Multiscale modeling and computation of flow through porous media

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Introduction

- Subsurface flows and transport are affected by heterogeneities at multiple scales (pore scale, core scale, field scale).
- Because of wide range of scales direct numerical simulations are not affordable.
- Upscaling of flow and transport parameters is commonly used in practice.
- Multiscale approaches are developed as an alternative to perform upscaling in the solution space.

Outline

- Porous media and heterogeneity
- Multiscale finite element methods (MsFEM) on coarse-grid
- Applications of multiscale finite element methods to porous media flows
- Multiscale finite element methods using limited global information
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- Generalizations of MsFEM to nonlinear problems and homogenization of nonlinear parabolic equation with random fluxes.
  $D_t u_\epsilon = div(a_\epsilon(x,t,u_\epsilon,D_x u_\epsilon)) + a_{0,\epsilon}(x,t,u_\epsilon,D_x u_\epsilon)$.
  ****
- Upscaling of transport equtions.
- Upscaling of two-phase flow in flow-based coordinate system.
- Uncertainty quantification using upscaled models

Darcy's law and permeability

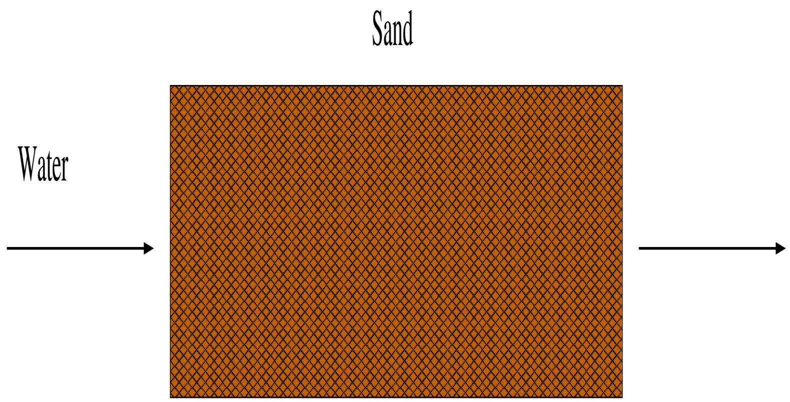
Darcy's empirical law, 1856: The volumetric flux $u(x,t)$ (Darcy velocity) is proportional to the pressure gradient

$$u = -\frac{k}{\mu}\nabla p = -K\nabla p,$$

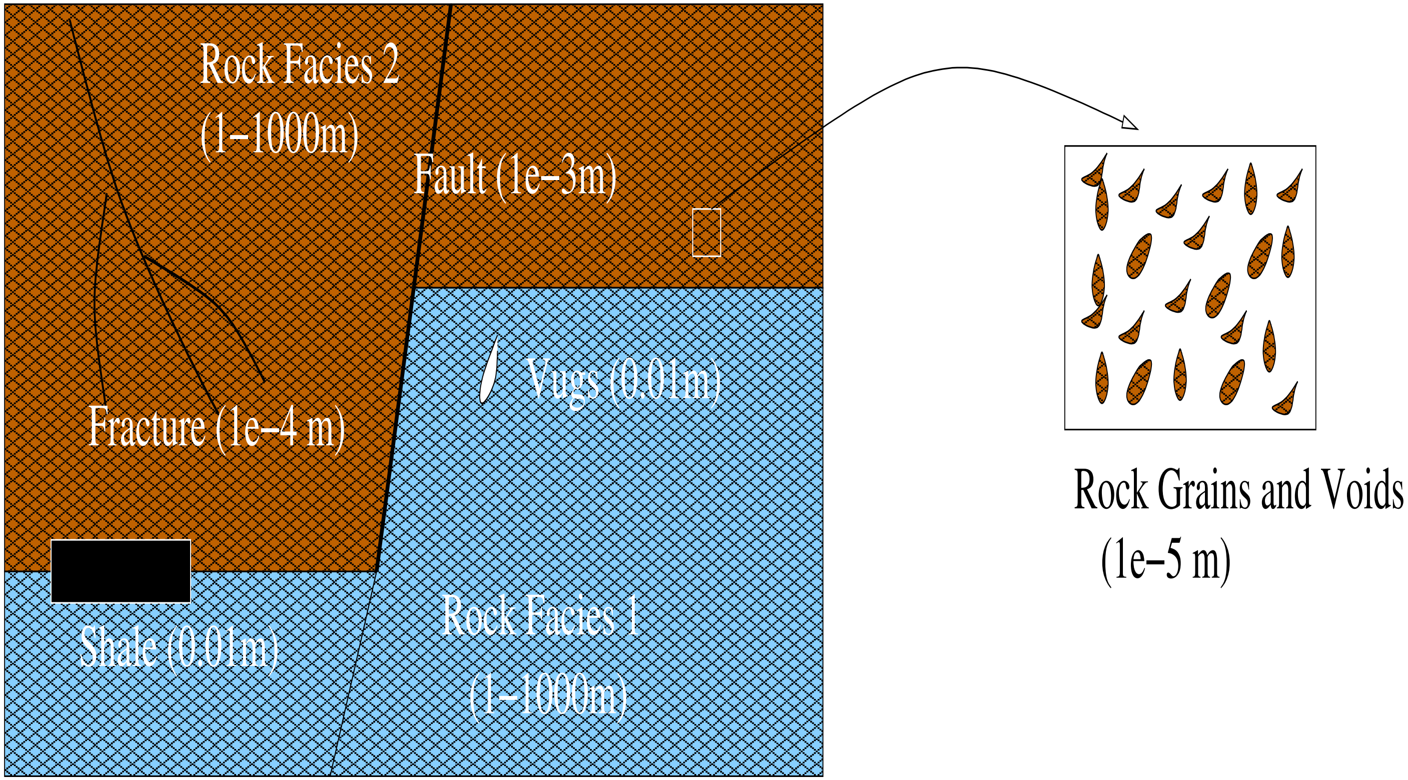
where $k(x)$ is the measured permeability of the rock, μ is the fluid viscosity, $p(x)$ is the fluid pressure, $u(x)$ is the Darcy velocity.
We obtain the second order elliptic system

$$u = -K\nabla p \quad \text{in } Q \quad \text{Darcy's Law}$$

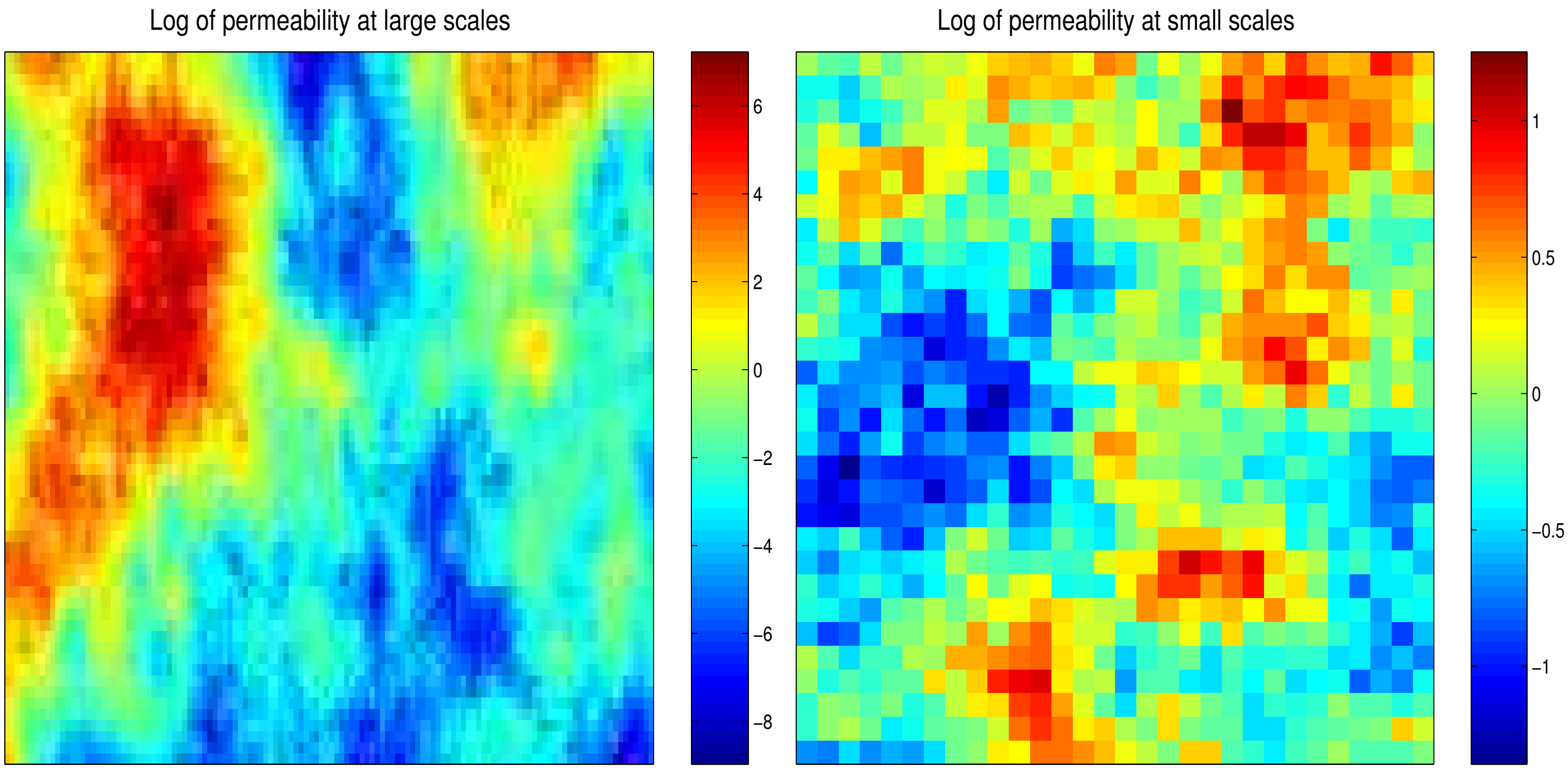
$$div(u) = f \quad \text{in } Q \quad \text{conservation}$$



A natural porous media



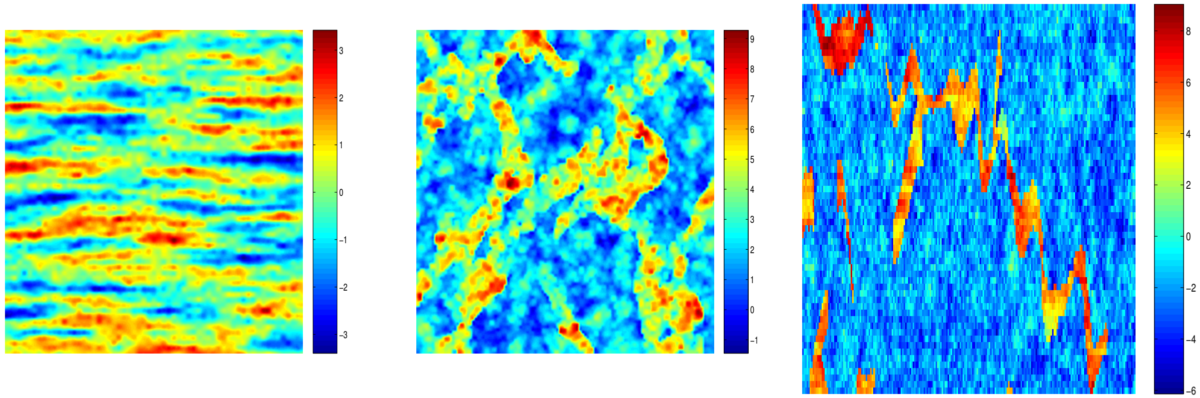
Heterogeneities



Upscaling: The system must be represented on a larger scale by incorporating the fine details in an average sense.

Requirements/Challenges

- Accuracy and Robustness
- Retain geological realism in flow simulation
- Valid for different types of subsurface heterogeneity



- Applicable for varying flow scenarios
- Heterogeneities. Conditional distributions. Inverse problems.

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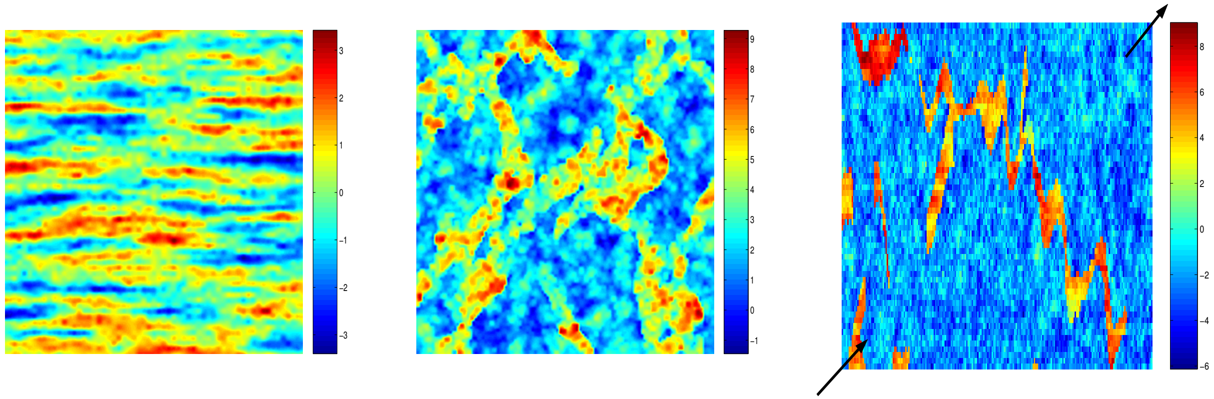

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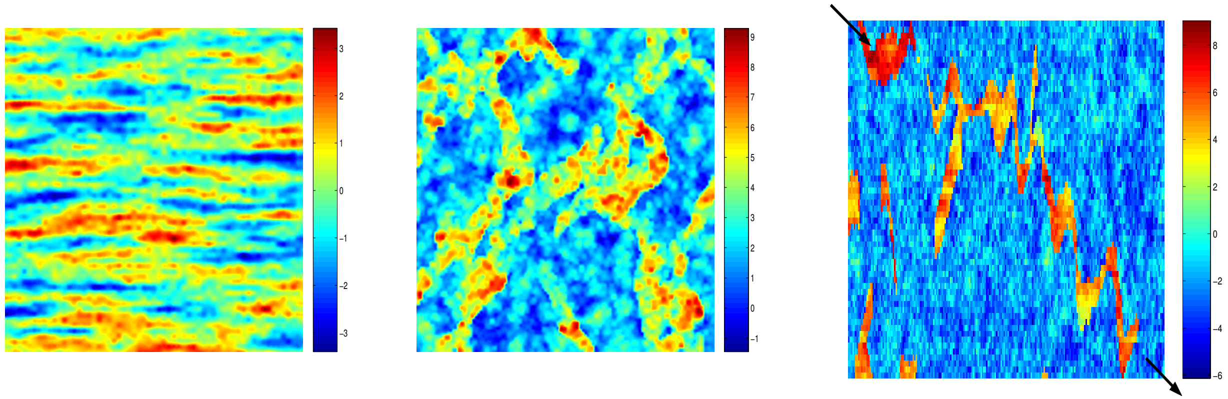

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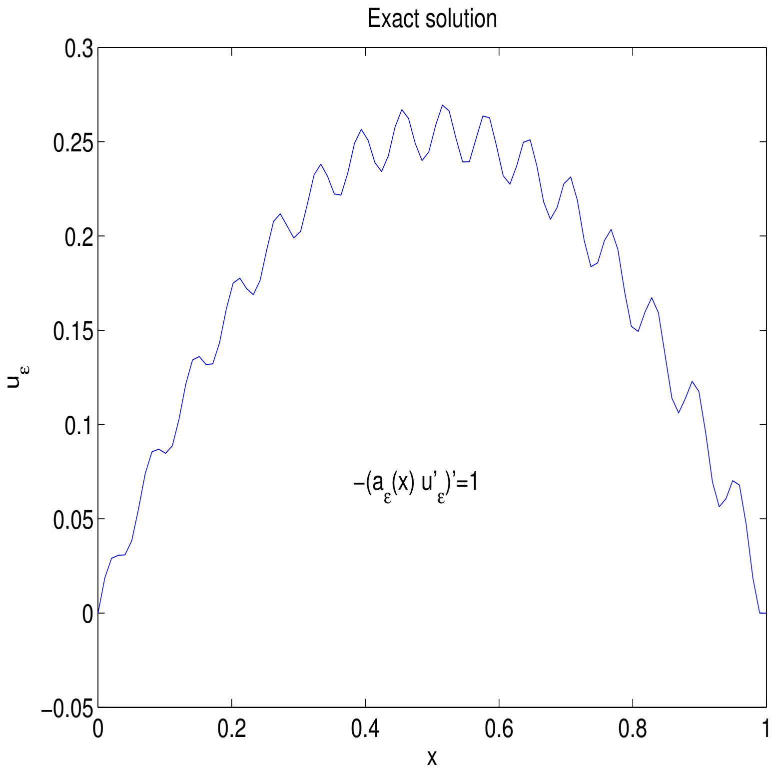
Kolmogorov n-width

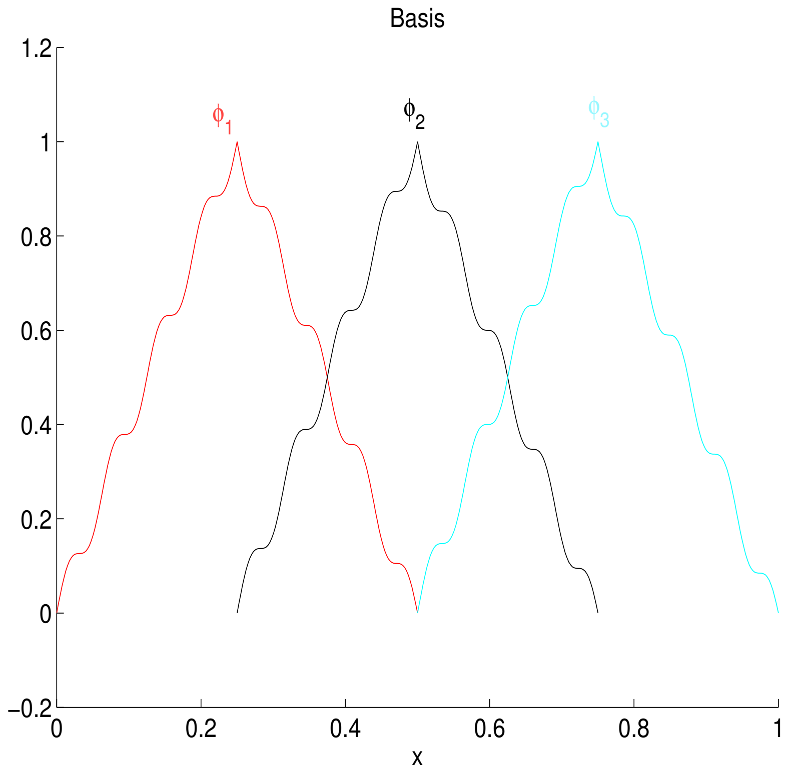
Given a Banach space V and a fixed integer n, find the best vector subspace V_n of dimension n that approximates V.
Find V_n such that

$$\sup_{u \in V, \|u\|=1} \inf_{u_n \in V_n} \|u - u_n\|$$

in minimized (this number is called Kolmogorov n-width).

A simple example





$a_\epsilon(x) = 1/(2 + 1.99\cos(x/\epsilon))$, $\epsilon = 0.01$.

Multiscale Finite Element Methods

Hou and Wu (1997).
Consider

$$div(k_\epsilon(x)\nabla p_\epsilon) = f,$$

where ϵ is a small parameter.

- The central idea is to incorporate the small scale information into the finite element bases
- Basis functions are constructed by solving the leading order homogeneous equation in an element K (coarse grid or Representative Elementary Volume (RVE))

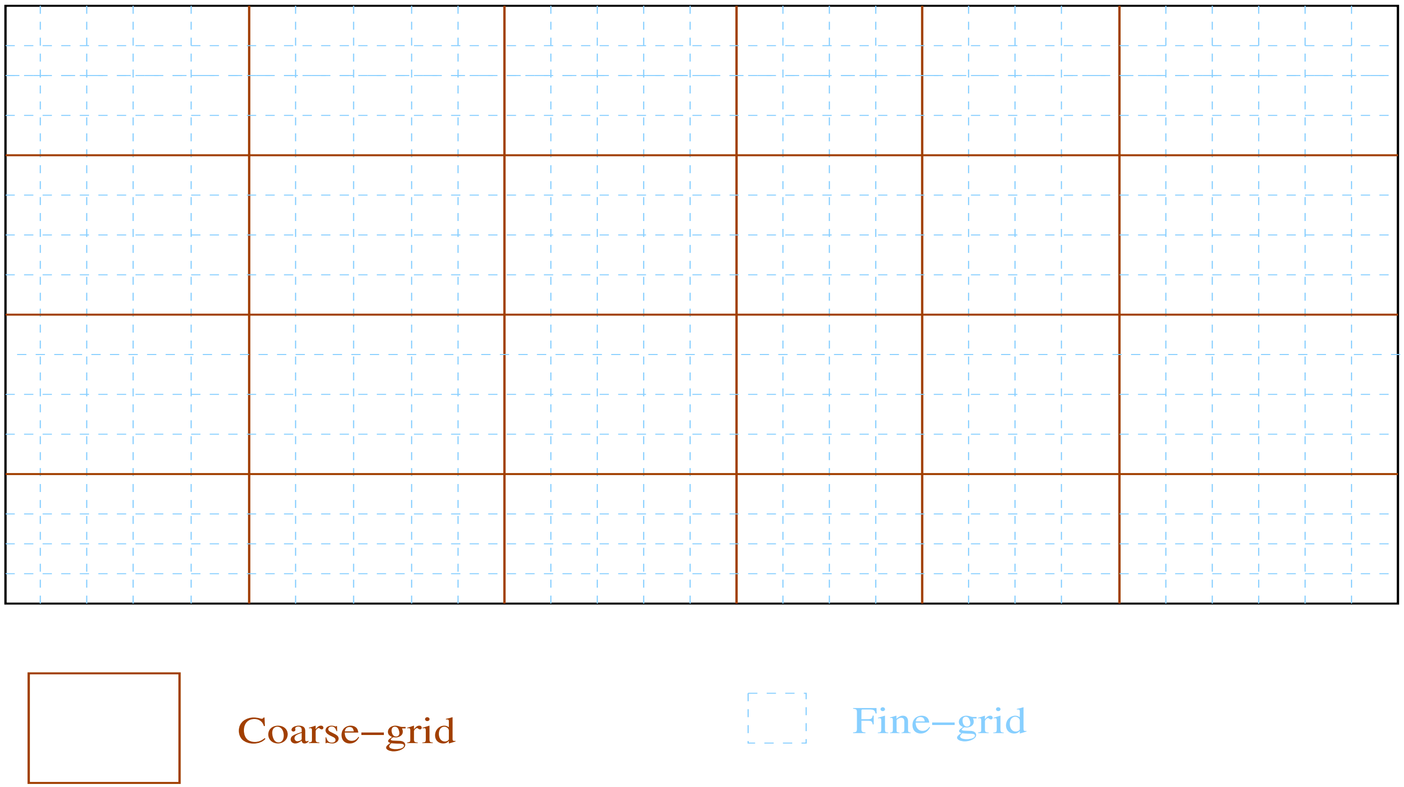
$$div(k_\epsilon(x)\nabla \phi^i) = 0 \quad \text{in} \quad K$$

- It is through the basis functions that we capture the local small scale information of the differential operator.
- 1-D example.

Multiscale Finite Element Methods

- Boundary conditions?

$$\phi^i = \text{linear function on } \partial K, \quad \phi^i(x_j) = \delta^{ij}$$



Multiscale Finite Element Methods

- Except for the multiscale basis functions, MsFEM is the same as the traditional FEM (finite element method). Find $p_\epsilon^h \in V^h = \{\phi^i\}$ such that

$$k(p_\epsilon^h, v^h) = f(v^h) \quad \forall v^h \in V^h,$$

  where

$$k(u,v) = \int_Q k_{ij}^\epsilon(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx, \quad f(v) = \int_Q f v dx$$

- The coupling of the small scales is through the variational formulation
- Solution has the form $p_\epsilon = p_0(x) + \epsilon N_k(x/\epsilon) \frac{\partial}{\partial x_k} p_0(x)$.
- If standard finite element method is used (linear basis functions):

$$\|p_\epsilon - p_\epsilon^h\|_{H^1(Q)} \le C h^\alpha \|p_\epsilon\|_{H^{1+\alpha}(Q)} = O(\frac{h^\alpha}{\epsilon^\alpha}).$$

  Thus, $h \ll \epsilon$, which is not affordable in practice.

Subgrid modeling (by T. Arbogast, I. Babuska, T. Hughes and others)
Subgrid stabilization (by F. Brezzi, L Franco, J.L. Guermond, T. Hughes, A. Russo, and others).

Brief introduction to homogenization

$p_\epsilon \in H_0^1(Q)$

$$div(k(x, \frac{x}{\epsilon})\nabla p_\epsilon) = f,$$

where $k(x, y)$ is a periodic function with respect to y. Consider formal expansion

$$p_\epsilon = p_0(x, y) + \epsilon p_1(x, y) + \epsilon^2 p_2(x, y) + ...$$

Taking into account

$$\nabla A(x, \frac{x}{\epsilon}) = \nabla_x A + \frac{1}{\epsilon}\nabla_y A$$

we have

$$(div_x + \frac{1}{\epsilon}div_y)[k(x, y)(\nabla_x + \frac{1}{\epsilon}\nabla_y)(p_0(x, y) + \epsilon p_1(x, y) + \epsilon^2 p_2(x, y) + ...) = f.$$

$$\epsilon^{-2} : \; div_y(k(x, y)\nabla_y p_0(x, y)) = 0.$$

From here, $p_0(x, y) = p_0(x)$.

Brief introduction to homogenization

$$\epsilon^{-1}: \; div_y(k(x,y)\nabla_y p_1(x,y)) = -div_y(k(x,y))\nabla_x p_0.$$

From here, $p_1(x,y) = N_l(x,y)\frac{\partial}{\partial x_l}p_0$, where

$$div_y(k(x,y)\nabla_y N_l) = -\nabla_{x_i} k_{il}(x,y).$$

$$\epsilon^0: \; div_y(k(x,y)\nabla_y p_2) + div_y(k(x,y)\nabla_x p_1) + div_x(k(x,y)\nabla_y p_1) + div_x(k(x,y)\nabla_x p_0) = f.$$

Taking the average and noting that

$$\langle div_y A(x,y)\rangle = \int_Y div_y A(x,y)dy = 0,$$

we get

$$div_x\langle k(x,y)\nabla_y p_1\rangle + div_x(\langle k(x,y)\rangle\nabla_x p_0) = f.$$

From here, we conclude that

$$div_x(k^*(x)\nabla_x p_0) = f,$$

where $k^*(x) = \langle k(x,y) + k(x,y)\nabla_y N\rangle$.

Basic convergence in homogenization

For bounded domains, we have $p_\epsilon = p_0(x) + \epsilon N(x,y) \cdot \nabla p_0 + \theta + \epsilon^2 p_2(x,y) + ...$, where

$$div(k\nabla\theta) = 0$$

$$\theta = -\epsilon N(x,y) \cdot \nabla p_0.$$

It can be shown that (e.g., JKO 94) $\|\theta\|_{H^1(Q)} \le C\sqrt{\epsilon}$.

Convergence property of MsFEM

Consider $k_\epsilon(x) = k(x/\epsilon)$, where $k(y)$ is periodic in y.
h - computational mesh size.
Theorem Denote p_ϵ^h the numerical solution obtained by MsFEM, and p_ϵ the solution of the original problem. Then,
If $h >> \epsilon$,

$$\|p_\epsilon - p_\epsilon^h\|_{1,Q} \le C(h + \sqrt{\frac{\epsilon}{h}})$$

- This theorem shows that MsFEM converges to the correct solution as $\epsilon \longrightarrow 0$
- The ratio ϵ/h reflects two intrinsic scales. We call ϵ/h the resonance error
- The theorem shows that there is a scale resonance when $h \approx \epsilon$. Numerical experiments confirm the scale resonance.

Resonance errors

- For problems with scale separation, we can choose $h \gg \epsilon$ in order to avoid the resonance, but for problems with continuous spectrum of scales, we cannot avoid this resonance.
- To demonstrate the influence of the boundary condition of the basis function on the overall accuracy of the method we perform multiscale expansion of ϕ^i
- Multiscale expansion of ϕ^i

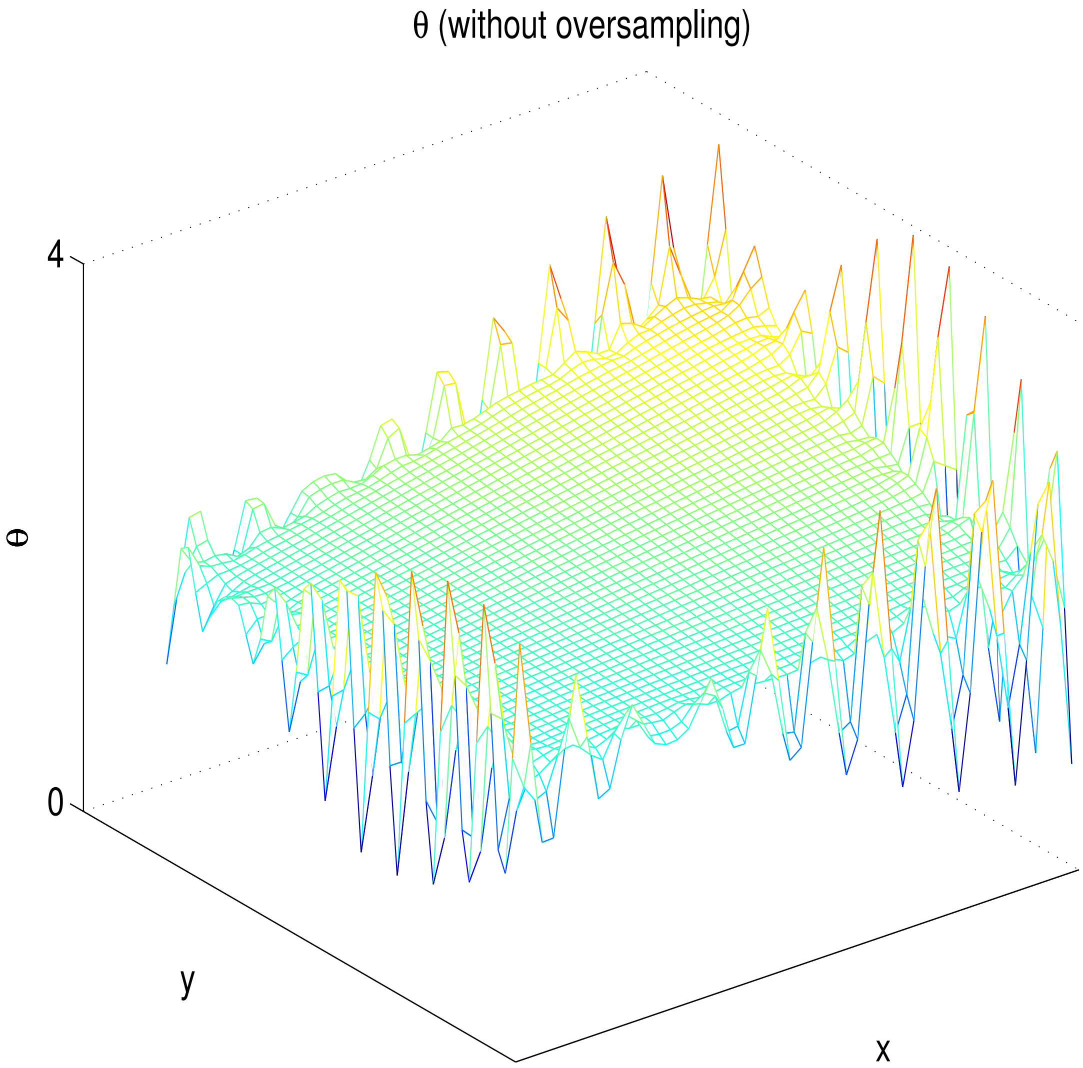
$$\phi^i = \phi_0(x) + \epsilon\phi_1(x, x/\epsilon) + \epsilon\theta + \ldots,$$

- $\phi_1(x, x/\epsilon) = N^k(x/\epsilon)\frac{\partial}{\partial x_k}\phi_0$, where $N^k(x/\epsilon)$ is a periodic function which depends on $k(x/\epsilon)$.
- θ satisfies

$$div(k_\epsilon \nabla\theta) = 0 \text{ in } K, \quad \theta^i = -\phi_1(x, x/\epsilon) + (\phi^i - \phi_0)/\epsilon \text{ on } \partial K$$

- Oscillations near the boundaries (in ϵ vicinity) of θ^i lead to the resonance error

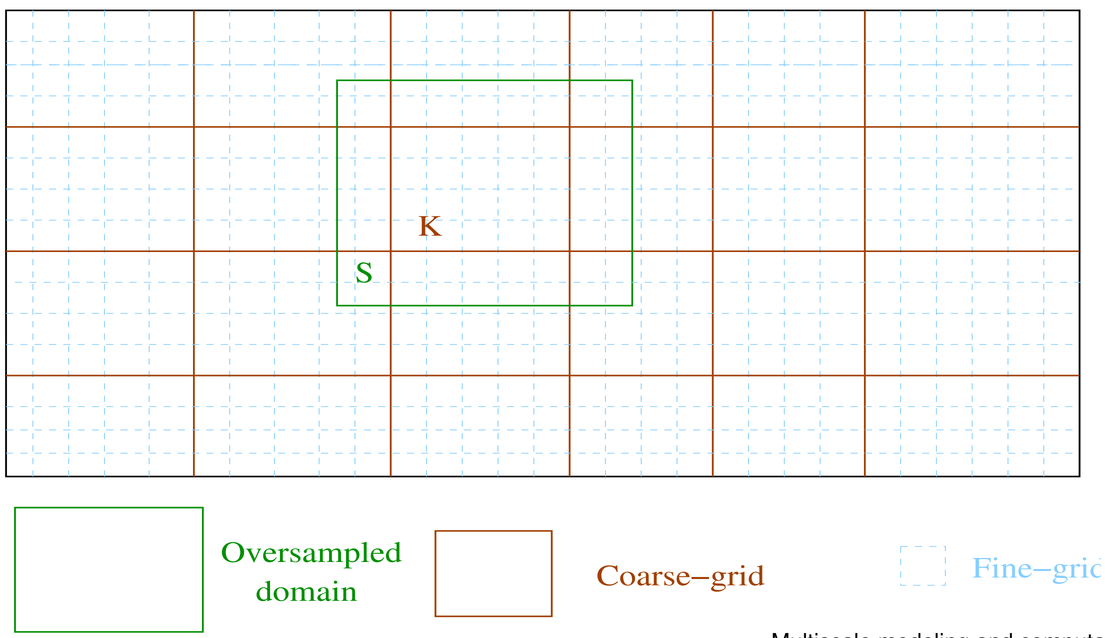
Illustration of θ



Oversampling technique

- To capture more accurately the small scale information of the problem, the effect of θ needs to be moderated
- Since the boundary layer of θ is thin ($O(\epsilon)$) we can sample in a domain with size larger than $h + \epsilon$ and use only interior sampled information to construct the basis functions.
- Let ψ^k be the functions in the domain S,

$$div(k_\epsilon(x)\nabla\psi^k) = 0 \text{ in } S, \quad \psi^k = \text{linear function on } \partial S, \; \psi^k(s_i) = \delta_{ik}.$$



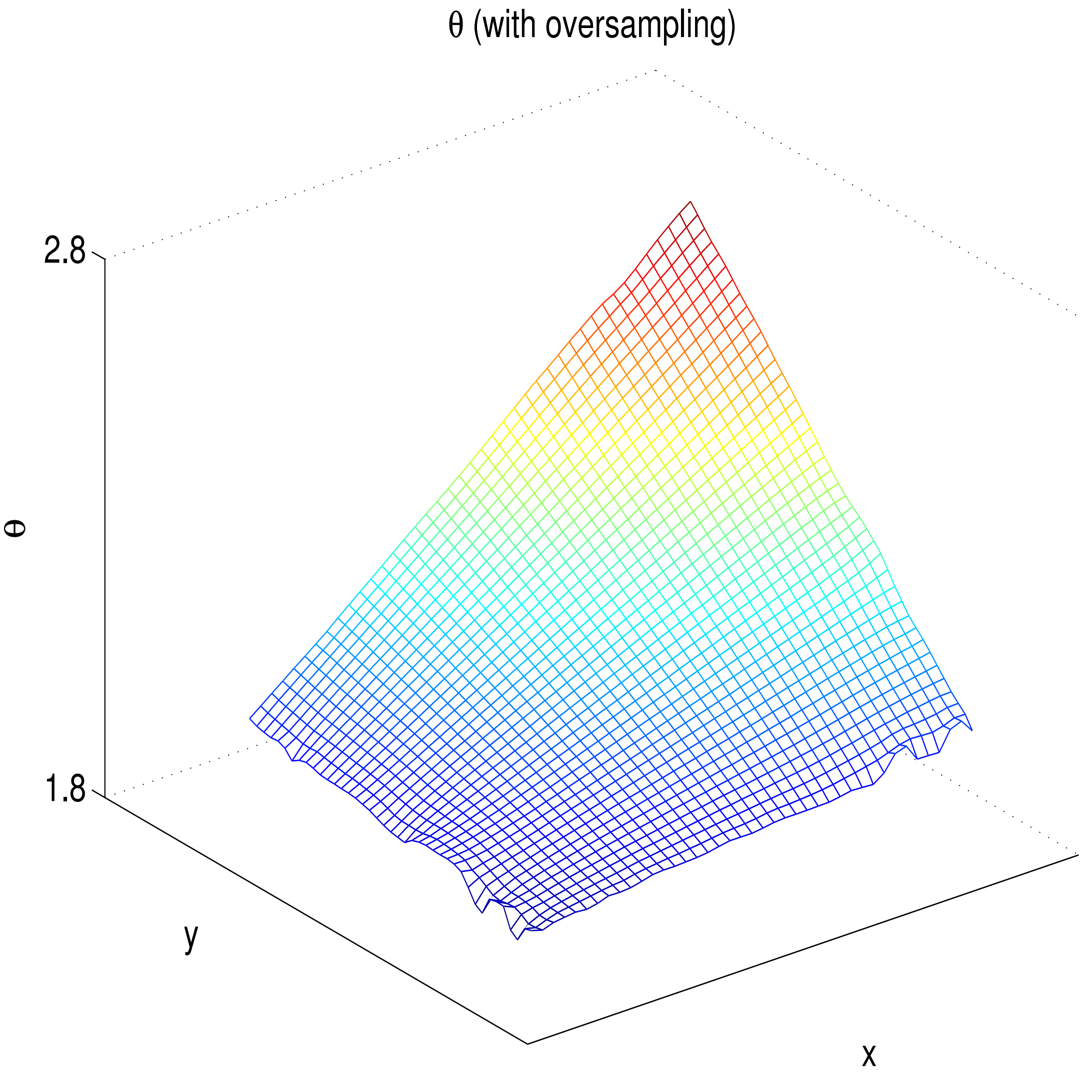
Oversampling technique

- The base functions in a domain $K \subset S$ constructed as

$$\phi^i|_K = \sum c_{ij} \psi^j|_K, \quad \phi^i(x_k) = \delta^{ik}$$

- The method is non-conforming.
- The derivation of the convergence rate uses the homogenization method combined with the techniques of non-conforming finite element method (Efendiev et al., SIAM Num. Anal. 1999)
- By a correct choice of the boundary condition of the basis functions we can reduce the effects of the boundary layer in θ.

Illustration of θ with oversampling



Numerical Results

Table 0: $\|U_\epsilon^h - U_0^h\|_{l_2}$, $\epsilon/h = 0.64$

h	MsFEM		MsFEM-os		Resolved FEM	
	l_2	rate	l_2	rate	h_{fine}	l_2
1/16	3.54e-4		7.78e-5		1/256	1.34e-4
1/32	3.90e-4	-0.14	3.38e-5	1.02	1/512	1.34e-4
1/64	4.00e-4	-0.05	1.97e-5	0.96	1/1024	1.34e-4
1/128	4.10e-4	-0.02	1.03e-5	0.95	1/2048	1.34e-4

The convergence of MsFEM

- The convergence of MsFEM for problems with multiple scales $\epsilon_1 \ll \epsilon_2 \ll ... \ll \epsilon_n$. If $\epsilon_k \ll h \ll \epsilon_{k+1}$, then the convergence rate in H^1 is $C\left(\frac{h}{\epsilon_{k+1}}\right)^s + \sqrt{\frac{\epsilon_k}{h}}$.
- The convergence of MsFEM for random coefficients (continuous ϵ-scales).
- The expansion of the base function, $\phi_\epsilon^i(\mathbf{x},\omega) = \phi_0(\mathbf{x}) + \epsilon\phi_1(\mathbf{x},\mathbf{x}/\epsilon,\omega) + \epsilon\theta$, where $\phi_1(\mathbf{x},\mathbf{x}/\epsilon,\omega) = N^k(\mathbf{x}/\epsilon,\omega)\nabla_k\phi_0(\mathbf{x})$.
- The estimates for stationary fields approximating $N(\mathbf{x}/\epsilon,\omega)$ have been derived under the strong mixing condition for the coefficients (Yurinskii, 86) (power decay of two point correlation).
- The convergence rate of MsFEM remains the same as in the periodic case if the coefficients are quasi-periodic or almost periodic subject to some conditions.

Various global formulations

- Once basis functions are constructed, various global formulation (mixed, control volume finite element, DG and etc) can be used to couple the subgrid effects.
- Control volume finite element: Find $p_h \in V_h$ such that

$$\int_{\partial V_z} k(x) \nabla p_h \cdot \mathbf{n} \, dl = \int_{V_z} q \, dx \quad \forall V_z \in Q,$$

  where V_z is control volume.
- Mixed finite element: In each coarse block K, we construct basis functions for the velocity field

$$div(k(x) \nabla w_i^K) = \frac{1}{|K|} \quad \text{in} \quad K$$

$$k(x) \nabla w_i^K \mathbf{n}^K = \begin{cases} \frac{1}{|e_i^K|} & \text{on } e_i^K \\ 0 & \text{else.} \end{cases}$$

  For the pressure, the basis functions are taken to be constants.

MsFEM for problems with scale separation

For periodic problems or problems with scale separation, multiscale finite element methods can take an advantage of scale separation. Local problems can be solved in RVE

$$div(k\nabla\phi^i) = 0$$

$\phi^i = \phi_0^i$ on $\partial\ RVE$.
Basis functions can be also approximated

$$\phi^i = \phi_0^i + N_\epsilon \cdot \nabla\phi_0^i,$$

where ϕ_i^0 is linear basis functions and N is the periodic solution of auxiliary problem in ϵ-size period

$$-div(k_\epsilon(x)(\nabla N + I)) = 0.$$

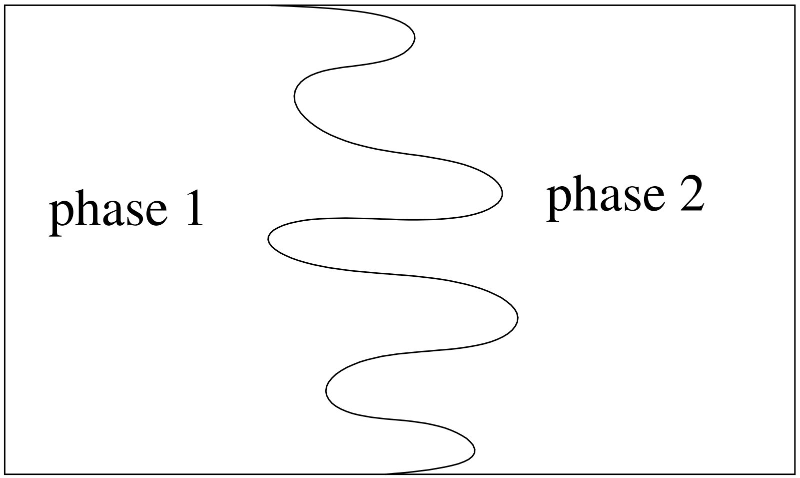
(cf. Durlofsky 1981, etc.).
Note, the above procedure works when “homogenization by periodization” is applicable (e.g., random homogeneous case).

Applications of MsFEM to subsurface flow simulations

Two-phase flow model. Darcy's law for each phase

$$v_i = -\frac{k k_i(S_i)}{\mu_i} \nabla p_i,$$

i=1,2. Here k - permeability field representing the heterogeneities (micro-level information), p_i - the pressure, v_i - velocity, k_i - relative permeability, S_i -saturation, μ_i - viscosity



Two-phase flow model

- $p_1 = p_2 = p$ if the capillary effects are neglected. The total velocity v is given by

$$v = v_1 + v_2 = -\lambda(S) k \nabla p, \quad \lambda(S) = \frac{k_1(S)}{\mu_1} + \frac{k_2(S)}{\mu_2}.$$

  where $S = S_1$, $S_2 = 1 - S_1$.

- Incompressibility of the total velocity implies

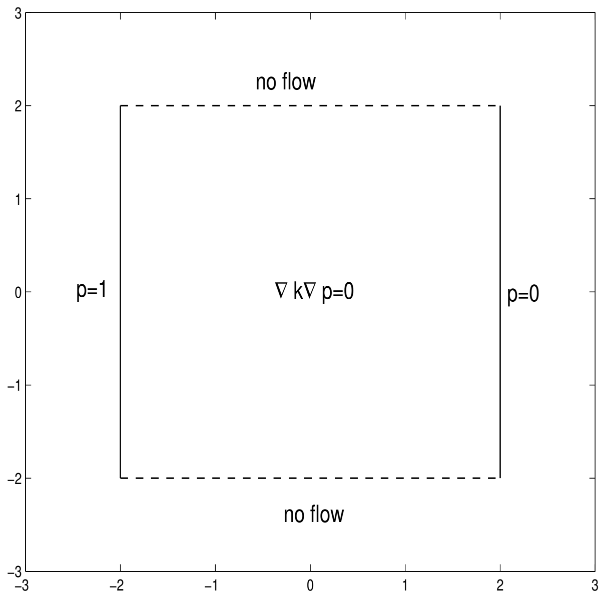
$$div(\lambda(S) k \nabla p) = 0,$$

- From the conservation of mass $S_t + div(v_1) = 0$ we can derive

$$\frac{\partial S}{\partial t} + v \cdot \nabla f(S) = 0, \quad f(S) = \frac{\frac{k_1(S)}{\mu_1}}{\frac{k_1(S)}{\mu_1} + \frac{k_2(S)}{\mu_2}}$$

Existing upscaling techniques

- $-div(\lambda(S)k\nabla p) = 0$, $S_t + v \cdot \nabla f(S) = 0$, $v = -\lambda(S)k\nabla p$.
- Single-phase upscaling: $(k \longrightarrow k^*)$, $\mathbf{k}^* = \frac{\overline{\mathbf{k}\nabla p}}{\overline{\nabla p}}$.



- Multiphase upscaling $\lambda \longrightarrow \lambda^*$, $f \longrightarrow f^*$.

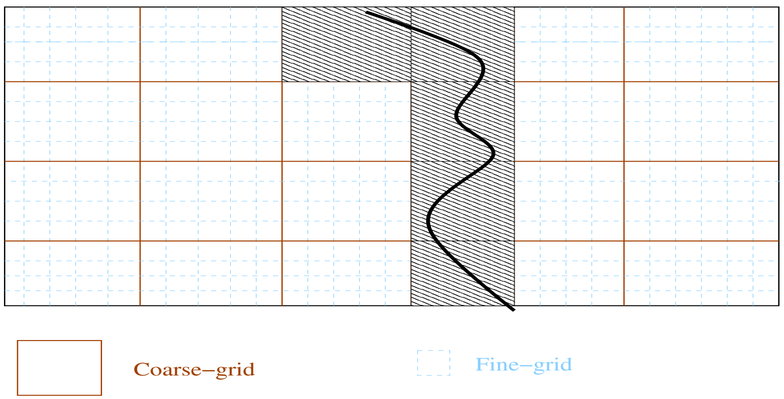
Applications of MsFEM

At least two way one can apply MsFEM
1) Solve the pressure equation on the coarse-grid and solve the saturation equation on the fine-grid

$$-div(\lambda(S)k\nabla p) = 0$$

$$\frac{\partial}{\partial t}S + v \cdot \nabla f(S) = 0,$$

where $v = -\lambda(S)k\nabla p$. Basis functions are updated only near sharp fronts.



MsFVEM applied to two-phase flow problem

(IM)plicit (P)ressure (E)xplicit (S)aturation:

Given S^0, for $n = 1, 2, 3, \cdots$, do the following:

- find $p_h^{n-1} \in V_h$ such that

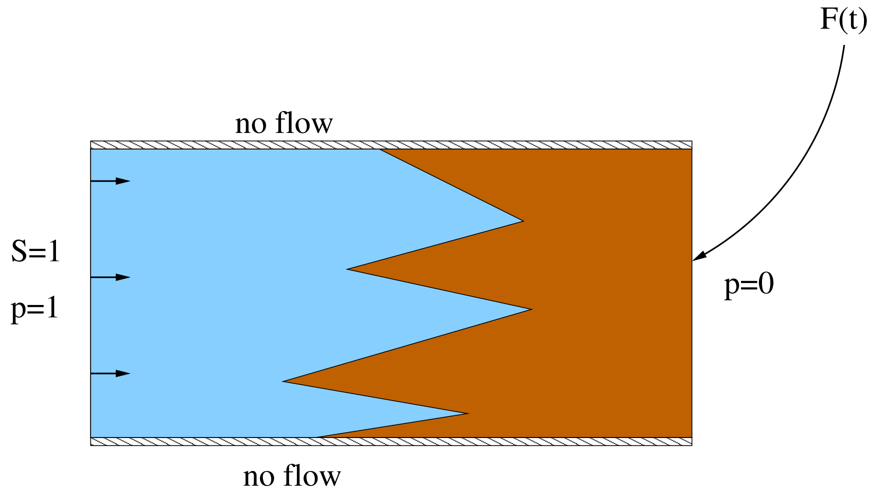
$$\int_{\partial V_z} \lambda(S^{n-1}) k(x) \nabla p_h^{n-1} \cdot \mathbf{n} \, dl = \int_{V_z} q \, dx \quad \forall V_z \in Q$$

- compute $\mathbf{v}^{n-1} = -\lambda(S^{n-1}) k(x) \nabla p_h^{n-1}$
- time march on the saturation equation:

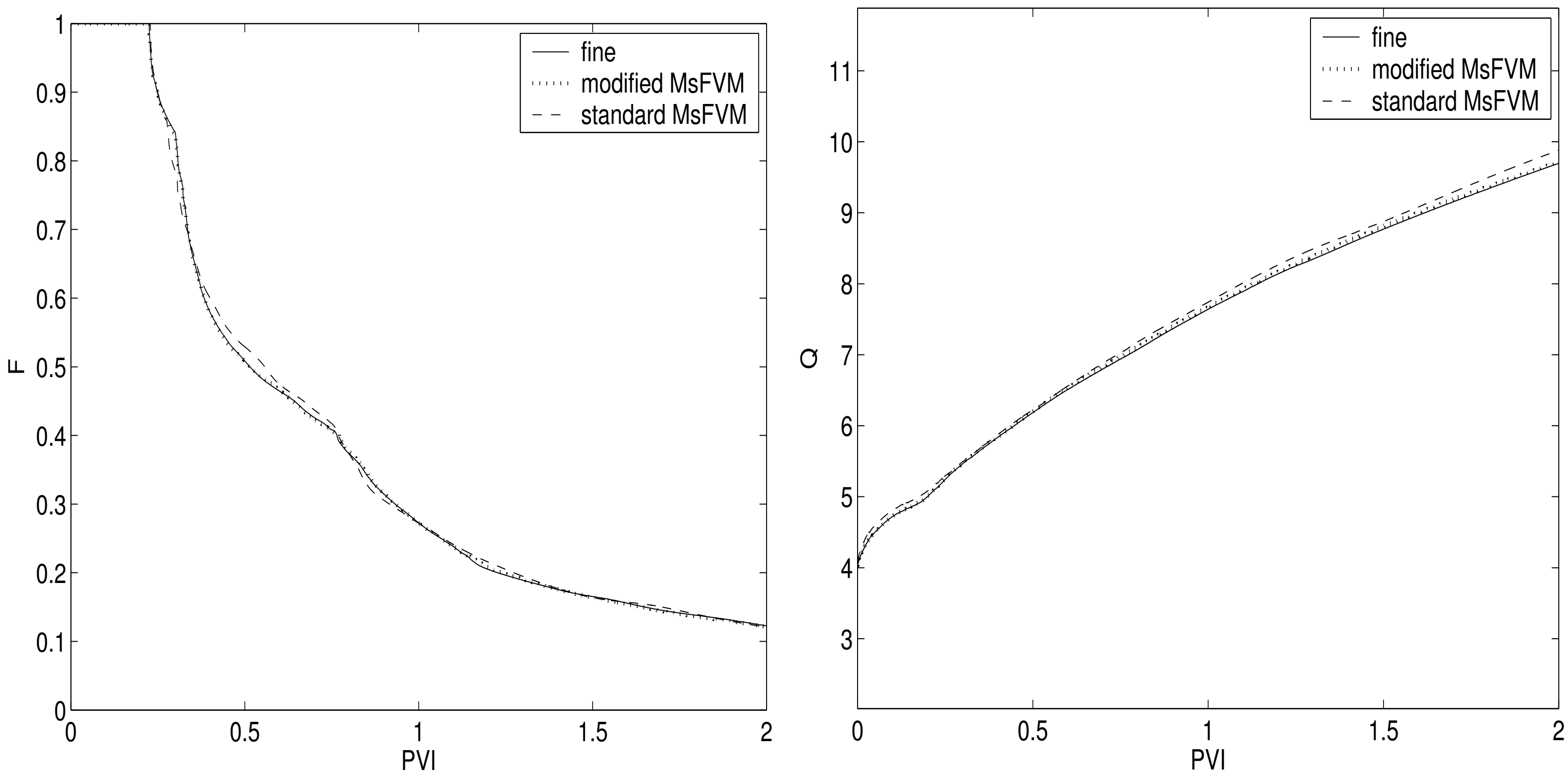
$$\int_{c_z} \left(S^n - S^{n-1}\right) \, dx + \Delta t^{n-1} \int_{\partial c_z} f(S^{n-1}) \mathbf{v}^{n-1} \cdot \mathbf{n} \, dl = \Delta t^{n-1} \int_{c_z} q\tilde{S} \, dx$$

Numerical Setting

- Rectangular domain is considered. The permeability field is generated using geostatistical libraries.
- The boundary conditions: no flow on top and bottom boundaries, a fixed pressure and saturation ($S = 1$) at the inlet (left edge), fixed pressure at the outlet (right edge).
- The production rate $F = q_0/q$, where q_0 the volumetric flow rate of oil produced at the outlet edge and q the volumetric flow rate of the total fluid produced at the outlet edge. The dimensionless time is defined as $PVI = qt/V_p$, where t is time, V_p is the total pore volume of the system.

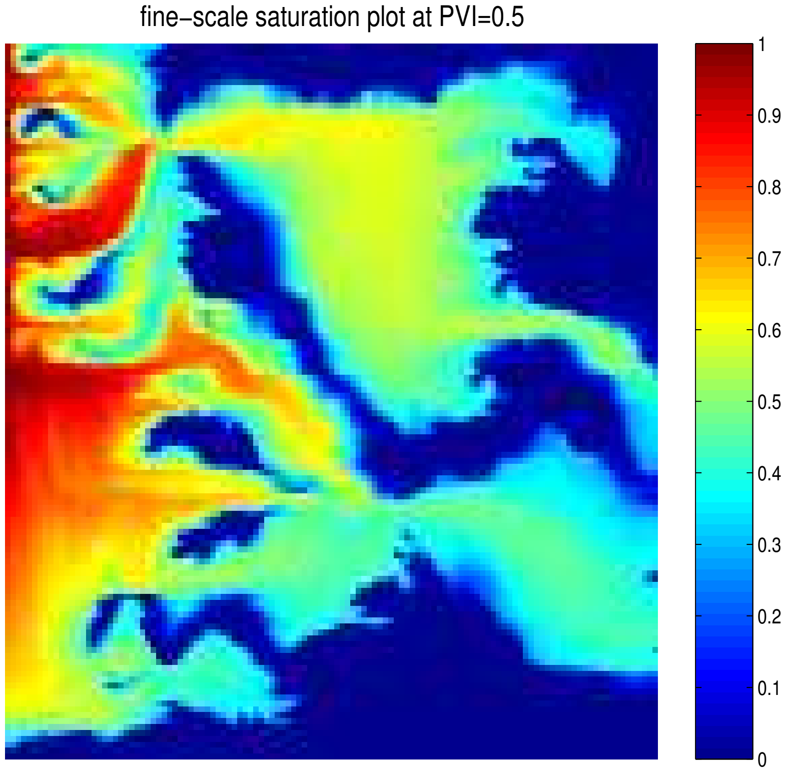


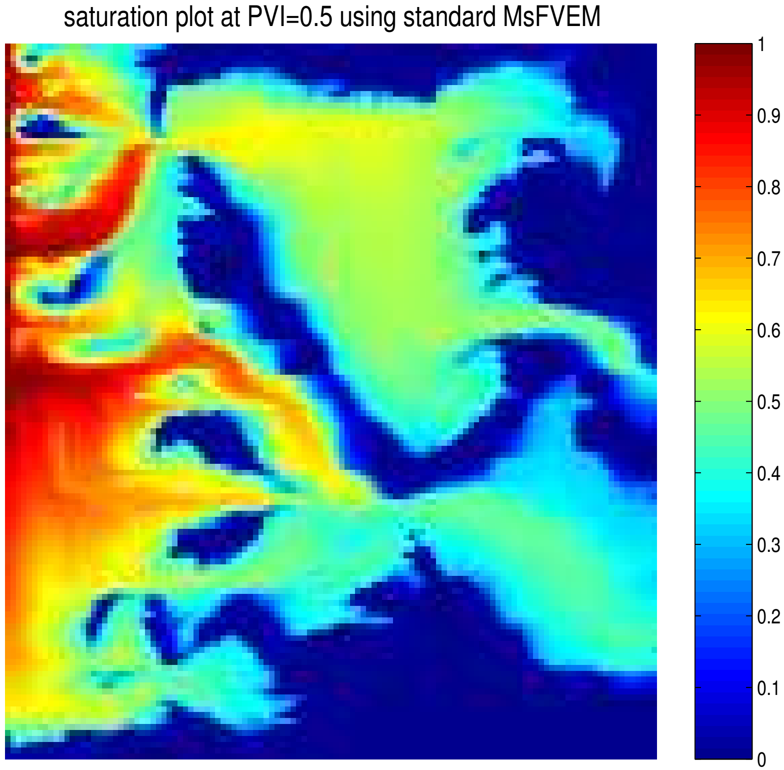
Two-point geostatistics



Fractional flow and total flow for a realization of permeability field with exponential variogram and $l_x = 0.4$, $l_z = 0.02$, $\sigma = 1.5$.

Two-point geostatistics





Applications of MsFEM

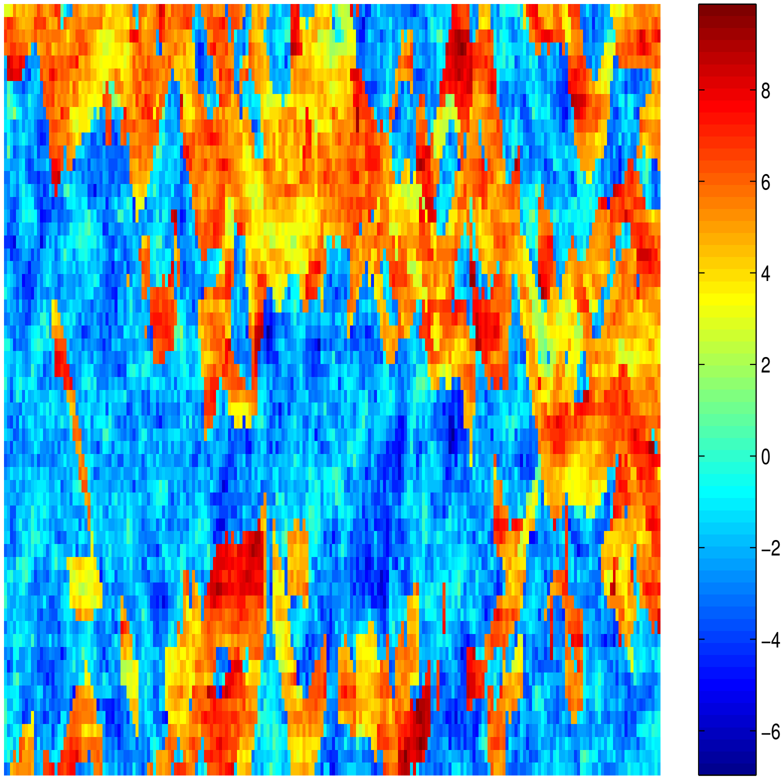
2) Obtain coarse-scale equations for the saturation equation. The approximate macro scale equation is (Efendiev et al., 2000, 2002, 2004)

$$\frac{\partial \overline{S}}{\partial t} + \overline{v} \cdot \nabla f(\overline{S}) = \nabla_i f'(\overline{S})^2 D^{ij} \nabla_j \overline{S}$$

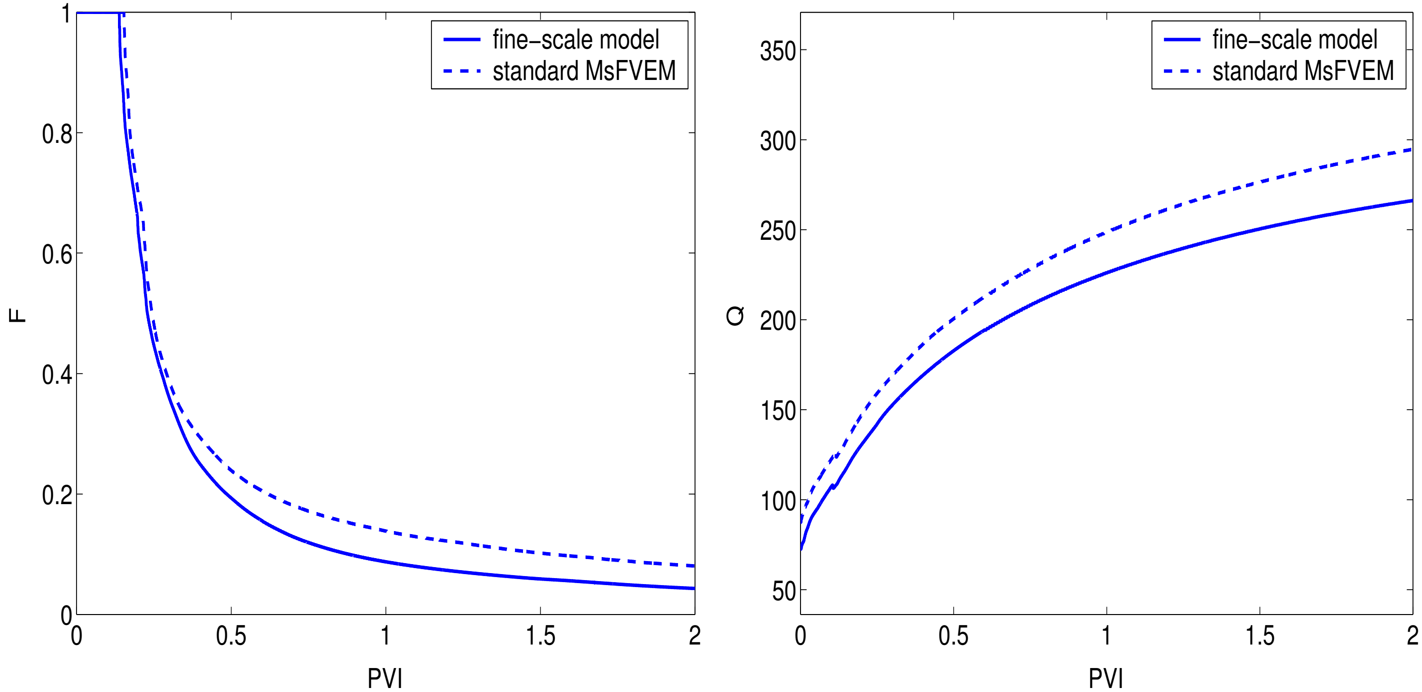
- D^{ij} depends on two point correlation of the velocity field and $\overline{S}$.
- The overall approach is obtained by combining the saturation equation with the pressure equation in the form $div(\lambda(\overline{S})k\nabla p) = 0$.
- The multiscale base functions are constructed once. The two-point correlation of the velocity can be found using the multiscale base functions. This approach is very efficient and can predict the quantity of interest on a highly coarsened grid.

Channelized permeability fields

Benchmark tests: SPE 10 Comparative Project

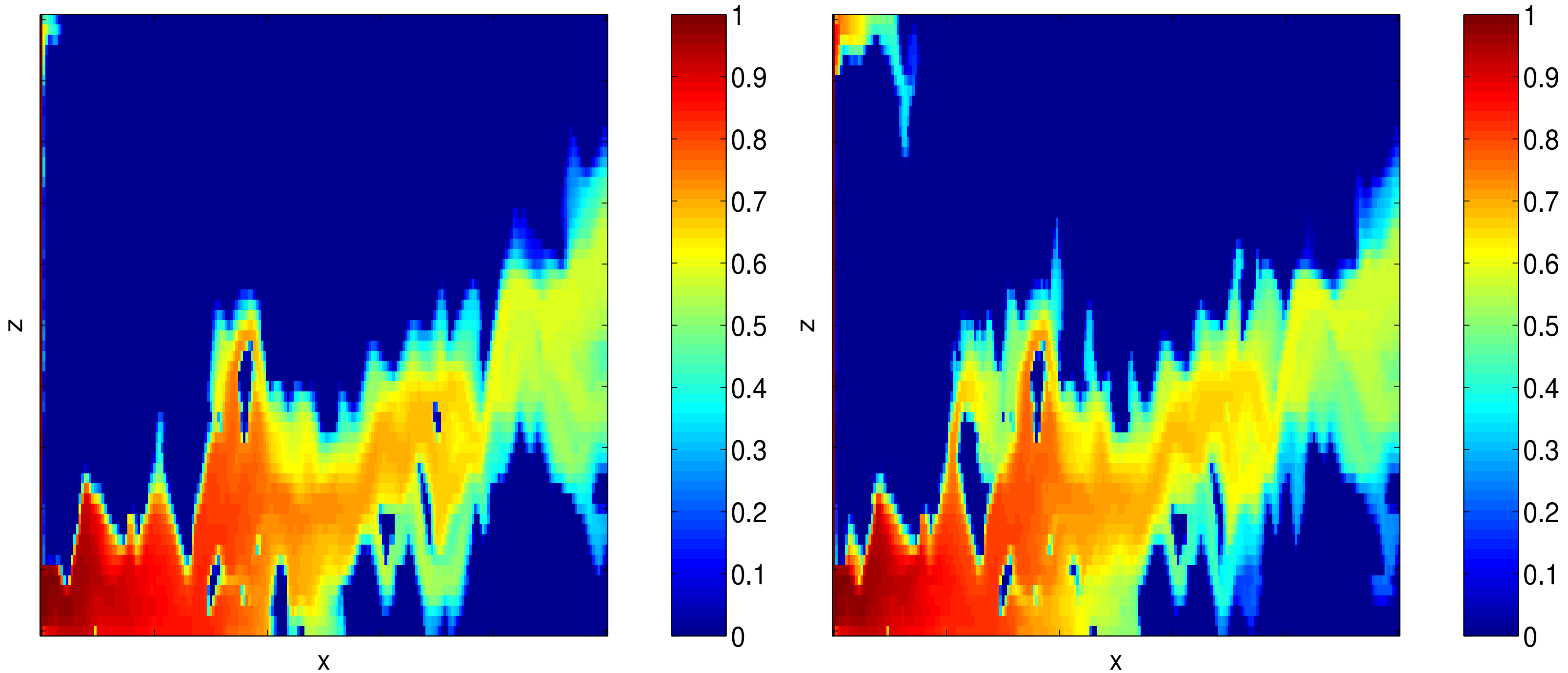


Channelized reservoir



Comparison of upscaled quantities (Layer 43)

Channelized reservoir



Comparison of saturation profile at PVI=0.5: (left) fine-scale model, (right) standard MsFVEM

No-scale separation

- Let $k_\epsilon(x)$ be a general heterogeneous field (no-scale separation).
- With these basis functions we would like to approximate

$$div(\lambda(x)k_\epsilon(x)\nabla p) = g(x),$$

for any $\lambda(x)$ and $g(x)$ that vary on the coarse grid (smooth function).

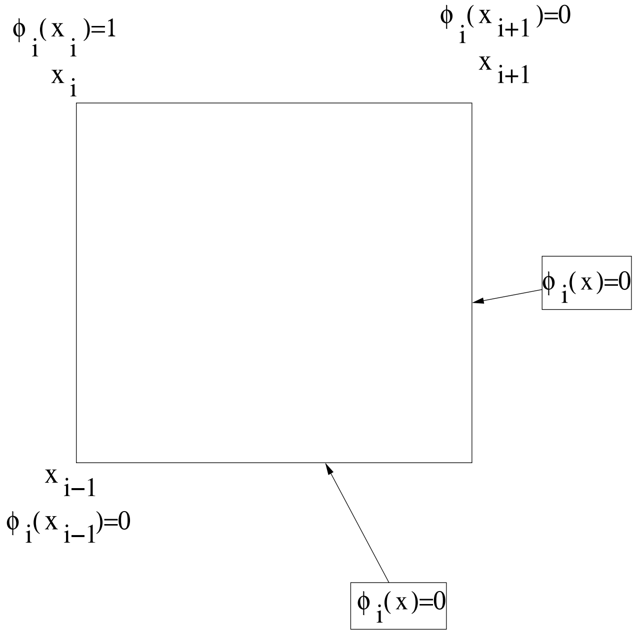
- Some type of “one-time” global information is required, e.g.,

$$div(k_\epsilon(x)\nabla p_1) = f(x),$$

$p_1(x) = p(x)$ on ∂Q.

MsFVEM utilizing global information

- The numerical tests using strongly channelized permeability fields (such as SPE 10 Comparative) show that local basis functions can not accurately capture the long-range information. There is a need to incorporate a global information.
- The main idea is to use the solution of the fine-scale problem at time zero, p^0, to determine the boundary conditions for the multiscale basis formulation.



- These approach is different from oversampling technique.
- Previous related work: J. Aarnes; L. Durlofsky et al.

MsFVEM utilizing global information

- If $p^0(x_i) \neq p^0(x_{i+1})$

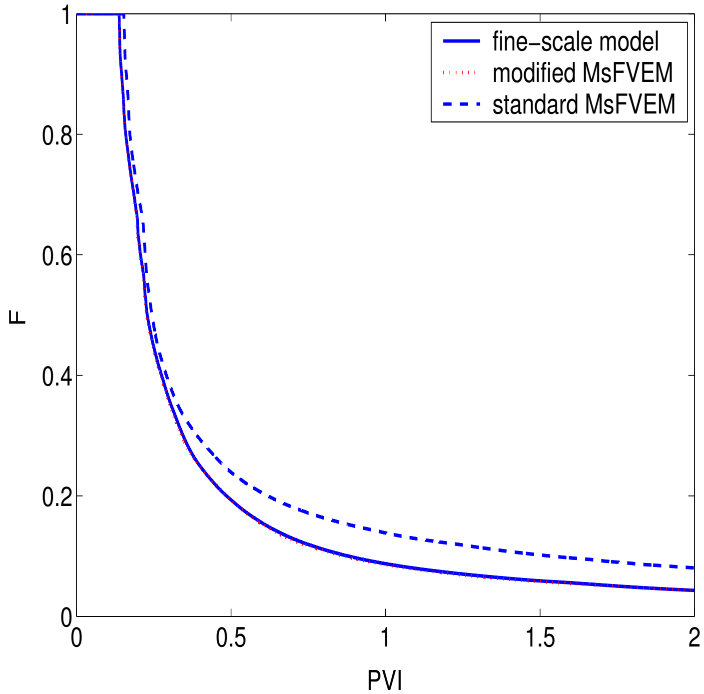
$$g_i(x)|_{[x_i,x_{i+1}]} = \frac{p^0(x) - p^0(x_{i+1})}{p^0(x_i) - p^0(x_{i+1})}, \quad g_i(x)|_{[x_i,x_{i-1}]} = \frac{p^0(x) - p^0(x_{i-1})}{p^0(x_i) - p^0(x_{i-1})}.$$

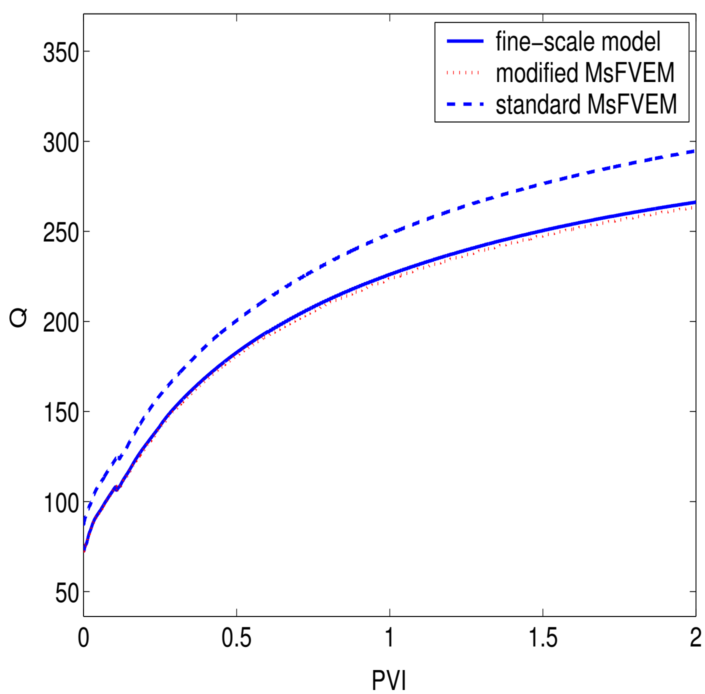
  If $p^0(x_i) = p^0(x_{i+1}) \neq 0$ then

$$g_i|_{[x_i,x_{i+1}]} = \psi_i(x) + \frac{1}{2p^0(x_i)}(p^0(x) - p^0(x_{i+1})),$$

  where $\psi_i(x)$ is a linear function on $[x_i, x_{i+1}]$ such that $\psi_i(x_i) = 1$ and $\psi_i(x_{i+1}) = 0$.
- The modified MsFVEM is exact for linear elliptic problem.
- When global boundary changes, then reevaluation of the basis might be needed.

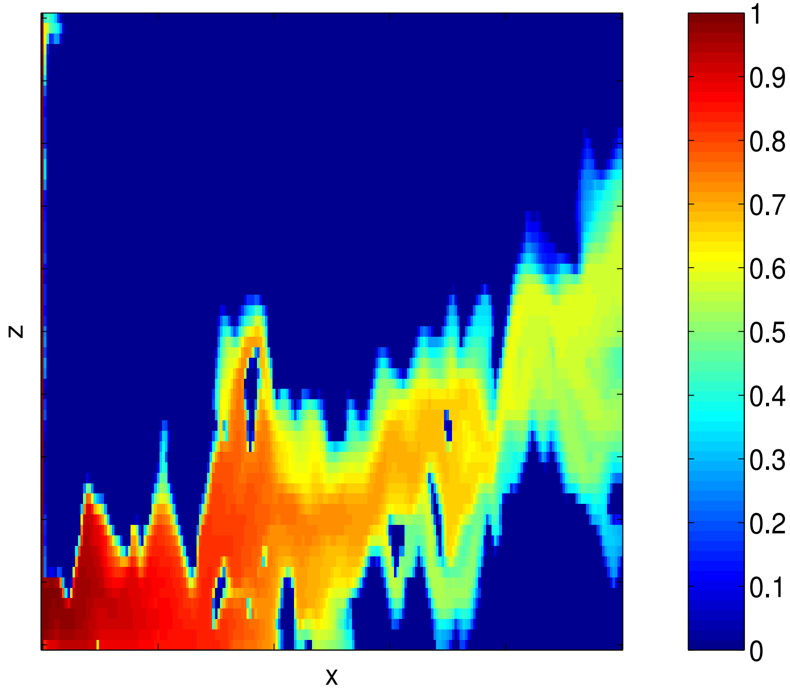
Channelized reservoir

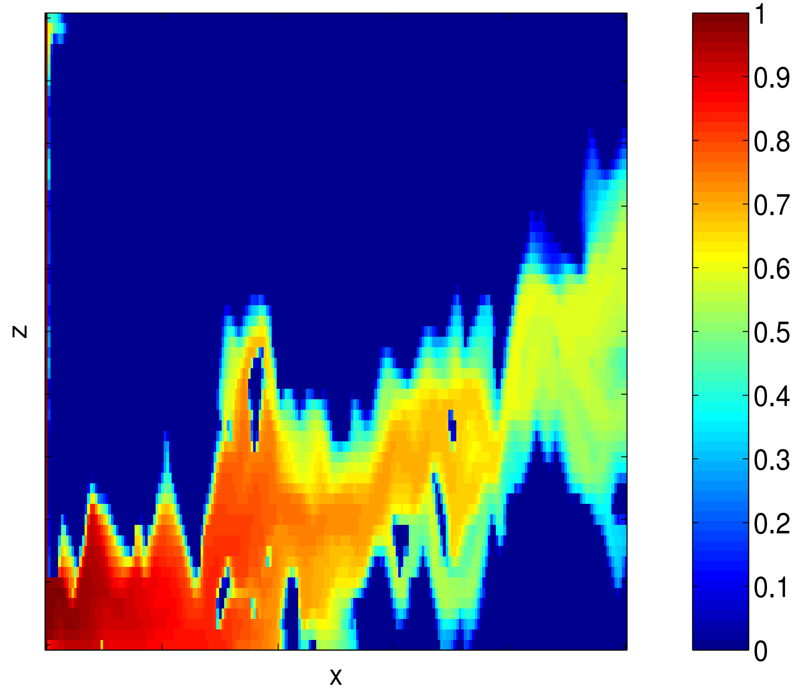




Comparison of upscaled quantities

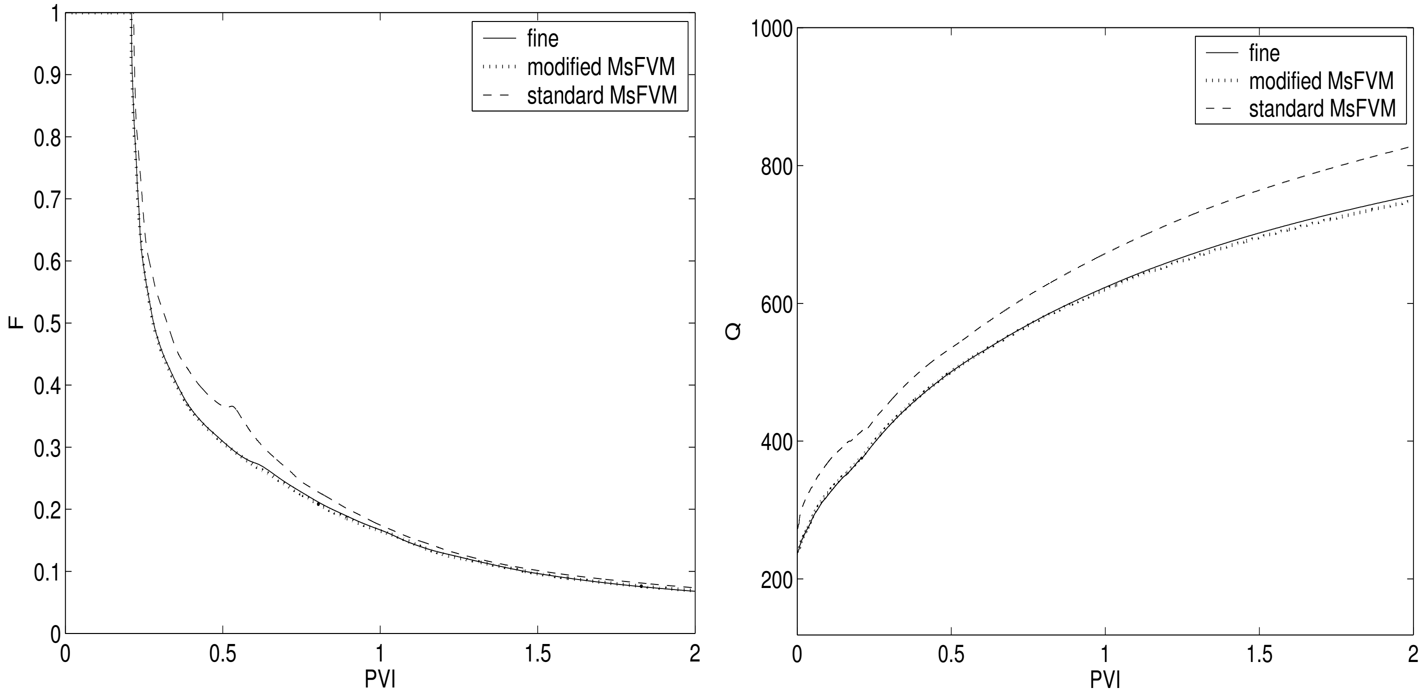
Channelized reservoir





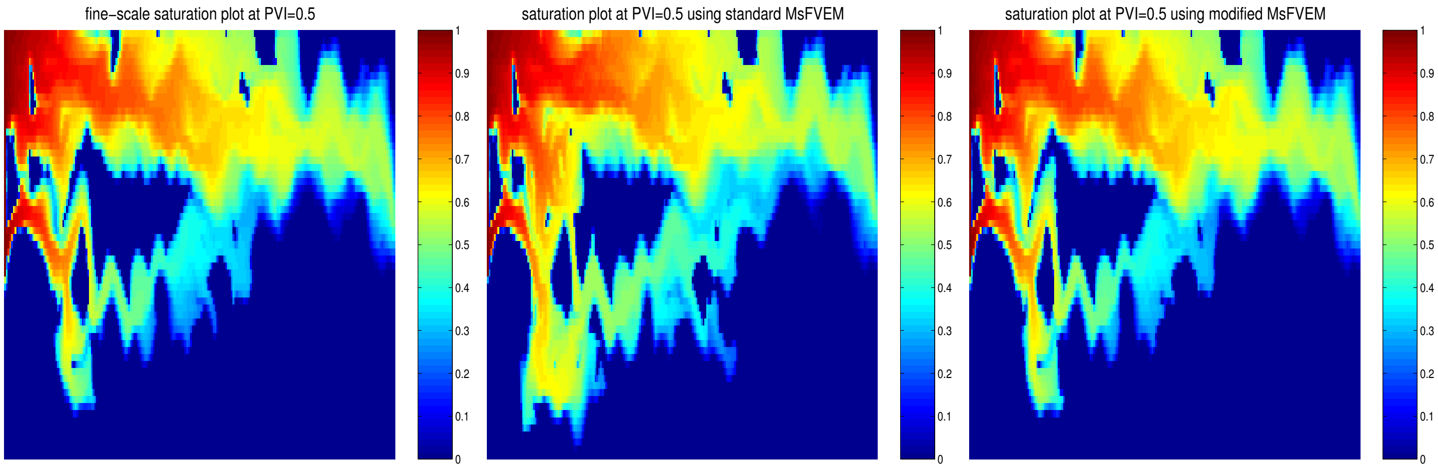
Comparison of saturation profile at PVI=0.5: (left) fine-scale model, (right) modified MsFVEM

Channelized reservoir



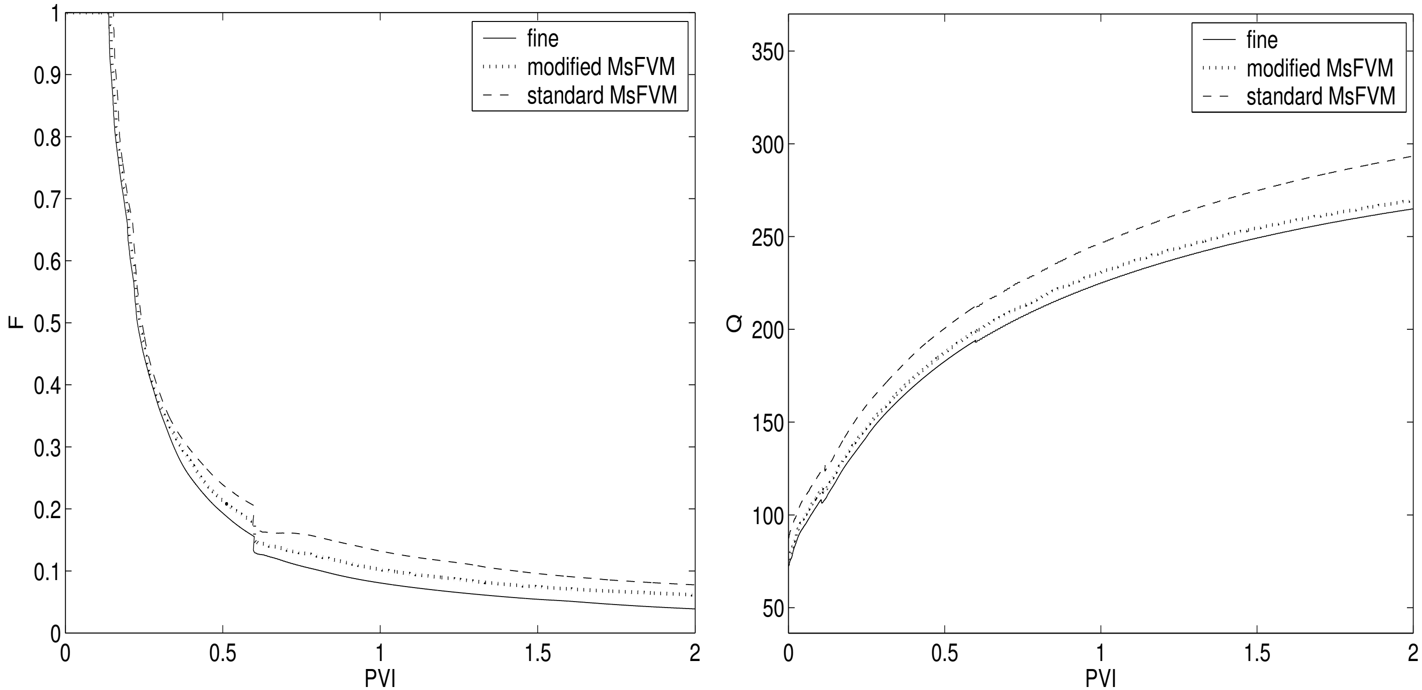
Comparison of upscaled quantities (Layer 59)

Channelized reservoir



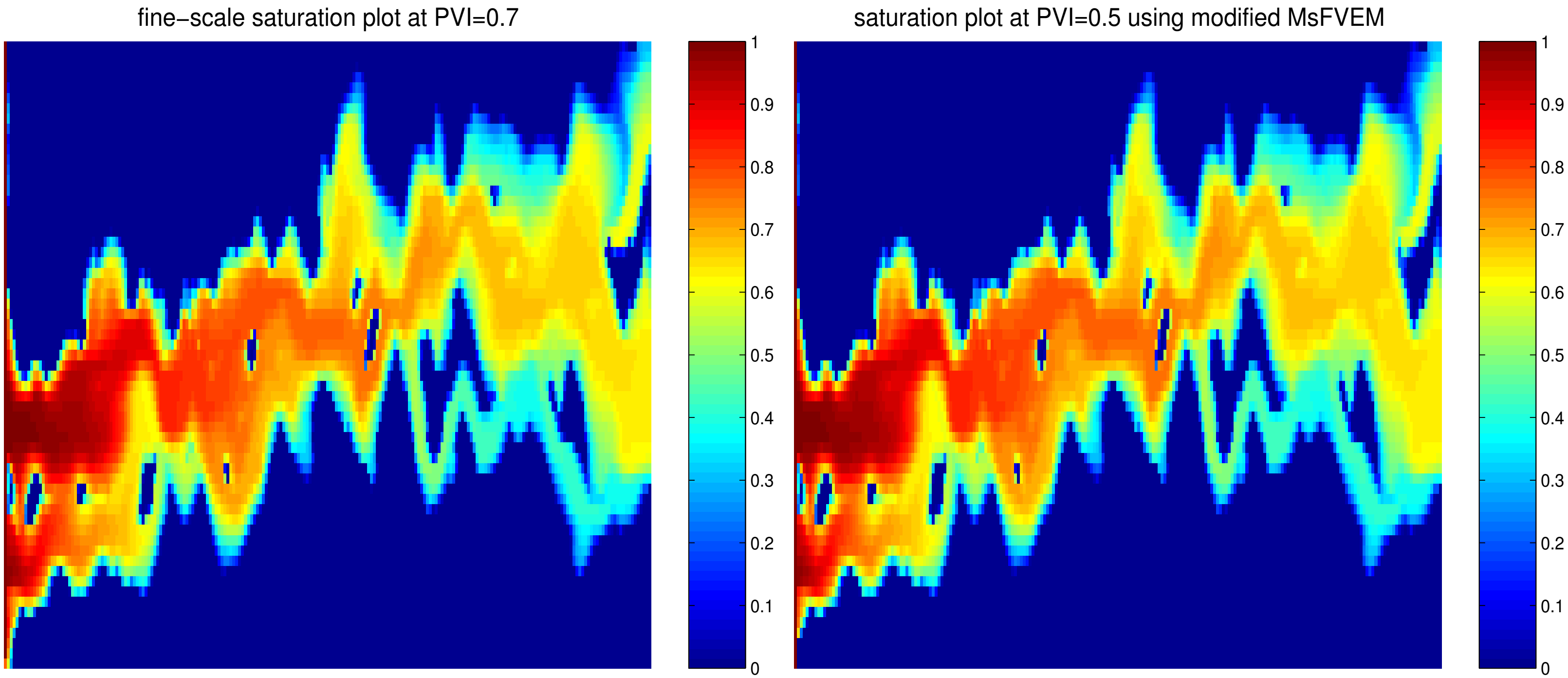
Comparison of saturation profile at PVI=0.5: (left) fine-scale model, (middle) standard MsFVEM (right) modified MsFVEM

Channelized reservoir



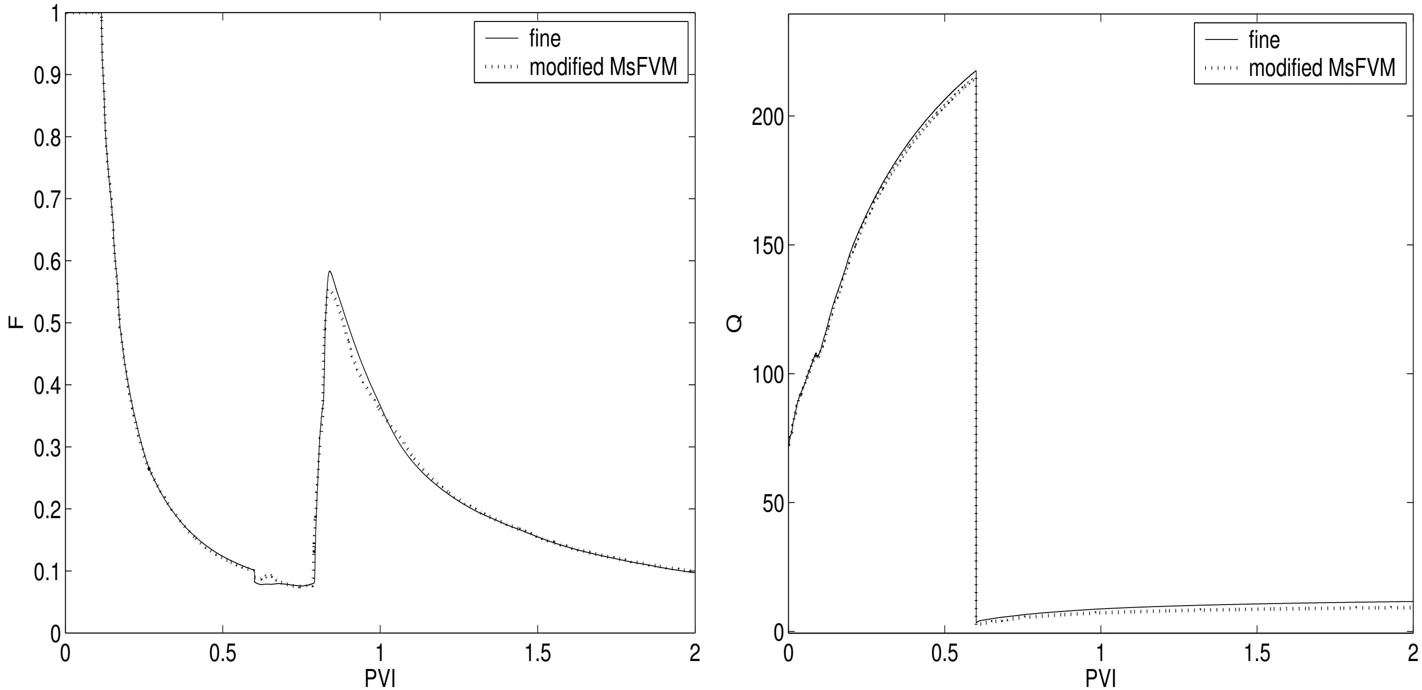
Comparison of upscaled quantities (Layer 43, changing boundary conditions)

Channelized reservoir



Comparison of saturation profile at PVI=0.5: (left) fine-scale model (right) modified MsFVEM (changing boundary condition)

Channelized reservoir



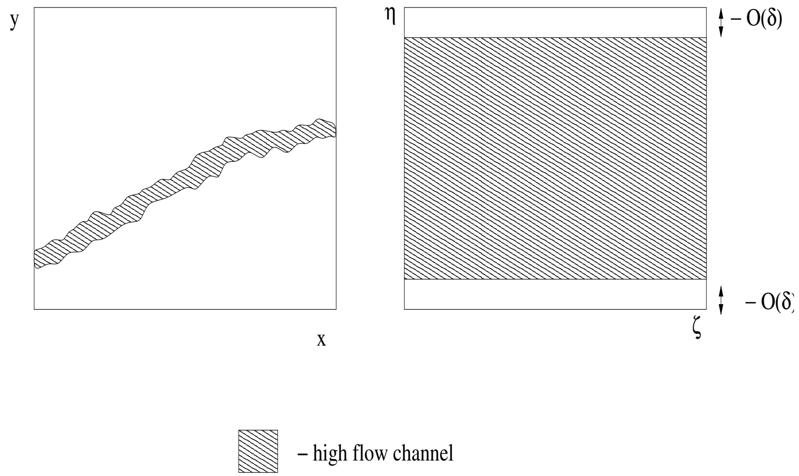
Comparison of upscaled quantities (Layer 43, changing boundary conditions)

Brief Analysis

- Main goal is to show that time-varying pressure is strongly influenced by the initial pressure field.
- Use the streamline-pressure coordinates:

$$\partial\psi/\partial x_1 = -v_2, \quad \partial\psi/\partial x_2 = v_1$$

- Set $\eta = \psi(x, t=0)$ and $\zeta = p(x, t=0)$ and transform as follows:



Brief Analysis

- The transformed pressure equation:

$$\frac{\partial}{\partial \eta}\left(|k|^2\lambda(S)\frac{\partial p}{\partial \eta}\right)+\frac{\partial}{\partial \zeta}\left(\lambda(S)\frac{\partial p}{\partial \zeta}\right)=0$$

- The transformed saturation equation:

$$\frac{\partial S}{\partial t}+(\mathbf{v}\cdot\nabla\eta)\frac{\partial f(S)}{\partial \eta}+(\mathbf{v}\cdot\nabla\zeta)\frac{\partial f(S)}{\partial \zeta}=0$$

- $|k|^2\lambda(S)=|k_0|^2\lambda_0(\zeta,t)1_{Q_{1-\delta}}+|k_1|^2\lambda_1(\eta,\zeta,t)1_{Q_\delta},\ \lambda(S)=\lambda_0(\zeta,t)1_{Q_{1-\delta}}+\lambda_1(\eta,\zeta,t)1_{Q_\delta}$.
- The pressure has the following expansion:

$$p(\eta,\zeta,t)=p_0(\zeta,t)+\delta p_1(\eta,\zeta,t)+\ldots,$$

$\frac{\partial}{\partial \zeta}\left(\lambda_0(\zeta,t)\frac{\partial p_0}{\partial \zeta}\right)=0.$

- Modified basis functions can exactly recover the initial pressure.

Analysis

Assumption G. There exists a sufficiently smooth scalar valued function $G(\eta)$ ($G \in C^3$), such that

$$|p - G(p^{sp})|_{1,Q} \leq C\delta,$$

where δ is sufficiently small.
Under Assumption G and $p^{sp} \in W^{1,s}(Q)$ ($s > 2$), multiscale finite element method converges with the rate given by

$$|p - p_h|_{1,Q} \leq C\delta + Ch^{1-2/s}|p^{sp}|_{W^{1,s}(Q)} \leq C\delta + Ch^{1-2/s}.$$

Mixed finite element methods

In each coarse block K, we construct basis functions for the velocity field

$$div(k(x)\nabla w_i^K) = \frac{1}{|K|} \quad \text{in} \quad K$$

$$k(x)\nabla w_i^K \mathbf{n}^K = \begin{cases} g_i^K & \text{on } e_i^K \\ 0 & \text{else,} \end{cases}$$

For the pressure, the basis functions are taken to be constants. In Chen and Hou, $g_i^K = \frac{1}{|e_i^K|}$ and e_i^K are the edges of K.

Mixed multiscale finite element methods using single-phase flow information is given in the following way (Aarnes, 2004).

Suppose that p^{sp} solves the single-phase flow equation. We set $b_i^K = (k\nabla p^{sp}|_{e_i^K}) \cdot \mathbf{n}^K$ and assume that b_i^K is uniformly bounded. Then the new basis functions for velocity is constructed by solving the following local problems with $g_i^K = b_i^K/\beta_i^K$, where $\beta_i^K = \int_{e_i^K} k\nabla p^{sp} \cdot \mathbf{n}^K ds$.

Lemma. Inf-sup condition holds.

Mixed finite element methods. Analysis

Assume

$$\|\mathbf{u} - A(x)\mathbf{u}^{sp}\|_{0,Q} \leq \delta$$

and

$$|\sum_i A_i \int_{\partial e_i^K} \mathbf{u}^{sp}\mathbf{n}^K ds| \leq C\delta_1 h^2.$$

Then

$$\|\mathbf{u} - \mathbf{u}_h\|_{H(div,Q)} + \|p - p_h\|_{0,Q} \leq C\delta + C\delta_1 + Ch^\gamma.$$

Extensions

Assume there exists a sufficiently smooth scalar valued function $G(\eta)$, $\eta \in R^N$ ($G \in C^3$), such that

$$|p - G(u_1, ..., u_N)|_{1,Q} \leq C\delta,$$

where δ is sufficiently small.

Let ω_i be a patch, and define ϕ_i^0 to be piecewise linear basis function in patch ω_i, such that $\phi_i^0(x_j) = \delta_{ij}$. For simplicity of notation, denote $u_1 = 1$. Then, the multiscale finite element method for each patch ω_i is constructed by

$$\psi_{ij} = \phi_i^0 u_j$$

where $j = 1, .., N$ and i is the index of nodes. First, we note that in each K, $\sum_{i=1}^n \psi_{ij} = u_j$ is the desired single-phase flow solution.

Theorem. Assume $u_i \in W^{1,s}(Q)$, $s > 2$, $i = 1, ..., N$. Then

$$|p - p_h|_{1,Q} \leq C\delta + Ch^{1-2/s}.$$

An approach to general 2nd order pdes

(H. Owhadi and L. Zhang, CPAM, 2006)

$$div(\lambda(x)k(x)\nabla p) = 0,$$

where $\lambda(x)$ is smooth function, while $k(x)$ is rough (e.g., $k(x) = k(x/\epsilon)$).
Take u_1 and u_2 that satisfy

$$-div(k(x)\nabla u_i) = 0 \text{ in } Q,$$

$u_i = x_i$ on ∂Q. Then, $p(u_1, u_2) \in W^{2,p}$ because it satisfies

$$a_{ij}\frac{\partial^2 p}{\partial u_i \partial u_j} \approx 0.$$

(e.g., $p = p_0 + \epsilon^2 N(x/\epsilon)...$).
Owhadi and Zhang showed that the non-conforming method with basis functions that span u_1 and u_2 converge.

Conclusions

- MsFEM on coarse grid. Analysis. Oversampling
- Some applications of MsFEM to porous media flows.
- MsFEM using limited global information.