The relativistic Vlasov-Maxwell system for laser-plasma interaction

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Plan

1) The reduced Vlasov-Maxwell model ;

2) Mild solutions ;

3) Global existence and uniqueness of the mild solution ;

4) The one dimensional Nordström-Vlasov system ;

5) Numerical scheme.

1) The reduced Vlasov-Maxwell model

$$\partial_t F + v(p) \cdot \nabla_x F - e(E(t,x) + v(p) \wedge B(t,x)) \cdot \nabla_p F = 0$$

$$\partial_t E - c^2 \mathrm{curl} B = \frac{e}{\varepsilon_0} j, \ \partial_t B + \mathrm{curl} E = 0, \ \mathrm{div} E = \frac{e}{\varepsilon_0}(\rho_{ext} - \rho), \ \mathrm{div} B = 0$$

$$\rho(t,x) = \int_{\mathbb{R}^3} F(t,x,p)\, dp, \ \ j(t,x) = \int_{\mathbb{R}^3} v(p) F(t,x,p)\, dp$$

$$v(p) = \frac{p}{m}\left(1 + \frac{|p|^2}{m^2 c^2}\right)^{-1/2}$$

Assumptions :

- all unknowns depend on only one space variable x_1
- the electrons are monokinetic in the directions p_2, p_3 :

$$F(t,x,p) = f(t,x_1,p_1)\delta(p_2 - p_2(t,x_1))\delta(p_3 - p_3(t,x_1))$$

$$B = \text{curl} A, \quad E = -\partial_t A - \nabla_x \Phi$$

$$B_1 = 0, \quad B_2 = -\partial_{x_1} A_3, \quad B_3 = \partial_{x_1} A_2$$

$$E_1 = -\partial_t A_1 - \partial_{x_1}\Phi, \quad E_2 = -\partial_t A_2, \quad E_3 = -\partial_t A_3$$

$$\frac{dX}{ds} = v(P(s)), \quad \frac{dP}{ds} = -e(E(s,X(s)) + v(P(s)) \wedge B(s,X(s)))$$

$$\frac{dP_2}{ds} = -e(-\partial_t A_2 - v_1(P(s))\ \partial_{x_1} A_2) = e\frac{d}{ds}A_2(s, X_1(s))$$

$$\frac{dP_3}{ds} = -e(-\partial_t A_3 - v_1(P(s))\ \partial_{x_1} A_3) = e\frac{d}{ds}A_3(s, X_1(s))$$

$$\partial_t f + \frac{p_1}{m\gamma}\partial_{x_1} f - e(E_1 + \frac{e}{m\gamma}A_2\partial_{x_1}A_2 + \frac{e}{m\gamma}A_3\partial_{x_1}A_3)\partial_{p_1} f = 0$$

$$\gamma(t, x_1, p_1) = \left(1 + \frac{|p_1|^2}{m^2c^2} + \frac{e^2}{m^2c^2}(|A_2(t,x_1)|^2 + |A_3(t,x_1)|^2)\right)^{\frac{1}{2}}$$

Dimensionless equations :

$$\partial_t f + \frac{p}{\gamma_1}\partial_x f - \left(E(t,x) + \frac{A(t,x)}{\gamma_2}\partial_x A\right)\partial_p f = 0,\ (t,x,p) \in]0,T[\times\mathbb{R}^2$$

$$\partial_t^2 A - \partial_x^2 A = -\rho_{\gamma_2}(t,x)A(t,x),\ (t,x) \in]0,T[\times\mathbb{R}$$

$$\partial_t E = j(t,x),\ (t,x) \in]0,T[\times\mathbb{R}$$

$$\partial_x E = \rho_{ext}(x) - \rho(t,x),\ (t,x) \in]0,T[\times\mathbb{R}$$

$$\{\rho, \rho_{\gamma_2}, j\}(t,x) = \int_{\mathbb{R}} \{1, \frac{1}{\gamma_2}, \frac{p}{\gamma_1}\} f(t,x,p)\, dp$$

$$f(0,\cdot,\cdot) = f_0,\ \ (E, A, \partial_t A)(0,\cdot) = (E_0, A_0, A_1)$$

$$\text{Non}-\text{relativistic model NR}: \ \gamma_1 = \gamma_2 = 1 \ ;$$

$$\text{Quasi}-\text{relativistic model QR}: \ \gamma_1 = \sqrt{1+|p|^2}, \gamma_2 = 1 \ ;$$

$$\text{Fully}-\text{relativistic model FR}: \ \gamma_1 = \gamma_2 = \sqrt{1+|p|^2+|A(t,x)|^2} \ ;$$

J.A. Carrillo and S. Labrunie, *Global solutions for the one-dimensional Vlasov-Maxwell system for laser-plasma interaction*, Math. Models Methods Appl. Sci. 16(2006) 19-57

2) Mild solutions

$$E \in L^\infty(]0,T[;W^{1,\infty}(\mathbb{R})), \quad A \in L^\infty(]0,T[;W^{2,\infty}(\mathbb{R}))$$

$$\frac{dX}{ds} = \frac{P(s)}{\gamma_1}, \quad \frac{dP}{ds} = -E(x,X(s)) - \frac{A(s,X(s))}{\gamma_2}\partial_x A(s,X(s))$$

$$f(t,x,p) = f_0(X(0;t,x,p),P(0;t,x,p))$$

$$\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} f\psi \, dp\, dx\, dt = \int_{\mathbb{R}}\int_{\mathbb{R}} f_0 \int_0^T \psi(s,X(s;0,x,p),P(s;0,x,p))\, ds\, dp\, dx$$

Continuous dependence of characteristics with respect to (E, A)

$$E, \tilde{E} \in L^{\infty}(]0,T[; W^{1,\infty}(\mathbb{R})), A, \tilde{A} \in L^{\infty}(]0,T[; W^{2,\infty}(\mathbb{R}))$$

$$(t,x,p), (t,\tilde{x},\tilde{p}) \in [0,T] \times \mathbb{R}^2$$

$$(E,A) \rightsquigarrow (X,P)(s;t,x,p), \quad (\tilde{E},\tilde{A}) \rightsquigarrow (\tilde{X},\tilde{P})(s;t,\tilde{x},\tilde{p})$$

$$\left(|X(s) - \tilde{X}(s)|^2 + |P(s) - \tilde{P}(s)|^2\right)^{\frac{1}{2}} \leq (|x - \tilde{x}|^2 + |p - \tilde{p}|^2)^{\frac{1}{2}}$$
$$+ \quad C \left| \int_t^s \{\|(E - \tilde{E})(\tau)\|_{\infty} + \|(A - \tilde{A})(\tau)\|_{1,\infty}\} d\tau \right| .$$

3) Global existence and uniqueness of the mild solution

$$E \in L^\infty(]0,T[; W^{1,\infty}(\mathbb{R})), A \in L^\infty(]0,T[; W^{2,\infty}(\mathbb{R}))$$

$$(E, A) \to f_{E,A} \to (\tilde{E}, \tilde{A}) =: \mathcal{F}(E, A)$$

$$\int_{\mathbb{R}} \tilde{E}(t,x)\varphi(x)\, dx = \int_{\mathbb{R}} E_0 \varphi\, dx + \int_{\mathbb{R}}\int_{\mathbb{R}} f_0 \int_x^{X(t;0,x,p)} \varphi(u) du\, dp\, dx, \ \varphi \in L^1(\mathbb{R})$$

$$\begin{aligned} \tilde{A}(t,x) &= \frac{1}{2}(A_0(x+t) + A_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} A_1(y)\, dy \\ &- \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} (\rho_{\gamma_2} A)(s,y)\, dy\, ds \end{aligned}$$

$$
\begin{aligned}
\int_{\mathbb{R}} \tilde{E}(t,x)\varphi(x)\,dx &= \int_{\mathbb{R}} E_0\varphi\,dx + \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} f_{E,A}(s,x,p)\frac{p}{\gamma_1}\varphi(x)\,dp\,dx\,ds \\
&= \int_{\mathbb{R}} E_0\varphi\,dx + \int_{\mathbb{R}}\int_{\mathbb{R}} f_0 \int_0^t \frac{dX}{ds}\varphi(X(s))\,ds\,dp\,dx \\
&= \int_{\mathbb{R}} E_0\varphi\,dx + \int_{\mathbb{R}}\int_{\mathbb{R}} f_0 \int_x^{X(t;0,x,p)} \varphi(u)\,du\,dp\,dx
\end{aligned}
$$

$$
\|\tilde{E}(t)\|_{L^\infty(\mathbb{R})} \le \|E_0\|_{L^\infty(\mathbb{R})} + \|f_0\|_{L^1(\mathbb{R}^2)}, \quad t \in [0,T]
$$

Invariance of the domain

$$D_T = \{(E,A) \in W^{1,\infty}(]0,T[\times\mathbb{R}) \times W^{2,\infty}(]0,T[\times\mathbb{R}) \mid \|E\|_{W^{1,\infty}(]0,T[\times\mathbb{R})} + \|A\|_{W^{2,\infty}(]0,T[\times\mathbb{R})} \le C\},$$

Estimate of $\mathcal{F}(E_1,A_1)-\mathcal{F}(E_2,A_2)$ w.r.t. $(E_1,A_1)-(E_2,A_2)$

$$|||\mathcal{F}(E_1,A_1)(t) - \mathcal{F}(E_2,A_2)(t)||| \le C\,\|A_1(t)-A_2(t)\|_{L^\infty} + C\int_0^t |||(E_1-E_2, A_1-A_2)(s)|||\,ds$$

$$|||(E,A)(t)||| = \|E(t)\|_{L^\infty} + \|A(t)\|_{L^\infty} + \|\partial_x A(t)\|_{L^\infty} + \|\partial_t A(t)\|_{L^\infty}$$

Hypotheses : $\exists\, n_0 : \mathbb{R} \to [0, +\infty[$ nondecreasing on $\mathbb{R}^-$ and nonincreasing on $\mathbb{R}^+$ such that

$$f_0(x,p) \le n_0(p), \ \ \forall\, (x,p) \in \mathbb{R}^2$$

$$M_0 + M_1 := \int_{\mathbb{R}} n_0(p)\, dp + \int_{\mathbb{R}} |p| n_0(p)\, dp < +\infty$$

$$M_\infty := \|n_0\|_{L^\infty(\mathbb{R})} < +\infty$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} (1 + |p|) f_0(x,p)\, dp\, dx < +\infty$$

$$E_0 \in W^{1,\infty}(\mathbb{R}), A_0 \in W^{2,\infty}(\mathbb{R}), A_1 \in W^{1,\infty}(\mathbb{R})$$

Lemma

For any $(E, A) \in D_T$ we have

$$\left\| \int_{\mathbb{R}} (1 + |p|) f_{E,A}(\cdot, \cdot, p) \, dp \right\|_{L^\infty} \leq C \, (M_0 + M_1 + M_\infty),$$

$$\int_{\mathbb{R}} \int_{\mathbb{R}} (1+|p|) f_{E,A}(t, x, p) \, dp \, dx \leq C \int_{\mathbb{R}} \int_{\mathbb{R}} (1+|p|) f_0 \, dp \, dx, \quad t \in [0, T],$$

where C is a constant depending on T and the initial conditions.

Estimates for the second derivatives of $\tilde{A}$

$$\begin{aligned}\partial_x \tilde{A}(t,x) &= \frac{1}{2}\{A_0'(x+t) + A_0'(x-t)\} + \frac{1}{2}\{A_1(x+t) - A_1(x-t)\} \\ &+ \frac{1}{2}D^-(t,x) - \frac{1}{2}D^+(t,x)\end{aligned}$$

$$\begin{aligned}\partial_t \tilde{A}(t,x) &= \frac{1}{2}\{A_0'(x+t) - A_0'(x-t)\} + \frac{1}{2}\{A_1(x+t) + A_1(x-t)\} \\ &- \frac{1}{2}D^+(t,x) - \frac{1}{2}D^-(t,x)\end{aligned}$$

$$D^{\pm}(t,x) = \int_0^t (\rho_{\gamma_2}\, A)(s, x \pm (t-s))\, ds$$

$$\int_{\mathbb{R}} D^{\pm}(t,x)\varphi'(x)\,dx \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p) I^{\pm}(t,x,p)\,dp\,dx$$

$$I^{\pm}(t,x,p) = \int_0^t \frac{A(s,X(s;0,x,p))}{\gamma_2(s)}\varphi'(X(s;0,x,p) \mp (t-s))\,ds$$

$$\gamma_2(s) = (1 + |P(s;0,x,p)|^2 + |A(s,X(s;0,x,p))|^2)^{1/2}$$

$$\left|\frac{dX}{ds}\right| = \left|\frac{P(s;0,x,p)}{\gamma_1(s)}\right| < 1$$

$$
\begin{aligned}
I^{\pm}(t,x,p) &= \int_0^t \frac{A(s,X(s))}{\gamma_2(s)(X'(s)\pm 1)}\frac{d}{ds}\varphi(X(s)\mp(t-s))\,ds \\
&= \frac{A(s,X(s))}{\gamma_2(s)(X'(s)\pm 1)}\varphi(X(s)\mp(t-s))\Big|_{s=0}^{s=t} \\
&- \int_0^t \frac{d}{ds}\left\{\frac{A(s,X(s))}{\gamma_2(s)(X'(s)\pm 1)}\right\}\varphi(X(s)\mp(t-s))\,ds
\end{aligned}
$$

$$
\frac{1}{|\gamma_2(s)(X'(s)\pm 1)|} \le 2(1+|P(s)|)
$$

$$
\left|\frac{d}{ds}\left\{\frac{A(s,X(s))}{\gamma_2(s)(X'(s)\pm 1)}\right\}\right| \le C(1+|P(s)|)
$$

$$\left| \int_{\mathbb{R}} D^{\pm}(t,x)\varphi'(x)\, dx \right| \leq C\, \|\varphi\|_{L^1}$$

$$\|\partial_x D^{\pm}\|_{L^\infty} \leq C$$

$$\|\partial_x^2 \tilde{A}\|_{L^\infty} \leq \|A_0''\|_{L^\infty} + \|A_1'\|_{L^\infty} + C$$

$$\|\partial_{xt}^2 \tilde{A}\|_{L^\infty} \leq \|A_0''\|_{L^\infty} + \|A_1'\|_{L^\infty} + C$$

$$\|\partial_t^2 \tilde{A}(t)\|_{L^\infty} \;\leq\; \|\partial_x^2 \tilde{A}(t)\|_{L^\infty} + \|\rho_{\gamma_2}(t)\|_{L^\infty}\, \|A(t)\|_{L^\infty}$$

Estimate for $\tilde{E}_1 - \tilde{E}_2$

$$\left|\int_{\mathbb{R}} (\tilde{E}_1(t,x) - \tilde{E}_2(t,x))\varphi \, dx\right| \le \int_{\mathbb{R}}\int_{\mathbb{R}} f_0 \left|\int_{X_2(t;0,x,p)}^{X_1(t;0,x,p)} |\varphi(u)| du\right| \, dp \, dx$$

$$\le \; C\|\varphi\|_{L^1(\mathbb{R})} \int_0^t |||(E_1 - E_2, A_1 - A_2)(s)||| ds$$

$$\|(\tilde{E}_1 - \tilde{E}_2)(t)\|_{L^\infty} \le C \int_0^t |||(E_1 - E_2, A_1 - A_2)(s)||| \, ds, \quad \forall \, t \in [0, T]$$

Estimates for $\tilde{A}_1 - \tilde{A}_2, \nabla_{(t,x)}\tilde{A}_1 - \nabla_{(t,x)}\tilde{A}_2$

$$\begin{aligned} \|\nabla_{(t,x)}\tilde{A}_1(t) - \nabla_{(t,x)}\tilde{A}_2(t)\|_{L^\infty} \; &\le \; C \, \|A_1(t) - A_2(t)\|_{L^\infty} \\ &+ \; C \int_0^t |||(E_1 - E_2, A_1 - A_2)(s)||| ds \end{aligned}$$

Theorem

There is a unique global mild solution (f, E, A) for the reduced relativistic Vlasov-Maxwell system. Moreover the solution preserves the total energy

$$\frac{d}{dt}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}} f(t,x,p)\sqrt{1+|p|^2+|A(t,x)|^2}\,dp\,dx + \frac{1}{2}\int_{\mathbb{R}}\{|E(t,x)|^2+|\partial_x A(t,x)|^2+|\partial_t A(t,x)|^2\}\,dx\Big) = 0.$$

4) The one dimensional Nordström-Vlasov system

$$\partial_t f + v(p)\cdot\nabla_x f - \Big((S\phi)p + (1+|p|^2)^{-\frac{1}{2}}\nabla_x \phi\Big)\cdot\nabla_p f = (N+1)f(S\phi)$$

$$\partial_t^2 \phi - \Delta_x \phi = -\mu(t,x)$$

$$\mu(t,x) = \int_{\mathbb{R}^N} \frac{f(t,x,p)}{(1+|p|^2)^{\frac{1}{2}}}\,dp$$

$$f(0,\cdot,\cdot) = f_0,\ \ \phi(0,\cdot) = \varphi_0,\ \ \partial_t\phi(0,\cdot) = \varphi_1$$

$$S = \partial_t + v(p)\cdot\nabla_x$$

S. Calogero and G. Rein, *On classical solutions of the Nordström-Vlasov system*, Comm. Partial Differential Equations **28** (2003) 1863-1885.

S. Calogero and G. Rein, *Global weak solutions to the Nordström-Vlasov system*, J. Differential Equations **204** (2004) 323-338.

Theorem (Calogero and Rein)
Initial data $f_0 \in C^1_c(\mathbb{R}^2)$, $\varphi_0 \in C^2_b(\mathbb{R})$, $\varphi_1 \in C^1_b(\mathbb{R})$ with $f_0 \geq 0$ launch a unique global classical solution $(f, \phi) \in C^1([0, +\infty[\times\mathbb{R}^2) \times C^2([0, +\infty[\times\mathbb{R})$ of the one dimensional Nordström-Vlasov system.

Theorem (MB)

Assume that $\varphi_0 \in W^{2,\infty}(\mathbb{R})$, $\varphi_1 \in W^{1,\infty}(\mathbb{R})$, $f_0 \in L^1(\mathbb{R}^2)$ and that there is some function $g_0 \in L^\infty(\mathbb{R})$ such that

$0 \le f_0(x,p) \le g_0(p)$, $(x,p) \in \mathbb{R}^2$, sign$(\cdot)g_0(\cdot)$ is nonincreasing on $\mathbb{R}$,

and

$$\int_{\mathbb{R}} (1+|p|) g_0(p)\, dp < +\infty.$$

Then there is a unique mild solution $(f \ge 0, \phi) \in L^\infty(]0,T[; L^1(\mathbb{R}^2)) \times W^{2,\infty}(]0,T[\times\mathbb{R})$, $\forall\ T > 0$ for the one dimensional Nordström-Vlasov system.

5) Numerical scheme. The 1D relativistic Vlasov-Maxwell equations

$$\partial_t f + v(p)\partial_x f + E(t,x)\partial_p f = 0, \quad (t,x,p) \in]0,T[\times\mathbb{R}\times\mathbb{R}$$

$$\partial_t E = -j(t,x), \quad \partial_x E = \rho(t,x), \quad (t,x) \in]0,T[\times\mathbb{R}$$

$$\int_{\mathbb{R}} E(t,x)\theta(x)\,dx = \int_{\mathbb{R}} E_0\theta\,dx - \int_{\mathbb{R}}\int_{\mathbb{R}} f_0 \int_x^{X(t;0,x,p)} \theta(u)\,du\,dp\,dx$$

Hypotheses

H1) $\exists\, g_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ nondecreasing on $\mathbb{R}^-$ and nonincreasing on $\mathbb{R}^+$ such that $0 \leq f_0(x,p) \leq g_0(p)$, $(x,p) \in \mathbb{R}^2$

H2) $f_0 \in L^1(\mathbb{R}^2)$

H3) $E_0 \in L^\infty(\mathbb{R})$, $E_0' = \rho_0 := \int_{\mathbb{R}} f_0 \, dp$

Theorem Assume that (f_0, E_0) satisfy the hypotheses H1, H2, H3. Then there is a global unique solution $(f, E) \in L^\infty(]0;T[;L^1(\mathbb{R}^2)) \times W^{1,\infty}(]0,T[\times\mathbb{R})$, $\forall T > 0$.

Let $(t^n, x_i, p_j) = (n\Delta t, i\Delta x, j\Delta p)$, $(n,i,j) \in \mathbb{N} \times \mathbb{Z}^2$ where $\Delta t, \Delta x, \Delta p > 0$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be a compactly supported nonnegative function satisfying $\int_{\mathbb{R}} \varphi(u)\, du = 1$.

$$f_{ij}^0 = \frac{1}{\Delta x \Delta p} \int_{|x-x_i|<\frac{\Delta x}{2}} \int_{|p-p_j|<\frac{\Delta p}{2}} f_0(x,p)\, dp\, dx,\ \forall\, (i,j) \in \mathbb{Z}^2$$

$$\sum_{(i,j)\in\mathbb{Z}^2} \Delta x \Delta p f_{ij}^0 = \|f_0\|_{L^1(\mathbb{R}^2)}$$

Numerical scheme

- start with $(X_{ij}^0, P_{ij}^0) = (x_i, p_j),\ \forall\,(i,j) \in \mathbb{Z}^2$
- for any $n \in \mathbb{N}$ compute

$$X_{ij}^{n+1} = X_{ij}^n + \Delta t\, v(P_{ij}^n),\ \ (i,j) \in \mathbb{Z}^2$$

$$P_{ij}^{n+1} = P_{ij}^n + \Delta t\, E^n(X_{ij}^n),\ \ (i,j) \in \mathbb{Z}^2$$

$$E^n(x) = E_0(x) - \sum_{(i,j)\in\mathbb{Z}^2} f_{ij}^0 \Delta p \int_{x_i}^{X_{ij}^n} \varphi\left(\frac{u-x}{\Delta x}\right)\, du,\ \ x \in \mathbb{R}$$

Theorem

Denote by (f, E) the unique global solution of the 1 D relativistic VM equations, by (X, P) the characteristics associated to the electric field E and by $(X_{ij}^n, P_{ij}^n)_{(n,i,j)\in\mathbb{N}\times\mathbb{Z}^2}$, $(E^n)_{n\in\mathbb{N}}$ the numerical solution. Then

$$\sup_{(i,j)\in\mathbb{Z}^2} (|X_{ij}^n - X(t^n; 0, x_i, p_j)| + |P_{ij}^n - P(t^n; 0, x_i, p_j)|) + \|E^n(\cdot) - E(t^n, \cdot)\|_{L^\infty(\mathbb{R})} \leq C(\Delta t + \Delta x + \Delta p)$$

Approximation of the derivatives

H4) $\exists h_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ nondecreasing on $\mathbb{R}^-$ and nonincreasing on $\mathbb{R}^+$ such that $|\partial_x f_0(x,p)| + |\partial_p f_0(x,p)| \le h_0(p), (x,p) \in \mathbb{R}^2$

H5) $\partial_x f_0, \partial_p f_0 \in L^1(\mathbb{R}^2)$

Theorem

Let (f, E) be the unique global solution of the 1D relativistic Vlasov-Maxwell equations. Then

$$\|E\|_{L^\infty(]0,T[\times\mathbb{R})} + \|\nabla_{(t,x)} E\|_{L^\infty(]0,T[\times\mathbb{R})} + \|\nabla^2_{(t,x)} E\|_{L^\infty(]0,T[\times\mathbb{R})} \le C_T,$$

$$\|f\|_{L^\infty(]0,T[;L^1(\mathbb{R}^2))} + \|\nabla_{(t,x,p)} f\|_{L^\infty(]0,T[;L^1(\mathbb{R}^2))} \le C_T.$$

$$(g_1, g_2, g_3) := (\partial_t f, \partial_x f, \partial_p f),\ (F_1, F_2) := (\partial_t E, \partial_x E)$$

$$\partial_t g_1 + v(p)\partial_x g_1 + E(t,x)\partial_p g_1 = -F_1(t,x) g_3(t,x,p)$$

$$\partial_t g_2 + v(p)\partial_x g_2 + E(t,x)\partial_p g_2 = -F_2(t,x) g_3(t,x,p)$$

$$\partial_t g_3 + v(p)\partial_x g_3 + E(t,x)\partial_p g_3 = -v'(p) g_2(t,x,p)$$

$$\partial_t F_1 = -\int_{\mathbb{R}} v(p) g_1(t,x,p)\, dp,\ \ \partial_t F_2 = -\int_{\mathbb{R}} v(p) g_2(t,x,p)\, dp$$

$$g_1(0,\cdot,\cdot) = -v\partial_x f_0 - E_0 \partial_p f_0,\ \ g_2(0,\cdot,\cdot) = \partial_x f_0,\ \ g_3(0,\cdot,\cdot) = \partial_p f_0$$

$$F_1(0,\cdot) = -\int_{\mathbb{R}} v(p) f_0(\cdot,p)\, dp,\ \ F_2(0,\cdot) = \int_{\mathbb{R}} f_0(\cdot,p)\, dp$$