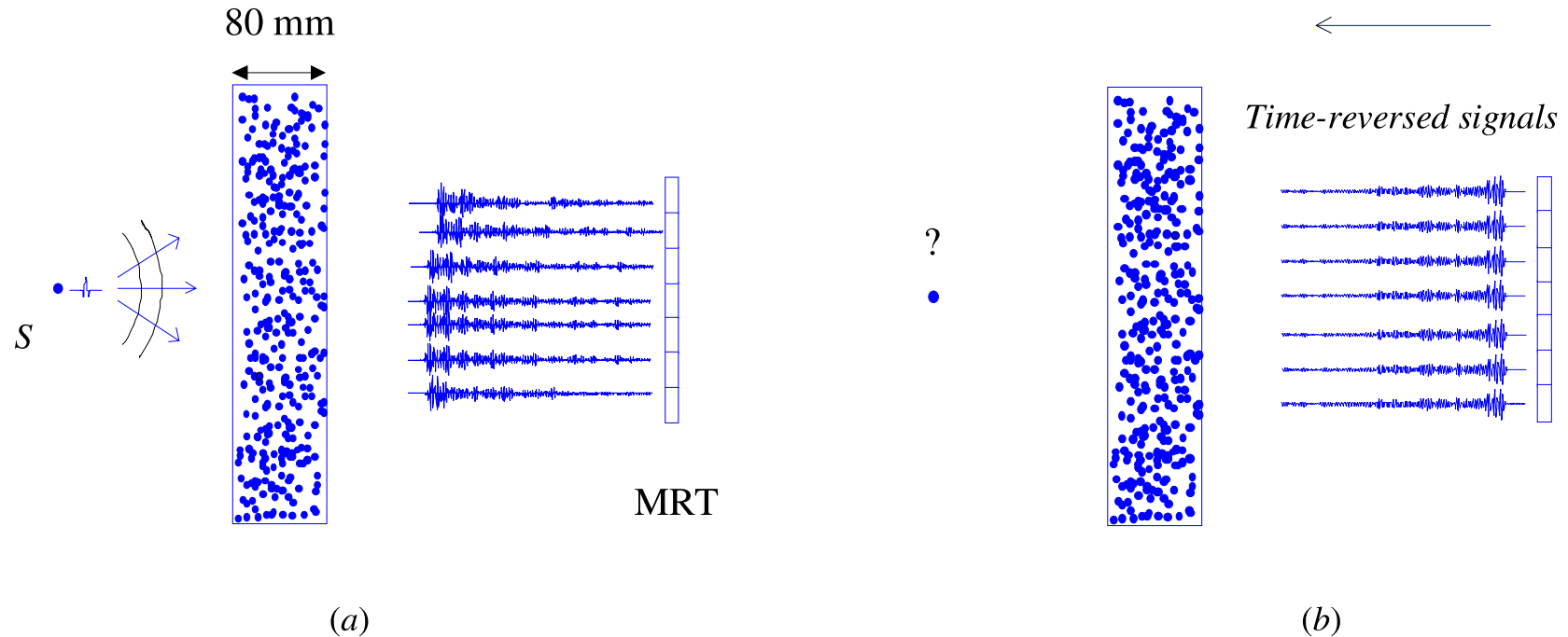


## Ultrasound experiment by M. Fink

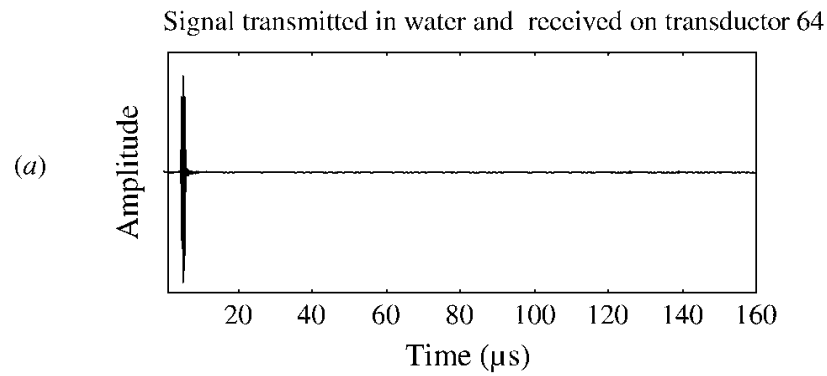
cf. A. Tourin, M. Fink, and A. Derode, Multiple scattering of sound, *Waves Random Media* **10** (2000), R31-R60.



Experimental set-up for a time-reversal experiment through a multiple-scattering medium:

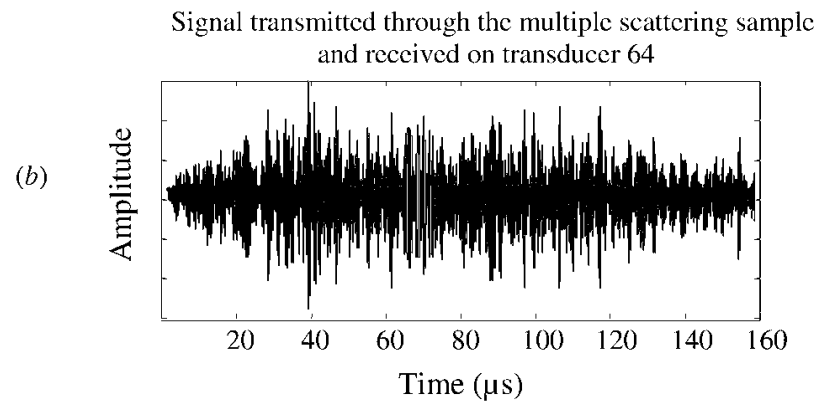
(a) first step, the source sends a pulse through the sample, the transmitted wave is recorded by the TRM.

(b) second step, the multiply scattered signals have been time-reverted, they are retransmitted by the TRM, and  $S$  records the reconstructed pressure field.

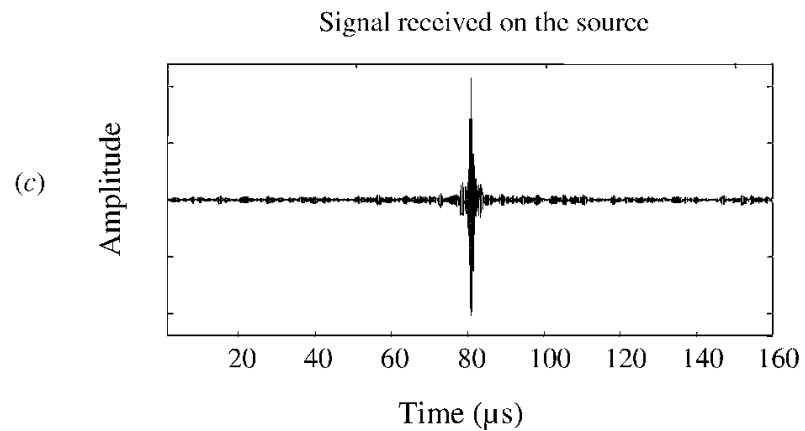


## Experimental observations

The source emits a short  $1\ \mu\text{s}$  pulse.



The TRM records a long scattered signal.



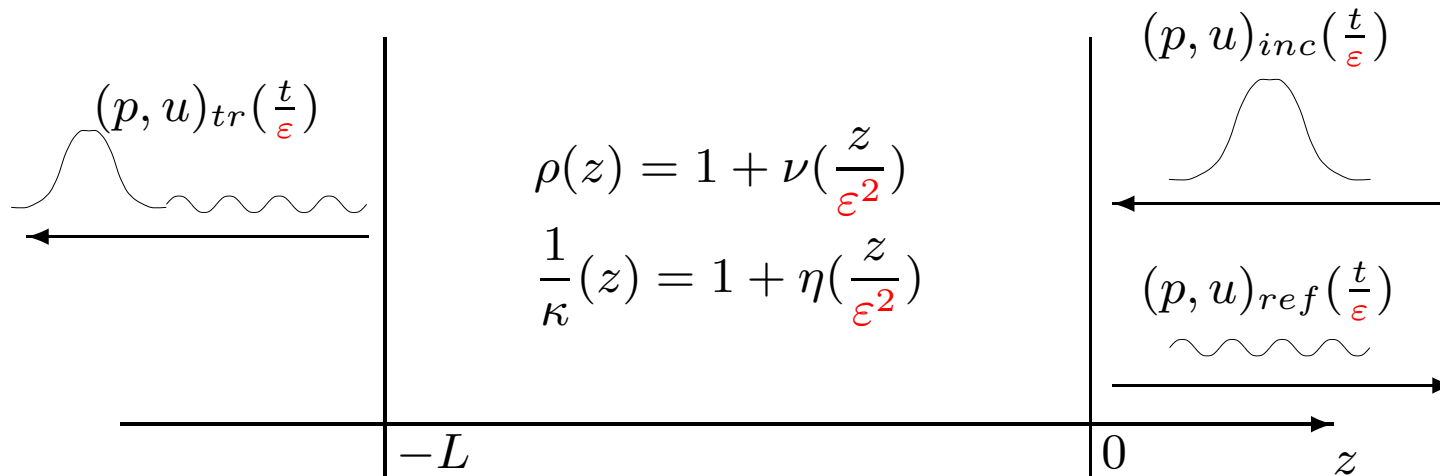
Recompression at the source location after propagation of the time-reversed wave.

## Scattering of an acoustic pulse in random media

Acoustic equations for pressure  $p$  and speed  $u$ :

$$\frac{\partial p}{\partial t} + \kappa(z) \frac{\partial u}{\partial z} = 0$$

$$\rho(z) \frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = 0$$



IC: left-going pulse incoming from the right homogeneous half-space.

$$m(z) = \eta(z) + \nu(z), \quad n(z) = \eta(z) - \nu(z)$$

Local velocity:  $c(z) = \sqrt{\kappa(z)/\rho(z)}$ .

Local impedance:  $I(z) = \rho(z)c(z)$ .

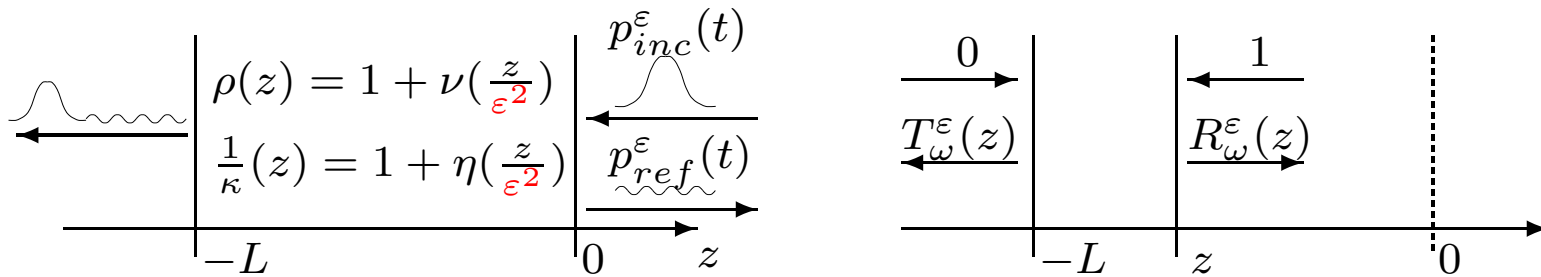
## Integral representation of the reflected signal

Send a left-going pulse  $f(\frac{t}{\varepsilon})$ :

$$p_{inc}^\varepsilon(t, z = 0) = f(\frac{t}{\varepsilon}) = \int e^{\frac{i\omega t}{\varepsilon}} \hat{f}(\omega) d\omega$$

Reflected signal:

$$p_{ref}^\varepsilon(t, z = 0) = \int e^{\frac{i\omega t}{\varepsilon}} \hat{f}(\omega) R_\omega^\varepsilon(0) d\omega$$



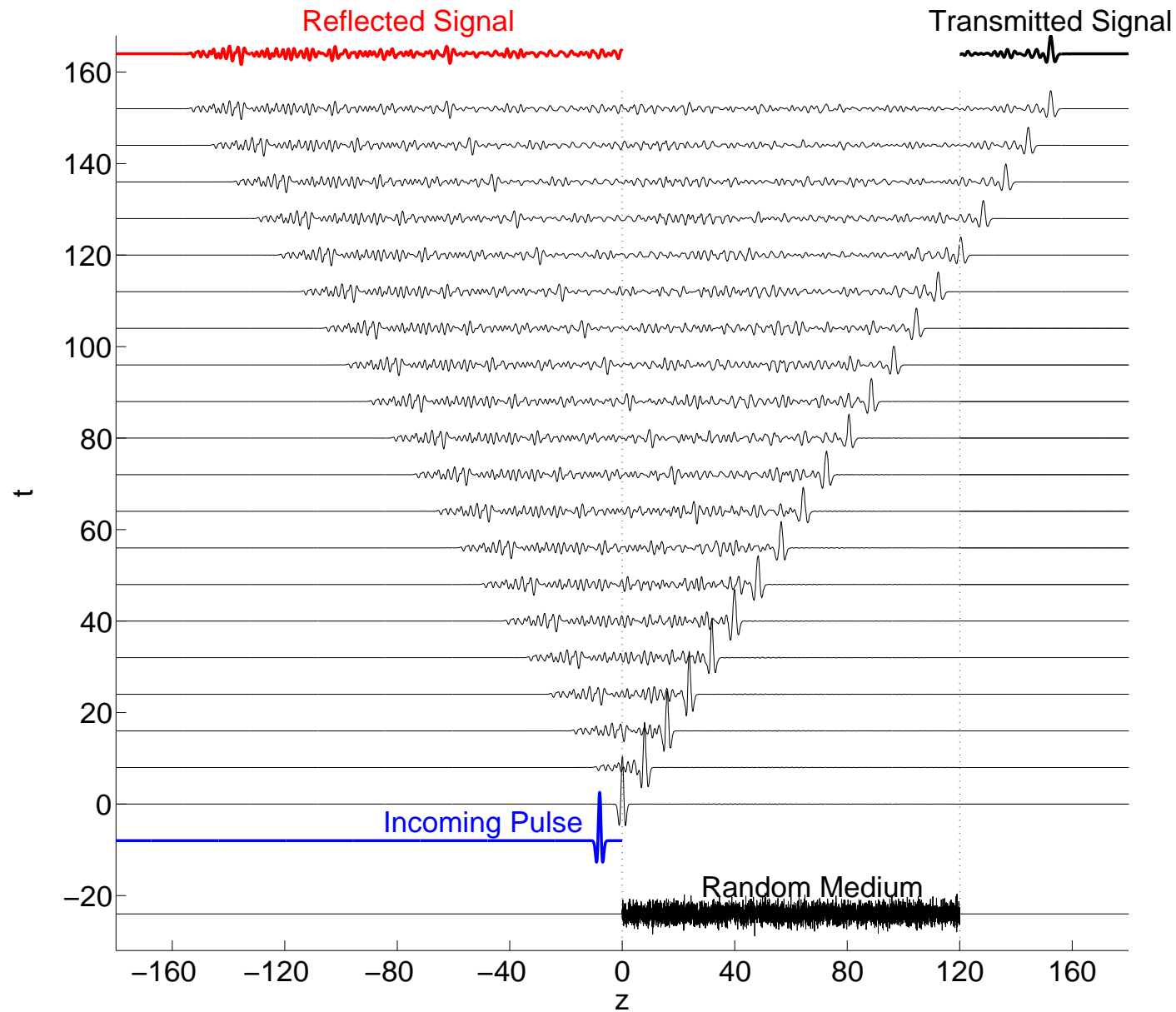
$R_\omega^\varepsilon(z)$  is the reflection coefficient for a random slab  $[-L, z]$ :

$$\frac{dR_\omega^\varepsilon}{dz} = -\frac{i\omega}{\varepsilon} m(\frac{z}{\varepsilon^2}) R_\omega^\varepsilon - \frac{i\omega}{2\varepsilon} n(\frac{z}{\varepsilon^2}) e^{-\frac{2i\omega z}{\varepsilon}} (R_\omega^\varepsilon)^2 - \frac{i\omega}{2\varepsilon} n(\frac{z}{\varepsilon^2}) e^{\frac{2i\omega z}{\varepsilon}},$$

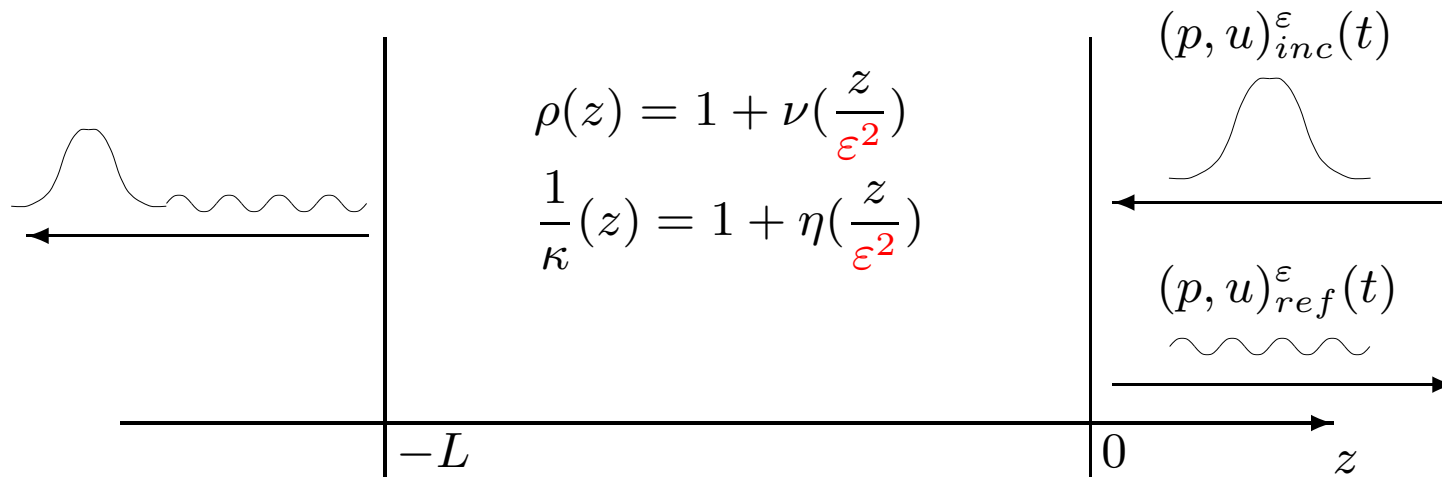
with the initial condition at  $z = -L$ :  $R_\omega^\varepsilon(z = -L) = 0$ .

Energy conservation  $|R_\omega^\varepsilon|^2 + |T_\omega^\varepsilon|^2 = 1 \rightarrow$  uniform boundedness of  $R_\omega^\varepsilon$ .

# Numerical simulation



## Time reversal in reflection (TR)

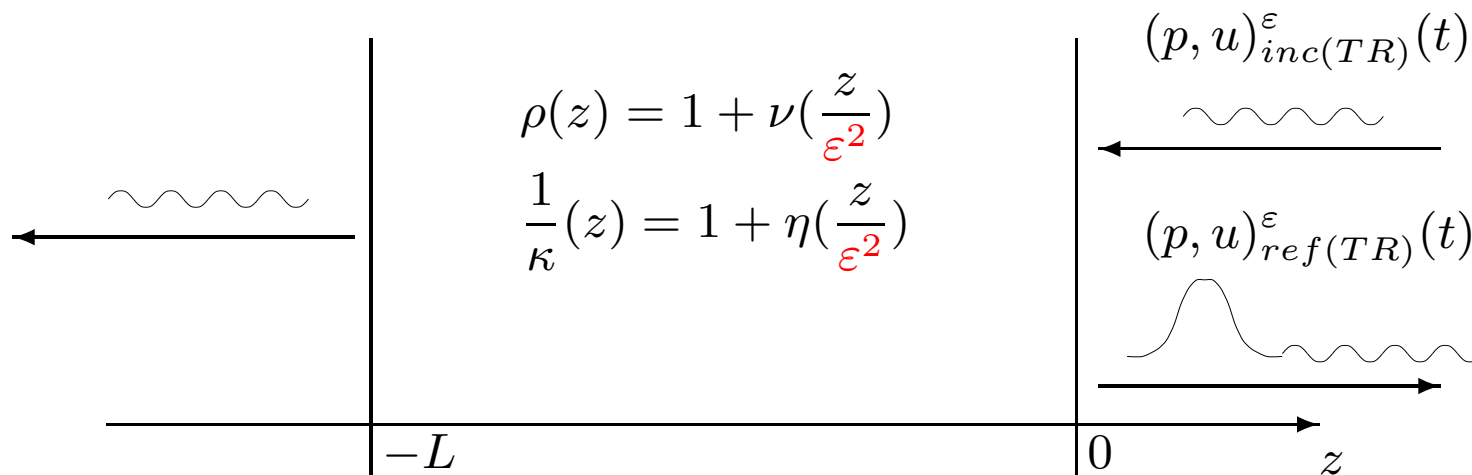


Denote  $p_{inc}^\varepsilon(t) = f\left(\frac{t}{\varepsilon}\right)$ .

Record  $p_{ref}^\varepsilon(t)$  up to time  $t_1$ .

Cut a piece  $p_{ref,cut}^\varepsilon(t) = p_{ref}^\varepsilon(t)G_1(t)$ , with  $\text{supp}(G_1) \subset [0, t_1]$ .

Time reverse and send back  $p_{inc(TR)}^\varepsilon(t) = p_{ref,cut}^\varepsilon(t_1 - t)$ .



## Expression of the refocused pulse

The incoming signal  $f(\frac{t}{\varepsilon})$

$$p_{inc}(t, z = 0) = \int e^{\frac{i\omega t}{\varepsilon}} \hat{f}(\omega) d\omega$$

propagates into the medium and generates the reflected signal:

$$p_{ref}^{\varepsilon}(t, z = 0) = \int e^{\frac{i\omega t}{\varepsilon}} \hat{f}(\omega) R_{\omega}^{\varepsilon}(0) d\omega$$

Record up to time  $t_1$  and cut a piece of the recorded signal (i.e.

$$G_1(t) = \mathbf{1}_{[0, t_1]}(t))$$

$$p_{ref, cut}^{\varepsilon}(t) = p_{ref}^{\varepsilon}(t, z = 0) G_1(t)$$

Time-reverse and send back into the medium:

$$\begin{aligned} p_{inc}^{\varepsilon}(TR)(t, z = 0) &= p_{ref}^{\varepsilon}(t_1 - t) G_1(t_1 - t) \\ &= \frac{1}{\varepsilon} \int \int e^{\frac{i\omega(t_1 - t)}{\varepsilon}} \hat{p}_{ref}^{\varepsilon}(\omega') \hat{G}_1\left(\frac{\omega - \omega'}{\varepsilon}\right) d\omega' d\omega \end{aligned}$$

The signal is real-valued:

$$p_{inc}^{\varepsilon}(TR)(t, z = 0) = \frac{1}{\varepsilon} \int \int e^{\frac{i\omega(t - t_1)}{\varepsilon}} \overline{\hat{p}_{ref}^{\varepsilon}(\omega')} \overline{\hat{G}_1\left(\frac{\omega - \omega'}{\varepsilon}\right)} d\omega' d\omega$$

The new signal propagates into the same medium and generates a new reflected signal observed at time  $t_2 + \varepsilon t$ :

$$p_{ref(TR)}^\varepsilon(t_2 + \varepsilon t, z = 0) = \int \hat{p}_{inc(TR)}^\varepsilon(\omega) R_\omega^\varepsilon(0) e^{\frac{i\omega t_1}{\varepsilon} + i\omega t} d\omega$$

Substituting the expression of  $\hat{p}_{inc(TR)}^\varepsilon$  into this equation:

$$\begin{aligned} p_{ref(TR)}^\varepsilon(t_2 + \varepsilon t, z = 0) &= \frac{1}{\varepsilon} \int \int e^{i\omega t} e^{\frac{i\omega(t_2 - t_1)}{\varepsilon}} \overline{\hat{f}(\omega')} \overline{\hat{G}_1\left(\frac{\omega - \omega'}{\varepsilon}\right)} \\ &\quad \times R_\omega^\varepsilon(0) \overline{R_{\omega'}^\varepsilon(0)} d\omega' d\omega. \end{aligned}$$

Change of variables  $\omega' = \omega - \varepsilon h$ :

$$\begin{aligned} p_{ref(TR)}^\varepsilon(t_2 + \varepsilon t, z = 0) &= \int \int e^{i\omega t} e^{\frac{i\omega(t_2 - t_1)}{\varepsilon}} \overline{\hat{f}(\omega - \varepsilon h)} \overline{\hat{G}_1(h)} \\ &\quad \times \overline{R_\omega^\varepsilon(0)} R_{\omega - \varepsilon h}^\varepsilon(0) dh d\omega. \end{aligned}$$

The autocorrelation function  $\overline{R_{\omega'}^\varepsilon(0)} R_\omega^\varepsilon(0)$  plays a primary role.

Refocusing at  $t_2 = t_1$ .



We have obtained:

$$\mathbb{E} \left[ R_{\omega + \frac{\varepsilon h}{2}}^{\varepsilon}(0) \overline{R_{\omega - \frac{\varepsilon h}{2}}^{\varepsilon}(0)} \right] \xrightarrow{\varepsilon \rightarrow 0} \int V_1(0, \omega, \tau) e^{-i h \tau} d\tau$$

In the time domain:

$$\begin{aligned} \mathbb{E} [p_{ref(T_R)}^{\varepsilon}(t_1 + \varepsilon t)] &\xrightarrow{\varepsilon \rightarrow 0} \int \int e^{i\omega t} \overline{\hat{f}}(\omega) \overline{\hat{G}_1(h)} \\ &\quad \times \left[ \int V_1(0, \omega, \tau) e^{-i h \tau} d\tau \right] dh d\omega \\ &= \int \int e^{i\omega t} \overline{\hat{f}}(\omega) G_1(\tau) V_1(0, \omega, \tau) d\tau d\omega \\ &= \int e^{i\omega t} \overline{\hat{f}}(\omega) \hat{K}_{TR}(\omega) d\omega \\ &= (f(-\cdot) * K_{TR})(t) \end{aligned}$$

where

$$\hat{K}_{TR}(\omega) = \int G_1(\tau) V_1(0, \omega, \tau) d\tau$$

This results only holds true in average (averaging over all possible realizations of the medium) !

## Frequency correlation of $R_\omega^\varepsilon$

Let us consider the forth-order moment at 4 different frequencies of  $R_\omega^\varepsilon$

$$\left| \mathbb{E} \left[ R_{\omega_1 + \frac{\varepsilon h_1}{2}}^\varepsilon \overline{R_{\omega_1 - \frac{\varepsilon h_1}{2}}^\varepsilon} R_{\omega_2 + \frac{\varepsilon h_2}{2}}^\varepsilon \overline{R_{\omega_2 - \frac{\varepsilon h_2}{2}}^\varepsilon} \right] \right. \\ \left. - \mathbb{E} \left[ R_{\omega_1 + \frac{\varepsilon h_1}{2}}^\varepsilon \overline{R_{\omega_1 - \frac{\varepsilon h_1}{2}}^\varepsilon} \right] \mathbb{E} \left[ R_{\omega_2 + \frac{\varepsilon h_2}{2}}^\varepsilon \overline{R_{\omega_2 - \frac{\varepsilon h_2}{2}}^\varepsilon} \right] \right| \\ \xrightarrow{\varepsilon \rightarrow 0} 0$$

if  $\omega_1 \neq \omega_2$ .

$$p_{ref(TR)}^\varepsilon(t_1 + \varepsilon t) = \int \int e^{i\omega t} g(\omega, h) R_{\omega + \frac{\varepsilon h}{2}}^\varepsilon \overline{R_{\omega - \frac{\varepsilon h}{2}}^\varepsilon} d\omega dh$$

$$\begin{aligned} \mathbb{E} [p_{ref(TR)}^\varepsilon(t_1 + \varepsilon t)^2] &= \int \int d\omega_1 d\omega_2 dh_1 dh_2 e^{i(\omega_1 + \omega_2)t} g(\omega_1, h_1) g(\omega_2, h_2) \\ &\quad \times \mathbb{E} \left[ R_{\omega_1 + \frac{\varepsilon h_1}{2}}^\varepsilon \overline{R_{\omega_1 - \frac{\varepsilon h_1}{2}}^\varepsilon} R_{\omega_2 + \frac{\varepsilon h_2}{2}}^\varepsilon \overline{R_{\omega_2 - \frac{\varepsilon h_2}{2}}^\varepsilon} \right] \\ &\stackrel{\varepsilon \rightarrow 0}{\simeq} \int \int d\omega_1 d\omega_2 dh_1 dh_2 e^{i(\omega_1 + \omega_2)t} g(\omega_1, h_1) g(\omega_2, h_2) \\ &\quad \times \mathbb{E} \left[ R_{\omega_1 + \frac{\varepsilon h_1}{2}}^\varepsilon \overline{R_{\omega_1 - \frac{\varepsilon h_1}{2}}^\varepsilon} \right] \mathbb{E} \left[ R_{\omega_2 + \frac{\varepsilon h_2}{2}}^\varepsilon \overline{R_{\omega_2 - \frac{\varepsilon h_2}{2}}^\varepsilon} \right] \\ &\stackrel{\varepsilon \rightarrow 0}{\simeq} \mathbb{E} [p_{ref(TR)}^\varepsilon(t_1 + \varepsilon t)]^2 \end{aligned}$$

$\text{Var} (p_{ref(TR)}^\varepsilon(t_1 + \varepsilon t)) \xrightarrow{\varepsilon \rightarrow 0} 0 \implies$  Convergence in  $L^2$  and in probability of  $p_{ref(TR)}^\varepsilon(t_1 + \varepsilon t)$ .

Decorrelation in frequency of  $R_\omega^\varepsilon \implies$  Self-averaging in time of  $p_{ref(TR)}^\varepsilon$ .

## Convergence of the refocused pulse

The refocused signal  $(p_{ref(TR)}^\varepsilon(t_1 + \varepsilon t, z = 0))_{t \in (-\infty, \infty)}$  converges in probability as  $\varepsilon \rightarrow 0$  to

$$P_{ref(TR)}(t) = (f(-\cdot) * K_{TR}(\cdot))(t)$$

$$\hat{K}_{TR}(\omega) = \int G_1(\tau) V_1(0, \omega, \tau) d\tau$$

where  $V_1(0, \omega, \tau)$  is the **deterministic** density given by the system of transport equations.

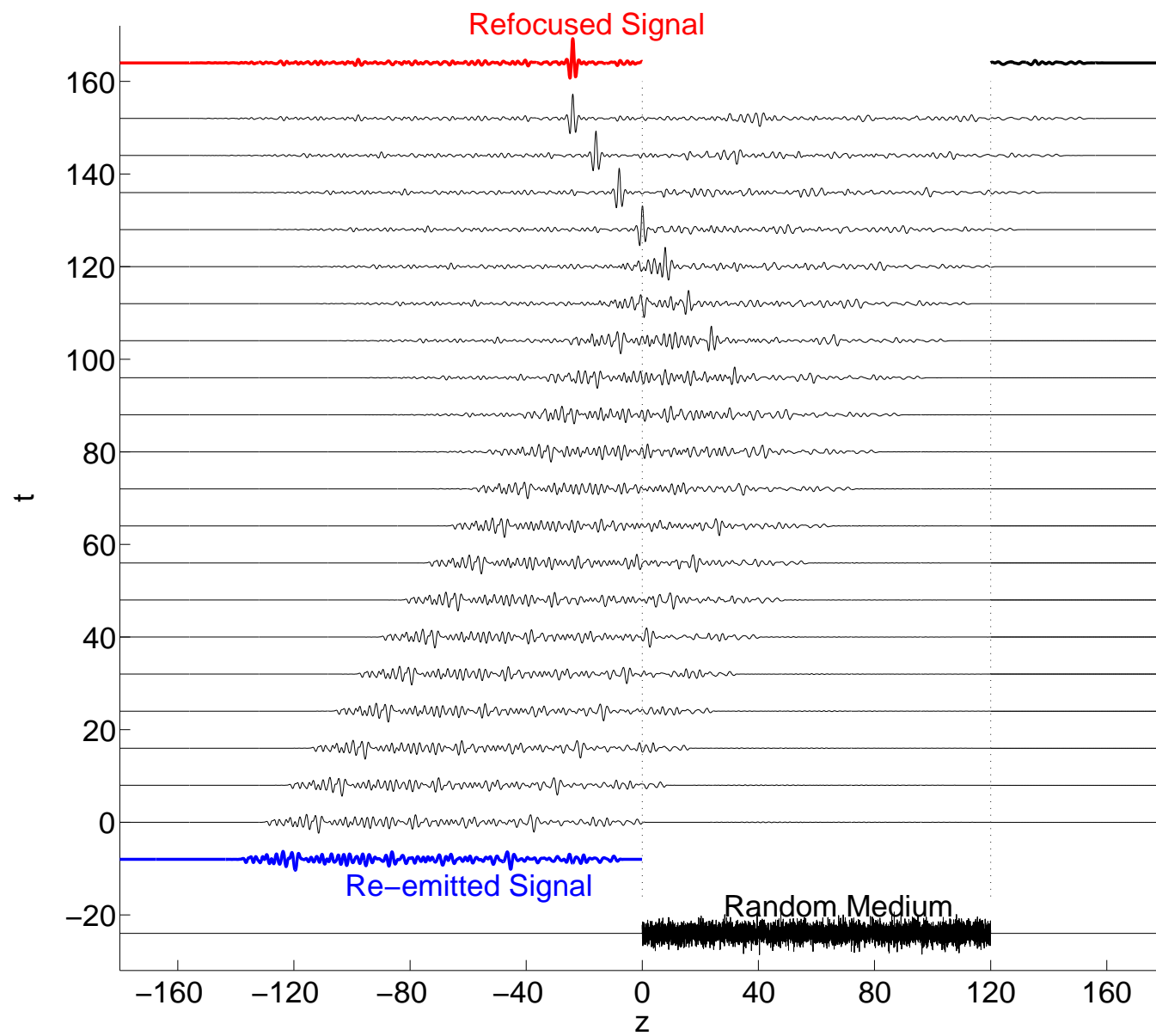
In particular, if  $L \rightarrow \infty$ :

$$V_1(0, \omega, \tau) = \frac{4\alpha_n \omega^2}{(4 + \alpha_n \omega^2 \tau)^2}$$

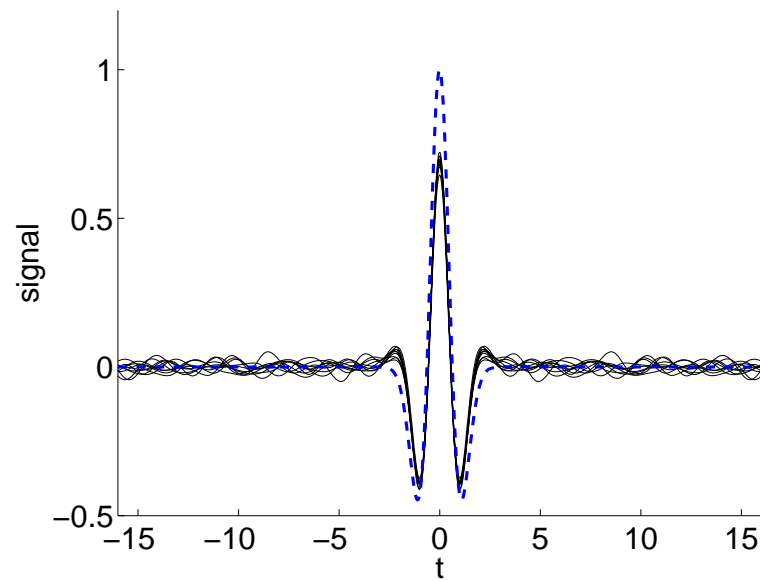
$\hookrightarrow$  The refocused pulse has deterministic center and shape.

$\hookrightarrow$  There is statistical stability.

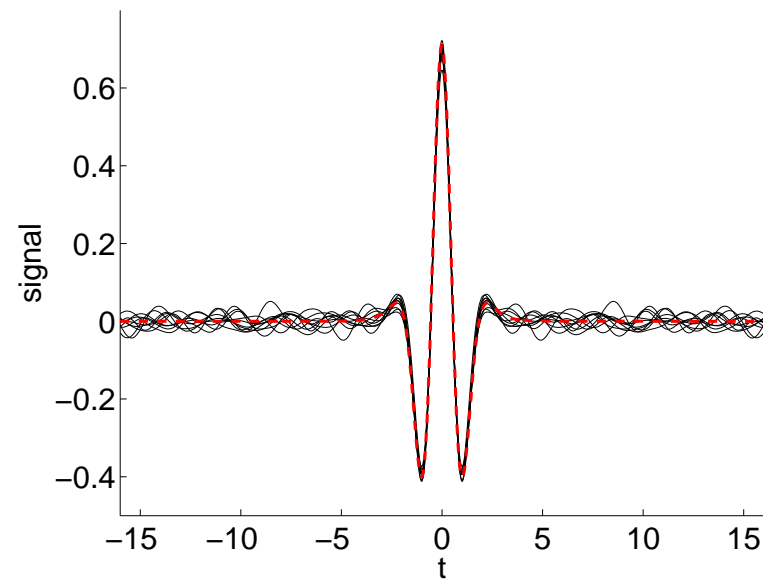
# Numerical simulation



## Comparisons simulations - theory



Comparison with  
the input pulse  
 $f(t)$



Comparison with  
the asymptotic formula  
 $K_{\text{TR}} * f(-t)$

Profiles of the refocused pulse at  $z = 0$

## Application to imagery

Goal: extract the information about the large-scale properties of the medium  $(\rho_0, \kappa_0)$

$$\rho(z) = \rho_0(z) \left(1 + \nu\left(\frac{z}{\varepsilon^2}\right)\right)$$
$$\frac{1}{\kappa(z)} = \frac{1}{\kappa_0(z)} \left(1 + \eta\left(\frac{z}{\varepsilon^2}\right)\right)$$

This information is contained in  $V_1(0, \omega, \tau)$ . The problem is to find a statistically stable estimator of  $V_1$ .

Method:

- 1) Compare the input pulse  $f(t)$  and the refocused pulse  $K_{\text{TR}} * f(-t)$ .  
 $\hookrightarrow$  Extract  $(\hat{K}_{\text{TR}}(\omega))_\omega$ .

$$\hat{K}_{\text{TR}}(\omega) = \int G_1(\tau) V_1(0, \omega, \tau) d\tau$$

- 2) Use different truncation functions  $G_1$  to get  $(V_1(0, \omega, \tau))_{\omega, \tau}$ .  
 $\hookrightarrow$  large-scale variations of the medium.

Statistically stable method, no local average is needed.

## Application to communications

*How to send a message  $f(t)$  from  $E$  to  $R$  in a highly scattering medium ?*

1)  $R$  emits a short, broadband pulse  $f_0(t)$ .

2)  $E$  receives and records a noisy signal  $G(t)$ .

3)  $E$  emits  $[f * G(-\cdot)](t)$ .

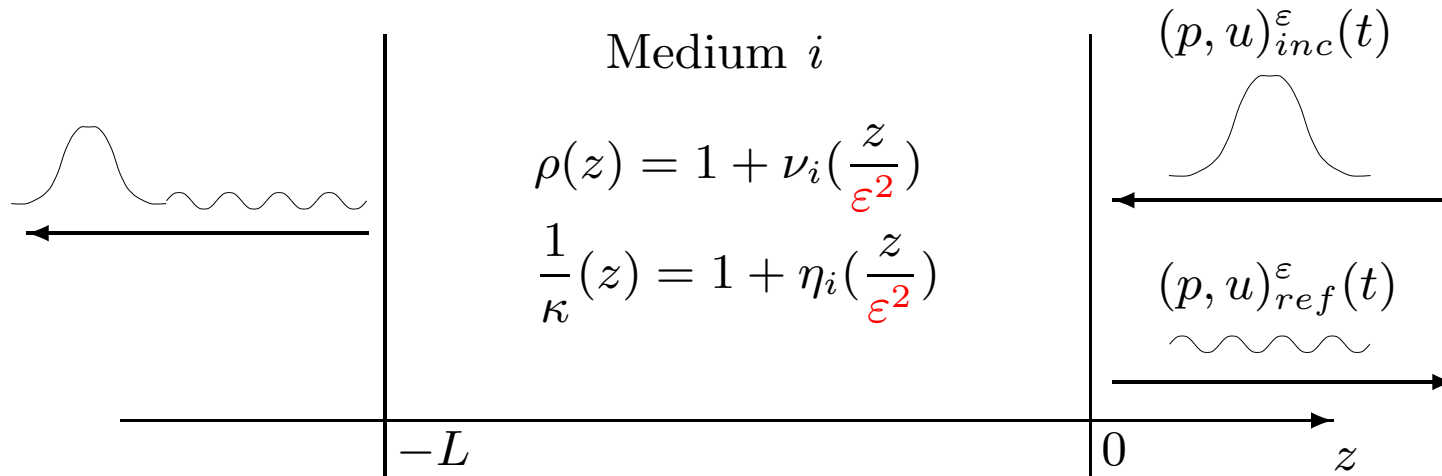
4)  $R$  receives  $[K_{\text{TR}} * f_0(-\cdot) * f](t)$ .

$\hookrightarrow R$  can extract  $f$ .

(OK in ocean acoustics, difficult in electromagnetics).



## Time reversal in changing media

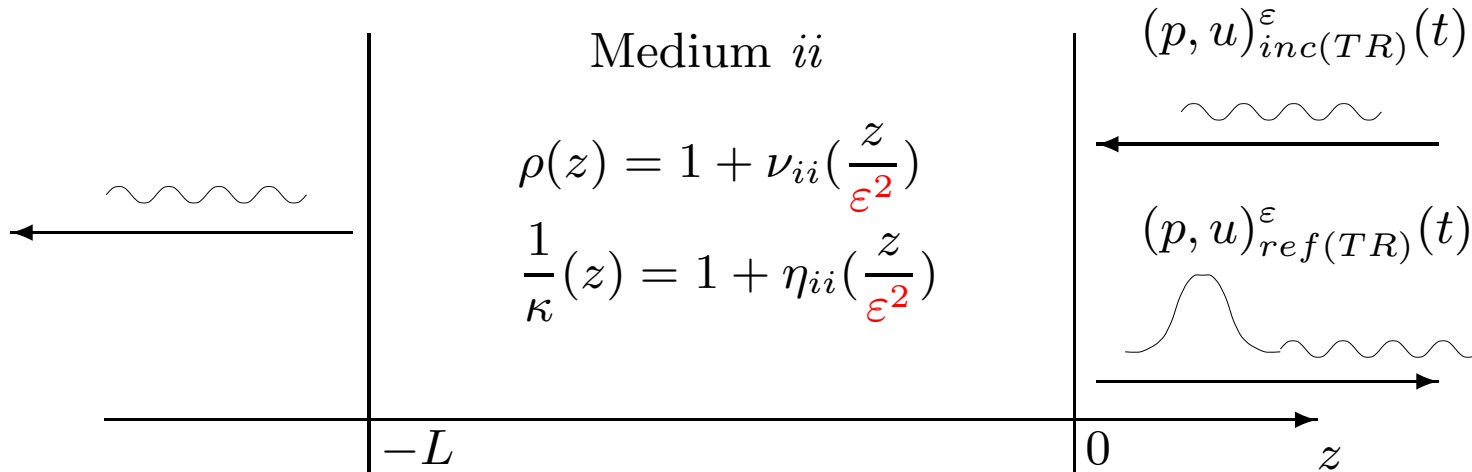


Denote  $p_{inc}^\varepsilon(t) = f\left(\frac{t}{\varepsilon}\right)$ .

Record  $p_{ref}^\varepsilon(t)$  up to time  $t_1$ .

Cut a piece  $p_{ref,cut}^\varepsilon(t) = p_{ref}^\varepsilon(t)G_1(t)$ , with  $\text{supp}(G_1) \subset [0, t_1]$ .

Time reverse and send it back  $p_{inc(TR)}^\varepsilon(t) = p_{ref,cut}^\varepsilon(t_1 - t)$ .



## Changing media

The density and compressibility fluctuations  $(\nu_i, \eta_i)$  and  $(\nu_{ii}, \eta_{ii})$  are identically distributed.

Integrated correlation functions:

$$\alpha_n = \int_0^\infty \mathbb{E}[n(0)n(t)]dt, \quad \alpha_m = \int_0^\infty \mathbb{E}[m(0)m(t)]dt$$

Degree of correlation  $\delta \in [-1, 1]$ :

$$\delta_m = \frac{\int_0^\infty \mathbb{E}[m_i(0)m_{ii}(t)]dt}{\int_0^\infty \mathbb{E}[m_i(0)m_i(t)]dt}, \quad \delta_n = \frac{\int_0^\infty \mathbb{E}[n_i(0)n_{ii}(t)]dt}{\int_0^\infty \mathbb{E}[n_i(0)n_i(t)]dt}$$

$\delta = 1 \leftrightarrow$  complete correlation.  $\delta = 0 \leftrightarrow$  complete decorrelation.

Refocused pulse:

$$p_{ref(TR)}^\varepsilon(t_1 + \varepsilon t, z = 0) = \int \int e^{i\omega t} \bar{\hat{f}}(\omega - \varepsilon h) \bar{\hat{G}_{t_1}}(h) \\ \times R_\omega^{\varepsilon, ii}(0) \overline{R_{\omega - \varepsilon h}^{\varepsilon, i}(0)} dh d\omega$$

where the reflection coefficient  $R_\omega^{\varepsilon, i}$  satisfies the Ricatti equation:

$$\frac{dR_\omega^{\varepsilon, i}}{dz} = -i\frac{\omega}{\varepsilon} m_i\left(\frac{z}{\varepsilon^2}\right) R_\omega^{\varepsilon, i} - \frac{i\omega}{2\varepsilon} n_i\left(\frac{z}{\varepsilon^2}\right) e^{-\frac{2i\omega z}{\varepsilon}} (R_\omega^{\varepsilon, i})^2 - \frac{i\omega}{2\varepsilon} n_i\left(\frac{z}{\varepsilon^2}\right) e^{\frac{2i\omega z}{\varepsilon}}$$

## Asymptotics of the refocused pulse

Expectation of the autocorrelation function  $\overline{R_{\omega}^{\varepsilon,i}} R_{\omega}^{\varepsilon,ii}$ :

$$\mathbb{E} \left[ R_{\omega + \frac{\varepsilon h}{2}}^{\varepsilon,ii}(0) \overline{R_{\omega - \frac{\varepsilon h}{2}}^{\varepsilon,i}(0)} \right] \xrightarrow{\varepsilon \rightarrow 0} \int v_1(0, \omega, \tau) e^{-ih\tau} d\tau$$

where  $v_1$  is given by a system of transport equations for  $(v_p)_{p \in \mathbb{N}}$ :

$$\begin{aligned} \frac{\partial v_p}{\partial z} + 2p \frac{\partial v_p}{\partial \tau} &= \frac{1}{2} \delta_n \alpha_n \omega^2 p^2 (v_{p+1} + v_{p-1} - 2v_p) \\ &\quad - (1 - \delta_n) \alpha_n \omega^2 p^2 v_p - 2\alpha_m (1 - \delta_m) \omega^2 p^2 v_p \\ v_p(z = -L, \omega, \tau) &= \delta_0(\tau) \mathbf{1}_0(p) \end{aligned}$$

$\Rightarrow$  Convergence of the expectation of the refocused pulse:

$$\mathbb{E}[p_{ref(T_R)}^{\varepsilon}(t_1 + \varepsilon t)] \xrightarrow{\varepsilon \rightarrow 0} \int \int e^{i\omega t} \bar{\hat{f}}(\omega) G_1(\tau) v_1(0, \omega, \tau) d\tau d\omega$$

But:

$$\mathbb{E}[p_{ref(T_R)}^{\varepsilon}(t_1 + \varepsilon t)^2] \not\xrightarrow{\varepsilon \rightarrow 0} \left[ \int \int e^{i\omega t} \bar{\hat{f}}(\omega) G_1(\tau) v_1(0, \omega, \tau) d\tau d\omega \right]^2$$

Equivalently  $v_p(z, \omega, \tau, z) = \mathbf{E}[V_p(z, \omega, \tau)]$  where  $(V_p)_{p \in \mathbb{N}}$  is solution of

$$\begin{aligned} dV_p + 2p \frac{\partial V_p}{\partial \tau} dz &= 2ip\omega \sqrt{(1 - \delta_m)\alpha_m} V_p \circ dW_z + (\mathcal{L}_\omega V)_p dz \\ V_p(z = -L, \omega, \tau) &= \delta_0(\tau) \mathbf{1}_0(p) \end{aligned}$$

$W_z$  is a standard Brownian motion.

Thus

$$\mathbb{E}[p_{ref}^\varepsilon(T_R)(t_1 + \varepsilon t)] \xrightarrow{\varepsilon \rightarrow 0} \int \int e^{i\omega t} \overline{\hat{f}}(\omega) G_1(\tau) \mathbf{E}[V_1(0, \omega, \tau)] d\tau d\omega$$

- Convergence of the finite-dimensional distributions:

$$\begin{aligned} &\mathbb{E} \left[ p_{ref}^\varepsilon(T_R)(t_1 + \varepsilon s_1)^{p_1} \dots p_{ref}^\varepsilon(T_R)(t_1 + \varepsilon s_k)^{p_k} \right] \xrightarrow{\varepsilon \rightarrow 0} \\ &\mathbf{E} \left[ \prod_{1 \leq j \leq k} \left( \int \int V_1(0, \omega, \tau) \overline{\hat{f}}(\omega) e^{i\omega s_j} G_1(\tau) d\omega d\tau \right)^{p_j} \right] \end{aligned}$$

- Tightness of  $(p_{ref}^\varepsilon(T_R)(t_1 + \varepsilon t))_{t \in (-\infty, \infty)}$  in the space of continuous functions.

- Conclusion:  $p_{ref}^\varepsilon(T_R)(t_1 + \varepsilon t)$  converges in distribution to  $\int \int V_1(0, \omega, \tau) \overline{\hat{f}}(\omega) e^{i\omega t} G_1(\tau) d\omega d\tau$

## Probabilistic representation of the transport equations

Consider our familiar jump (Markov) process  $(N_z)_{z \geq -L}$  with state space  $\mathbb{N}$  and generator

$$\mathcal{L}\phi(N) = \frac{1}{2}\alpha_n\omega^2 N^2 (\phi(N+1) + \phi(N-1) - 2\phi(N))$$

Therefore (Feynman-Kac) :

$$\begin{aligned} \int_{\tau_0}^{\tau_1} V_1(0, \omega, \tau) d\tau &= \mathbb{E} \left[ \exp \left( 2i\sqrt{\alpha_m(1-\delta_m)}\omega \int_{-L}^0 N_{-L-s} dW_s \right) \right. \\ &\quad \left. \times \mathbf{1}_0(N_0) \mathbf{1}_{[\tau_0, \tau_1]} \left( \int_{-L}^0 2N_s ds \right) \mid N_{-L} = 1 \right] \end{aligned}$$

where  $\mathbb{E}$  is the expectation w.r.t the distribution of  $(N_z)_{z \geq -L}$

## Convergence of the refocused pulse

The refocused signal  $(p_{ref(TR)}^\varepsilon(t_1 + \varepsilon t, z = 0))_{t \in (-\infty, \infty)}$  converges in distribution as  $\varepsilon \rightarrow 0$  to

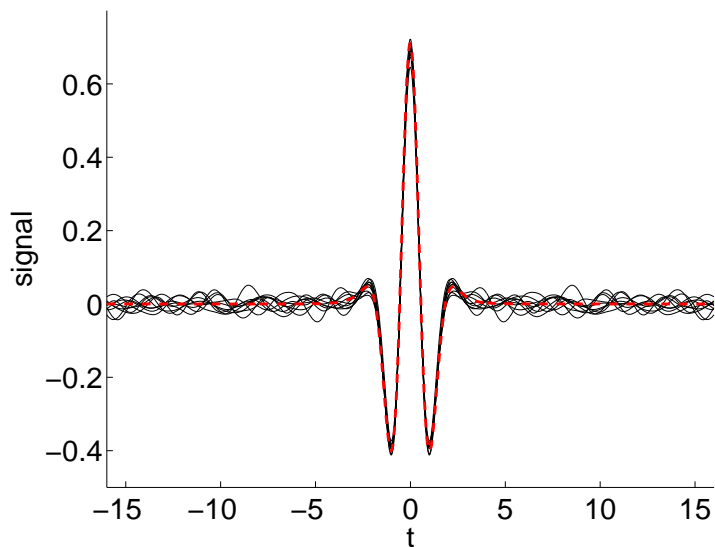
$$P_{ref(TR)}(t) = (f(-\cdot) * K_{TR}(\cdot))(t)$$

$$\hat{K}_{TR}(\omega) = \int G_1(\tau) V_1(0, \omega, \tau) d\tau$$

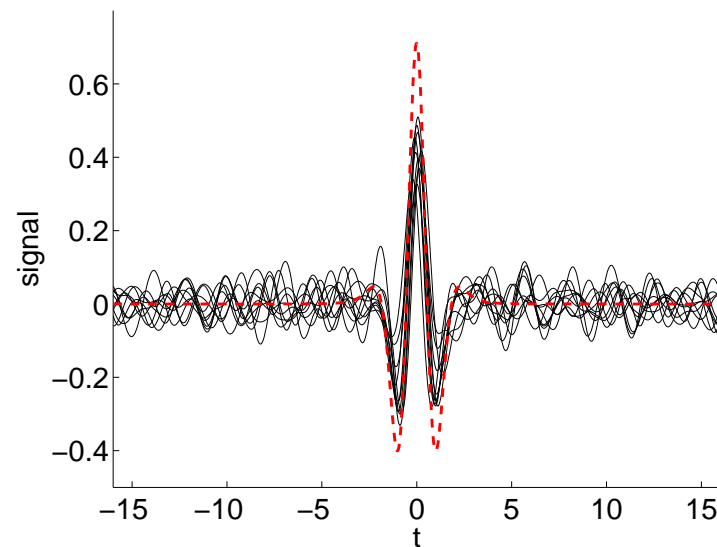
where  $V_1(0, \omega, \tau)$  is the **random** density given by the system of transport equations driven by the Brownian motion  $W_z$ .

$\hookrightarrow$  The refocused pulse has random center and shape.

$\hookrightarrow$  There is no statistical stability.



Fixed medium  
 $\delta_m = \delta_n = 1$



Changing medium  
 $\delta_m = \delta_n = 0.75$

### Refocused pulses

Comparison of the refocused pulses from the numerical experiments and the expected refocused pulse shape obtained by the asymptotic theory.

Only the density is random  $\kappa_i = \kappa_{ii} \equiv \kappa_0 = 1 \Rightarrow \alpha_m = \alpha_n$ .

## Mean refocused shape

$$\mathbb{E}[P_{ref}(TR)(t)] = (f(- \cdot) * \mathbb{E}[K_{TR}](\cdot))(t)$$

$$\mathbb{E}[\hat{K}_{TR}](\omega) = \int G_1(\tau) \mathbb{E}[V_1(0, \omega, \tau)] d\tau$$

For large slab  $L \rightarrow \infty$ :

$$\mathbb{E}[V_1(0, \omega, \tau)] = \frac{\alpha_0 \omega^2 \delta_0}{4} \frac{1 - \tanh^2 \left( \frac{\sqrt{1 - \delta_0^2} \alpha_0 \omega^2 \tau}{4} \right)}{\left[ 1 + \frac{1}{\sqrt{1 - \delta_0^2}} \tanh \left( \frac{\sqrt{1 - \delta_0^2} \alpha_0 \omega^2 \tau}{4} \right) \right]^2}$$

where

$$\alpha_0 = \alpha_n + 2(1 - \delta_m) \alpha_m \quad \delta_0 = \frac{\delta_n \alpha_n}{\alpha_n + 2(1 - \delta_m) \alpha_m}$$

If  $G_1(t) = \mathbf{1}_{[0, t_1]}(t)$ ,

$$\mathbb{E}[\hat{K}_{TR}](\omega) = \delta_0 \frac{\tanh \left( \frac{\sqrt{1 - \delta_0^2} \alpha_0 \omega^2 t_1}{4} \right)}{\sqrt{1 - \delta_0^2} + \tanh \left( \frac{\sqrt{1 - \delta_0^2} \alpha_0 \omega^2 t_1}{4} \right)}$$



## Conclusion

Statistical stability of the refocused pulse depends on the statistical properties of the reflection coefficient  $R$ .

Asymptotic framework  $\varepsilon \rightarrow 0$ :

- Frequency decorrelation of  $R$ .
- Moments of  $R$  satisfy a system of transport equations.
- Representation in terms of a canonical jump Markov process on the nonnegative integers.

Refocusing is robust. Observation of a refocused pulse as soon as the correlation degree is positive.

No statistical stability if the medium is changing (except in special cases).