Multiscale modeling and computation of flow through porous media

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Generalizations of multiscale finite element methods

- Homogenization of nonlinear pdes. Non-periodic homogenization.
- Generalizations of multiscale finite element methods to nonlinear partial differential equations.
- Convergence.
- Oversampling technique.
- Applications.

Nonlinear elliptic and parabolic equations

$$\frac{\partial}{\partial t}u_\epsilon - div(a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)) + a_{0,\epsilon}(x,t,u_\epsilon,\nabla u_\epsilon) = f.$$

Assumptions:

$$(a_\epsilon(\cdot,\cdot,\eta,\xi_1) - a_\epsilon(\cdot,\cdot,\eta,\xi_2), \xi_1-\xi_2) \geq C|\xi_1-\xi_2|^p$$

$$|a_\epsilon(\cdot,\cdot,\eta,\xi)| + |a_{0,\epsilon}(\cdot,\cdot,\eta,\xi)| \leq C(1+|\eta|+|\xi|)^{p-1}$$

$$|a_\epsilon(\cdot,\cdot,\eta,\xi_1) - a_\epsilon(\cdot,\cdot,\eta,\xi_2)| \leq C(1+|\eta|^{p-1-s}+|\xi_i|^{p-1-s})|\xi_1-\xi_2|^s$$

$$|a_\epsilon(\cdot,\cdot,\eta_1,\xi) - a_\epsilon(\cdot,\cdot,\eta_2,\xi)| \leq C(1+|\eta_i|^{p-1}+\xi|^{p-1})\nu(|\eta_1-\eta_2|)$$

$$(a_\epsilon(\cdot,\cdot,\eta,\xi),\xi) + a_{0,\epsilon}(\cdot,\cdot,\eta,\xi)\eta \geq C|\xi|^p - C_0$$

$p \geq 2$, $s \in (0, \min(p-1,1)]$

Random homogeneous case

Extensions of periodic case: Quasiperiodic; Almost periodic; Random Homogeneous.

(Ω, Σ, μ) - a probability space. Assume $a(x, \omega)$ is strictly stationary field. Then it can be represented as $a(x, \omega) = a(T(x)\omega)$, $x \in R^d$ where $a(\omega)$ is a fixed r.v., $T(x) : \Omega \to \Omega$ is a measure preserving transformation, s.t., $T(0) = I$, and $T(x_1 + x_2) = T(x_1)T(x_2)$; 3) $T(x) : \Omega \to \Omega$ preserve the measure μ on Ω;
Assume $T(x)$ is ergodic (i.e., any invariant function is constant almost everywhere).
Birkhoff Ergodic Theorem:

$$f_\omega(x/\varepsilon) \to M\{f_\omega\}$$

as $\varepsilon \to 0$ weakly in $L^p_{loc}(\mathbb{R}^d)$.

Periodic and almost periodic cases are special cases.

Random case

Auxiliary problem.
Periodic: $div(a(y)(\xi + \nabla N_\xi)) = 0$, $N \in H^1_{per}$, $a^*\xi = \langle a(y)(\xi + \nabla N_\xi)\rangle$.
Random: $a(\omega)(\xi + v(\omega)) \in L^2_{sol}$, $v(\omega) \in L^2_{pot}$.

$$a^*\xi = E[a(\omega)(\xi + v(\omega))]$$

Homogenized solution

$$-div(a^*\nabla u^*) = f.$$

First order corrector: $u_{1,\epsilon} = u^* + N_k^\epsilon(x)\frac{\partial u^*}{\partial x_k}$, where

$$\nabla N_k^\epsilon(x) = v_k(x/\epsilon).$$

Note that $\|N_k^\epsilon\|_{L^2(Q)} = o(1)$ as $\epsilon \longrightarrow 0$.
"Desirable" convergence rate for $\|u_\epsilon - u_{1,\epsilon}\|_{H^1(Q)}$ is open question (for periodic problems, the convergence rate is $\sqrt{\epsilon}$).

Nonlinear parabolic equations

$$A_\epsilon u_\epsilon = D_t u_\epsilon - div(a_\epsilon(x,t,u_\epsilon,D_x u_\epsilon)) + a_{0,\epsilon}(x,t,u_\epsilon,D_x u_\epsilon) = f.$$

$a(\cdot,\cdot,\eta,\xi)$ and $a_0(\cdot,\cdot,\eta,\xi)$ are Carathéodory functions on $Q \times \mathbb{R} \times \mathbb{R}^n$, with values in $\mathbb{R}^n$ and $\mathbb{R}$ respectively, satisfying:

$$(a_\epsilon(\cdot,\cdot,\eta,\xi_1) - a_\epsilon(\cdot,\cdot,\eta,\xi_2), \xi_1 - \xi_2) \geq C(1 + |\xi_1| + |\xi_2|)^{p-\beta}|\xi_1 - \xi_2|^\beta,$$

$$|a_\epsilon(\cdot,\cdot,\eta,\xi)| + |a_{0,\epsilon}(\cdot,\cdot,\eta,\xi)| \leq C(1 + |\eta| + |\xi|)^{p-1},$$

$$|a_\epsilon(\cdot,\cdot,\eta,\xi_1) - a_\epsilon(\cdot,\cdot,\eta,\xi_2)| \leq C(1 + |\eta,\xi_i|^{p-1-s})|\xi_1 - \xi_2|^s,$$

$$(a_\epsilon(\cdot,\eta,\xi),\xi) + a_{0,\epsilon}(\cdot,\eta,\xi)\eta \geq C|\xi|^p - C_0$$

$$|a_\epsilon(\cdot,\cdot,\eta_1,\xi) - a_\epsilon(\cdot,\cdot,\eta_2,\xi)| \leq C(1 + |\eta_i,\xi|^{p-1})\nu(|\eta_1 - \eta_2|)$$

It is known that up there exists a parabolic operator A^*, such that $A^\epsilon \stackrel{G}{\Longrightarrow} A^*$ (up to a subsequence). This means that $u_\epsilon \to u$ weakly in $L^p(W^{1,p})$, where $A^* u = f$.

G-convergence

Introduce

$$V = L^p(0,T,W_0^{1,p}(Q_0)), \quad \overline{V} = L^p(0,T,W^{1,p}(Q_0)),$$
$$W = \{u \in V, D_t u \in L^q(0,T,W^{-1,q}(Q_0))\},$$
$$\overline{W} = \{u \in \overline{V}, D_t u \in L^q(0,T,W^{-1,q}(Q_0))\}, \quad W_0 = \{u \in W, u(0) = 0\}$$

and $L^1(u,v) = D_t u - div\big(a(x,t,v,D_x u)\big)$.
Let $L_k^1(u_k,v) = L^1(u,v)$. The sequence $\mathcal{L}_k$ is called G-*convergent* to $\mathcal{L}$ if for every $v \in V$ and $u \in W_0$ we have that

$$\lim u_k = u$$

weakly in W_0 and

$$\begin{aligned} \lim \Gamma^k(u,v) &= \Gamma(u,v), \\ \lim \Gamma_0^k(u,v) &= \Gamma_0(u,v) \end{aligned}$$

weakly in $L^q(Q)^n$ and $L^q(Q)$, respectively, as $k \to \infty$. Here Γ's denote the nonlinear fluxes.

Homogenization in random media

(Ω, Σ, μ) - a probability space. Assume $a(\omega, \eta, \xi)$ is strictly stationary field for each $\eta \in R, \xi \in R^n$. Then it can be represented as $a(z, \omega, \eta, \xi) = a(T_z\omega, \eta, \xi)$, $z \in R^{d+1}$ where $a(\omega, \eta, \xi)$ is a fixed r.v., $T(z) : \Omega \to \Omega$ is a measure preserving transformation, s.t., $T(0) = I$, and $T(z_1 + z_2) = T(z_1)T(z_2)$; 3) $T(z) : \Omega \to \Omega$ preserve the measure μ on Ω;

$$D_t u_\epsilon = div\ a(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon) - a_0(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon) + f$$

in $Q = Q_0 \times [0, T]$.
$U(z)f(\omega) = f(T(z)\omega)$ defines a $(d+1)$-parameter group of isometries in the space of $L_p(\Omega)$. Denote by $\partial_{full} = (\partial_1, \cdots, \partial_{d+1})$ the collection of generators of the group $U(z)$.

Auxiliary problems

$$-div(a(x/\epsilon, u_\epsilon, D_x u_\epsilon)) = f,\ u_\epsilon \in W_0^{1,p}(Q_0)$$

Periodic case. The auxiliary problem: find $N_{\eta,\xi}(y)$ periodic for every η, ξ, such that

$$-div(a(y, \eta, \xi + D_y N_{\eta,\xi}(y))) = 0.$$

Then $u_\epsilon \to u$ weakly in $W^{1,p}$, where

$$-div(a^*(u, D_x u)) = f$$

and $a^*(\eta, \xi) = \langle a(y, \eta, \xi + D_x N_{\eta,\xi}(y)) \rangle$.
Random case. The auxiliary problem: find $w_{\eta,\xi} \in L^p_{pot}(\Omega)$, $\langle w_{\eta,\xi} \rangle = 0$, such that

$$a(\omega, \eta, \xi + w_{\eta,\xi}(\omega)) \in L^{p'}_{sol}(\Omega).$$

Then $a^*(\eta, \xi) = \langle a(\omega, \eta, \xi + w_{\eta,\xi}(\omega)) \rangle$.
Note that if we define N, such that $\partial N = w$, then N is no longer strictly stationary (periodic case is an exception).

Auxiliary problem for nonlinear parabolic equations

$$\mu D_\tau N^\mu_{\eta,\xi} - div_y\ a(y,\tau,\eta,\xi + D_y N^\mu_{\eta,\xi}) = 0.$$

$\mu = \epsilon^{2\beta-\alpha}$. Depending on the relation between α and β, the auxiliary problem is different.
(1) Self-similar case $\alpha = 2\beta$:

$$D_\tau N_{\eta,\xi} - div_y\ a(y,\tau,\eta,\xi + D_y N_{\eta,\xi}) = 0.$$

(2) $\alpha < 2\beta$:

$$-diva(\tau,y,\eta,\xi + D_y N_{\eta,\xi}) = 0.$$

(3) $\alpha > 2\beta$:

$$-div\overline{a}(y,\eta,\xi + D_y N_{\eta,\xi}) = 0,$$

where $\overline{a}(y,\eta,\xi) = \langle a(y,\tau,\eta,\xi)\rangle_\tau$.
(4) $\alpha = 0$ - spatial homogenization:

$$-div_y a(t,y,\eta,\xi + N_{\eta,\xi}) = 0.$$

(5) $\beta = 0$ - temporal homogenization:

$$\hat{a}(x,\eta,\xi) = \langle a(\tau,x,\eta,\xi)\rangle_\tau.$$

Auxiliary problem for nonlinear parabolic equations

$$\mu\sigma N^{\mu}_{\eta,\xi} - \mathbf{div}\, a(\omega,\eta,\xi+\partial N^{\mu}_{\eta,\xi}) = 0.$$

$S \subset L_p(\Omega)$ that is contained in the domains of all operators $\partial^{\alpha}_{full} = \partial_1^{\alpha_1}\cdots\partial_{n+1}^{\alpha_{n+1}}$, $\alpha \in Z_+^{n+1}$.
$\mathcal{V} = \mathcal{V}^p$ (analog of $L^p(W^{1,p})$) the completion of S with respect to the semi-norm $\|f\|_{\mathcal{V}} = \left(\sum_{i=1}^{n} \|\partial_i f\|^p_{L_p(\Omega)}\right)^{1/p}$.
$\mathbf{div} : L^q(\Omega)^n \to \mathcal{V}'$ is dual of ∂.
σ can be defined as a closed linear operator from $\mathcal{V}$ into $\mathcal{V}'$, with domain $\mathcal{W} = D(\sigma)$.
We need near solutions that approximate N by random fields with smooth realizations.
Near solutions are defined by $\mu\sigma N^{\mu}_{\delta} + AN^{\mu}_{\delta} = \partial\rho_{\delta}$. It can be shown that $\lim_{\delta\to 0}\langle|\rho_\delta|^p\rangle = 0$.

Homogenization result

The homogenized operator is defined for a.e. realization by

$$L^* u = D_t u - div(a^*(\omega, x, t, u, D_x u)) + a_0^*(\omega, x, t, u, D_x u).$$

a^* and a_0^* are defined as follows (Efendiev and Pankov 2005).
For self-similar case ($\alpha = 2\beta$),

$$a^*(\eta, \xi) = \langle a(\omega, \eta, \xi + \partial w_{\eta,\xi}) \rangle, {a_0}^*(\eta, \xi) = \langle a_0(\omega, \eta, \xi + \partial w_{\eta,\xi}) \rangle,$$

where $w_{\eta,\xi} = w^{\mu=1} \in \mathcal{W}$ is the unique solution of

$$\sigma w^{\mu=1} - div\ a(\omega, \eta, \xi + \partial w^{\mu=1}) = 0.$$

For non self-similar case ($\alpha < 2\beta$),

$$a^*(\eta, \xi) = \langle a(\omega, \eta, \xi + \partial w_{\eta,\xi}) \rangle, {a_0}^*(\eta, \xi) = \langle a_0(\omega, \eta, \xi + \partial w_{\eta,\xi}) \rangle,$$

where $w_{\eta,\xi} = w^0 \in \mathcal{V}$ is the unique solution of

$$-div\ a(\omega, \eta, \xi + \partial w^0) = 0.$$

Homogenization result

Denote $M_t\{f\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(T(0,\tau)\omega)d\tau$,
$M_x\{f\} = \lim_{|K|\to\infty} \frac{1}{|K|}\int_K f(T(y,0)\omega)dy$

$$\overline{a}(\omega,\eta,\xi) = M_t(a(\omega,\eta,\xi)).$$

$\mathcal{V}_s$ is obtained by completing the elements of S

$$f(\omega) = M_t\{f(T_1(t)\omega\}.$$

with respect to the norm $\|f\| = (\sum_{i=1}^n \|\partial_i f\|_{L_p(\Omega)}^p)^{1/p}$.
For spatial case ($\alpha = 0$),

$$a(\omega,\eta,\xi) = M_x\{a(T_2(x)\omega,\eta,\xi + \partial w_{\eta,\xi}(T_2(x)\omega))\},$$
$$a_0(\omega,\eta,\xi) = M_x\{a_0(T_2(x)\omega,\eta,\xi + \partial w_{\eta,\xi}(T_2(x)\omega))\},$$

where $w_{\eta,\xi} = w_x \in \mathcal{V}$

$$-div\ a(\omega,\eta,\xi + \partial w_x) = 0.$$

Homogenization result

For temporal case ($\beta = 0$), the homogenized fluxes are defined by

$$a^*(\omega, \eta, \xi) = P_1 a(\omega, \eta, \xi), a_0^*(\omega, \eta, \xi) = P_1 a_0(\omega, \eta, \xi).$$

Individual homogenization

The homogenization result is for a.e. $\omega \in \Omega$. We consider almost periodic coefficients $a(x,t,\eta,\xi)$ (in the sense of Besicovitch).
Individual homogenization takes place for the operator

$$\mathcal{L}_\varepsilon^m u = D_t u - div\, a^m(\frac{t}{\epsilon^\alpha}, \frac{x}{\epsilon^\beta}, u, D_x u) + a_0^m(\frac{t}{\epsilon^\alpha}, \frac{x}{\epsilon^\beta}, u, D_x u),$$

we have $\mathcal{L}_\varepsilon^m \overset{G}{\Longrightarrow} \hat{\mathcal{L}}^m$. Comparison theorem

$$\sup_{|\eta|,|\xi|\le R} |\hat{a}^m(\eta,\xi) - a^G(t,x,\eta,\xi)| \le \overline{g}(t,x,r) + c(R)\left[\varphi(r)^{1/q} + \varphi^\gamma(r) + (1+r)^\gamma \overline{g}(t,x,r)^\gamma\right],$$

where

$$\overline{g}(t,x,r) = \limsup_{\rho\to 0} \limsup_{\varepsilon\to 0} \frac{1}{|K_\rho(t,x)|} \int_{K_\rho(t,x)} \sup_{|\eta|,|\xi|\le r} |a(t/\epsilon^\alpha, x/\epsilon^\beta, \eta, \xi) - a^G(t/\epsilon^\alpha, x/\epsilon^\beta, \eta, \xi)$$

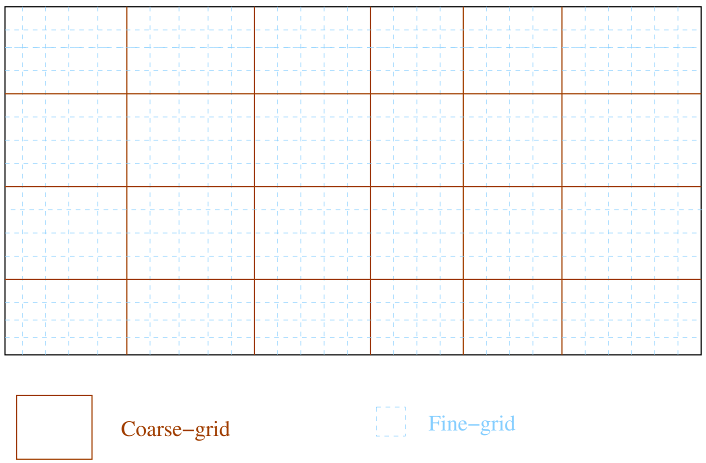
$\gamma = \frac{s}{q^2(\beta-1)}$, $\varphi(r) = r^{-p} + r^{-\alpha p/(p+\alpha)}, \quad r > 0.$
Pass to the limit as $m \to \infty$, then $r \to \infty$ gives $a^G(t,x,\eta,\xi) = \hat{a}(\eta,\xi)$.

Multiscale finite element methods for nonlinear problems

Consider $u_\epsilon \in W_0^{1,p}(Q)$, $-div(a_\epsilon(x, u_\epsilon, \nabla u_\epsilon)) + a_{0,\epsilon}(x, u_\epsilon, \nabla u_\epsilon) = f$.
Let S^h be "usual" finite dimensional space defined on a coarse-grid ($1 \gg h \gg \epsilon$).
Multiscale map: Define $E : S^h \to V_\epsilon^h$ such that for any $u_h \in S^h$, $u_{\epsilon,h} = E u_h$ is defined by

$$-div(a_\epsilon(x, \eta^{u_h}, \nabla u_{\epsilon,h})) = 0 \text{ in } K,$$

$\eta^{u_h} = 1/|K| \int_K u_h dx$ and $u_{\epsilon,h} - u_h \in W_0^{1,p}(K)$ in each K (coarse grid or RVE).
For the linear case, V_ϵ^h is a linear space whose basis can be obtained by mapping the basis of S^h. This is precisely MsFEM for linear problems.



Multiscale finite element methods for nonlinear problems

Multiscale Formulation

MsFEM

Find $u_h \in S^h$ ($u_{\epsilon,h} = Eu_h \in V_\epsilon^h$) such that

$$A(u_h, v_h) = \int_Q f v_h dx, \ \forall v_h \in S^h,$$

where

$$A(u_h, v_h) = \sum_K \int_K ((a_\epsilon(x, \eta^{u_h}, \nabla u_{\epsilon,h}), \nabla v_h) + a_{0,\epsilon}(x, \eta^{u_h}, \nabla u_{\epsilon,h}) v_h) dx.$$

Multiscale finite element methods for nonlinear problems

Multiscale Formulation

MsFEM

Find $u_h \in S^h$ ($u_{\epsilon,h} = Eu_h \in V_\epsilon^h$) such that

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where

$$A(u_h, v_h) = \sum_K \int_K ((a_\epsilon(x, \eta^{u_h}, \nabla u_{\epsilon,h}), \nabla v_h) + a_{0,\epsilon}(x, \eta^{u_h}, \nabla u_{\epsilon,h}) v_h) dx.$$

MsFVEM

$$-\int_{\partial V_z} a_\epsilon \left(x, \eta^{u_h}, \nabla u_{\epsilon,h}\right) \cdot n \, dS + \int_{V_z} a_{0,\epsilon} \left(x, \eta^{u_h}, \nabla u_{\epsilon,h}\right) \, dx = \int_{V_z} f,$$

where V_z is control volume.

Convergence Theorems

(1) General heterogeneities (up to a subsequence) (Efendiev and Pankov, 2004)

$$\lim_{h \to 0} \lim_{\epsilon \to 0} \|u_h - u\|_{W^{1,p}(Q)} = 0$$

(2) Periodic heterogeneities (up to a subsequence) (Efendiev, Hou and Ginting, 2004)

$$\lim_{\epsilon/h \to 0} \|u_h - u\|_{W^{1,p}(Q)} = 0$$

Explicit convergence rates for strongly monotone operators are obtained.

Proof. Periodic Case

Consider, $u_\epsilon \in W_0^{1,p}(Q)$, $-div(a(\frac{x}{\epsilon}, u_\epsilon, \nabla u_\epsilon)) = f$.

Homogenization. For each $\eta \in R$, $\xi \in R^d$, $N_{\eta,\xi} \in W_{per}^{1,p}(Y)$

$$-div(a(y, \eta, \xi + \nabla_y N_{\eta,\xi}(y))) = 0.$$

The homogenized fluxes are computed by $a^*(\eta, \xi) = \langle a(y, \eta, \xi + \nabla_x N_{\eta,\xi}(y)) \rangle$, and the homogenized equation is given by $-div(a^*(u, \nabla_x u)) = f$.

Proof. Periodic Case

Theorem.

$$\lim_{\epsilon/h \to 0} \|u_h - u\|_{W^{1,p}(Q)} = 0,$$

where $h = h(\epsilon) \gg \epsilon$, and $h(\epsilon) \to 0$, as $\epsilon \to 0$.

Lemma. Coercivity: $\|u_h\|_{W^{1,p}(Q)} \leq C$.

Proof. Periodic Case

Theorem.

$$\lim_{\epsilon/h \to 0} \|u_h - u\|_{W^{1,p}(Q)} = 0,$$

where $h = h(\epsilon) \gg \epsilon$, and $h(\epsilon) \to 0$, as $\epsilon \to 0$.

Lemma. Coercivity: $\|u_h\|_{W^{1,p}(Q)} \le C$.

$$\langle A_{\epsilon,h} u_h, v_h \rangle = \sum_K \int_K (a(\frac{x}{\epsilon}, \eta^{u_h}, \nabla u_{\epsilon,h}), \nabla v_h) dx = \langle f, v_h \rangle$$

$$\langle A^* u_h, v_h \rangle = \sum_K \int_K (a^*(u_h, \nabla u_h), \nabla v_h) dx$$

$$\begin{aligned}\langle A^* u_h - A^* P_h u, u_h - P_h u \rangle &= \langle A^* u_h - A_{\epsilon,h} u_h, u_h - P_h u \rangle + \langle A_{\epsilon,h} u_h - A^* P_h u, u_h - P_h u \rangle \\ &= \langle A^* u_h - A_{\epsilon,h} u_h, u_h - P_h u \rangle,\end{aligned}$$

where $P_h u$ is a Galerkin solution, $\langle A^* P_h u, v_h \rangle = \langle f, v_h \rangle, \forall v_h \in S^h$.

Proof. Periodic Case

Introduce $\mathcal{P} = \nabla_x u_h + \nabla_y N_{\eta^{u_h},\nabla u_h}(y)$ in each K, where $-div_y a(y, \eta^{u_h}, \mathcal{P}) = 0$.

$$\langle A_{\epsilon,h} u_h - A^* u_h, v_h \rangle = \sum_K \int_K (a(\frac{x}{\epsilon}, \eta^{u_h}, \nabla u_{\epsilon,h}) - a(\frac{x}{\epsilon}, \eta^{u_h}, \mathcal{P}), \nabla v_h) dx +$$

$$\sum_K \int_K (a(\frac{x}{\epsilon}, \eta^{u_h}, \mathcal{P}) - a^*(\eta^{u_h}, \nabla u_h), \nabla v_h) dx +$$

$$\sum_K \int_K (a^*(\eta^{u_h}, \nabla u_h) - a^*(u_h, \nabla u_h), \nabla v_h) dx = I + II + III$$

Lemma. $\|\nabla u_{\epsilon,h} - \mathcal{P}\|_{p,Q} \leq C \left(\frac{\epsilon}{h}\right)^{\frac{1}{p(p-s)}} \left(|Q| + \|u_h\|_{p,Q}^p + \|\nabla u_h\|_{p,Q}^p\right)^{\frac{1}{p}}$

Lemma. $III \to 0$ as $h \to 0$ if $\|u_h\|_{W^{1,p+\alpha}(Q)} \leq C$, for some $\alpha > 0$ (Meyers type estimate, Efendiev and Pankov, Num.Math., 2004).

Proof. Periodic case

$$u_{\epsilon,h} = u_\epsilon^h(x) + \epsilon\, N_{\eta^{u_h}, D_x u_\epsilon^h u}(x/\epsilon) + \theta(x, x/\epsilon).$$

It can be shown that

$$\|D_x\theta\|_{p,K}^p \le C\, H_0\, \|D_x(u_h - u_h - \epsilon\, \tilde{N}_{\eta^{u_h}, D_x u_h})\|_{p,K}^{\frac{p}{p-s}} \le C\, H_0\, \|\epsilon\, D_x \tilde{N}_{\eta^{u_h}, D_x u_h}\|_{p,K}^{\frac{p}{p-s}},$$

where

$$H_0 = \left(|K| + \|\eta^{u_h}\|_{p,K}^p + \|D_x u_h\|_{p,K}^p + \|D_x(u_h + \epsilon\, \tilde{N}_{\eta^{u_h}, D_x u_h})\|_{p,K}^p\right)^{\frac{p-s-1}{p-s}}.$$

Then

$$\|\epsilon\, D_x \tilde{N}_{\eta^{u_h}, D_x u_h}\|_{p,K}^p \le \epsilon^p \sum_{i \in J_\epsilon^K} \int_{Y_\epsilon^i} \left(|D_x\, N_{\eta^{u_h}, D_x u_h}|^p\, |\varphi|^p + |N_{\eta^{u_h}, D_x u_h}|^p\, |D_x \varphi|^p\right)\, dx,$$

Proof. Periodic case

Note

$$\|D_y\, N_{\eta^{u_h},D_x u_h}\|^p_{p,Y^i_\epsilon} \leq C(1+|\eta^{u_h}|^p + |D_x u_h|^p)\,|Y^i_\epsilon|.$$

Because φ is sufficiently smooth and $|D_x\varphi| \leq C/\epsilon$, we have

$$\begin{aligned}\|\epsilon\, D_x \tilde{N}_{\eta^{u_h},D_x u_h}\|^p_{p,K} &\leq C\,\epsilon^p\,(1+|\eta^{u_h}|^p + |D_x u_h|^p)\,\sum_{i\in J^K_\epsilon}(1+\epsilon^{-p})\,|Y^i_\epsilon| \\ &\leq C\,(1+|\eta^{u_h}|^p + |D_x u_h|^p)\,\sum_{i\in J^K_\epsilon}|Y^i_\epsilon|.\end{aligned}$$

Because all Y^i_ϵ, $i\in J^K_\epsilon$, cover the strip S_ϵ, we know that $\sum_{i\in J^K_\epsilon}|Y^i_\epsilon| \leq C\,h^{d-1}\,\epsilon$, we have

$$\begin{aligned}\|\epsilon\, D_x \tilde{N}_{\eta^{u_h},D_x u_h}\|^p_{p,K} &\leq C\,\frac{h^d}{h^d}\,(1+|\eta^{u_h}|^p + |D_x u_h|^p)\,h^{d-1}\,\epsilon \\ &\leq C\,\frac{\epsilon}{h}\,\left(|K| + \|\eta^{u_h}\|^p_{p,K} + \|D_x u_h\|^p_{p,K}\right).\end{aligned}$$

Proof. Periodic case

$$\|D_x \theta\|_{p,K}^p \leq C \left(\frac{\epsilon}{h}\right)^{\frac{1}{p-s}} \left(|K| + \|\eta^{u_h}\|_{p,K}^p + \|u_h\|_{p,K}^p + \|D_x u_h\|_{p,K}^p\right).$$

Summing over all K,

$$\|D_x \theta\|_{p,\Omega}^p \leq C \left(\frac{\epsilon}{h}\right)^{\frac{1}{p-s}} \left(|\Omega| + \|u_h\|_{p,\Omega}^p + \|D_x u_h\|_{p,\Omega}^p\right).$$

Proof. Periodic Case

$$\langle A^* u_h - A^* P_h u, u_h - P_h u \rangle \leq c \left(\left(\frac{\epsilon}{h} \right)^{\frac{s}{p(p-s)}} + \frac{\epsilon}{h} \right) \left(|Q| + \|u_h\|_{p,Q}^p + \|\nabla u_h\|_{p,Q}^p \right)^{\frac{1}{q}} \times$$

$$\|\nabla(u_h - P_h u)\|_{p,Q} + e(h) \|\nabla(u_h - P_h u)\|_{p,Q}.$$

Proof. Periodic Case

$$\langle A^* u_h - A^* P_h u, u_h - P_h u\rangle \le c\left(\left(\frac{\epsilon}{h}\right)^{\frac{s}{p(p-s)}} + \frac{\epsilon}{h}\right)\left(|Q| + \|u_h\|^p_{p,Q} + \|\nabla u_h\|^p_{p,Q}\right)^{\frac{1}{q}} \times$$

$$\|\nabla(u_h - P_h u)\|_{p,Q} + e(h)\|\nabla(u_h - P_h u)\|_{p,Q}.$$

If A^* is a monotone operator, explicit convergence rate can be obtained.

$$\|D_x(u_h - u)\|^p_{p,\Omega} \le c\left(\frac{\epsilon}{h}\right)^{\frac{s}{(p-1)(p-s)}} + c\left(\frac{\epsilon}{h}\right)^{\frac{p}{p-1}} + e(h).$$

Proof. Periodic Case

$$\langle A^* u_h - A^* P_h u, u_h - P_h u \rangle \leq c \left(\left(\frac{\epsilon}{h} \right)^{\frac{s}{p(p-s)}} + \frac{\epsilon}{h} \right) \left(|Q| + \|u_h\|_{p,Q}^p + \|\nabla u_h\|_{p,Q}^p \right)^{\frac{1}{q}} \times$$
$$\|\nabla (u_h - P_h u)\|_{p,Q} + e(h) \|\nabla (u_h - P_h u)\|_{p,Q}.$$

If A^* is a monotone operator, explicit convergence rate can be obtained.

$$\|D_x(u_h - u)\|_{p,\Omega}^p \leq c \left(\frac{\epsilon}{h} \right)^{\frac{s}{(p-1)(p-s)}} + c \left(\frac{\epsilon}{h} \right)^{\frac{p}{p-1}} + e(h).$$

Approximation of the gradients

Theorem. If u_h is a MsFEM solution, then $u_{\epsilon,h} = E u_h$ converges to u_ϵ in $W^{1,p}(Q)$ as $\epsilon/h \to 0$.

Multiscale finite element methods of parabolic eqns

For any $u_h \in S^h$ define $u_{\epsilon,h}(x,t) = Eu_h$ such that $E : S^h \to V_\epsilon^h$ and

$$\frac{\partial}{\partial t} u_{\epsilon,h} = div(a_\epsilon(x,t,\eta^{u_h}, \nabla u_{\epsilon,h}))\ in\ K \times [t_n, t_{n+1}],$$

$u_{\epsilon,h}(x, t = t_n) = u_h(x)$, $u_{\epsilon,h} = u_h$ on ∂K.
Find $u_h \in S^h$ such that

$$\int_Q (u_h(t = t_{n+1}) - u_h(t = t_n)) v_h dx + A_{\epsilon,h}(u_h, v_h) = \int_{t_n}^{t_{n+1}} f v_h dx dt$$

where

$$A_{\epsilon,h}(u_h, v_h) = \int_{t_n}^{t_{n+1}} [(a_\epsilon(x,t,\eta^{u_h}, \nabla u_{\epsilon,h}), \nabla v_h) dx dt +$$

$$a_{0,\epsilon}(x,t,\eta^{u_h}, \nabla u_{\epsilon,h}) v_h] dx dt$$

Explicit if $u_{\epsilon,h} = Eu_h(t = t_n)$
Implicit if $u_{\epsilon,h} = Eu_h(t = t_{n+1})$

Convergence result

Theorem. (General heterogeneities)

$$\lim_{h\to 0}\lim_{\epsilon\to 0}\|u_h - u\|_{L^p(0,T,W_0^{1,p}(Q))} = 0$$

(up to a subsequence).

Numerical Correctors

Define M_h in the following way

$$M_h \phi(x,t) = \sum_i 1_{Q^i} \frac{1}{|Q^i|} \int_{Q^i} \phi(y,\tau) dy d\tau$$

Denote

$$P(T(y,\tau)\omega, \eta, \xi) = \xi + w_{\eta,\xi}(T(y,\tau)\omega),$$

where $w_{\eta,\xi} = \partial N$ and N is the solution of auxiliary problem.

Theorem.

$$\lim_{h\to 0} \lim_{\epsilon \to 0} \int_\Omega \int_Q |P(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, M_h u, M_h D_x u) - D_x u_\epsilon|^p \, dx dt d\mu(\omega) \to 0,$$

From this Theorem, it follows that
$\lim_{h\to 0} \lim_{\epsilon\to 0} \int_\Omega \int_Q |D_x u_{\epsilon,h} - D_x u_\epsilon|^p dx dt d\mu(\omega) \to 0.$

Numerical Correctors

$$\int_\Omega \int_Q |P - D_x u_\epsilon|^p dxdtd\mu(\omega) \leq C \int_\Omega \int_Q (\mathbf{a}(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, P)$$

$$- \mathbf{a}(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon), P - D_x u_\epsilon) dxdtd\mu(\omega) \leq$$

$$C| \int_\Omega \int_Q (\mathbf{a}(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, M_h u, P) -$$

$$\mathbf{a}(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon), P - D_x u_\epsilon) dxdtd\mu(\omega)| +$$

$$C| \int_\Omega \int_Q (\mathbf{a}(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, P) -$$

$$\mathbf{a}(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, M_h u, P), P - D_x u_\epsilon) dxdtd\mu(\omega)| =: I_1 + I_2,$$

Numerical Correctors

$$I_1 = \int_\Omega \int_Q (\mathbf{a}(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, M_h u, P) -$$

$$\mathbf{a}(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon), P - D_x u_\epsilon) dx dt d\mu(\omega) =$$

$$C \int_\Omega \int_Q (\mathbf{a}(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, M_h u, P), P) dx dt d\mu(\omega) -$$

$$C \int_\Omega \int_Q (\mathbf{a}(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, M_h u, P), D_x u_\epsilon) dx dt d\mu(\omega) -$$

$$C \int_\Omega \int_Q (\mathbf{a}(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon), P) dx dt d\mu(\omega) +$$

$$C \int_\Omega \int_Q (\mathbf{a}(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon, D_x u_\epsilon), D_x u_\epsilon) dx dt d\mu(\omega).$$

Numerical Correctors

$$\lim_{\epsilon \to 0} I_1 \le C\Big(\int_Q (\mathbf{a}^*(M_h u, M_h D_x u), M_h D_x u) dx dt -$$

$$\int_Q (\mathbf{a}^*(M_h u, M_h D_x u), D_x u) dx dt -$$

$$\int_Q (\mathbf{a}^*(u, D_x u), M_h D_x u) dx dt$$

$$+ \int_Q (\mathbf{a}^*(u, D_x u), D_x u) dx dt\Big) =$$

$$\int_Q (\mathbf{a}^*(u, D_x u) - \mathbf{a}^*(M_h u, M_h D_x u), D_x u - M_h D_x u) dx dt$$

Remarks

- In the periodic case the problem in a period can be solved to approximate the solution of the local problem by periodicity. Solve

$$-div(a_\epsilon(x, \eta^{u_h}, \nabla u_{\epsilon,h})) = 0 \quad \text{in a period}$$

$\eta^{u_h} = 1/|K| \int_K u_h dx$ and $u_{\epsilon,h} - u_h \in W^{1,p}_{per}$.

- Oversampling techniques both in space and time. The local problems are solved in S ($K \subset S$, K - target coarse block) to avoid “pollution” from artificial boundary conditions.
- If $a_\epsilon(x, t, \eta, \xi) = k_\epsilon(x) k_r(\eta) \xi$ then the local problems are solved only once.
- In general one can avoid solving the local parabolic problems in $K \times [t_n, t_{n+1}]$. Assume $a_\epsilon(x, t, \eta, \xi) = a_\epsilon(x/\epsilon^\beta, t/\epsilon^\alpha, \eta, \xi)$. 1) if $a_\epsilon(x, t, \eta, \xi) = a_\epsilon(x, \eta, \xi)$, then the following local problems can be considered: for each $v_h \in S^h$, $div(a_\epsilon(x, \eta^{v_h}, \nabla v_{\epsilon,h})) = 0$ in K. 2) if $\alpha < 2\beta$, $-div(a(x/\epsilon^\beta, t/\epsilon^\alpha, \eta^{v_h}, \nabla v_\epsilon)) = 0$ in K.

Oversampling technique

In general, given $v^h \in S^h$, where v^h is defined in K, we want to find $v_{\epsilon,h}$ that satisfies

$$div(a_\epsilon(x, \eta^{v_h}, \nabla v_{\epsilon,h})) = 0 \quad \text{in} \quad S$$

such that $v_{\epsilon,h}(z_i) = v^h(z_i)$, where z_i are the nodal points of the target coarse element K.
Special cases: $a_\epsilon(x, \eta, \xi) = a_\epsilon(x, \eta)\xi$. Given $v^h \in S^h$, we define

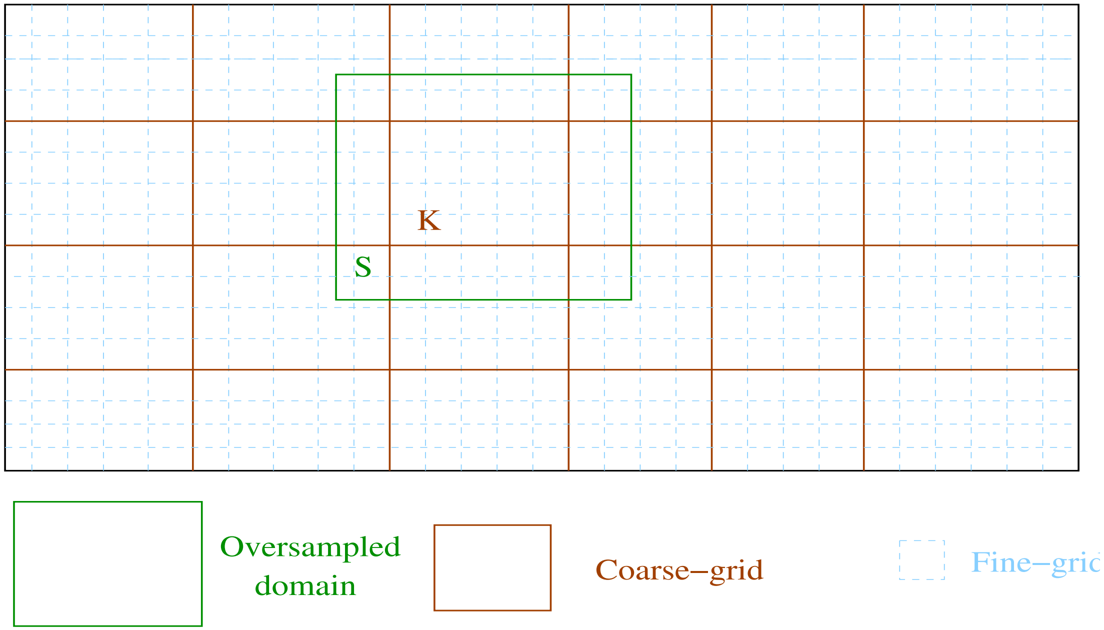
$$v_{\epsilon,h} = \sum_{i=1}^{3} c_i \, \phi_\epsilon^i,$$

where ϕ_ϵ^i satisfies

$$div(a(x/\epsilon, \eta^{v_h}) \, \nabla \phi_\epsilon^i) = 0 \quad \text{in} \quad S, \; \phi_\epsilon^i = \phi^i \quad \text{on} \quad \partial S.$$

The constants c_i, $i = 1, 2, 3$ are determined by imposing the conditions
$v_{\epsilon,h}(z_j) = v^h(z_j) \qquad j = 1, 2, 3.$

Oversampling. Illustration



Numerical Examples

- Enhanced diffusion due to nonlinear heterogeneous convection

$$\frac{\partial}{\partial t}u_\epsilon - \frac{1}{\epsilon}v_\epsilon(x,t)) \cdot \nabla F(u_\epsilon) - d\Delta_{xx}u_\epsilon = f,$$

  where $div(v) = 0$.
- $-div(a_\epsilon(x,u_\epsilon)\nabla u_\epsilon) = f$.
- Richards' equation

Enhanced diffusion

$$D_t u_\epsilon - \frac{1}{\epsilon}\mathbf{v}(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega) \cdot D_x F(u_\epsilon) - d\Delta_{xx} u_\epsilon = f,$$

where $div\mathbf{v} = 0$. Assuming that there exists homogeneous stream function $\mathbf{H}(T(x/\epsilon^\beta, t/\epsilon^\alpha)\omega)$, $div\mathbf{H} = \mathbf{v}$.

$$D_t u_\epsilon - div(\mathbf{a}((x/\epsilon^\beta, t/\epsilon^\alpha)\omega, u_\epsilon) D_x u_\epsilon) = f,$$

where

$$\mathbf{a} = \begin{pmatrix} d & H((x/\epsilon^\beta, t/\epsilon^\alpha)\omega)F'(u) \\ -H((x/\epsilon^\beta, t/\epsilon^\alpha)\omega)F'(u) & d \end{pmatrix}.$$

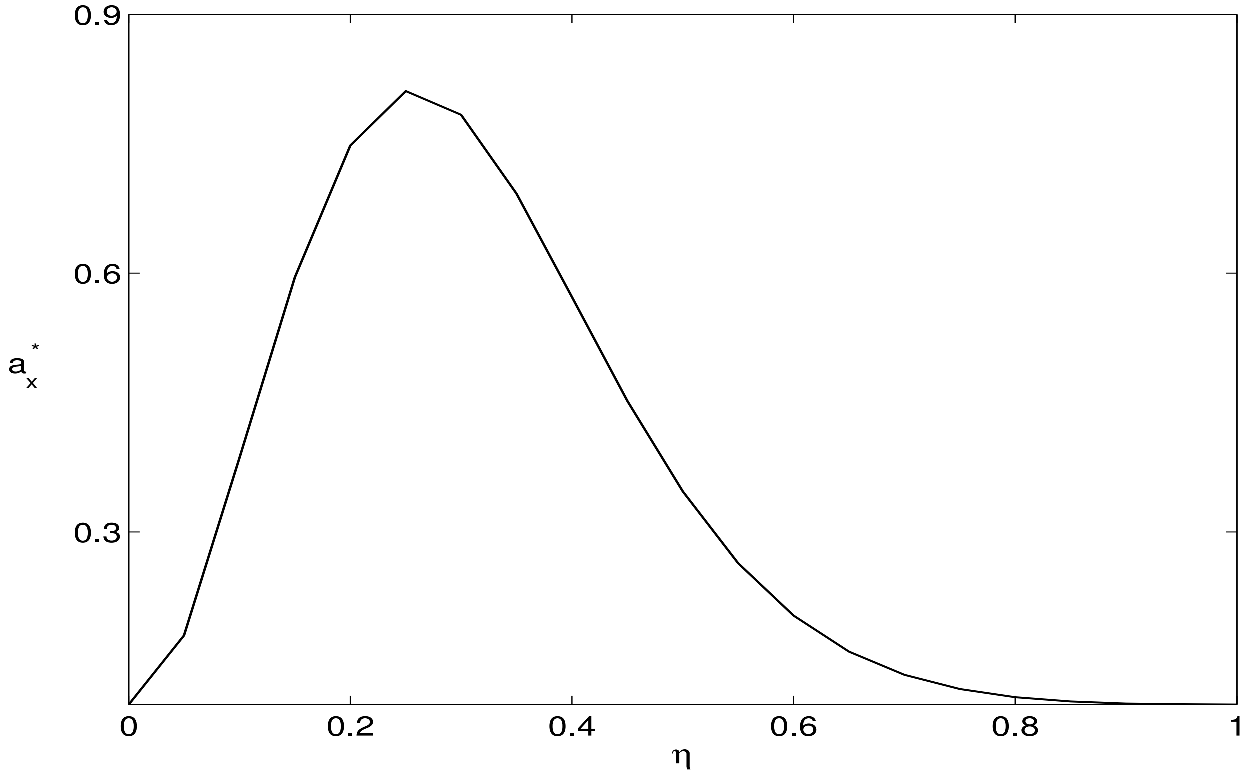
Enhanced diffusion

$$D_t u = div(\mathbf{a}^*(u) D_x u),$$

$a^*_{ij}(\eta) = d\delta_{ij} + \langle H_{ik} F'(\eta) W^{kj}_\eta \rangle$, where $w^i_\eta = W^{ij}_\eta \xi_i$, $w_\eta = \partial N_\eta$.

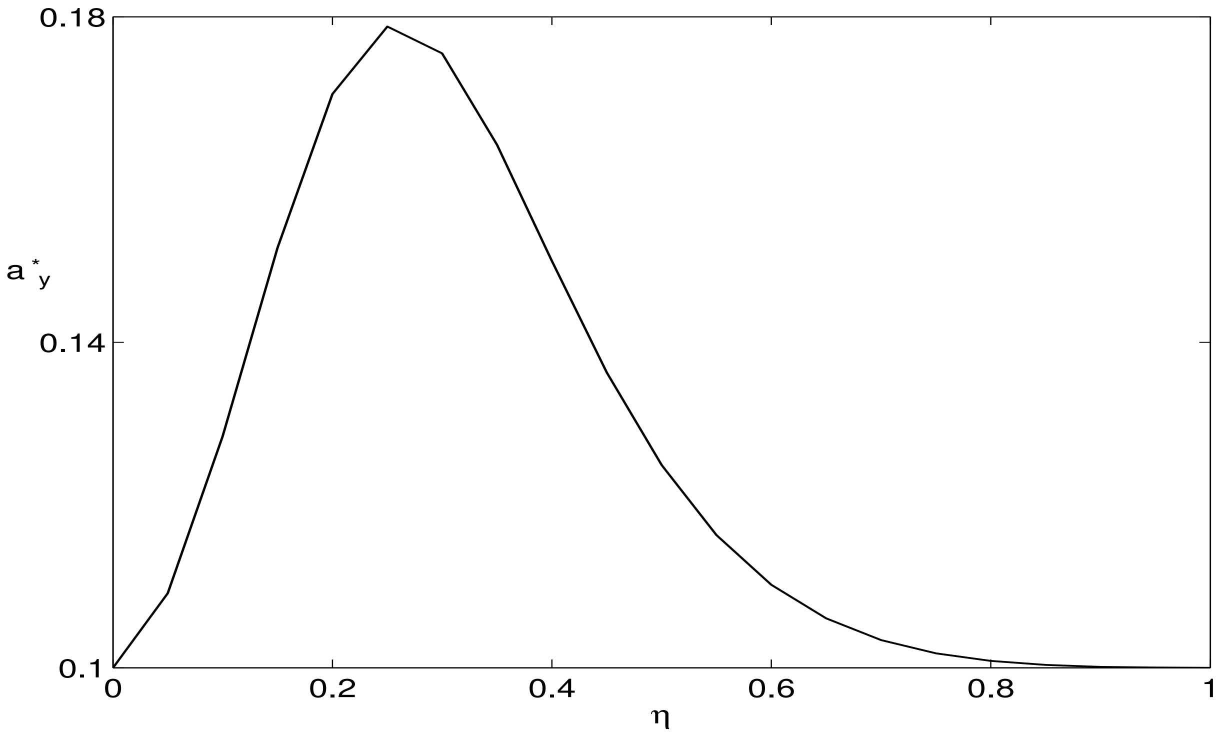
Numerical examples:$H = 0.5(\sin(t/\epsilon^\alpha) + \sin(t\sqrt(2)/\epsilon^\alpha))(\sin(2\pi y/\epsilon) + \sin(2\sqrt(2)\pi y/\epsilon))$, $\epsilon = 0.1$ and $d = 0.1$ (molecular diffusion) and vary α, $\alpha = 1, 2$. The flux function is chosen to be Buckley-Leverett function $F(u) = u^2/(u^2 + 0.2(1-u)^2))$, also the case - H is a Gaussian field is considered.

Numerical Results



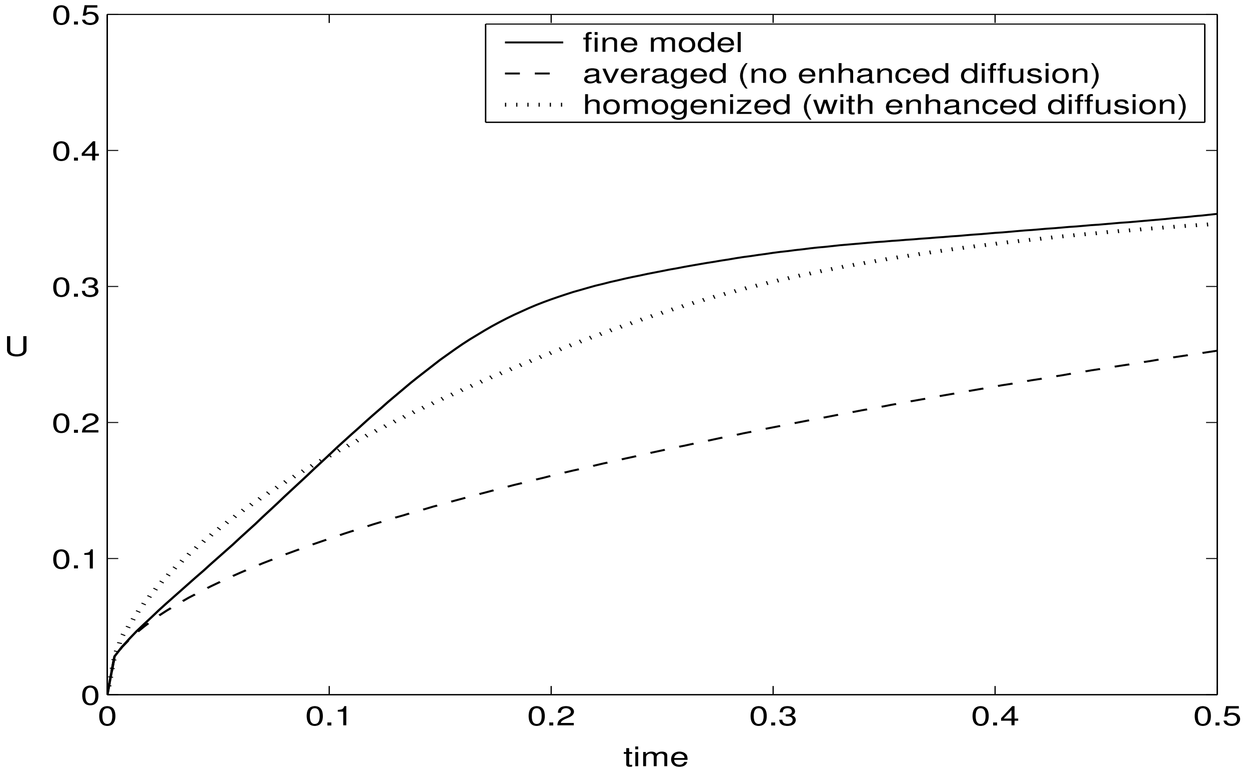
Enhanced diffusion for horizontal and vertical directions, quasi periodic layered flow, $\alpha = \beta = 1$.

Numerical Results



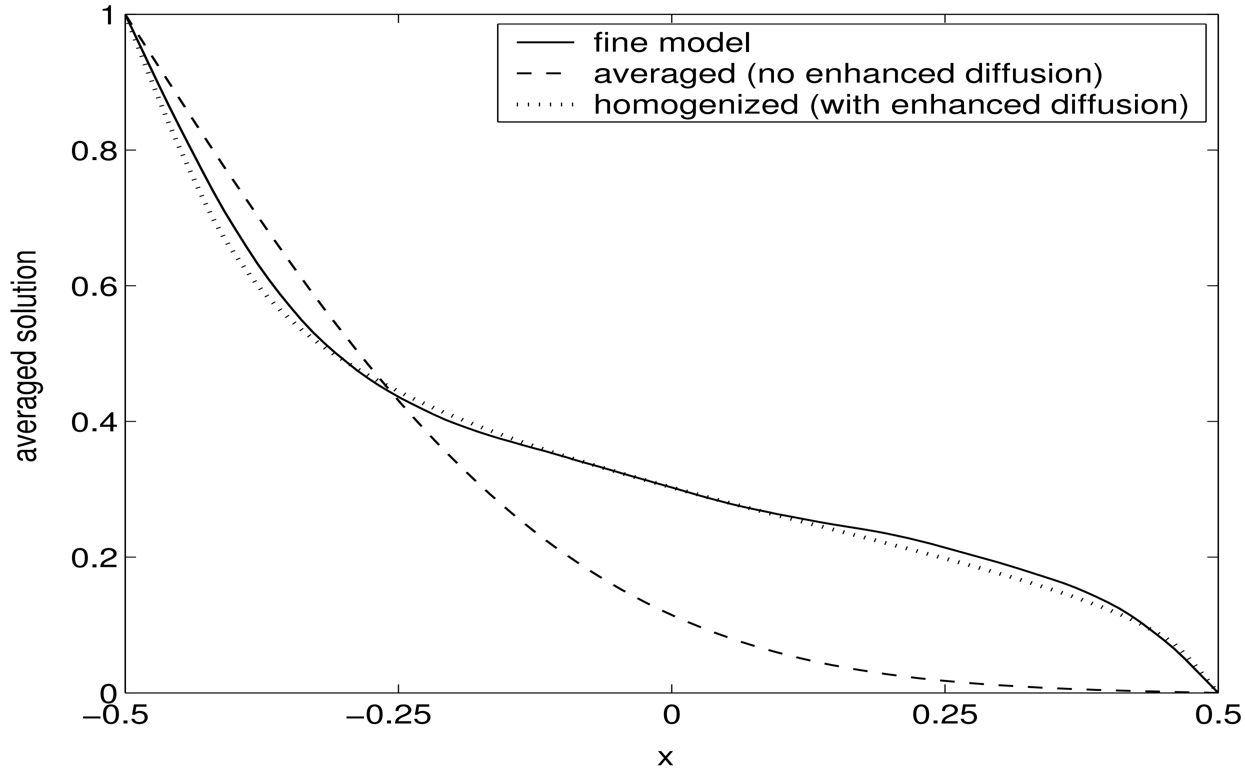
Enhanced diffusion for horizontal and vertical directions, quasi periodic layered flow

Numerical Results



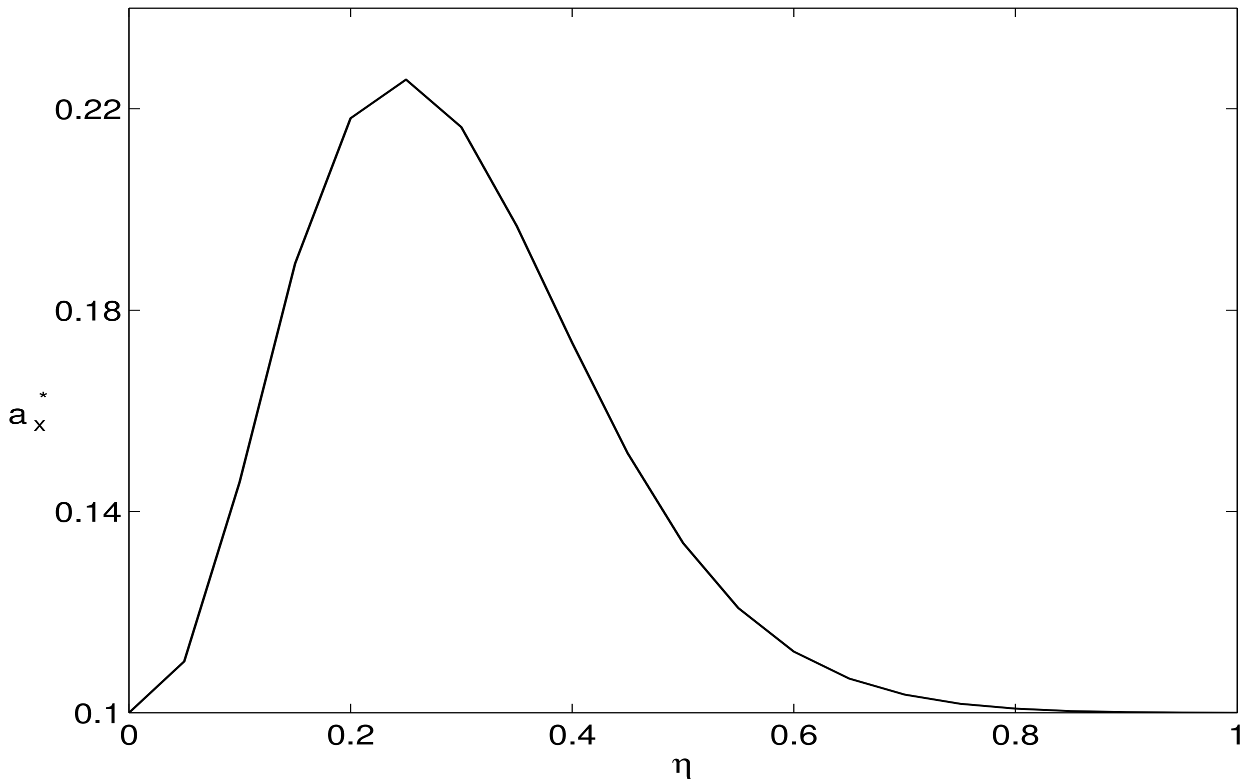
The solution comparison

Numerical Results



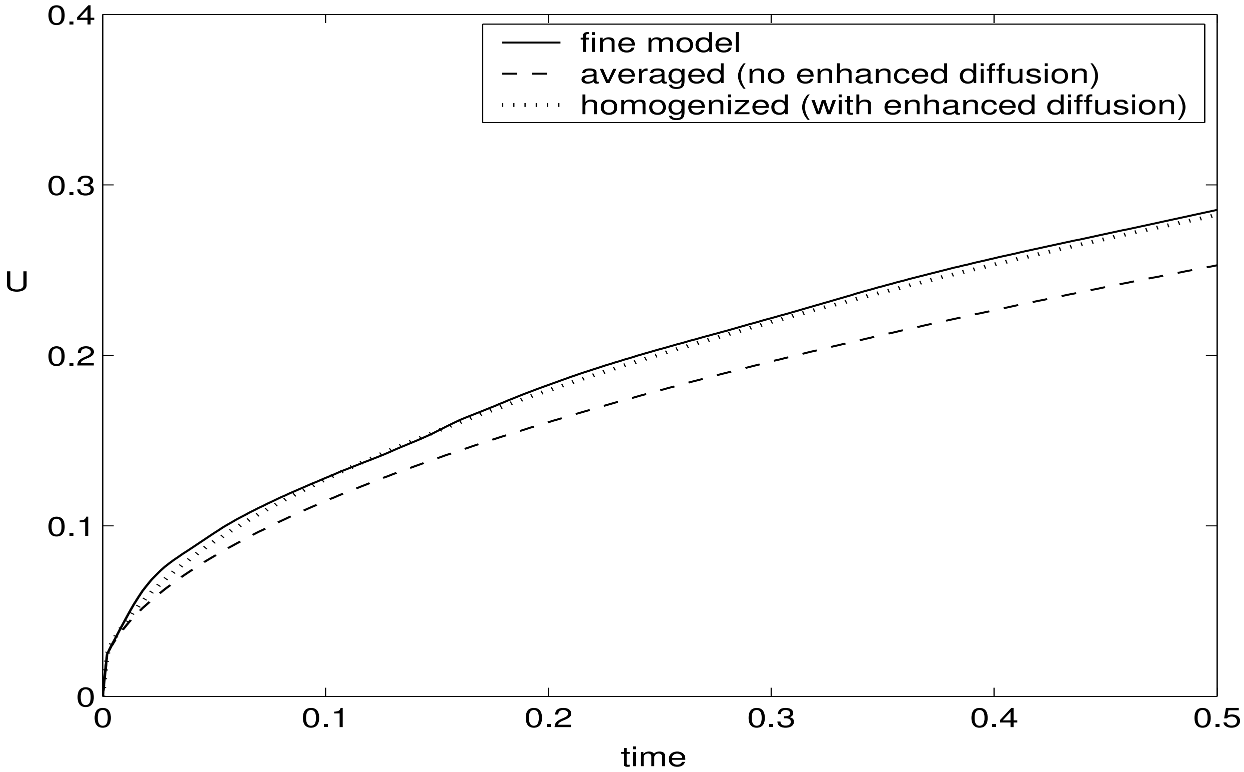
The solution comparison

Numerical Results



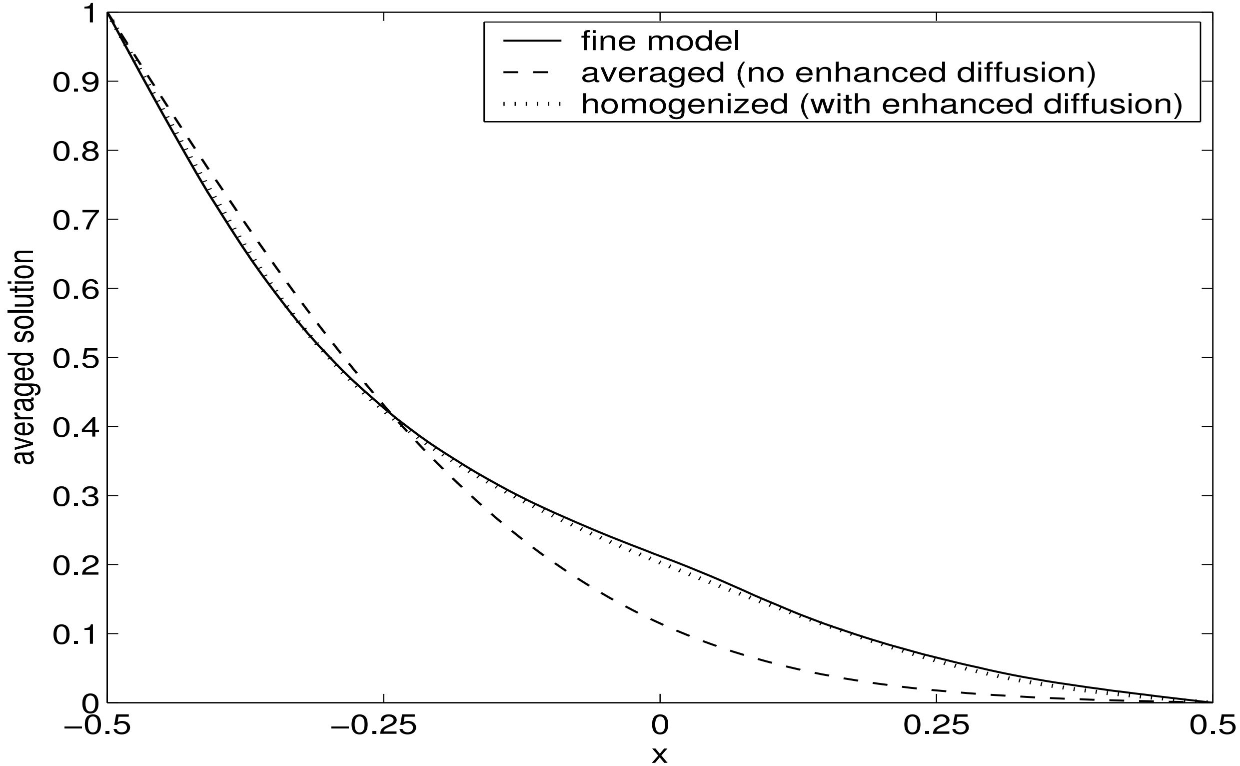
Enhanced diffusion for horizontal and vertical directions, Gaussian spatial field, $\alpha = 2$, $\beta = 1$.

Numerical Results



The solution comparison

Numerical Results



The solution comparison

Elliptic case

$$-div(a_\epsilon(x, u_\epsilon)\nabla u_\epsilon) = f.$$

$a_\epsilon(x, \eta) = k_\epsilon(x)/(1+\eta)^{\alpha_\epsilon(x)}$. $k_\epsilon(x) = \exp(\beta_\epsilon(x))$ is chosen such that $\beta_\epsilon(x)$ is a realization of a random field.

Convergence

Relative MsFEM Errors without Oversampling

N	L^2-norm		H^1-norm		L^∞-norm	
	Error	Rate	Error	Rate	Error	Rate
32	0.029		0.115		0.03	
64	0.053	-0.85	0.156	-0.44	0.0534	-0.94
128	0.10	-0.94	0.234	-0.59	0.10	-0.94

Relative MsFEM Errors with Oversampling

N	L_2-norm		H^1-norm		L_∞-norm	
	Error	Rate	Error	Rate	Error	Rate
32	0.002		0.038		0.005	
64	0.003	-0.43	0.021	0.87	0.003	0.72
128	0.001	1.10	0.009	1.09	0.001	1.08

Convergence

Relative MsFEM Errors for random heterogeneities, exponential variogram, $l_x = 0.20$, $l_z = 0.02$, $\sigma = 1.0$

N	L_2-norm		H^1-norm		L_∞-norm		hor. flux	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate
32	0.0006		0.0515		0.0025		0.027	
64	0.0002	1.58	0.029	0.81	0.0013	0.94	0.018	0.58
128	0.0001	1	0.016	0.85	0.0005	1.38	0.012	0.58

Convergence

Relative MsFEM Errors for random heterogeneities, spherical variogram, $l_x = 0.20$, $l_z = 0.02$, $\sigma = 1.0$, aligned discontinuity, jump = $\exp(1)$

N	L^2-norm		H^1-norm		L^∞-norm		hor. flux	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate
32	0.0006		0.0641		0.0020		0.039	
64	0.0002	1.58	0.0382	0.75	0.0010	1.00	0.027	0.53
128	0.0001	1.00	0.0210	0.86	0.0005	1.00	0.018	0.59

Relative MsFEM Errors for random heterogeneities, spherical variogram, $l_x = 0.20$, $l_z = 0.02$, $\sigma = 1.0$, aligned discontinuity, jump = $\exp(2)$

N	L^2-norm		H^1-norm		L^∞-norm		hor. flux	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate
32	0.0008		0.0817		0.0040		0.061	
64	0.0004	1.00	0.0493	0.73	0.0023	0.80	0.041	0.57
128	0.0002	1.00	0.0256	0.95	0.0011	1.06	0.025	0.71

Convergence

Relative MsFEM Errors for random heterogeneities, spherical variogram, $l_x = 0.20$, $l_z = 0.02$, $\sigma = 1.0$, aligned discontinuity, jump = $\exp(4)$

N	L^2-norm		H^1-norm		L^∞-norm		hor. flux	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate
32	0.0011		0.1010		0.0068		0.195	
64	0.0006	0.87	0.0638	0.66	0.0045	0.59	0.109	0.84
128	0.0003	1.00	0.0349	0.87	0.0024	0.91	0.063	0.79

Convergence

Relative MsFEM Errors for random heterogeneities, spherical variogram, $l_x = 0.20$, $l_z = 0.02$, $\sigma = 1.0$, nonaligned discontinuity, jump = $\exp(1)$

N	L^2-norm		H^1-norm		L^∞-norm		hor. flux	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate
32	0.0006		0.0623		0.0023		0.035	
64	0.0002	1.58	0.0366	0.77	0.0014	0.72	0.024	0.54
128	0.0001	1.00	0.0203	0.85	0.0006	1.22	0.016	0.59

Relative MsFEM Errors for random heterogeneities, spherical variogram, $l_x = 0.20$, $l_z = 0.02$, $\sigma = 1.0$, nonaligned discontinuity, jump = $\exp(2)$

N	L^2-norm		H^1-norm		L^∞-norm		hor. flux	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate
32	0.0010		0.0785		0.0088		0.052	
64	0.0003	1.74	0.0440	0.84	0.0052	0.76	0.031	0.75
128	0.0001	1.59	0.0239	0.88	0.0022	1.24	0.017	0.87

Convergence

Relative MsFEM Errors for random heterogeneities, spherical variogram, $l_x = 0.20$, $l_z = 0.02$, $\sigma = 1.0$, nonaligned discontinuity, jump = $\exp(4)$

N	L^2-norm		H^1-norm		L^∞-norm		hor. flux	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate
32	0.0067		0.1775		0.1000		0.164	
64	0.0016	2.07	0.0758	1.23	0.0288	1.80	0.077	1.09
128	0.0009	0.83	0.0687	0.14	0.0423	-0.55	0.039	0.98

Richards' Equation

$$\frac{\partial}{\partial t}\theta(u) - div K(x,u)\nabla(u + x_3) = 0,$$

where $\theta(u)$ is volumetric water content (soil moisture) and u is the pressure.

Haverkamp model - $\theta(u) = \frac{\alpha(\theta_s - \theta_r)}{\alpha + |u|^\beta} + \theta_r$, $K(x,u) = K_s(x)\frac{A}{A+|u|^\gamma}$;

van Genuchten model (M. T. van Genuchten, 1980) - $\theta(u) = \frac{\alpha(\theta_s - \theta_r)}{[1+(\alpha|u|)^n]^m} + \theta_r$,

$K(x,u) = K_s(x)\frac{\{1-(\alpha|u|)^{n-1}[1+(\alpha|u|)^n]^{-m}\}^2}{[1+(\alpha|u|)^n]^{m/2}}$;

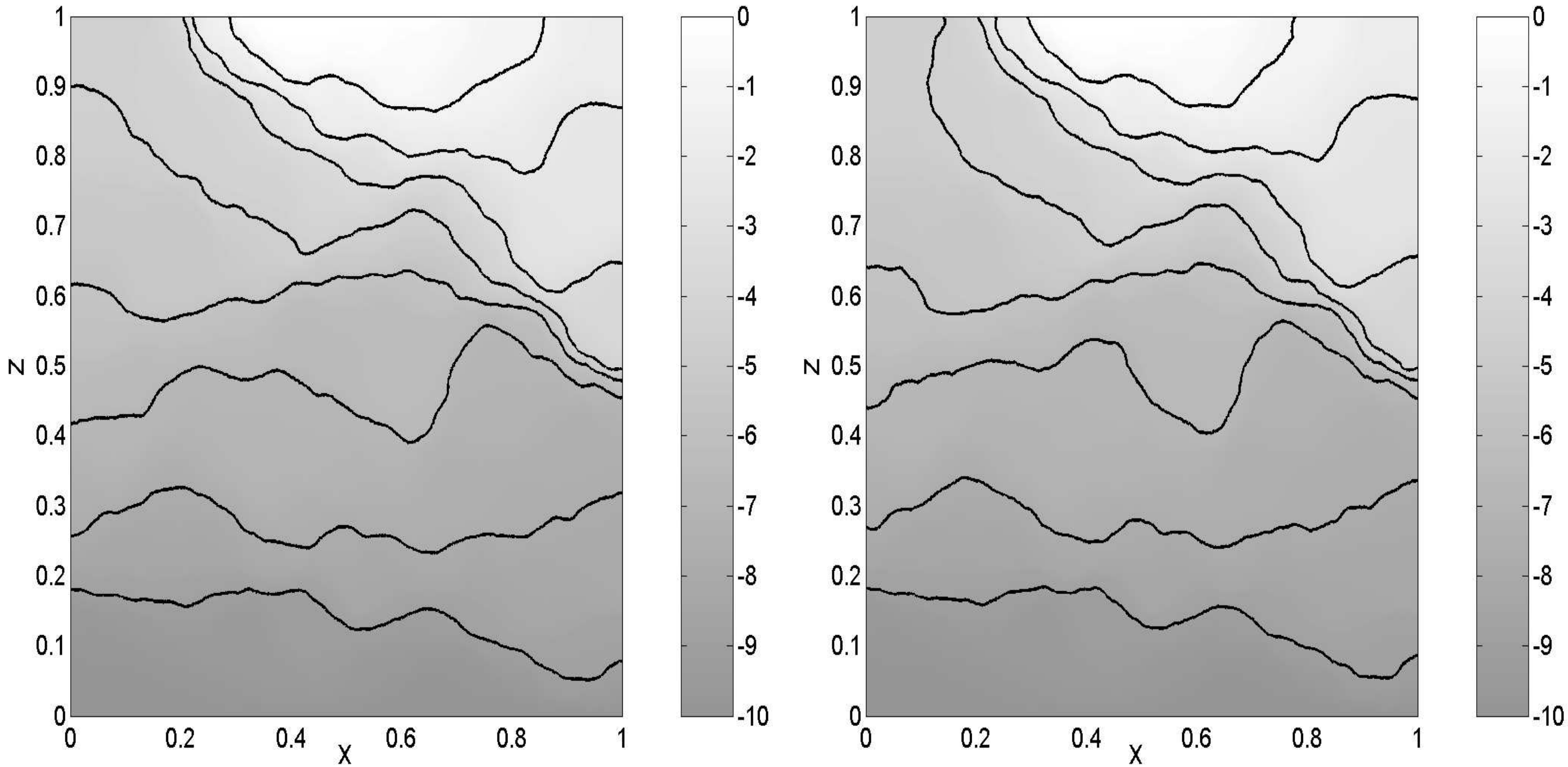
Exponential model (A. W. Warrick, 1976) - $\theta(u) = \theta_s\, e^{\beta u}$, $K(x,u) = K_s(x)\, e^{\alpha u}$.

Numerical setting

Exponential model

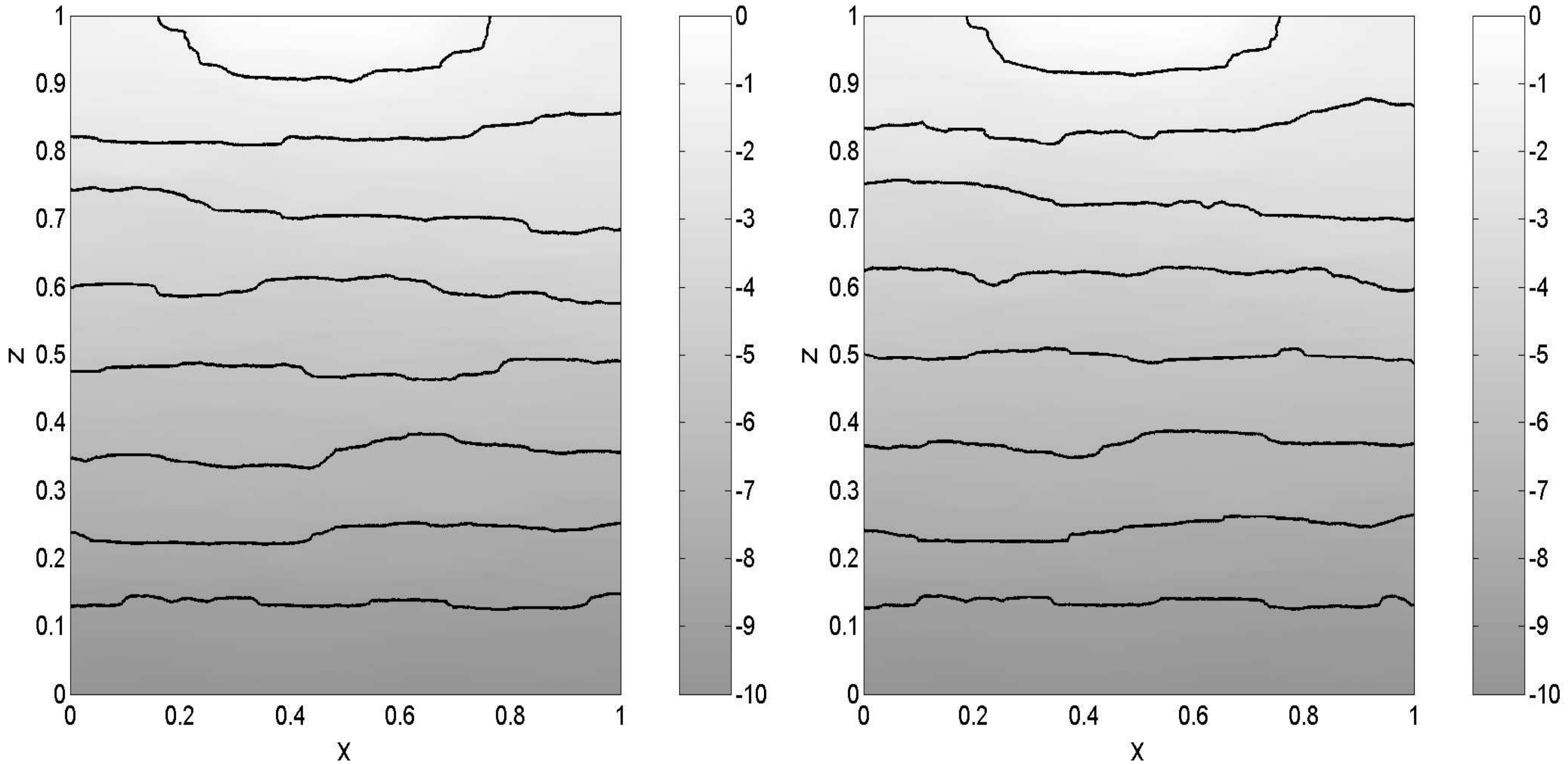
- BC: no flow on the lateral sides and $u_B = -10$ on the bottom. The top boundary is divided into three equal parts with prescribed u_T in the middle and no flow on other two.
- The other parameters: $\beta = 0.01$, $\theta_s = 1$, $\overline{Ks} = 1$, and $\overline{\alpha} = 0.01$.
- The heterogeneity comes from $K_s(x)$ and $\alpha(x)$.
- Isotropic and anisotropic heterogeneities are considered with $l_x = l_z = 0.1$ and $l_x = 0.20$, $l_z = 0.01$, respectively.
- Backward Euler scheme is used with $\Delta t = 2$.

Numerical results



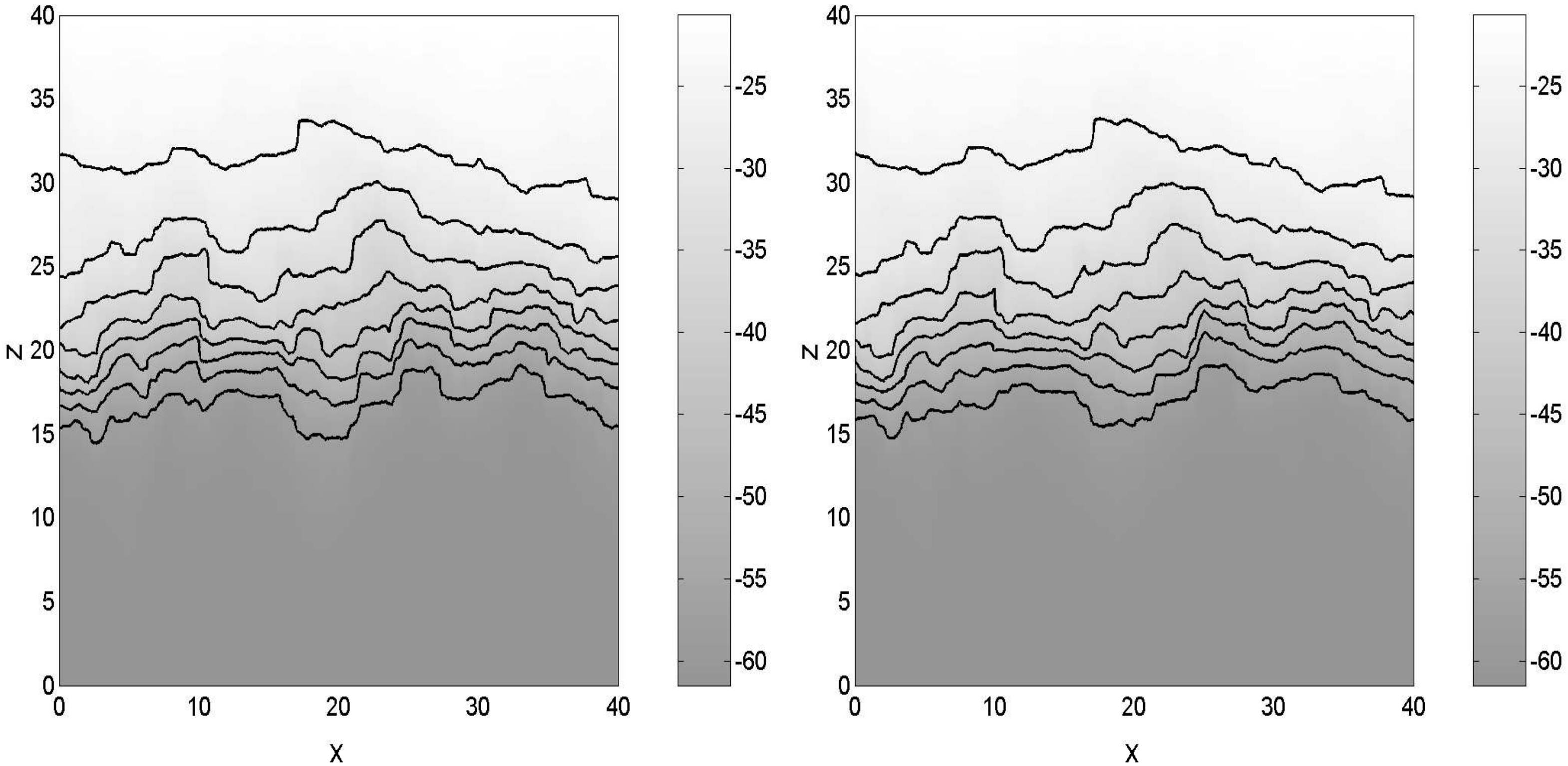
Exponential model with isotropic heterogeneity. Comparison of water pressure between the fine model (left) and the coarse model (right).

Numerical results



Exponential model with anisotropic heterogeneity. Comparison of water pressure between the fine model (left) and the coarse model (right).

Numerical Results



Haverkamp model
The use of limited global information for strongly channelized media.