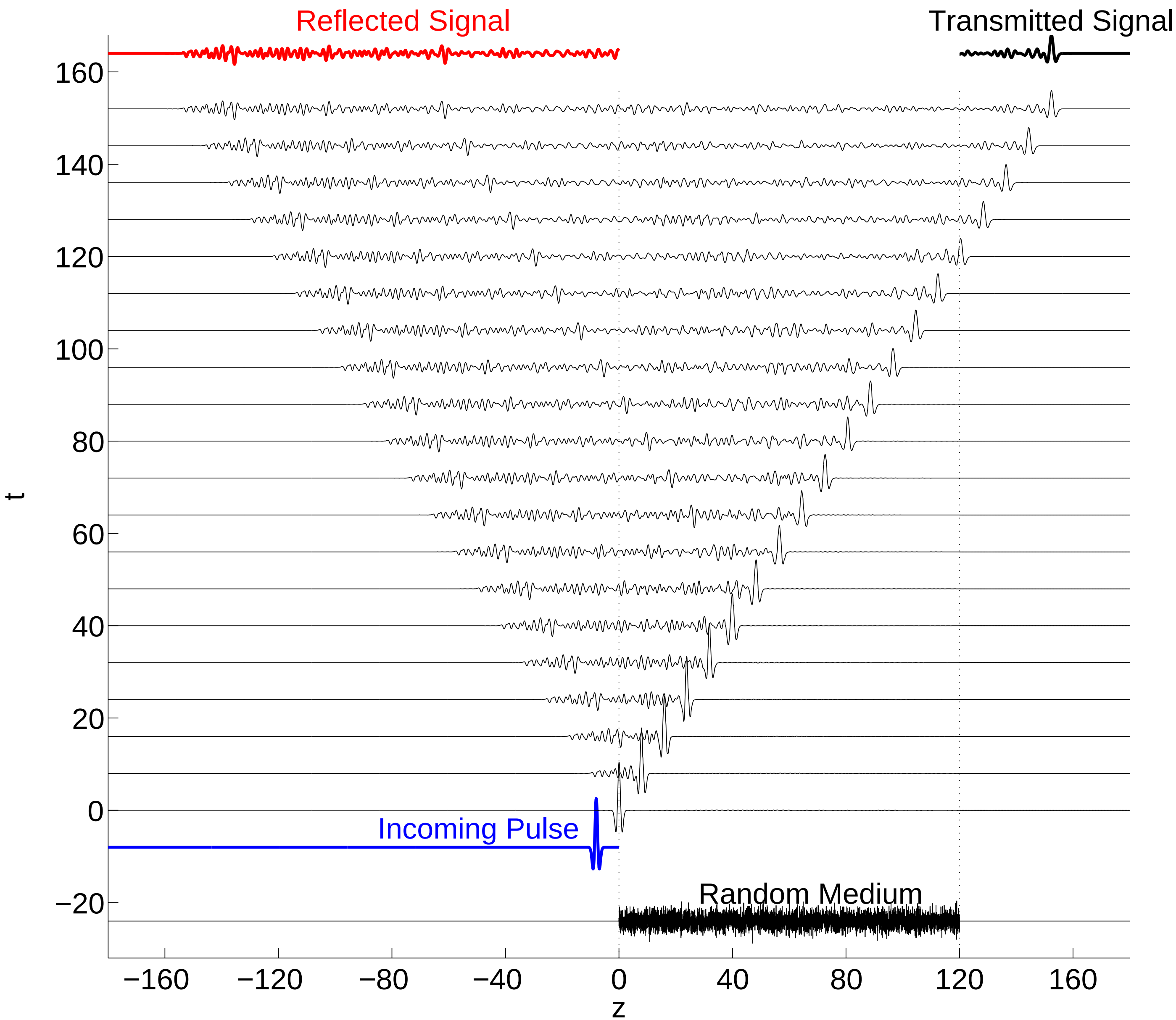
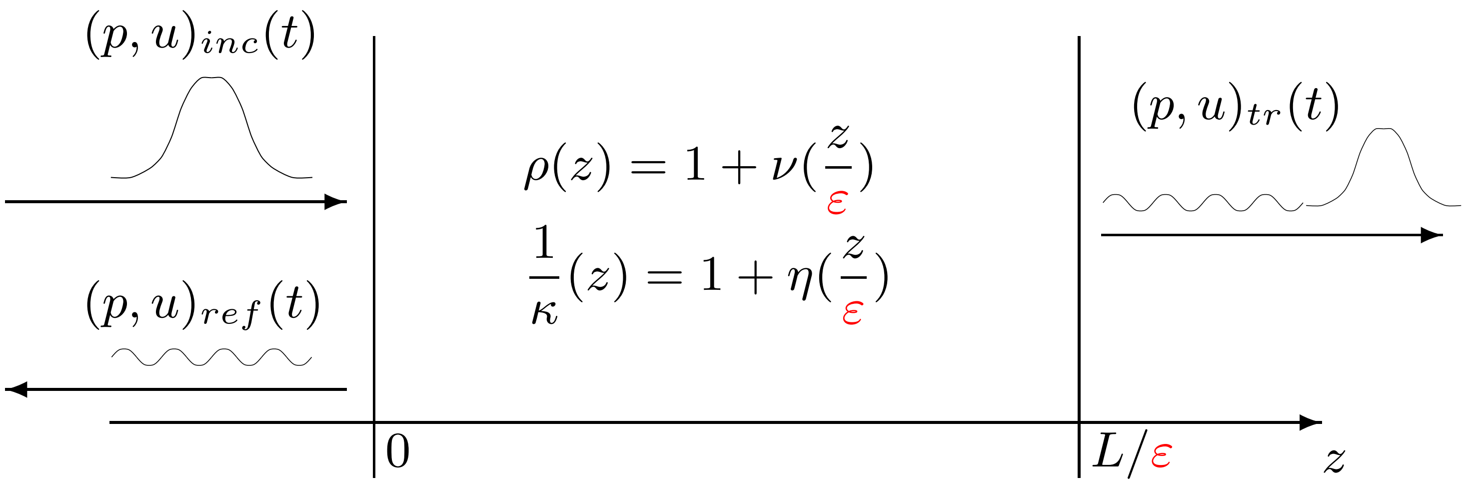
Long distance propagation



Long distance propagation $l_c \ll \lambda \ll L$

Acoustic equations for pressure p and velocity u:

$$\frac{\partial p}{\partial t} + \kappa(z)\frac{\partial u}{\partial z} = 0$$
$$\rho(z)\frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = 0$$



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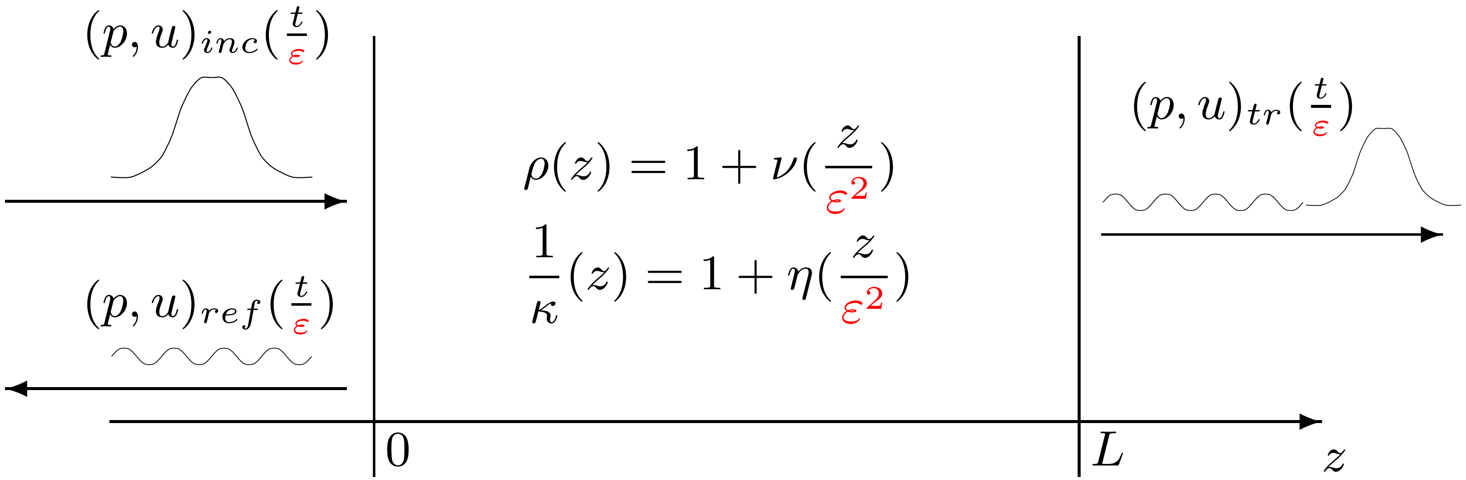

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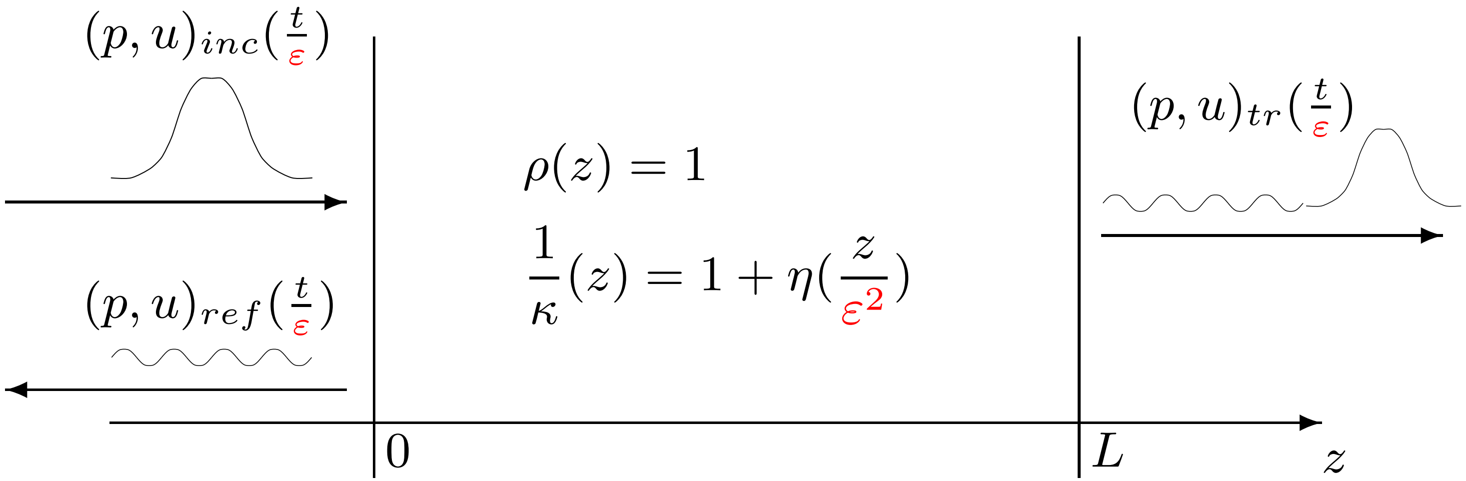


IC: right-going pulse incoming from the left homogeneous half-space $f(\frac{t}{\varepsilon})$.

Introduce the right-going mode $A = u + p$ and left-going mode $B = u - p$ that satisfy:

$$\frac{\partial}{\partial z}\begin{pmatrix} A \\ B \end{pmatrix} = \left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\eta(\frac{z}{\varepsilon^2})\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\right)\frac{\partial}{\partial t}\begin{pmatrix} A \\ B \end{pmatrix}$$

Correlation radius $\sim \varepsilon^2 \ll$ wavelength $\sim \varepsilon \ll$ propagation distance ~ 1.

$$A(0,t) = f(\frac{t}{\varepsilon}), \qquad B(L,t) = 0$$

Observe the transmitted wave around the expected arrival time $t = L$:

$$A(L, L + \varepsilon\sigma)_{\sigma \in (-\infty,\infty)}$$

Define

$$A^\varepsilon(z,\sigma) = A(z, z+\varepsilon\sigma), \qquad B^\varepsilon(z,\sigma) = B(z, -z+\varepsilon\sigma)$$

Take a scaled Fourier transform ε:

$$\hat{A}^\varepsilon(z,\omega) = \int e^{i\omega\sigma} A^\varepsilon(z,\sigma) d\sigma, \qquad \hat{B}^\varepsilon(z,\omega) = \int e^{i\omega\sigma} B^\varepsilon(z,\sigma) d\sigma$$

In the frequency domain:

$$\frac{d}{dz}\begin{pmatrix} \hat{A}^\varepsilon \\ \hat{B}^\varepsilon \end{pmatrix} = P^\varepsilon_\omega(z) \begin{pmatrix} \hat{A}^\varepsilon \\ \hat{B}^\varepsilon \end{pmatrix}, \qquad P^\varepsilon_\omega(z) = \frac{i\omega}{2\varepsilon}\eta(\frac{z}{\varepsilon^2}) \begin{pmatrix} 1 & -e^{-2i\omega\frac{z}{\varepsilon}} \\ e^{2i\omega\frac{z}{\varepsilon}} & -1 \end{pmatrix}$$

with the boundary conditions $\hat{A}^\varepsilon(0,\omega) = \hat{f}(\omega)$ and $\hat{B}^\varepsilon(L,\omega) = 0$.

Let $(\hat{a}^\varepsilon, \hat{b}^\varepsilon)$ solution of:

$$\frac{d}{dz}\begin{pmatrix} \hat{a}^\varepsilon \\ \hat{b}^\varepsilon \end{pmatrix} = P^\varepsilon_\omega(z)\begin{pmatrix} \hat{a}^\varepsilon \\ \hat{b}^\varepsilon \end{pmatrix},$$

with the initial conditions:

$$\hat{a}^\varepsilon(0,\omega) = 1, \qquad \hat{b}^\varepsilon(0,\omega) = 0$$

By symmetry $(\overline{\hat{b}^\varepsilon}, \overline{\hat{a}^\varepsilon})$ is another solution, and therefore $Y^\varepsilon_\omega(z) = \begin{pmatrix} \hat{a}^\varepsilon(z,\omega) & \overline{\hat{b}^\varepsilon}(z,\omega) \\ \hat{b}^\varepsilon(z,\omega) & \overline{\hat{a}^\varepsilon}(z,\omega) \end{pmatrix}$ is the fundamental matrix (propagator) of the system:

$$\frac{d}{dz}Y^\varepsilon_\omega = P^\varepsilon_\omega(z)Y^\varepsilon_\omega, \qquad Y^\varepsilon_\omega(0) = \mathrm{Id}_2$$

By linearity:

$$\begin{pmatrix} \hat{A}^\varepsilon(L,\omega) \\ \hat{B}^\varepsilon(L,\omega) \end{pmatrix} = Y^\varepsilon_\omega(L)\begin{pmatrix} \hat{A}^\varepsilon(0,\omega) \\ \hat{B}^\varepsilon(0,\omega) \end{pmatrix}$$

with $\hat{B}^\varepsilon(L,\omega) = 0$ and $\hat{A}^\varepsilon(0,\omega) = \hat{f}(\omega)$.

$$\hat{B}^\varepsilon(0,\omega) = R^\varepsilon_\omega(L)\hat{f}(\omega), \qquad \hat{A}^\varepsilon(L,\omega) = T^\varepsilon_\omega(L)\hat{f}(\omega)$$

$$R^\varepsilon_\omega(L) = -(\hat{b}^\varepsilon/\overline{\hat{a}^\varepsilon})(L,\omega), \qquad T^\varepsilon_\omega(L) = (1/\overline{\hat{a}^\varepsilon})(L,\omega),$$

Trace$(P_\omega^\varepsilon) = 0$, therefore $\det(Y_\omega^\varepsilon) = 1$ and $|\hat{a}^\varepsilon|^2 - |\hat{b}^\varepsilon|^2 = 1$:

$$|R_\omega^\varepsilon(L)|^2 + |T_\omega^\varepsilon(L)|^2 = 1$$

Transmitted field:

$$A(L, L+\varepsilon\sigma) = A^\varepsilon(L,\sigma) = \frac{1}{2\pi}\int e^{-i\omega\sigma}\hat{f}(\omega)T_\omega^\varepsilon(L)d\omega$$

The convergence of $(A^\varepsilon(L,\sigma))_{\sigma\in\mathbb{R}}$ requires:

- relative compacity,
- convergence of the finite-dimensional distributions.

The finite-dimensional distributions are characterized by the moments

$$\mathbb{E}[A^\varepsilon(L,\sigma_1)^{p_1}\ldots A^\varepsilon(L,\sigma_k)^{p_k}]$$

for any real $\sigma_1 < \ldots < \sigma_k$ and integer $p_1, \ldots, p_k$.

[1] J.-F. Clouet and J.-P. Fouque, Ann. Appl. Probab. **4** (1994), 1083-1097.

Lemma. *The transmitted field* $((A^\varepsilon(L,\sigma))_{-\infty<\sigma<\infty})_{\varepsilon>0}$ *is relatively compact in* $\mathcal{C}_b(\mathbb{R},\mathbb{R})$.

Proof. We must show that, $\forall h>0$, there is a compact K in $\mathcal{C}_b(\mathbb{R},\mathbb{R})$ such that

$$\sup_{\varepsilon>0} \mathbb{P}(A^\varepsilon(L,.)\in K) \geq 1-h$$

We have $A^\varepsilon(L,\sigma) = \dfrac{1}{2\pi}\displaystyle\int e^{-i\omega\sigma}\hat{f}(\omega)T^\varepsilon_\omega(L)d\omega.$
On the one hand $A^\varepsilon(L,\sigma)$ is bounded by:

$$|A^\varepsilon(L,\sigma)| \leq \frac{1}{2\pi}\int |\hat{f}(\omega)|d\omega$$

On the other hand the modulus of continuity

$$M^\varepsilon(\delta) = \sup_{|\sigma_1-\sigma_2|\leq\delta} |A^\varepsilon(L,\sigma_1) - A^\varepsilon(L,\sigma_2)|$$

is bounded by

$$M^\varepsilon(\delta) \leq \int \sup_{|\sigma_1-\sigma_2|\leq\delta} |1-\exp(i\omega(\sigma_1-\sigma_2))||\hat{f}(\omega)|d\omega$$

which goes to 0 as $\delta\to 0$.

First-order moment:

$$\mathbb{E}[A^\varepsilon(L,\sigma)] = \frac{1}{2\pi}\int e^{-i\omega\sigma}\hat{f}(\omega)\mathbb{E}[T^\varepsilon_\omega(L)]d\omega$$

with $T^\varepsilon_\omega(L) = 1/\overline{\hat{a}^\varepsilon}(L,\omega)$. Let us fix ω and denote $X_1^\varepsilon = \mathrm{Re}(\hat{a}^\varepsilon(.,\omega))$, $X_2^\varepsilon = \mathrm{Im}(\hat{a}^\varepsilon(.,\omega))$, $X_3^\varepsilon = \mathrm{Re}(\hat{b}^\varepsilon(.,\omega))$ and $X_4^\varepsilon = \mathrm{Im}(\hat{b}^\varepsilon(.,\omega))$. The process X^ε satisfies:

$$\frac{dX^\varepsilon(z)}{dz} = \frac{1}{\varepsilon}F_\omega\left(\eta(\frac{z}{\varepsilon^2}),\frac{z}{\varepsilon}\right)X^\varepsilon(z),$$

with the initial conditions $X_1^\varepsilon(0) = 1$ and $X_{j'}^\varepsilon(0) = 0$ if $j' = 2,3,4$, where

$$F_\omega(\eta,h) = \frac{\omega\eta}{2}\begin{pmatrix} 0 & -1 & -\sin(2\omega h) & \cos(2\omega h) \\ 1 & 0 & -\cos(2\omega h) & -\sin(2\omega h) \\ \sin(2\omega h) & \cos(2\omega h) & 0 & -1 \\ -\cos(2\omega h) & \sin(2\omega h) & 1 & 0 \end{pmatrix}$$

Diffusion-approximation: X^ε converges in distribution to X Markov with generator

$$\mathcal{L} = \sum_{i,j=1}^{4} a_{ij}(X)\frac{\partial^2}{\partial X_i \partial X_j}$$

$$a_{11} = \frac{\alpha\omega^2}{4}\left(X_2^2 + \frac{X_3^2+X_4^2}{2}\right), \quad a_{12} = \frac{\alpha\omega^2}{4}\left(-X_1X_2\right), \quad \alpha = \int_0^\infty \mathbb{E}[\eta(0)\eta(z)]dz$$

The moment $\phi(z) = \mathbb{E}[(X_1(x) - iX_2(z))^{-1}] = \lim_{\varepsilon\to 0} \mathbb{E}[1/\overline{\hat{a}^\varepsilon}(z,\omega)]$ satisfies

$$\frac{d\phi}{dz} = \mathcal{L}\phi = -\frac{\alpha\omega^2}{2}\phi, \quad \phi(0) = 1.$$

where

$$\alpha = \int_0^\infty \mathbb{E}[\eta(0)\eta(z)]dz$$

Solution: $\phi(L) = \exp(-\alpha\omega^2 L/2)$. The expectation $A^\varepsilon(L,\sigma)$ converges:

$$\mathbb{E}[A^\varepsilon(L,\sigma)] \overset{\varepsilon\to 0}{\longrightarrow} \frac{1}{2\pi}\int e^{-i\omega\sigma}\hat{f}(\omega)\exp(-\alpha\omega^2 L/2)d\omega$$

General moment:

$$\mathbb{E}\left[A^\varepsilon(L,\sigma_1)^{p_1}...A^\varepsilon(L,\sigma_k)^{p_k}\right] =$$

$$= \frac{1}{(2\pi)^n}\int\cdots\int \prod_{\substack{1\le j\le k\\ 1\le l\le p_j}} \hat{f}(\omega_{j,l})e^{-i\omega_{j,l}\sigma_j}\mathbb{E}\left[\prod_{\substack{1\le j\le k\\ 1\le l\le p_j}} T^\varepsilon_{\omega_{j,l}}(L)\right]\prod_{\substack{1\le j\le k\\ 1\le l\le p_j}} d\omega_{j,l}$$

One must compute the limit moments $\mathbb{E}[T^\varepsilon_{\omega_1}(L)\dots T^\varepsilon_{\omega_n}(L)]$ for n distinct frequencies $(\omega_i)_{i=1,\dots,n}$.

Introduce $X^\varepsilon_{4j+1} = \mathrm{Re}(\hat{a}^\varepsilon(\omega_j,.))$, $X^\varepsilon_{4j+2} = \mathrm{Im}(\hat{a}^\varepsilon(\omega_j,.))$, $X^\varepsilon_{4j+3} = \mathrm{Re}(\hat{b}^\varepsilon(\omega_j,.))$ and $X^\varepsilon_{4j+4} = \mathrm{Im}(\hat{b}^\varepsilon(\omega_j,.))$, $j = 1, \dots, n$. X^ε satisfies

$$\frac{dX^\varepsilon(z)}{dz} = \frac{1}{\varepsilon}\mathbf{F}\left(\eta(\frac{z}{\varepsilon^2}), \frac{z}{\varepsilon}\right) X^\varepsilon(z),$$

with $X^\varepsilon_{4j+j'}(0) = 1$ if $j' = 1$, $X^\varepsilon_{4j+j'}(0) = 0$ if $j' = 2, 3, 4$, where

$$\mathbf{F}(\eta, h) = \oplus_{j=1}^n F_{\omega_j}(\eta, h)$$

Diffusion-approximation: $X^\varepsilon \to X$ Markov with generator $\mathcal{L}$.
If we denote $\phi^\varepsilon(z) = \mathbb{E}[T^\varepsilon_{\omega_1}(z) \dots T^\varepsilon_{\omega_n}(z)]$ then $\phi(z) = \lim_{\varepsilon \to 0} \phi^\varepsilon(z)$ satisfies:

$$\frac{d\phi(z)}{dz} = -\frac{2\alpha \sum_k \omega_k^2 + \alpha \sum_{k \neq l} \omega_k \omega_l}{4} \phi(z), \quad \phi(0) = 1.$$

This equation is satisfied by: $\tilde{\phi}(z) = \mathbb{E}\left[\prod_k \tilde{A}(z, \omega_k)\right]$ with:

$$\tilde{A}(z, \omega) = \exp(i\frac{\omega\sqrt{\alpha}}{\sqrt{2}} W_z - \frac{\omega^2 \alpha}{4} z)$$

where W_z is a Brownian motion (W_z is a random variable, with Gaussian density, mean 0 and variance z).

Thus $\phi(L) = \tilde{\phi}(L)$ and the limit in distribution of $A^{\varepsilon}(L,\sigma)$ is:

$$\frac{1}{(2\pi)} \int e^{-i\omega\sigma} \hat{f}(\omega) \exp(i\frac{\omega\sqrt{\alpha}}{\sqrt{2}} W_L - \frac{\omega^2\alpha}{4} L) d\omega$$

Proposition. *$(A^{\varepsilon}(L,\sigma))_{\sigma\in\mathbb{R}}$ converge in distribution to $(\bar{A}(L,\sigma))_{\sigma\in\mathbb{R}}$*

$$\bar{A}(L,\sigma) = K_{\mathrm{ODA}} * f\left(\sigma - \sqrt{\frac{\alpha}{2}} W_L\right)$$

where

$$K_{\mathrm{ODA}}(t) = \frac{1}{\sqrt{\pi\alpha L}} \exp\left(-\frac{t^2}{\alpha L}\right)$$

The initial pulse f is modified in two ways:

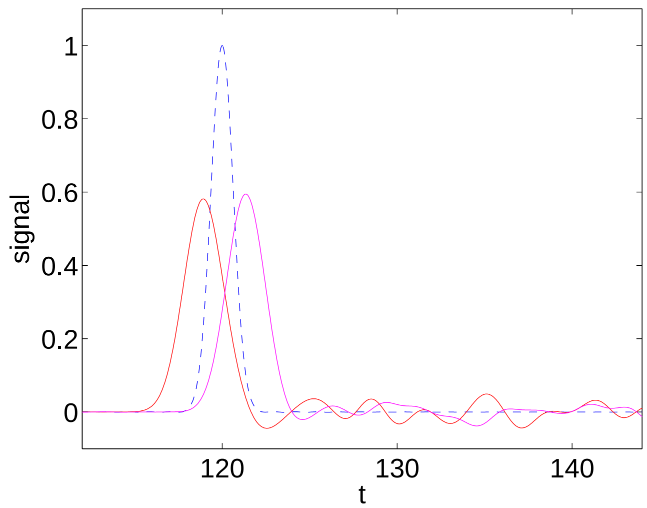
- deterministic spreading (convolution with a Gaussian kernel).
- random time delay $\sim W_L$.

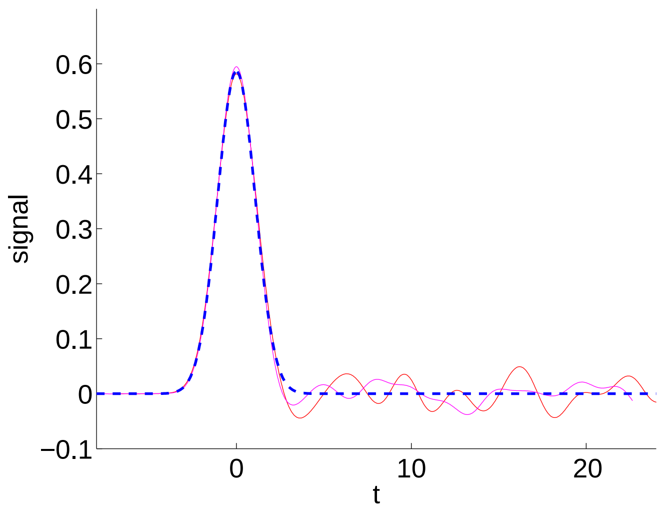
Effective convection-diffusion:

$$d\bar{A} = \frac{\sqrt{\alpha}}{\sqrt{2}} \frac{\partial \bar{A}}{\partial \sigma} \circ dW_z + \frac{\alpha}{4} \frac{\partial^2 \bar{A}}{\partial \sigma^2} dz$$

$$d\bar{A} = \frac{\sqrt{\alpha}}{\sqrt{2}} \frac{\partial \bar{A}}{\partial \sigma} dW_z + \frac{\alpha}{2} \frac{\partial^2 \bar{A}}{\partial \sigma^2} dz$$

Comparison theory - numerics





Time profiles of the transmitted wave at $z = L$

Without time shift

Red, purple: two numerical results obtained with two realizations of the random medium

Blue: original pulse $f(t)$

With time shift

Blue: theoretical (deterministic) transmitted pulse shape $K_{\mathrm{ODA}} * f(t)$

Remark on the mean field approach

Consider $f(z=0,t) = f_0(t) = \exp(-\frac{t^2}{2})$, and

$$f(z,t) = f_0(t - W_z)$$

where W_z is a Brownian motion, with pdf

$$p_z(w) = \frac{1}{\sqrt{2\pi z}} \exp\left(-\frac{w^2}{2z}\right)$$

For a realization: pulse shape preserved, random time delay.

For the mean field:

$$\begin{aligned} \bar{f}(z,t) := \mathbb{E}[f(z,t)] &= \int f_0(t-w) p_z(w) dw \\ &= \frac{1}{\sqrt{1+z}} \exp\left(-\frac{t^2}{2(1+z)}\right) \end{aligned}$$

which means that $\bar{f}$ satisfies a diffusion equation:

$$\frac{\partial \bar{f}}{\partial z} = \frac{1}{2} \frac{\partial^2 \bar{f}}{\partial t^2}$$

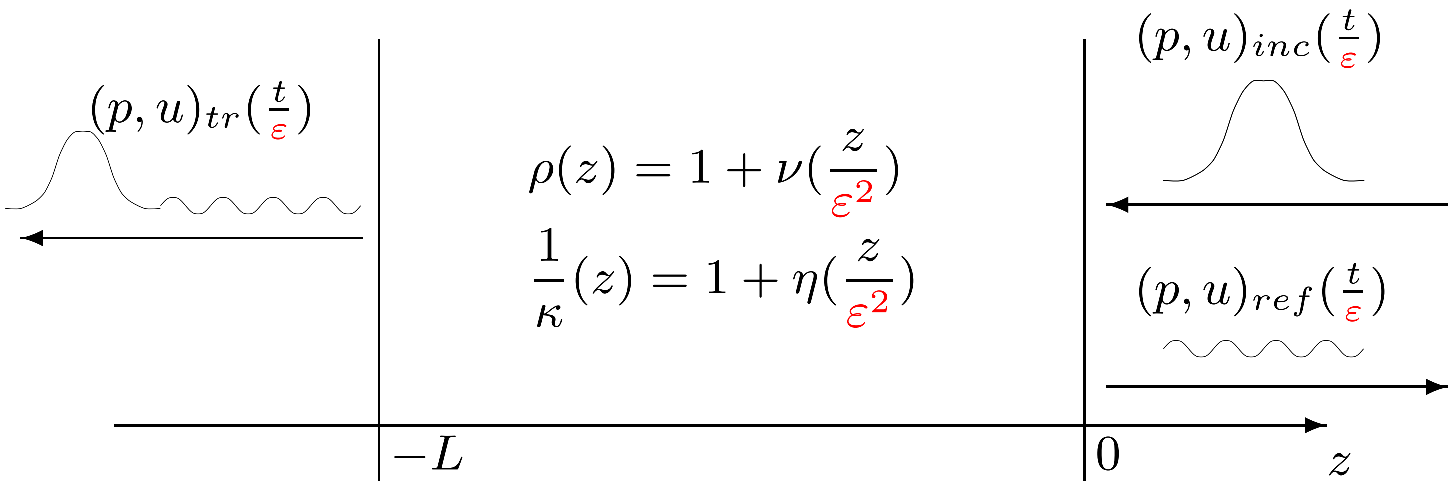
$\hookrightarrow$ the mean field can be very different from the “typical” field.

Scattering of an acoustic pulse in random media

Acoustic equations for pressure p and speed u:

$$\frac{\partial p}{\partial t} + \kappa(z)\frac{\partial u}{\partial z} = 0$$

$$\rho(z)\frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = 0$$



IC: left-going pulse incoming from the right homogeneous half-space.

$$m(z) = \eta(z) + \nu(z), \qquad n(z) = \eta(z) - \nu(z)$$

The propagator $Y_\omega^\varepsilon(z) = \begin{pmatrix} \hat{a}^\varepsilon(z,\omega) & \overline{\hat{b}^\varepsilon}(z,\omega) \\ \hat{b}^\varepsilon(z,\omega) & \overline{\hat{a}^\varepsilon}(z,\omega) \end{pmatrix}$ is the fundamental matrix of the system:

$$\frac{d}{dz} Y_\omega^\varepsilon = P_\omega^\varepsilon(z) Y_\omega^\varepsilon, \qquad P_\omega^\varepsilon(z) = \frac{i\omega}{2\varepsilon} \begin{pmatrix} -m(\frac{z}{\varepsilon^2}) & -n(\frac{z}{\varepsilon^2})e^{\frac{2i\omega z}{\varepsilon}} \\ n(\frac{z}{\varepsilon^2})e^{-\frac{2i\omega z}{\varepsilon}} & m(\frac{z}{\varepsilon^2}) \end{pmatrix}$$

starting from $Y_\omega^\varepsilon(-L) = \mathrm{Id}_2$. By linearity:

$$Y_\omega^\varepsilon(z) \begin{pmatrix} 0 \\ T_\omega^\varepsilon(z) \end{pmatrix} = \begin{pmatrix} R_\omega^\varepsilon(z) \\ 1 \end{pmatrix}$$

The reflection and transmission coefficients for the slab $[-L, z]$ are:

$$R_\omega^\varepsilon(z) = \frac{\overline{\hat{b}^\varepsilon(z,\omega)}}{\overline{\hat{a}^\varepsilon(z,\omega)}}, \qquad T_\omega^\varepsilon(z) = \frac{1}{\overline{\hat{a}^\varepsilon(z,\omega)}}.$$

$$\frac{dR_\omega^\varepsilon}{dz} = \overline{\frac{1}{\hat{a}^\varepsilon}\frac{d\hat{b}^\varepsilon}{dz} - \frac{\hat{b}^\varepsilon}{(\hat{a}^\varepsilon)^2}\frac{d\hat{a}^\varepsilon}{dz}}, \qquad \frac{dT_\omega^\varepsilon}{dz} = -\overline{\frac{1}{(\hat{a}^\varepsilon)^2}\frac{d\hat{a}^\varepsilon}{dz}}$$

From the equations satisfied by $(\hat{a}^\varepsilon, \hat{b}^\varepsilon)$ we get

$$\frac{\partial R_\omega^\varepsilon}{\partial z} = -\frac{i\omega}{\varepsilon} m(\frac{z}{\varepsilon^2}) R_\omega^\varepsilon - \frac{i\omega}{2\varepsilon} n(\frac{z}{\varepsilon^2}) e^{-\frac{2i\omega z}{\varepsilon}} (R_\omega^\varepsilon)^2 - \frac{i\omega}{2\varepsilon} n(\frac{z}{\varepsilon^2}) e^{\frac{2i\omega z}{\varepsilon}}$$

$$\frac{\partial T_\omega^\varepsilon}{\partial z} = -\frac{i\omega}{2\varepsilon} n(\frac{z}{\varepsilon^2}) R_\omega^\varepsilon e^{-\frac{2i\omega z}{\varepsilon}} T_\omega^\varepsilon - \frac{i\omega}{2\varepsilon} m(\frac{z}{\varepsilon^2}) T_\omega^\varepsilon$$

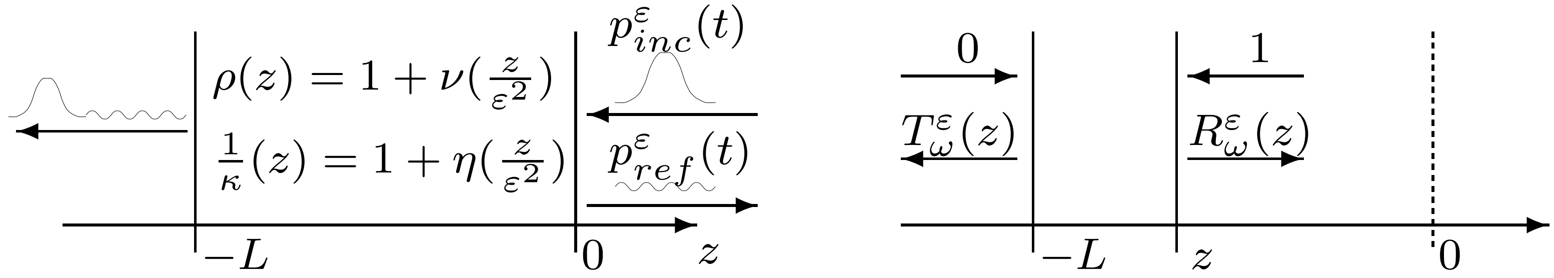
Integral representation of the reflected signal

Send a left-going pulse $f(\frac{t}{\varepsilon})$:

$$p^\varepsilon_{inc}(t) = f(\frac{t}{\varepsilon}) = \int e^{\frac{i\omega t}{\varepsilon}} \hat{f}(\omega) d\omega$$

Reflected signal:

$$p^\varepsilon_{ref}(t) = \int e^{\frac{i\omega t}{\varepsilon}} \hat{f}(\omega) R^\varepsilon_\omega(0) d\omega$$



$R^\varepsilon_\omega(\omega, z)$ is the reflection coefficient for a random slab $[-L, z]$:

$$\frac{\partial R^\varepsilon_\omega}{\partial z} = -\frac{i\omega}{\varepsilon} m(\frac{z}{\varepsilon^2}) R^\varepsilon_\omega - \frac{i\omega}{2\varepsilon} n(\frac{z}{\varepsilon^2}) e^{-\frac{2i\omega z}{\varepsilon}} (R^\varepsilon_\omega)^2 - \frac{i\omega}{2\varepsilon} n(\frac{z}{\varepsilon^2}) e^{\frac{2i\omega z}{\varepsilon}},$$

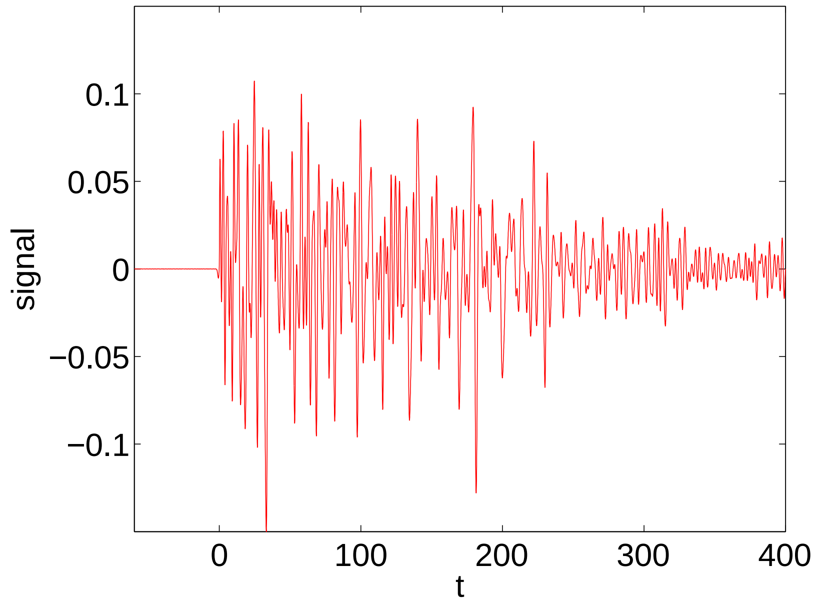
with the initial condition at $z = -L$: $R^\varepsilon_\omega(z = -L) = 0$.

Energy conservation $|R^\varepsilon_\omega|^2 + |T^\varepsilon_\omega|^2 = 1 \longrightarrow$ uniform boundedness of R^ε_ω.

cf M. Asch *et al.*, SIAM Review **33** (1991), 519-625.

The reflected wave

$$p_{ref}(t) = \int e^{\frac{i\omega t}{\varepsilon}} \hat{f}(\omega) R_\omega^\varepsilon(0) d\omega$$



We have $\mathbb{E}[p_{ref}(t)] = 0$ (no coherent signal).

Reflected intensity:

$$\begin{aligned} p_{ref}^2(t) &= \int\int e^{\frac{i(\omega-\omega')t}{\varepsilon}} R_\omega^\varepsilon(0)\overline{R_{\omega'}^\varepsilon(0)}\hat{f}(\omega)\overline{\hat{f}(\omega')}d\omega d\omega' \\ &= \varepsilon \int\int e^{iht} R_\omega^\varepsilon(0)\overline{R_{\omega-\varepsilon h}^\varepsilon(0)}\hat{f}(\omega)\overline{\hat{f}(\omega-\varepsilon h)}d\omega dh \end{aligned}$$

The autocorrelation function of the reflection coefficient

Let us fix ω.

$$U^\varepsilon_{1,1}(z,\omega,h) = R^\varepsilon_{\omega+\frac{\varepsilon h}{2}}(z)\overline{R^\varepsilon_{\omega-\frac{\varepsilon h}{2}}(z)}$$

We look for $\lim_{\varepsilon\to 0}\mathbb{E}[U^\varepsilon_{1,1}(0,\omega,h)]$.

For $p,q\in\mathbb{N}$, $z\in[-L,0]$ we introduce

$$U^\varepsilon_{p,q}(z,\omega,h) = \left(R^\varepsilon_{\omega+\frac{\varepsilon h}{2}}(z)\right)^p\left(\overline{R^\varepsilon_{\omega-\frac{\varepsilon h}{2}}(z)}\right)^q$$

From the Ricatti equation satisfied by R^ε_ω:

$$\begin{aligned}\frac{\partial U^\varepsilon_{p,q}}{\partial z} &= \frac{i\omega}{\varepsilon}m(\frac{z}{\varepsilon^2})(q-p)U^\varepsilon_{p,q}\\ &\quad -\frac{i\omega}{2\varepsilon}n(\frac{z}{\varepsilon^2})e^{\frac{2i\omega z}{\varepsilon}}\left(pe^{ihz}U^\varepsilon_{p-1,q}-qe^{-ihz}U^\varepsilon_{p,q+1}\right)\\ &\quad +\frac{i\omega}{2\varepsilon}n(\frac{z}{\varepsilon^2})e^{-\frac{2i\omega z}{\varepsilon}}\left(qe^{ihz}U^\varepsilon_{p,q-1}-pe^{-ihz}U^\varepsilon_{p+1,q}\right)\end{aligned}$$

with $U^\varepsilon_{p,q}(z=-L,\omega,h)=\mathbf{1}_0(p)\mathbf{1}_0(q)$.
Take a Fourier transform with respect to h:

$$V^\varepsilon_{p,q}(z,\omega,\tau) = \frac{1}{2\pi}\int e^{ih(\tau-(p+q)z)}U^\varepsilon_{p,q}(z,\omega,h)dh$$

$$
\begin{aligned}
\frac{\partial V^{\varepsilon}_{p,q}}{\partial z} \quad = \quad & -(p+q)\frac{\partial V^{\varepsilon}_{p,q}}{\partial \tau} + \frac{i\omega}{\varepsilon} m(\frac{z}{\varepsilon^2})(q-p)V^{\varepsilon}_{p,q} \\
& -\frac{i\omega}{2\varepsilon} n(\frac{z}{\varepsilon^2}) e^{\frac{2i\omega z}{\varepsilon}} \left(pV^{\varepsilon}_{p-1,q} - qV^{\varepsilon}_{p,q+1}\right) \\
& +\frac{i\omega}{2\varepsilon} n(\frac{z}{\varepsilon^2}) e^{-\frac{2i\omega z}{\varepsilon}} \left(qV^{\varepsilon}_{p,q-1} - pV^{\varepsilon}_{p+1,q}\right)
\end{aligned}
$$

starting from $V^{\varepsilon}_{p,q}(z=-L,\omega,\tau) = \delta(\tau)\mathbf{1}_0(p)\mathbf{1}_0(q)$.

Approximation-diffusion $\Longrightarrow$ $V^{\varepsilon}_{p,q}$ converges as $\varepsilon \to 0$ to a diffusion Markov process.

In particular $\mathbb{E}[V^{\varepsilon}_{p,p}(z,\omega,\tau)]$, $p \in \mathbb{N}$, converges to $V_p(z,\omega,\tau)$:

$$
\begin{aligned}
& \frac{\partial V_p}{\partial z} + 2p\frac{\partial V_p}{\partial \tau} = \frac{1}{2}\alpha_n \omega^2 p^2 \left(V_{p+1} + V_{p-1} - 2V_p\right) \\
& V_p(z=-L,\omega,\tau) = \delta(\tau)\mathbf{1}_0(p)
\end{aligned}
$$

where $\alpha_n = \int_0^\infty \mathbb{E}[n(0)n(z)]dz$.

We thus get the limit of the expectation of R^{ε}_{ω}:

$$
\mathbb{E}\left[R^{\varepsilon}_{\omega+\frac{\varepsilon h}{2}}(0)\overline{R^{\varepsilon}_{\omega-\frac{\varepsilon h}{2}}(0)}\right] \stackrel{\varepsilon\to 0}{\longrightarrow} \int V_1(0,\omega,\tau)e^{-ih\tau}d\tau
$$

Analysis of the transport equations

Let us introduce the jump Markov process $(N_z)_{z\geq -L}$ with state space $\mathbb{N}$ and infinitesimal generator

$$\mathcal{L}\phi(N) = \frac{1}{2}\alpha_n \omega^2 N^2 \left(\phi(N+1) + \phi(N-1) - 2\phi(N)\right)$$

We also define the process $\dfrac{\partial \mathcal{T}_z}{\partial z} = -2N_z$.

Then $(N_z, \mathcal{T}_z)$ is Markovian with generator: $\mathcal{L} - 2N\frac{\partial}{\partial \mathcal{T}}$.

The solution to

$$\frac{\partial \phi}{\partial z} + 2N\frac{\partial \phi}{\partial \tau} = \mathcal{L}\phi, \qquad \phi(z=-L, N, \tau) = \phi_0(N,\tau)$$

can be written as

$$\phi(z,N,\tau) = \mathbb{E}\left[\phi_0\left(N_z, \tau - 2\int_{-L}^{z} N_s ds\right) \mid N_{-L} = N\right]$$

Taking (formally) $\phi_0(N,\tau) = \mathbf{1}_0(N)\delta(\tau)$, we find $\phi(0,1,\tau) = V_1(0,\omega,\tau)$

$$\int_{\tau_0}^{\tau_1} V_1(0,\omega,\tau)d\tau = \mathbb{P}\left(N_0 = 0 \bigcap \int_{-L}^{0} 2N_s ds \in [\tau_0,\tau_1] \mid N_{-L} = 1\right)$$

Application 1: By taking $\tau_0 = 0$ and $\tau_1 = \infty$

$$\mathbb{E}\left[|R_\omega^\varepsilon|^2(0)\right] \stackrel{\varepsilon\to 0}{\longrightarrow} \mathbb{P}(N_0 = 0 \,|\, N_{-L} = 1).$$

Duality formula: Let us introduce the diffusion process $(\theta_z)_{z\geq -L}$:

$$d\theta_z = \sqrt{\alpha_n}\omega dW_z + \frac{1}{2}\alpha_n\omega^2 \coth(\theta_z)dz.$$

We have

$$\mathbb{E}\left[\xi^{N_z} \,|\, N_{-L} = p_0\right] = \mathbb{E}\left[\tanh(\frac{\theta_z}{2})^{2p_0} \mid \theta_{-L} = 2 \text{ argtanh}(\sqrt{\xi})\right],$$

and in particular

$$\mathbb{E}\left[|R_\omega^\varepsilon|^2(0)\right] \stackrel{\varepsilon\to 0}{\longrightarrow} \mathbb{P}(N_0 = 0 \,|\, N_{-L} = 1) = \mathbb{E}\left[\tanh(\frac{\theta_0}{2})^2 \mid \theta_{-L} = 0\right].$$

The pdf of θ_0 can be computed:

$$\mathbb{E}\left[|R_\omega^\varepsilon|^2(0)\right] \stackrel{\varepsilon\to 0}{\longrightarrow} 1 - \frac{4}{\sqrt{\pi}} \exp\left(-\frac{L}{l(\omega)}\right) \int_0^\infty \frac{x^2 e^{-x^2}}{\cosh\left(2\sqrt{L/l(\omega)}x\right)} dx$$

where $l(\omega) = \dfrac{8}{\alpha_n\omega^2}$.

Application 2: $L \to \infty$.

We study the limit distribution of the Markov $(N_0, \mathcal{T}_0)$ starting from $(N_{-L} = N, \mathcal{T}_{-L} = 0)$ when $L \to \infty$.

0 is an absorbing state of $(N_z)_{z \geq -L}$ and $(N_0, 2\int_{-L}^{0} N_z dz)$ converges to $(0, \mu)$ where μ is a random variable with density P_N (pdf of μ starting from $N_0 = N$). It satisfies

$$\frac{\partial P_N}{\partial \tau} = \frac{1}{4}\alpha_n \omega^2 N \left(P_{N+1} - 2P_N + P_{N-1}\right), \qquad P_0(\tau) = \delta(\tau)$$

After some algebra:

$$P_N(\omega, \tau) = \frac{\partial}{\partial \tau}\left[\left(\frac{\alpha_n \omega^2 \tau}{4 + \alpha_n \omega^2 \tau}\right)^N \mathbf{1}_{[0,\infty)}(\tau)\right]$$

Finally

$$\mathbb{E}\left[R^{\varepsilon}_{\omega + \frac{\varepsilon h}{2}}(0)\overline{R^{\varepsilon}_{\omega - \frac{\varepsilon h}{2}}(0)}\right] \stackrel{\varepsilon \to 0}{\longrightarrow} \int P_1(\omega, \tau) e^{-ih\tau} d\tau$$

where

$$P_1(\omega, \tau) = \frac{4\alpha_n \omega^2}{(4 + \alpha_n \omega^2 \tau)^2}\mathbf{1}_{[0,\infty)}(\tau)$$

The reflected wave

Mean reflected intensity:

$$\begin{aligned}
\frac{1}{\varepsilon}\mathbb{E}[p_{ref}^2(t)] &= \int\int e^{iht}\mathbb{E}[R^\varepsilon_{\omega+\frac{\varepsilon h}{2}}(0)\overline{R^\varepsilon_{\omega-\frac{\varepsilon h}{2}}(0)}] \\
&\quad \times \hat{f}(\omega+\frac{\varepsilon h}{2})\overline{\hat{f}}(\omega-\frac{\varepsilon h}{2})d\omega dh \\
&\overset{\varepsilon\to 0}{\longrightarrow} \int\int e^{iht}\int V_1(0,\omega,\tau)e^{-ih\tau}d\tau|\hat{f}(\omega)|^2 d\omega dh \\
&= 2\pi\int V_1(0,\omega,t)|\hat{f}(\omega)|^2 d\omega
\end{aligned}$$

Autocorrelation function:

$$\frac{1}{\varepsilon}\mathbb{E}[p_{ref}(t)p_{ref}(t+\varepsilon\tau)] \overset{\varepsilon\to 0}{\longrightarrow} 2\pi\int V_1(0,\omega,t)|\hat{f}(\omega)|^2 e^{i\omega\tau}d\omega \qquad (1)$$

Compute all moments:

a) For a fixed $t>0$, $\left(\varepsilon^{-1/2}p_{ref}(t+\varepsilon\tau)\right)_\tau$ converges to a zero-mean stationary Gaussian process with autocorrelation function (1).

b) For $0<t_1<\cdots<t_n$, the processes $\left(\varepsilon^{-1/2}p_{ref}(t_j+\varepsilon\tau)\right)_\tau$, $j=1,\cdots,n$, become independent as $\varepsilon\to 0$.

Applications to imaging

Assume the medium presents small-scale random fluctuations and large-scale deterministic variation that we want to image:

$$\frac{1}{\kappa(z)} = \frac{1}{\kappa_0(z)}\left(1+\eta(\frac{z}{\varepsilon^2})\right)$$
$$\rho(z) = \rho_0(z)\left(1+\nu(\frac{z}{\varepsilon^2})\right)$$

Idea: Perform a series of experiments where you probe the medium with a source f (or a set of sources).

Goal: extract information about the medium from the reflected waves.

Property: a Gaussian process is characterized by its autocorrelation function.

Consequence: All the information is in the autocorrelation function.

$$\mathbb{E}[p_{ref}(t+\varepsilon\tau)p_{ref}(t)] = 2\pi\varepsilon \int V_1(0,\omega,t)|\hat{f}(\omega)|^2 e^{i\omega\tau}d\omega$$

Fortunately, V_1 contains the information about the large-scale features of the medium.

Thus: if we get $\mathbb{E}[p_{ref}(t+\varepsilon\tau)p_{ref}(t)]$, then we get V_1, and we "know" how to solve the inverse problem

$$\text{“}(V_1(0,\omega,\tau))_{\omega,\tau} \mapsto \text{large-scale variations (such as } c_0(z))\text{”}$$

$$\frac{\partial V_p}{\partial z} + \frac{2p}{c_0(z)}\frac{\partial V_p}{\partial \tau} = \frac{1}{2}\alpha_n \omega^2 p^2 \left(V_{p+1} + V_{p-1} - 2V_p\right)$$

$$V_p(z=-L,\omega,\tau) = \delta(\tau)\mathbf{1}_0(p)$$

Problem: we only have a single realization of the medium !

How to estimate $\mathbb{E}[p_{ref}(t+\varepsilon\tau)p_{ref}(t)]$?

Answer: Local average,

$$\frac{1}{\Delta t}\int_{t-\Delta t/2}^{t-\Delta t/2} p_{ref}(s+\varepsilon\tau)p_{ref}(s)ds$$

with $\varepsilon \ll \Delta t \ll 1$ (optimal choice not easy).

Or: time-windowed Fourier transform, wavelet decomposition, ...

Not very efficient...