon$ for $n \ge n(\varepsilon)$, hence $x_n \in (\Delta_{s_n} \le \mu + \varepsilon)$ for $n \ge n(\varepsilon)$. This implies $x \in \limsup(\Delta_s \le \mu + \varepsilon)$ for every $\varepsilon > 0$, and by the above observation, these are in fact limits. Hence we find sequences $x_n^\varepsilon \to x$ such that $\Delta_{t_n}(x_n^\varepsilon) \le \mu + \varepsilon$. Using a diagonal procedure, we may now extract a sequence (y_k) of the form $y_k = x_k^{1/n}$ for $k(n) \le k < k(n+1)$ such that $y_k \to x$ and $\Delta_{t_k}(y_k) \le \mu + 1/n$ for $k(n) \le k < k(n+1)$. This proves $(x, \mu) \in \liminf \operatorname{epi}(\Delta_t)$.

(2) Conversely, assume that $\operatorname{epi}(\Delta_t)$ has a limit in the sense of Kuratowski. We show that, for fixed $\alpha > 0$, the level sets $(\Delta_t \le \alpha)$ are also convergent in the sense of Kuratowski. Let $x \in \limsup(\Delta_t \le \alpha)$. Fix a sequence $t_n \to 0^+$. By the definition of the limit superior we find $s_n \to 0^+$ and $x_n \to x$ having $\Delta_{s_n}(x_n) \le \alpha$. Using formula (2.2), the latter implies

$$\Delta_{s_n/(1-\varepsilon)}((1-\varepsilon)x_n) \le (1-\varepsilon)^2\alpha \tag{3.3}$$

for every fixed $\varepsilon > 0$. In other terms, $((1-\varepsilon)x, (1-\varepsilon)^2\alpha)$ is an element of $\limsup \operatorname{epi}(\Delta_t)$. By assumption, the latter is in fact a limit, so we find sequences $x_n^\varepsilon \to (1-\varepsilon)x$ and $\alpha_n^\varepsilon \to (1-\varepsilon)^2\alpha$ such that $\Delta_{t_n}(x_n^\varepsilon) \le \alpha_n^\varepsilon$.

Owing to the fact that $\alpha > 0$, we have $\alpha_n^\varepsilon \le \alpha$ from an index $n(\varepsilon)$. Using a diagonal procedure, we may now define a sequence (y_k) of the form $y_k = x_k^{1/n}$ for $k(n) \le k < k(n+1)$ which converges to x and satisfies $\Delta_{t_k}(y_k) \le \alpha$. This proves $x \in \liminf(\Delta_t \le \alpha)$ and, hence, the claim.

(3) As epi convergence of the functions Δ_t is known to be equivalent to convergence of their epigraphs in the sense of Kuratowski used above, we have established the first half of the statement. It remains to prove the representation (3.2). Suppose $\Delta_t \xrightarrow{e} q$ for some quadratic convex and lower semicontinuous function q. Then $\operatorname{epi}(\Delta_t) \to \operatorname{epi}(q)$ in the Kuratowski sense. Now part (2) of the proof gives $(\Delta_t \le \alpha) \to (q \le \alpha)$ for every positive α, and this proves formula (3.2). ■

Remark 1. It is known that for a sequence (f_n) of lower semicontinuous functions, convergence $(f_n \le \alpha) \to (f \le \alpha)$ of *all* level sets implies epi convergence $f_n \xrightarrow{e} f$, while the converse is not true in general. In our special situation of convergence of second order difference quotients, the above proof shows that both notions are almost the same. The only level sets which did not play a role in the reasoning above are $(\Delta_t \le 0) = (\Delta_t = 0)$. (Observe that by convexity, we always have $\Delta_t \ge 0$, so level sets with negative heights do not occur.) However, epi convergence $\Delta_t \xrightarrow{e} q$ does *not* imply convergence of the zero level sets, i.e. we do not have $(\Delta_t = 0) \to (q = 0)$ in general. For an example take $f(x) = x^4$, then the second derivative $q = q_{0,0}$ at 0 is $q \equiv 0$, so $(q = 0) = \mathbb{R}$, while the sets $(\Delta_t = 0)$ equal $\{0\}$ for all $t \ne 0$ and, hence, do not converge to $(q = 0)$. The reason for this lies in the fact that the function indentically zero is a kind of singular element with regard to the notion of convergence of level set (see also [20]).

Remark 2. As a consequence of theorem 3.1, Ind_{f,x,y^*} exists if and only if $\mathrm{Ind}_{f^*,y^*,x}$ exists, and these sets are polar to each other. Indeed, this follows from the invariance of epi convergence under Young-Fenchel conjugation in tandem with the representation (3.2). Notice that for any lower semicontinuous quadratic convex function q one has the equality $(q \le \frac{1}{2})^\circ = (q^* \le \frac{1}{2})$.

Remark 3. Notice that Ind_{f,x,y^*} is symmetric with respect to the origin precisely when the second epi derivative q_{x,y^*} is two-sided, and that Ind_{f,x,y^*} is a conic if and only if q_{x,y^*} is purely quadratic. In the latter case, Ind_{f,x,y^*} may be unbounded, which corresponds with $\mathrm{Ind}_{f^*,y^*,x}$ being degenerate, or it may be degenerate, which corresponds with its polar being unbounded.

Remark 4. The latter observation is true without the requirement that $\mathrm{Ind} = \mathrm{Ind}_{f,x,y^*}$ be a conic. Indeed, unboundedness of Ind means $q(h) = 0$ for some $h \neq 0$, where $q = q_{x,y^*}$. However, q^* must be degenerate, namely, we have $q^*(k) = \infty$ for all directions k having $\langle h, k\rangle > 0$. In fact $\mathrm{dom}(q^*)$ is the cone polar to $\mathrm{Ker}(q)$.

4. CONSEQUENCES

In this section we obtain two consequences of theorem 3.1, which have been discussed in [12] under stronger conditions guaranteeing the existence of the Dupin indicatrix.

Let f be convex, $x \in \mathbb{R}^d$, $y^* \in \partial f(x)$. Then the function $\underline{D}^2 f(x, y^*; \cdot)$ defined as

$$\underline{D}^2 f(x, y^*; h) = \liminf_{t \to 0^+ h' \to h} \Delta_{f,x,y^*,t}(h') \tag{4.1}$$

is lower semicontinuous (cf. [12]). It is closely connected to second order epi differentiability, namely, f is second order epi differentiable at x with respect to y^* if and only if the limit inferior (4.1) is independent of the limit $t \to 0^+$, i.e. fixing a sequence $t_n \to 0^+$ does not affect the value of (4.1).

The authors of [12] use another kind of one-sided second order derivative, the function $\bar{f}''(x, y^*; \cdot)$ defined as

$$\bar{f}''(x, y^*; h) = \limsup_{t \to 0^+} \Delta_{f,x,y^*,t}(h), \tag{4.2}$$

which is convex and quadratic. It gives rise to the definition of the *second order subdifferential*

$$\partial^2 f(x, y^*) = \{z \in \mathbb{R}^d : \langle z, h\rangle \leq \sqrt{\bar{f}''(x, y^*; h)}\ \forall h \in \mathbb{R}^d\}, \tag{4.3}$$

which has been discussed, for example, in [8, 12]. Observe that $\underline{D}^2 f \leq \bar{f}''$ in general, and that since $\underline{D}^2 f$ is lower semicontinuous, $\underline{D}^2 f \leq cl\bar{f}''$, where $cl\phi$ means the lower-semicontinuous hull of ϕ. The interest in the function $cl\bar{f}''$ lies in the fact that by (4.3), $(cl\bar{f}'')^{1/2}$ is the support function of the second order subdifferential. It is natural to ask under what conditions $\underline{D}^2 f = \bar{f}''$, or at least $\underline{D}^2 f = cl\bar{f}''$. We have the following proposition.

PROPOSITION 4.1. The equality

$$\underline{D}^2 f(x, y^*, \cdot) = cl\bar{f}''(x, y^*, \cdot) \tag{4.4}$$

implies that f is second order epi differentiable at x relative to y^*. Moreover, $q_{x,y^*} = \underline{D}^2 f(x, y^*, \cdot)$ in this case. The equality

$$\underline{D}^2 f(x, y^*; \cdot) = \bar{f}''(x, y^*; \cdot) \tag{4.5}$$

is equivalent to the existence of q_{x,y^*} in tandem with the equality $q_{x,y^*} = q^{\#}_{x,y^*}$. In other words, the epi limit exists and is a pointwise limit.

Proof. By [12, theorem 3.7], the equality (4.4) implies the existence of the Dupin indicatrix, hence second order epi differentiability of f at x relative to y^* by theorem 3.1.

Concerning the second part of the statement, it is clear by the first part that equation (4.5), which is stronger than (4.4), implies $q_{x,y^*} = q^{\#}_{x,y^*}$. The converse is clear. ∎

Remark. It should be noted that the converse of the first part of the statement is not true, i.e. second order epi diffentiability does not give the equality (4.4). This follows from example 5.3.

The following result even shows that (4.4) reduces to (4.5) in most cases, which means that (4.4) is tentatively stronger than the mere existence of the second epi derivative, resp. the Dupin indicatrix.

PROPOSITION 4.2. Suppose (4.4) is satisfied, and let $\mathrm{dom}(q_{x,y^*})$ be a linear subspace. Then equality (4.5) is satisfied. In particular, this is the case when q_{x,y^*} is two-sided.

Proof. Let $H = \mathrm{dom}(q_{x,y^*})$ be a linear subspace. We aim to prove that

$$\Delta_{f,x,y^*,t} \,|\, H = \Delta_{f|H,x,y^*|H,t}$$

converges to $q_{x,y^*} \,|\, H$ in the sense of epi convergence in H. It then follows from known results (cf. [2, 13, 17]) that the limit is pointwise on H, and this proves the statement.

We have to check conditions (α) and (β) for the epi convergence in H. Now condition (β) is trivially satisfied, since it holds for the epi convergence in the larger space. Concerning (α), we fix h and $\mu = q_{x,y^*}(h) = \underline{D}^2 f(x, y^*; h)$. By condition (4.4) we have $(h, \mu) \in cl\, \mathrm{epi}\, \bar{f}''(x, y^*; \cdot)$, so there exist $h_n \to h$ and $\mu_n \to \mu$ such that $\mu_n \geq \bar{f}''(x, y^*; h_n)$. Using $\underline{D}^2 f \leq \bar{f}''$ and the lower semicontinuity of $\underline{D}^2 f$, we deduce that $\bar{f}''(x, y^*; h_n) \to \mu$, so we may assume $\mu_n = \bar{f}''(x, y^*; h_n)$. Now we have

$$q_{x,y^*}(h_n) \leq \bar{f}''(x, y^*; h_n) < \infty,$$

which means $h_n \in \mathrm{dom}(q_{x,y^*}) = H$. Fix $t_n \to 0^+$. By definition (4.2) we find indices $k(n)$ such that

$$\Delta_{f,x,y^*,t_k}(h_n) \leq \bar{f}''(x, y^*; h_n) + \frac{1}{n} \tag{4.6}$$

for all $k \geq k(n)$. Setting $h^{(k)} = h_n$ for $k(n) \leq k < k(n+1)$, therefore, provides a sequence converging to h in H and such that

$$\limsup_{k \to \infty} \Delta_{f,x,y^*,t_k}(h^{(k)}) \leq \lim_{n \to \infty} \bar{f}''(x, y^*; h_n) = q_{x,y^*}(h).$$

This establishes condition (α) for epi convergence on H. ■

Remark. If we drop the assumption that $\mathrm{dom}(q_{x,y^*})$ is a linear subspace, the above reasoning still guarantees equality (4.5) on the relative interior of $\mathrm{dom}(q_{x,y^*})$. However, example 5.1 shows that this need not be the case on the relative boundary.

We end this section with the following application of theorem 3.1 combined with the results in [12].

PROPOSITION 4.3. Let f be convex, $x \in \mathbb{R}^d$, $y^* \in \partial f(x)$. Then the limit

$$\lim_{\varepsilon \to 0^+} \frac{\partial_\varepsilon f(x) - y^*}{\sqrt{2\varepsilon}} = \mathrm{Ind}_{f^*,y^*,x} \tag{4.7}$$

exists precisely when f is second order epi differentiable at x relative to y^*.

Proof. It was proved in [12] that

$$\liminf_{\varepsilon \to 0^+} \frac{\partial_\varepsilon f(x) - y^*}{\sqrt{2\varepsilon}} = \underline{\mathrm{Ind}}_{f^*,y^*,x}$$
$$\limsup_{\varepsilon \to 0^+} \frac{\partial_\varepsilon f(x) - y^*}{\sqrt{2\varepsilon}} = \overline{\mathrm{Ind}}_{f^*,y^*,x}, \tag{4.8}$$

hence the existence of the limit (4.7) is equivalent to the existence of the dual Dupin indicatrix $\mathrm{Ind}_{f^*,y^*,x}$, which by remark 2 following theorem 3.1 is equivalent to the existence of Ind, hence to second order epi differentiability of f. ■

Remark. Notice that the equality (4.7) is in contrast with the limiting behaviour of the difference quotient

$$\lim_{\varepsilon \to 0^+} \frac{f_\varepsilon'(x; h) - f'(x; h)}{\sqrt{2\varepsilon}}. \tag{4.9}$$

Here $f_\varepsilon'(x; \cdot)$ denotes the ε-approximate directional derivative at x (cf. [9, 10]), in other terms, the support function of the ε-subdifferential $\partial_\varepsilon f(x)$. Namely, as has been proved by Hiriart-Urruty [9, 12], the limit (4.9) exists and equals $\sqrt{2q^\#_{x,y^*}(h)}$ precisely when $q^\#_{x,y^*}(h)$ exists, whereas (4.7) reflects the epi information rather than the pointwise information of the limiting behaviour of the difference quotient of the ε-approximate subdifferential. This is no longer a surprise, however, when we observe that (4.9) is a one-dimensional notion which is completely determined by the behaviour of f along the ray $x + \mathbb{R}h$.

5. EXAMPLES

The following example has already been used in [13] to discuss possible pathological behaviour of the second epi derivative.

Example 5.1. Define a convex $f_1 : \mathbb{R}^2 \to \mathbb{R}$ by setting

$$f_1(x, y) = \sup_{|t| \le 1} z_t(x, y), \tag{5.1}$$

where $z_t(x, y) = -\frac{1}{2}t^2 - \frac{1}{2}|t|^{2-\alpha} + \frac{3}{2}ty$, and where $1 < \alpha < 2$ is fixed. Then we have $f_1(0, 0) = 0$, $\nabla f(0, 0) = (0, 0)$, so $f_1 \ge 0$. The following properties of f_1 are relevant for its second order behaviour at $(0, 0)$:

(1) on the y-axis, $f_1(0, y) = \frac{9}{8}y^2$;

(2) on the negative x-axis, $f_1(x, 0) = C_\alpha |x|^{2/\alpha}$ for a positive constant C_α;

(3) there exists a critical parabola $y = \pm K_\alpha x^{-\alpha}$ in the half plane $x \ge 0$ such that $f_1(x, y) = 0$ for the points (x, y) inside the parabola. Specially, $K_{3/2} = 2^{-2/3}$.

It follows that $q^\#$ exists at $(0, 0)$ with domain the half space $\{(h_1, h_2) : h_1 \ge 0\}$, and by (1), resp. (3), we have $q^\#(0, \pm 1) = \frac{9}{8}$, while $q^\#(h_1, h_2) = 0$ for $h_1 > 0$. For the (h_1, h_2) with the $h_1 < 0$ it can be shown that $q^\#(h_1, h_2) = \infty$.

Concerning the second epi derivative q at $(0, 0)$, (3) clearly shows $q(h_1, h_2) = 0$ for all $h_1 > 0$. Now the important point is that also $q(0, \pm 1) = 0 < q^\#(0, \pm 1)$. This follows by considering the second order difference quotient of f_1 along the critical parabola in tandem with fact 2 (3) from Section 2. Finally, one shows that $q(h_1, h_2) = \infty$ for $h_1 < 0$, hence $\mathrm{dom}(q) = \mathrm{dom}(q^\#)$.

The Dupin indicatrix Ind at (0, 0) is the half plane $\{(h_1, h_2): h_1 \geq 0\}$, while the pointwise indicatrix $\mathrm{Ind}^{\#}$ in the sense of [19] is $\{(h_1, h_2): h_1 > 0\} \cup \{0\} \times [-\frac{9}{8}, \frac{9}{8}]$, which is not a closed set. Indeed, Ind is the closure of $\mathrm{Ind}^{\#}$. Notice also that $\underline{D}^2 f_1 = cl\bar{f}''$. It follows that condition (4.4) need not imply (4.5) on the relative boundary of the domain of q.

Example 5.2. Let $g(x, y) = f_1(x, y)$ for $x \geq 0$, $g(x, y) = f_1(-x, y)$ for $x \leq 0$, then g is the minimum of two convex functions and, hence, is locally Lipschitz. Notice that g has both a pointwise second derivative $q^{\#}$ and a second epi derivative q at (0, 0) such that $\mathrm{dom}(q) = \mathrm{dom}(q^{\#}) = \mathbb{R}^2$, but we have $q(0, 1) = 0 < q^{\#}(0, 1) = \frac{9}{8}$.

Example 5.3. The drawback of f_1 is that its second epi derivative is not a quadratic form. Therefore, we construct the following modification. Define $f_2: \mathbb{R}^2 \to \mathbb{R}$ by

$$f_2(x, y) = \sup_{|s| \leq 1} \tilde{z}_s(x, y), \tag{5.2}$$

where $\tilde{z}_s(x, y) = -\frac{1}{2}s^2 + \frac{1}{2}|s|^{2-\alpha} + sy$. Then $f_2 \geq 0$, $\nabla f_2(0, 0) = (0, 0)$, and $f_2(0, 0) = 0$. Again $f_2(0, y) = \frac{9}{8}y^2$ along the y-axis, and $f_2(x, 0) = C_\alpha x^{2/\alpha}$ along the positive x-axis, with C_α the same as above. There exists a critical parabola sited in the half plane $x \leq 0$ such that $f_2 = 0$ inside the parabola. Setting $f = \max(f_1, f_2)$ now provides a convex function whose second order epi derivative II $= q$ and second order pointwise derivative $II^{\#} = q^{\#}$ at (0, 0) both exist as quadratic forms with domain the y-axis, and such that $II(0, \pm 1) < II^{\#}(0, \pm 1) = \frac{9}{8}$. The exact value $II(0, \pm 1)$ can be found, for example, for $\alpha = 3/2$ by calculating the curve $\gamma: (f_1 = f_2)$ in the half plane $x \geq 0$. It turns out that γ is of the form $x = cy^{3/2}$ with $c = 0.559713$, and we find the numerical value $II(0, \pm 1) = 0.78904311$.

Observe that Ind is the y-axis, while the pointwise indicatrix $\mathrm{Ind}^{\#}$ is the segment $\{0\} \times [-\frac{9}{8}, \frac{9}{8}]$. Here we have $\underline{D}^2 f(0, 0; (0, 1)) < \bar{f}''(0, 0; (0, 1))$, so condition (4.4) need not be satisfied when f is second order epi differentiable.

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