Wave propagation in random media

J. Garnier (Université Paris VII)

`http://www.proba.jussieu.fr/~garnier`

1. Homogenization theory, diffusion approximation, and asymptotic theory for random differential equations.
2. Wave propagation in one-dimensional random media: the coherent wave front, the incoherent wave fluctuations, time reversal.
3. Wave propagation and time reversal in a random waveguide.

What is a random medium ?

Problem: Wave propagation in a highly heterogeneous medium.

Stochastic modeling: the medium is a realization of a random medium (a set of possible media described statistically).
- takes into account the available data (mean, standard deviation of the fluctuations, ...)
- completes the modeling by a statistical description (Gaussian process, ...).

Statistical distribution of the random medium $\Longrightarrow$ statistical distribution of the wave (highly nonlinear problem).

What about a wave propagating in a "typical" realization ?
- Mean-field (or averaged) approach can be misleading.
- A complete statistical analysis is necessary.
- There exist statistically stable quantities.

Importance of scaled regimes and asymptotic theory.

Methodology

- Modeling:
  - Identification of the phenomena and equations.
  - Statistical description of the medium parameters.
  - Determination of the scales.
- Asymptotics:
  - Separation of scales.
  - Limit theorems.
- Limit problem:
  - Analysis of the physically relevant quantities.
  - Use of stochastic calculus.

The acoustic wave equations

The acoustic pressure $p(z,t)$ and velocity $u(z,t)$ satisfy the continuity and momentum equations

$$\rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = 0$$
$$\frac{\partial p}{\partial t} + \kappa \frac{\partial u}{\partial z} = 0$$

where $\rho(z)$ is the material density,

$\kappa(z)$ is the bulk modulus of the medium.

Propagation in homogeneous medium

Linear hyperbolic system with ρ, κ constant.

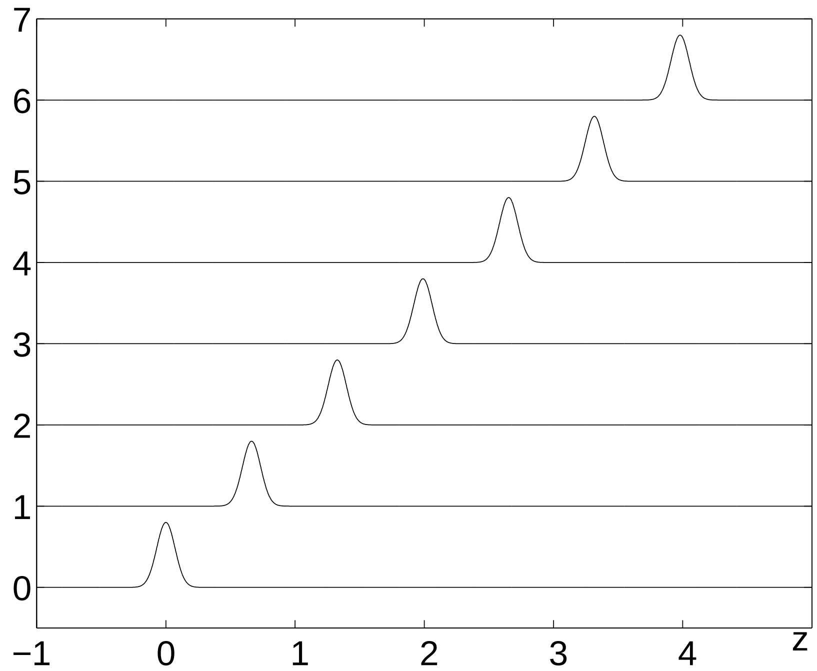
Impedance: $\zeta = \sqrt{\rho\kappa}$. Sound speed: $c = \sqrt{\kappa/\rho}$.

Right and left going modes:

$$A = \zeta^{1/2} u + \zeta^{-1/2} p, \qquad B = \zeta^{1/2} u - \zeta^{-1/2} p$$

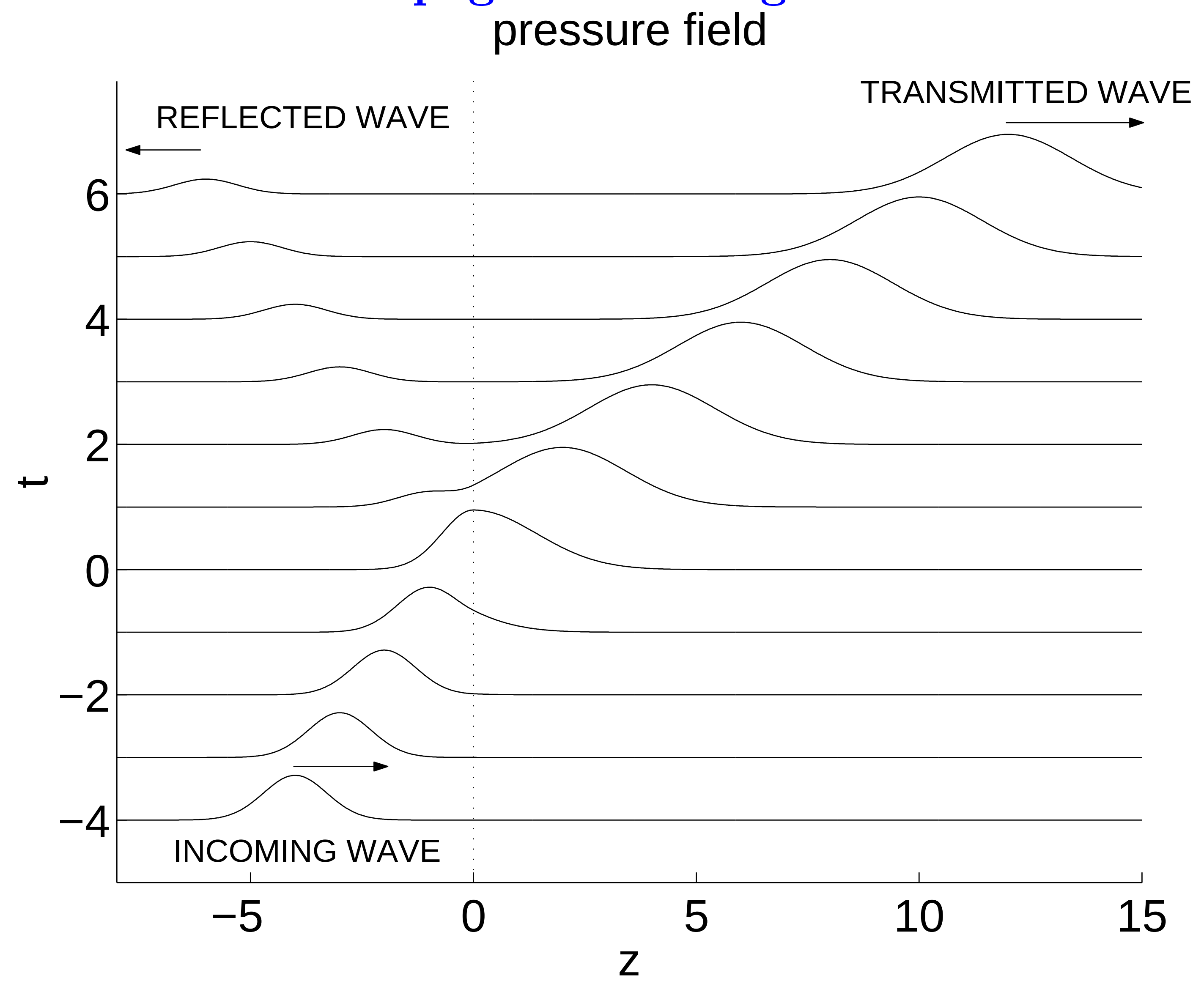
$$\frac{\partial A}{\partial t} + c\frac{\partial A}{\partial z} = 0, \qquad \frac{\partial B}{\partial t} - c\frac{\partial B}{\partial z} = 0$$

A: right-going wave $\qquad$ B: left-going wave.



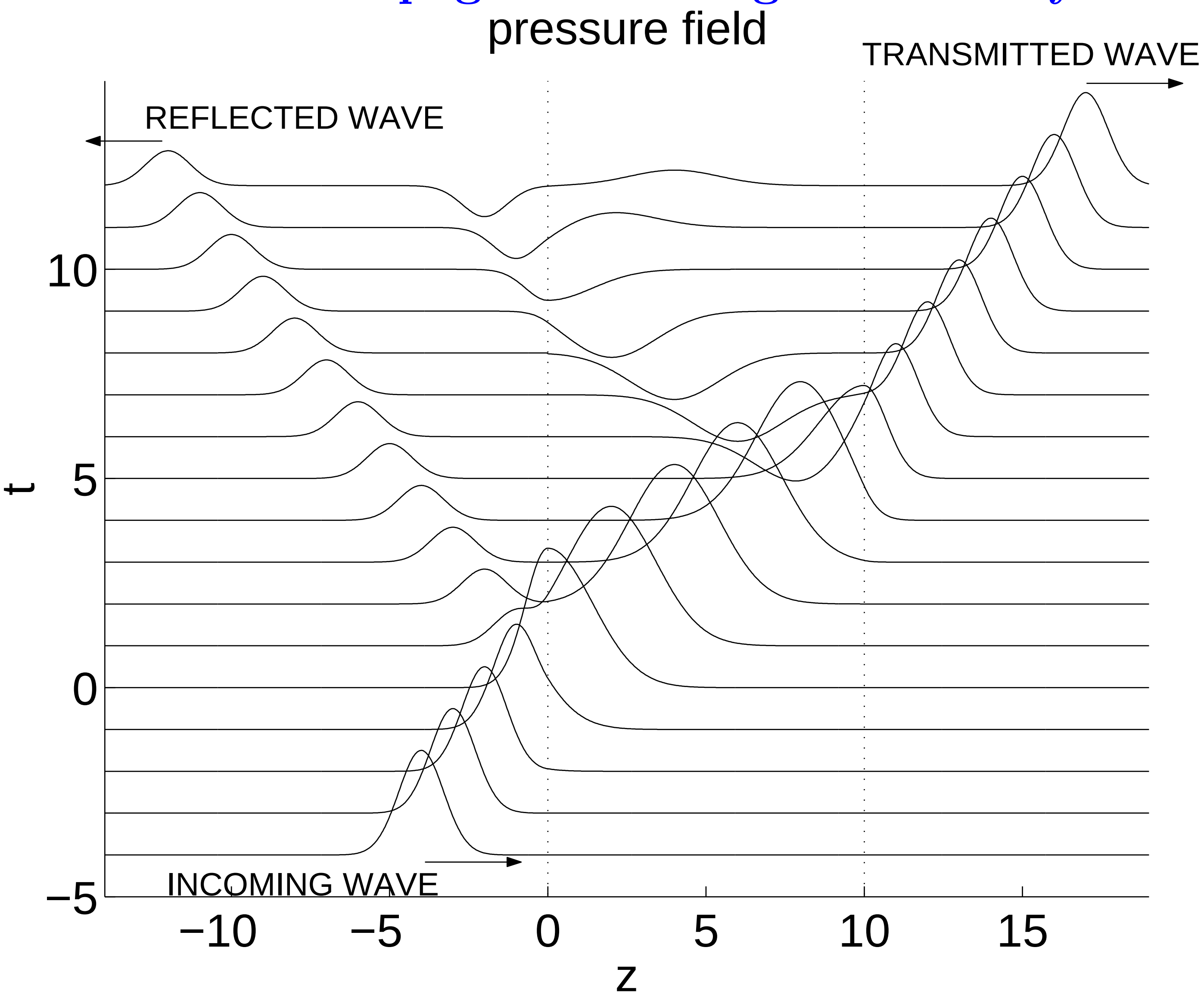
Spatial profiles of the wave at different times for a pure right-going wave

Propagation through an interface



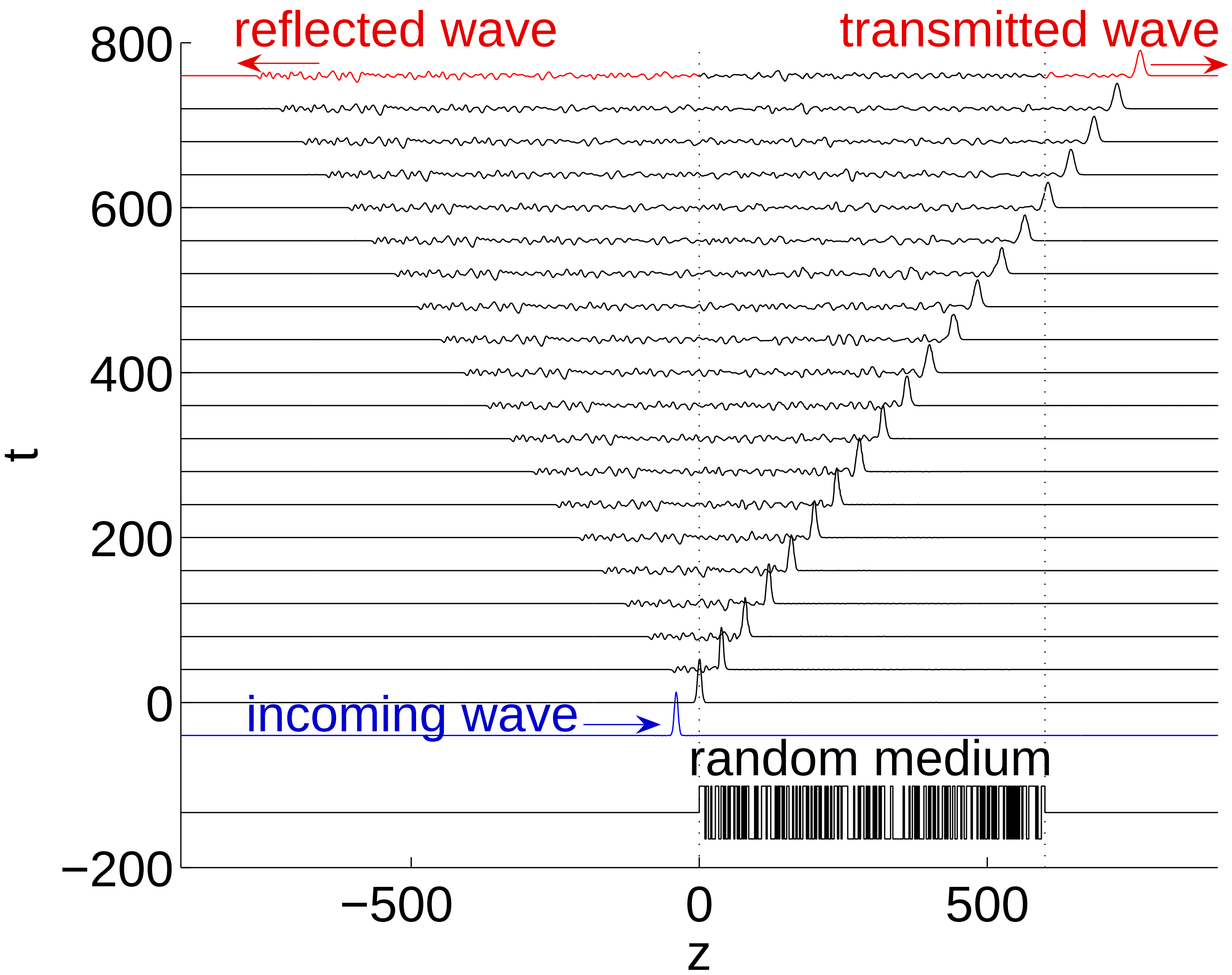
Medium $z < 0$: $c = 1$, $\zeta = 1$. Medium $z > 0$: $c = 2$, $\zeta = 2$.

Propagation through a thick layer



Medium $\begin{Bmatrix} z < 0 \\ z > 10 \end{Bmatrix}$: $c = 1$, $\zeta = 1$. Medium $0 < z < 10$: $c = 2$, $\zeta = 2$.

A numerical experiment in random medium



Random medium: stack of thin layers composed of two materials.

The three scales

The acoustic pressure $p(z,t)$ and velocity $u(z,t)$ satisfy the continuity and momentum equations

$$\rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = 0$$
$$\frac{\partial p}{\partial t} + \kappa \frac{\partial u}{\partial z} = 0$$

where $\rho(z)$ is the material density,

$\kappa(z)$ is the bulk modulus of the medium.

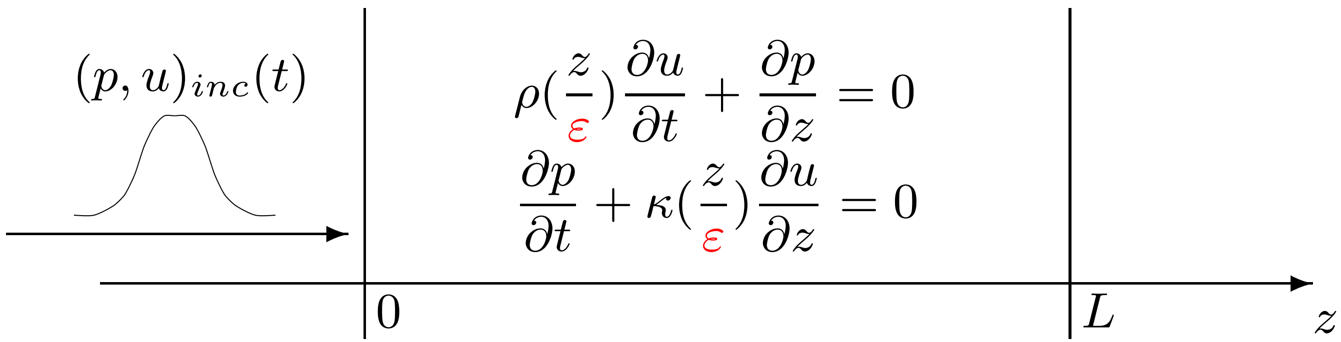
Three scales:

l_c: correlation radius of the random process ρ and κ.

λ: typical wavelength of the incoming pulse.

L: propagation distance.

Effective medium theory $L \sim \lambda \gg l_c$



Model: $\rho = \rho(z/\varepsilon)$ and $\kappa = \kappa(z/\varepsilon)$, where $0 < \varepsilon \ll 1$ and ρ, κ are stationary random functions.

Perform a Fourier transform with respect to t:

$$u(z,t) = \int \hat{u}(z,\omega)e^{i\omega t}d\omega, \quad p(z,t) = \int \hat{p}(z,\omega)e^{i\omega t}d\omega$$

so that we get a system of ordinary differential equations:

$$\frac{dX^\varepsilon}{dz} = F(\frac{z}{\varepsilon}, X^\varepsilon),$$

where

$$X^\varepsilon = \begin{pmatrix} \hat{p} \\ \hat{u} \end{pmatrix}, \quad F(z,X) = -i\omega \begin{pmatrix} 0 & \rho(z) \\ \frac{1}{\kappa(z)} & 0 \end{pmatrix} X$$

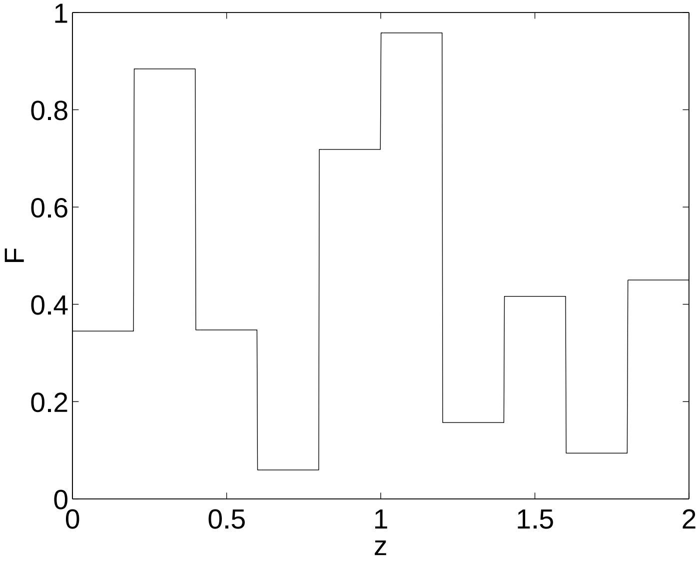
Method of averaging: Toy model

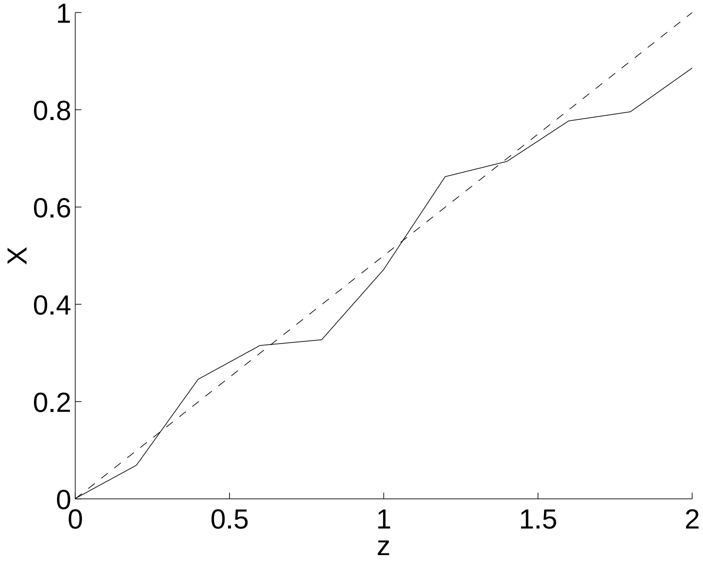
Let $X^\varepsilon(z) \in \mathbb{R}$ be the solution of

$$\frac{dX^\varepsilon}{dz} = F\left(\frac{z}{\varepsilon}\right)$$

with $F(z) = \sum_{i=1}^\infty F_i \mathbf{1}_{[i-1,i)}(z)$, F_i independent random variables $\mathbb{E}[F_i] = \bar{F}$ and $\mathbb{E}[(F_i - \bar{F})^2] = \sigma^2$.

($z \mapsto t$, particle in a random velocity field)



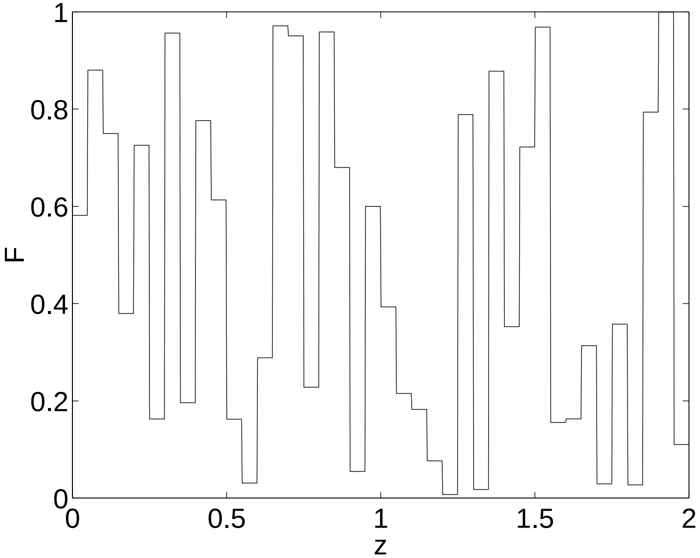


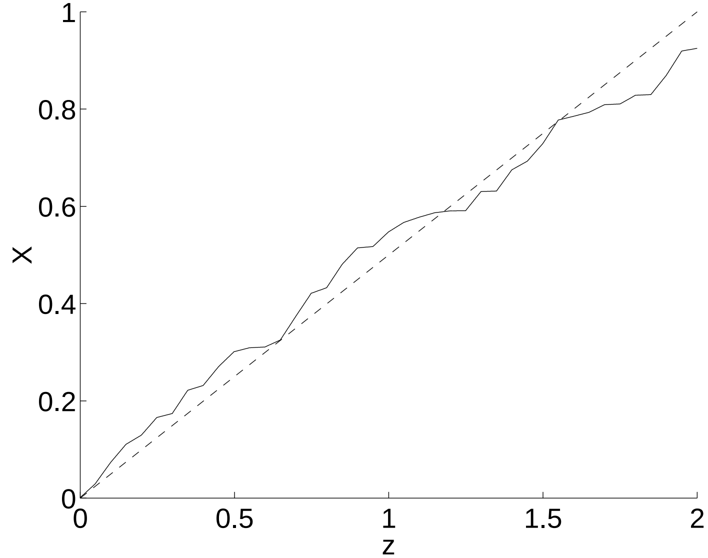
$\varepsilon = 0.2$

$$
\begin{aligned}
X^{\varepsilon}(z) &= \varepsilon \int_0^{\frac{z}{\varepsilon}} F(s)ds = \varepsilon \left(\sum_{i=1}^{\left[\frac{z}{\varepsilon}\right]} F_i \right) + \varepsilon \int_{\left[\frac{z}{\varepsilon}\right]}^{\frac{z}{\varepsilon}} F(s)ds \\
&= \underset{\substack{\varepsilon \to 0 \downarrow \\ z}}{\varepsilon \left[\frac{z}{\varepsilon}\right]} \times \underset{\substack{\text{a.s.} \downarrow (LLN) \\ \mathbb{E}[F(z)] = \bar{F}}}{\frac{1}{\left[\frac{z}{\varepsilon}\right]} \left(\sum_{i=1}^{\left[\frac{z}{\varepsilon}\right]} F_i \right)} + \underset{\substack{\text{a.s.} \downarrow \\ 0}}{\varepsilon \left(\frac{z}{\varepsilon} - \left[\frac{z}{\varepsilon}\right] \right) F_{\left[\frac{z}{\varepsilon}\right]}}
\end{aligned}
$$

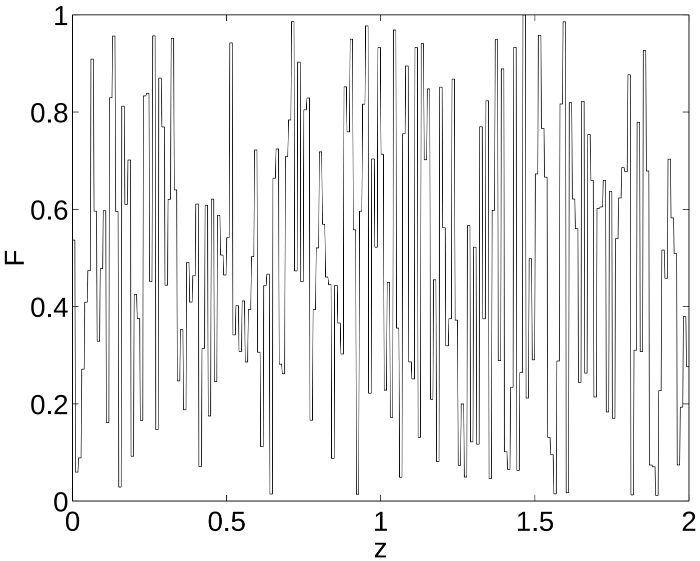
Thus:

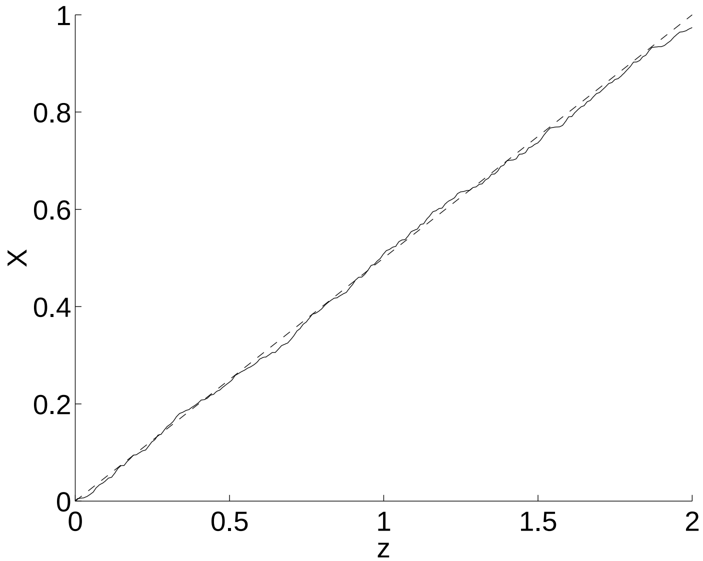
$$
X^{\varepsilon}(z) \xrightarrow{\varepsilon \to 0} \bar{X}(z), \qquad \frac{d\bar{X}}{dz} = \bar{F}.
$$





$$\varepsilon = 0.05$$





$$\varepsilon = 0.01$$

Method of averaging: Khasminskii theorem

$$\frac{dX^\varepsilon}{dz} = F(\frac{z}{\varepsilon}, X^\varepsilon), \quad X^\varepsilon(0) = x_0$$

$x \mapsto F(x,z)$ and $x \mapsto \bar{F}(x)$ are Lipschitz, $z \mapsto F(z,x)$ is stationary and ergodic

$$\bar{F}(x) = \mathbb{E}[F(z,x)]$$

Let $\bar{X}$ be the solution of

$$\frac{d\bar{X}}{dz} = \bar{F}(\bar{X}), \quad \bar{X}(0) = x_0$$

Theorem: for any $Z > 0$,

$$\sup_{z\in[0,Z]} \mathbb{E}\left[|X^\varepsilon(z) - \bar{X}(z)|\right] \stackrel{\varepsilon\to 0}{\longrightarrow} 0$$

[1] R. Z. Khasminskii, Theory Probab. Appl. **11** (1966), 211-228.

Equations for the Fourier components of the wave:

$$\frac{dX^\varepsilon}{dz} = F\left(\frac{z}{\varepsilon}, X^\varepsilon\right),$$

where

$$X^\varepsilon = \begin{pmatrix} \hat{p} \\ \hat{u} \end{pmatrix}, \qquad F(z, X) = -i\omega \begin{pmatrix} 0 & \rho(z) \\ \frac{1}{\kappa(z)} & 0 \end{pmatrix} X$$

Apply the method of averaging $\Longrightarrow X^\varepsilon(z,\omega)$ converges in $L^1(\mathbb{P})$ to $\bar{X}(z,\omega)$

$$\frac{d\bar{X}}{dz} = -i\omega \begin{pmatrix} 0 & \bar{\rho} \\ \frac{1}{\bar{\kappa}} & 0 \end{pmatrix} \bar{X}, \qquad \bar{\rho} = \mathbb{E}[\rho], \qquad \bar{\kappa} = \left(\mathbb{E}[\kappa^{-1}]\right)^{-1}$$

$\hookrightarrow$ deterministic “effective medium” with parameters $\bar{\rho}$, $\bar{\kappa}$.

Let $(\bar{p}, \bar{u})$ be the solution of the homogeneous effective system

$$\bar{\rho}\frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{p}}{\partial z} = 0$$
$$\frac{\partial \bar{p}}{\partial t} + \bar{\kappa}\frac{\partial \bar{u}}{\partial z} = 0$$

The propagation speed of $(\bar{p}, \bar{u})$ is $\bar{c} = \sqrt{\bar{\kappa}/\bar{\rho}}$.

Compare $u^\varepsilon(z,t)$ with $\bar{u}(z,t)$:

$$\begin{aligned}\mathbb{E}\left[|u^\varepsilon(z,t) - \bar{u}(z,t)|\right] &= \mathbb{E}\left[\left|\int e^{i\omega t}(\hat{u}^\varepsilon(z,\omega) - \hat{\bar{u}}(z,\omega))d\omega\right|\right] \\ &\leq \int \mathbb{E}\left[|\hat{u}^\varepsilon(z,\omega) - \hat{\bar{u}}(z,\omega)|\right] d\omega\end{aligned}$$

The dominated convergence theorem then gives the convergence in $L^1(\mathbb{P})$ of u^ε to $\bar{u}$ in the time domain.
$\hookrightarrow$ the effective speed of the acoustic wave $(p^\varepsilon, u^\varepsilon)$ as $\varepsilon \longrightarrow 0$ is $\bar{c}$.

This analysis is just a small piece of the homogenization theory.
cf lecture by A. Piatnitski

Example: bubbles in water

$\rho_a = 1.2\ 10^3$ g/m^3, $\kappa_a = 1.4\ 10^8$ g/s^2/m, $c_a = 340$ m/s.
$\rho_w = 1.0\ 10^6$ g/m^3, $\kappa_w = 2.0\ 10^{18}$ g/s^2/m, $c_w = 1425$ m/s.
If the typical pulse frequency is 10 Hz - 30 kHz, then the typical wavelength is 1 cm - 100 m. The bubble sizes are much smaller $\Longrightarrow$ the effective medium theory can be applied.

$$\bar{\rho} = \mathbb{E}[\rho] = \phi\rho_a + (1-\phi)\rho_w = \begin{cases} 9.9\ 10^5\ \text{g/m}^3 & \text{if } \phi = 1\% \\ 9\ 10^5\ \text{g/m}^3 & \text{if } \phi = 10\% \end{cases}$$

$$\bar{\kappa} = \left(\mathbb{E}[\kappa^{-1}]\right)^{-1} = \left(\frac{\phi}{\kappa_a} + \frac{1-\phi}{\kappa_w}\right)^{-1} = \begin{cases} 1.4\ 10^{10}\ \text{g/s}^2\text{/m} & \text{if } \phi = 1\% \\ 1.4\ 10^{9}\ \text{g/s}^2\text{/m} & \text{if } \phi = 10\% \end{cases}$$

where $\phi =$ volume fraction of air.
Thus, $\bar{c} = 120$ m/s if $\phi = 1\%$ and $\bar{c} = 37$ m/s if $\phi = 10$ %.

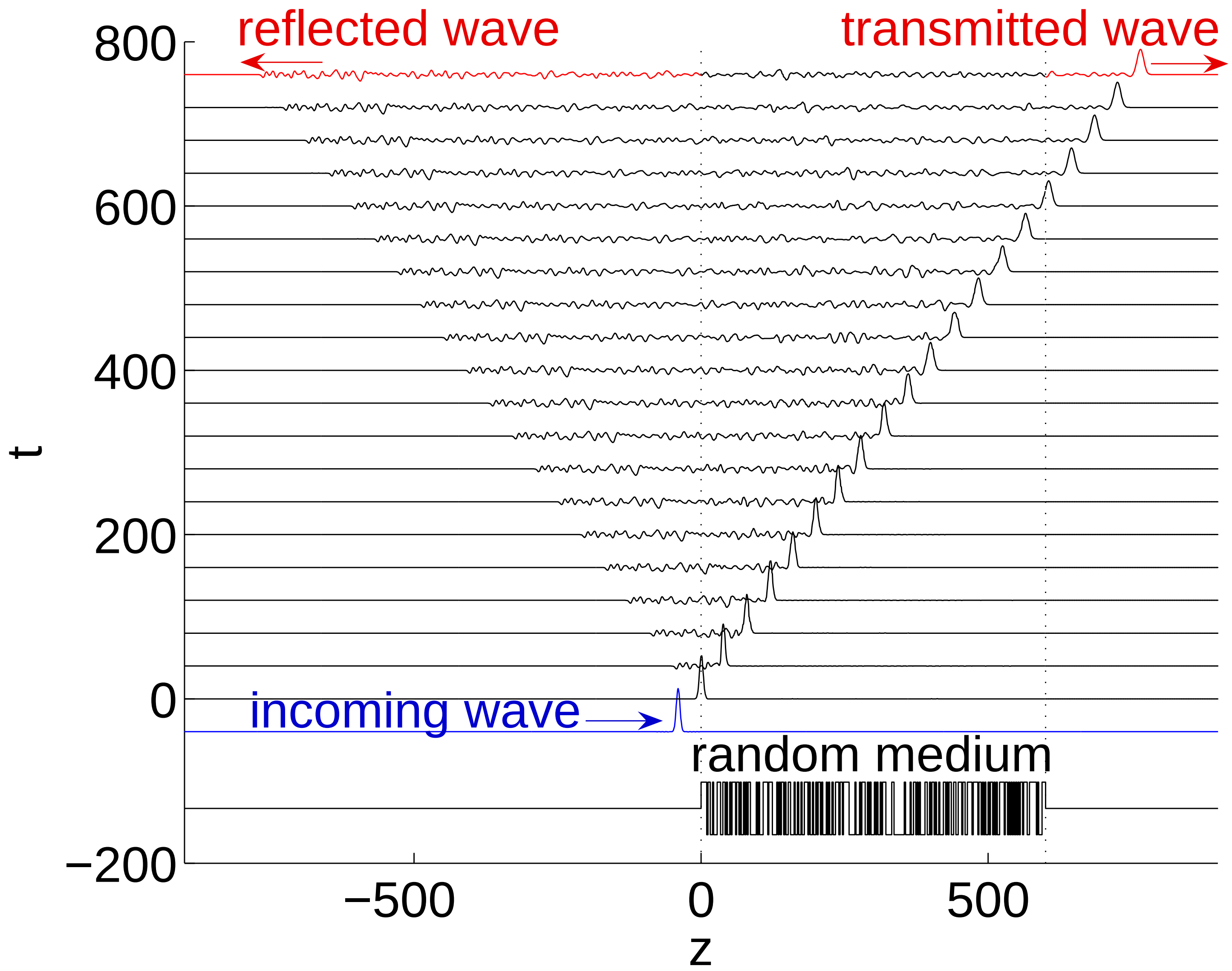
$\hookrightarrow$ the average sound speed $\bar{c}$ can be much smaller than ess inf(c).

The converse is impossible:

$$\mathbb{E}[c^{-1}] = \mathbb{E}\left[\kappa^{-1/2}\rho^{1/2}\right] \leq \mathbb{E}[\kappa^{-1}]^{1/2}\mathbb{E}[\rho]^{1/2} = \bar{c}^{-1}$$

Thus $\bar{c} \leq \mathbb{E}[c^{-1}]^{-1} \leq$ ess sup(c).

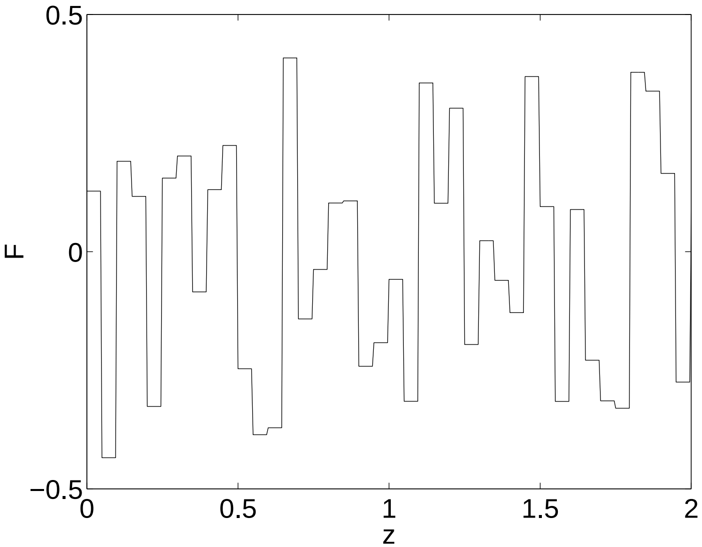
Long distance propagation

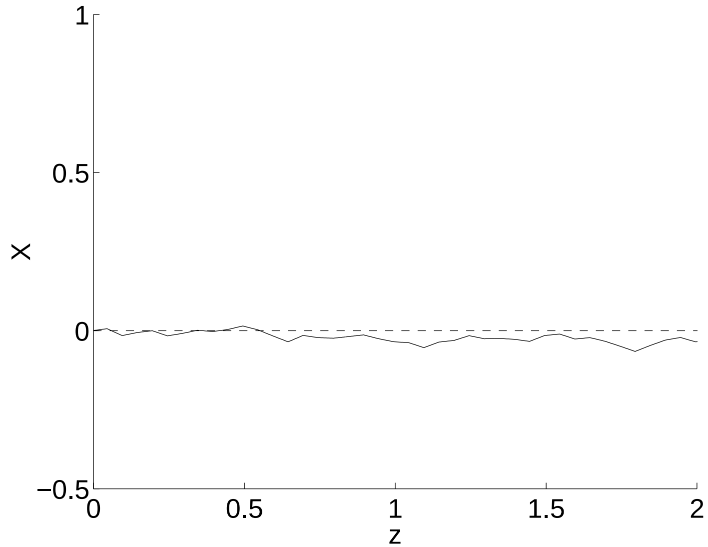


Toy model

$$\frac{dX^\varepsilon}{dz} = F(\frac{z}{\varepsilon})$$

with $F(z) = \sum_{i=1}^\infty F_i \mathbf{1}_{[i-1,i)}(z)$, F_i independent random variables $\mathbb{E}[F_i] = \bar{F} = 0$ and $\mathbb{E}[(F_i - \bar{F})^2] = \sigma^2$.





$\varepsilon = 0.05$

For any $z \in [0, Z]$, we have

$$X^\varepsilon(z) \xrightarrow{\varepsilon \to 0} \bar{X}(z), \qquad \frac{d\bar{X}}{dz} = \bar{F} = 0.$$

No macroscopic evolution is noticeable.

$\longrightarrow$ it is necessary to look at larger z to get an effective behavior

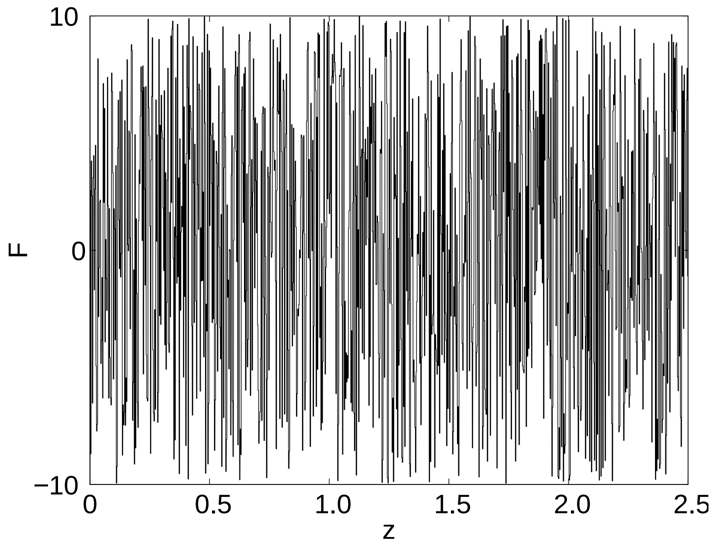
$$z \mapsto \frac{z}{\varepsilon}, \qquad \tilde{X}^\varepsilon(z) = X^\varepsilon(\frac{z}{\varepsilon})$$

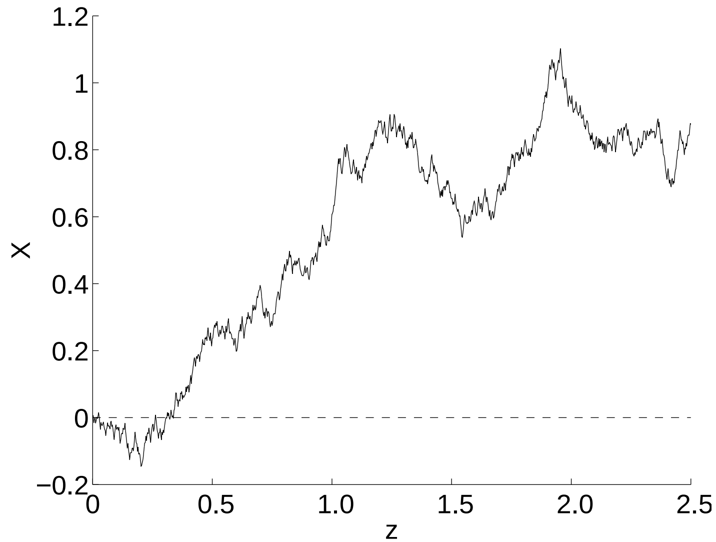
$$\frac{d\tilde{X}^\varepsilon}{dz} = \frac{1}{\varepsilon} F(\frac{z}{\varepsilon^2})$$

Diffusion-approximation: Toy model

$$\frac{dX^\varepsilon}{dz} = \frac{1}{\varepsilon} F(\frac{z}{\varepsilon^2})$$

with $F(z) = \sum_{i=1}^{\infty} F_i \mathbf{1}_{[i-1,i)}(z)$, F_i independent random variables $\mathbb{E}[F_i] = 0$ and $\mathbb{E}[F_i^2] = \sigma^2$.





$\varepsilon = 0.05$

$$
\begin{aligned}
X^{\varepsilon}(z) &= \varepsilon \int_0^{\frac{z}{\varepsilon^2}} F(s)ds = \varepsilon \left(\sum_{i=1}^{\left[\frac{z}{\varepsilon^2}\right]} F_i \right) + \varepsilon \int_{\left[\frac{z}{\varepsilon^2}\right]}^{\frac{z}{\varepsilon^2}} F(s)ds \\
&= \underbrace{\varepsilon \sqrt{\left[\frac{z}{\varepsilon^2}\right]}}_{\varepsilon \to 0 \,\downarrow\, \sqrt{z}} \times \underbrace{\frac{1}{\sqrt{\left[\frac{z}{\varepsilon^2}\right]}} \left(\sum_{i=1}^{\left[\frac{z}{\varepsilon^2}\right]} F_i \right)}_{\text{law} \,\downarrow\, (CLT)\ \mathcal{N}(0,\sigma^2)} + \underbrace{\varepsilon \left(\frac{z}{\varepsilon^2} - \left[\frac{z}{\varepsilon^2}\right] \right) F_{\left[\frac{z}{\varepsilon^2}\right]}}_{\text{a.s.} \,\downarrow\, 0}
\end{aligned}
$$

Thus: $X^{\varepsilon}(z)$ converges in distribution as $\varepsilon \to 0$ to the Gaussian statistics $\mathcal{N}(0, \sigma^2 z)$.

With some more work: The process $(X^{\varepsilon}(z))_{z \in \mathbb{R}^+}$ converges in distribution to a Brownian motion $\sigma W(z)$.

For the definition of a Brownian motion: ask Lenya.

Markov process

A stochastic process Y_z with state space S is Markov if $\forall 0 \le s < z$ and $f \in L^\infty(S)$

$$\mathbb{E}[f(Y_z)|Y_u, u \le s] = \mathbb{E}[f(Y_z)|Y_s]$$

"the state Y_s at time s contains all relevant information for calculating probabilities of future events".

For the definition of conditional expectation: ask Lenya.

The processus is stationary if $\mathbb{E}[f(Y_z)|Y_s = y] = \mathbb{E}[f(Y_{z-s})|Y_0 = y]$.

Define the family of operators on $L^\infty(S)$:

$$T_z f(y) = \mathbb{E}[f(Y_z)|Y_0 = y]$$

Proposition.

1) $T_0 = I_d$

2) $\forall s, z \ge 0$, $T_{z+s} = T_z T_s$

3) T_z is a contraction $\|T_z f\|_\infty \le \|f\|_\infty$.

Proof of 2):

$$\begin{aligned} T_{z+s} f(y) &= \mathbb{E}[f(Y_{z+s})|Y_0 = y] = \mathbb{E}[\mathbb{E}[f(Y_{z+s})|Y_u,\ u \le z]|Y_0 = y] \\ &= \mathbb{E}[\mathbb{E}[f(Y_{z+s})|Y_z]|Y_0 = y] = \mathbb{E}[T_s f(Y_z)|Y_0 = y] \\ &= T_z T_s f(y) \end{aligned}$$

Feller process: T_z is strongly continuous from $\mathcal{C}_0$ to $\mathcal{C}_0$ (for any $f \in \mathcal{C}_0$, $\|T_z f - f\|_\infty \xrightarrow{z \to 0} 0$).

The generator of the Markov process is:

$$Q := \lim_{z \searrow 0} \frac{T_z - I_d}{z}$$

It is defined on a subset of $\mathcal{C}^0$, supposed to be dense.

Proposition. If $f \in \mathrm{Dom}(Q)$, then the function $u(z, y) = T_z f(y)$ satisfies the Kolmogorov equation

$$\frac{\partial u}{\partial z} = Qu, \qquad u(z = 0, y) = f(y)$$

Proof.

$$\frac{u(z+h, y) - u(z, y)}{h} = \frac{T_{z+h} f(y) - T_z f(y)}{h} = T_z \frac{T_h - I_d}{h} f(y) \xrightarrow{h \to 0} T_z Q f(y)$$

because $f \in \mathrm{Dom}(Q)$ and T_z is continuous. This shows that u is differentiable and $\partial_z u = T_z Q f$. Besides

$$\frac{T_h - I_d}{h} T_z f(y) = \frac{T_{z+h} f(y) - T_z f(y)}{h} = \frac{u(z+h, y) - u(z, y)}{h}$$

has a limit as $h \to 0$, which shows that $T_z f \in \mathrm{Dom}(Q)$ and $\partial_z u = Q T_z f = Qu$.

Example: Brownian motion

W_z: Gaussian process with independent increments

$$\mathbb{E}[(W_{z+h} - W_z)^2] = h$$

The semi-group T_z is the heat kernel:

$$\begin{aligned} T_z f(x) &= \mathbb{E}[f(x+W_z)] = \int f(x+w) \frac{1}{\sqrt{2\pi z}} \exp\left(-\frac{w^2}{2z}\right) dz \\ &= \int f(y) \frac{1}{\sqrt{2\pi z}} \exp\left(-\frac{(y-x)^2}{2z}\right) dy \end{aligned}$$

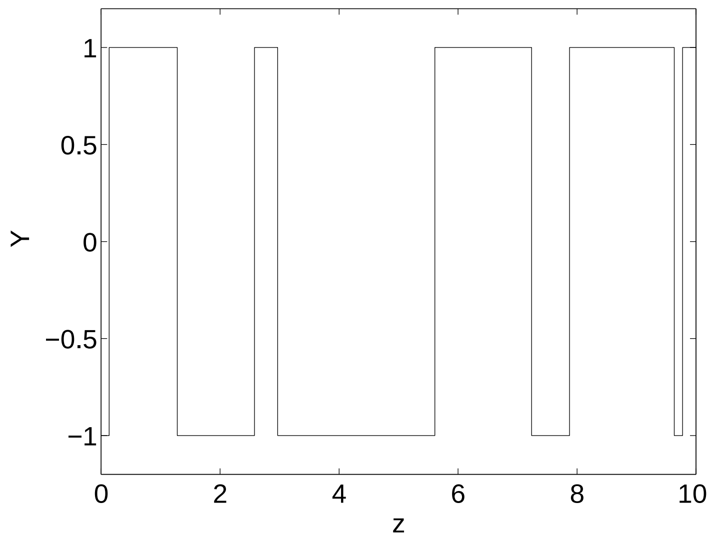
It is a Markov process with the generator:

$$Q = \frac{1}{2} \frac{\partial^2}{\partial x^2}$$

Proof.

$$Qf(x) = \lim_{z\to 0} \frac{T_z f(x) - f(x)}{z} = \frac{f'(x)\mathbb{E}[W_z] + \frac{1}{2} f''(x)\mathbb{E}[W_z^2] + \dots}{z}$$

Example: Two-state Markov process



The process Y_z takes values in $S = \{-1, 1\}$.

The time intervals are independent with the common exponential distribution with mean 1.

Functions $f \in L^\infty(S)$ are vectors. The semigroup $(T_z)_{z\geq 0}$ is a family of matrices:

$$T_z = \begin{pmatrix} \mathbb{P}(Y_z = 1|Y_0 = 1) & \mathbb{P}(Y_z = 1|Y_0 = -1) \\ \mathbb{P}(Y_z = -1|Y_0 = 1) & \mathbb{P}(Y_z = -1|Y_0 = -1) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}e^{-2z} & \frac{1}{2} - \frac{1}{2}e^{-2z} \\ \frac{1}{2} - \frac{1}{2}e^{-2z} & \frac{1}{2} + \frac{1}{2}e^{-2z} \end{pmatrix}$$

The generator is a matrix:

$$Q = \lim_{z\to 0} \frac{T_z - I}{z} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Martingale property

For any function $f \in \mathrm{Dom}(Q)$, the process

$$M_f(z) := f(Y_z) - \int_0^z Qf(Y_u)du$$

is a martingale.

Denoting $\mathcal{F}_s = \sigma(Y_u, 0 \leq u \leq s)$,

$$\begin{aligned}
\mathbb{E}[M_f(z)|\mathcal{F}_s] &= M_f(s) + \mathbb{E}\left[f(Y_z) - f(Y_s) - \int_s^z Qf(Y_u)du|Y_s\right] \\
&= M_f(s) + T_{z-s}f(Y_s) - f(Y_s) - \int_s^z T_{u-s}Qf(Y_s)du \\
&= M_f(s) + T_{z-s}f(Y_s) - f(Y_s) - \int_0^{z-s} T_uQf(Y_s)du
\end{aligned}$$

The function $T_z f(y)$ satisfies the Kolmogorov equation, which shows that the last three terms of the r.h.s. cancel:

$$\mathbb{E}[M_f(z)|\mathcal{F}_s] = M_f(s)$$

Reciprocal: If Q is non-degenerate, and M_f is a martingale for all test functions f, then Y is a Markov process with generator Q.

Ordinary differential equation driven by a Feller process

Proposition. Let Y be a S-valued Feller process with generator Q and X be the solution of:

$$\frac{dX}{dz} = F(Y_z, X(z)), \qquad X(0) = x \in \mathbb{R}^d$$

where $F : S \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded Borel function such that $x \mapsto F(y,x)$ has bounded derivatives uniformly with respect to $y \in S$. Then $\tilde{X} = (Y, X)$ is a Markov process with generator:

$$\mathcal{L} = Q + \sum_{j=1}^{d} F_j(y,x) \frac{\partial}{\partial x_j}$$

Formal proof. Let f be a test function.

$$\begin{aligned}
&\frac{d}{dz}\mathbb{E}[f(Y_z, X(z))|Y_0 = y, X(0) = x] \\
&\quad = \mathbb{E}[Qf(Y_z, X(z))|Y_0 = y, X(0) = x] \\
&\qquad + \mathbb{E}[\nabla_x f(Y_z, X(z))F(Y_z, X(z))|Y_0 = y, X(0) = x] \\
&\quad = \mathbb{E}[\mathcal{L}f(Y_z, X(z))|Y_0 = y, X(0) = x]
\end{aligned}$$

Ergodic Markov process

Ergodicity is related to the **null space of** Q**.**
Since $T_z 1 = 1$, we have $Q1 = 0$, so that $1 \in \mathrm{Null}(Q)$.
A Markov process is ergodic iff $\mathrm{Null}(Q) = \mathrm{Span}(\{1\})$ iff there is a unique invariant probability measure $\mathbb{P}$ satisfying $Q^*\mathbb{P} = 0$, i.e.

$$\forall f \in \mathrm{dom}(Q)\,, \qquad \int Qf(y)d\mathbb{P}(y) = 0 \Longleftrightarrow \mathbb{E}_{\mathbb{P}}[Qf(Y_0)] = 0$$

$$\int T_z f(y)d\mathbb{P}(y) = \int f(y)d\mathbb{P}(y) \Longleftrightarrow \mathbb{E}_{\mathbb{P}}[f(Y_z)] = \mathbb{E}_{\mathbb{P}}[f(Y_0)]$$

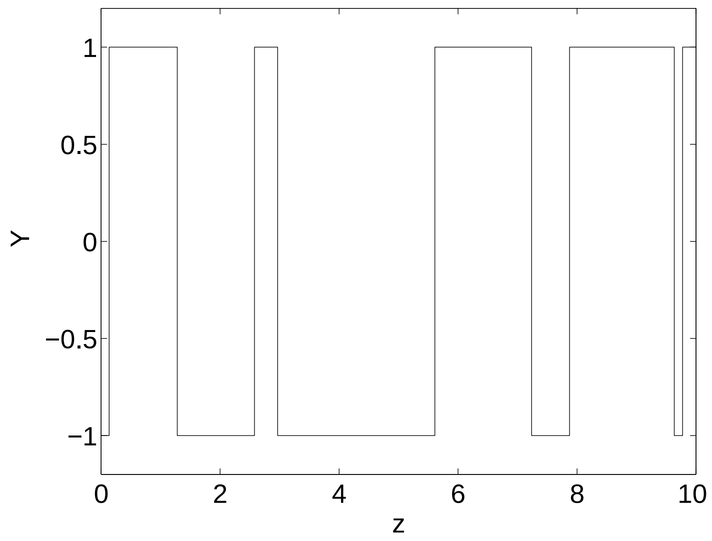
Ergodicity: $T_z f(y)$ converges to $\mathbb{E}_{\mathbb{P}}[f(Y_0)]$ as $z \to \infty$. The spectrum of Q gives the convergence (mixing) rate. The existence of a spectral gap

$$\inf_{f, \int f d\mathbb{P} = 0} \frac{-\int f Q f d\mathbb{P}}{\int f^2 d\mathbb{P}} > 0$$

ensures the exponential convergence of $T_z f(y)$ to $\mathbb{E}[f(Y_0)]$.

Example: a reversible Markov process with finite state space S.
Then Q is a symmetric matrix, with nonpositive eigenvalues and at least one zero eigenvalue since $Q1 = 0$. If all other eigenvalues are negative, the process is ergodic and exponentially mixing.

Example: Two-state Markov process



The process Y_z takes values in $S = \{-1, 1\}$.

The time intervals are independent

with the common exponential distribution

with mean 1.

The semigroup $(T_z)_{z \geq 0}$ is a family of matrices:

$$T_z = \begin{pmatrix} \mathbb{P}(Y_z = 1 | Y_0 = 1) & \mathbb{P}(Y_z = 1 | Y_0 = -1) \\ \mathbb{P}(Y_z = -1 | Y_0 = 1) & \mathbb{P}(Y_z = -1 | Y_0 = -1) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2} e^{-2z} & \frac{1}{2} - \frac{1}{2} e^{-2z} \\ \frac{1}{2} - \frac{1}{2} e^{-2z} & \frac{1}{2} + \frac{1}{2} e^{-2z} \end{pmatrix}$$

The generator is a matrix:

$$Q = \lim_{z \to 0} \frac{T_z - I}{z} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

It is ergodic. The invariant probability ($Q^T \bar{p} = 0$) is the uniform probability $\bar{p} = (1/2, 1/2)^T$ over S.

Example: Brownian motion

W_z: Gaussian process with independent increments

$$\mathbb{E}[(W_{z+h} - W_z)^2] = h$$

The semi-group T_z is the heat kernel:

$$T_z f(x) = \int f(y) p_z(x,y) dy\,, \qquad p_z(x,y) = \frac{1}{\sqrt{2\pi z}} \exp\left(-\frac{(y-x)^2}{2z}\right)$$

It is a Markov process with the generator:

$$Q = \frac{1}{2}\frac{\partial^2}{\partial x^2}$$

It is not ergodic. Its invariant measure is the Lebesgue measure (not a finite measure).

Example: Ornstein-Uhlenbeck process

Solution of the stochastic differential equation $dX(z) = -\lambda X(z) + dW_z$:

$$X(z) = X_0 e^{-\lambda z} + \int_0^z e^{-\lambda(z-s)} dW_s$$

where W_z is a Brownian motion, $\lambda > 0$.
(if $z \mapsto t$, this process describes the motion of a particle in a quadratic potential)
The semi-group T_z is

$$T_z f(x) = \int f(y) p_z(x,y) dy$$

$y \mapsto p_z(x,y)$ is a Gaussian density with mean $xe^{-\lambda z}$ and variance $\sigma^2(z)$:

$$p_z(x,y) = \frac{1}{\sqrt{2\pi\sigma(z)^2}} \exp\left(-\frac{(y - xe^{-\lambda z})^2}{2\sigma^2(z)}\right), \quad \sigma^2(z) = \frac{1 - e^{-2\lambda z}}{2\lambda}$$

The generator is:

$$Q = \frac{1}{2}\frac{\partial^2}{\partial x^2} - \lambda x \frac{\partial}{\partial x}$$

$X(z)$ is ergodic. Its invariant probability density ($Q^* \bar{p} = 0$) is

$$\bar{p}(y) = \sqrt{\frac{\lambda}{\pi}} \exp\left(-\lambda y^2\right)$$

Diffusion processes

• Let σ and b be $\mathcal{C}^1(\mathbb{R},\mathbb{R})$ functions with bounded derivatives.
Let W_z be a Brownian motion.
The solution $X(z)$ of the 1D stochastic differential equation:

$$dX(z) = \sigma(X(z))dW_z + b(X(z))dz$$

is a Markov process with the generator

$$Q = \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}$$

• Let $\sigma \in \mathcal{C}^1(\mathbb{R}^n,\mathbb{R}^m)$ and $b \in \mathcal{C}^1(\mathbb{R}^n,\mathbb{R}^n)$ with bounded derivatives.
Let W_z be a m-dimensional Brownian motion.
The solution $X(z)$ of the stochastic differential equation:

$$dX(z) = \sigma(X(z))dW_z + b(X(z))dz$$

is a Markov process with the generator

$$Q = \frac{1}{2}\sum_{ij} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x)\frac{\partial}{\partial x_i}$$

with $a = \sigma\sigma^T$.

Poisson equation $Qu = f$

Let us consider an ergodic Markov process with generator Q.

$\mathrm{Null}(Q^*)$ has dimension 1 and is spanned by the invariant probability $\mathbb{P}$.

By Fredholm alternative, the Poisson equation has a solution iff $f \perp \mathrm{Null}(Q^*)$, i.e. $\int f d\mathbb{P} = 0$ or $\mathbb{E}[f(Y_0)] = 0$ where $\mathbb{E}$ is the expectation w.r.t. the invariant probability $\mathbb{P}$.

Proposition. *If* $\mathbb{E}[f(Y_0)] = 0$, *a solution of* $Qu = f$ *is*

$$u(y) = -\int_0^\infty T_z f(y) dz$$

The following expressions are equivalent:

$$T_z f(y) = e^{zQ} f(y) = \mathbb{E}[f(Y_z) | Y_0 = y]$$

Proof.

$$u(y) = -\int_0^\infty T_z f(y) dz = -\int_0^\infty \{T_z f(y) - \mathbb{E}[f(Y_0)]\}\, dz$$

The convergence of this integral requires some mixing.

Formally $T_z = e^{zQ}$

$$Qu = -\int Qe^{zQ} f dz = -\int_0^\infty \frac{de^{zQ}}{dz} f dz = -\left[e^{zQ} f\right]_0^\infty = f - \mathbb{E}[f(Y_0)] = f$$

Moreover $\mathbb{E}[u(Y_0)] = 0$ because $\mathbb{E}[f(Y_z)] = \mathbb{E}[f(Y_0)] = 0$.

Finally:
$\left[-\int_0^\infty dz e^{zQ}\right] : \mathcal{D} \to \mathcal{D}$ *is the inverse of* Q *on* $\mathcal{D} = (\mathrm{Null}(Q^*))^\perp$.

Diffusion-approximation

$$\frac{dX^\varepsilon}{dz}(z) = \frac{1}{\varepsilon} F\left(Y(\frac{z}{\varepsilon^2}), X^\varepsilon(z)\right), \qquad X^\varepsilon(0) = x_0 \in \mathbb{R}^d.$$

Y stationary and ergodic, F centered: $\mathbb{E}[F(Y(0),x)] = 0$.

Theorem: *The processes $(X^\varepsilon(z))_{z\geq 0}$ converge in distribution in $\mathbf{C}^0([0,\infty),\mathbb{R}^d)$ to the diffusion (Markov) process X with generator $\mathcal{L}$.*

$$\mathcal{L}f(x) = \int_0^\infty du \mathbb{E}\left[F(Y(0),x).\nabla\left(F(Y(u),x).\nabla f(x)\right)\right].$$

$$\mathcal{L} = \sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^d b_j(x)\frac{\partial}{\partial x_j}$$

with

$$a_{ij}(x) = \int_0^\infty du \mathbb{E}\left[F_i(Y(0),x)F_j(Y(u),x)\right]$$

$$b_j(x) = \sum_{i=1}^d \int_0^\infty du \mathbb{E}\left[F_i(Y(0),x)\partial_{x_i}F_j(Y(u),x)\right]$$

Formal proof. Assume that Y is Markov, with generator Q, ergodic (+ technical conditions for the Fredholm alternative).

The joint process $\tilde{X}^\varepsilon(z) := (Y(z/\varepsilon^2), X^\varepsilon(z))$ is Markov with

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon^2} Q + \frac{1}{\varepsilon} F(y,x).\nabla$$

The Kolmogorov backward equation for this process is

$$\frac{\partial U^\varepsilon}{\partial z} = \mathcal{L}^\varepsilon U^\varepsilon \tag{1}$$

Let us take an initial condition at $z = 0$ independent of y:

$$U^\varepsilon(z=0,y,x) = f(x)$$

where f is a smooth test function. We solve (1) as $\varepsilon \to 0$ by assuming the multiple scale expansion:

$$U^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n U_n(z,y,x) \tag{2}$$

Then Eq. (1) becomes

$$\frac{\partial U^\varepsilon}{\partial z} = \frac{1}{\varepsilon^2} Q U^\varepsilon + \frac{1}{\varepsilon} F.\nabla U^\varepsilon \tag{3}$$

We obtain a hierarchy of equations:

$$QU_0 = 0 \tag{4}$$

$$QU_1 + F.\nabla U_0 = 0 \tag{5}$$

$$QU_2 + F.\nabla U_1 = \frac{\partial U_0}{\partial z} \tag{6}$$

$Y(z)$ is ergodic i.e. $\mathrm{Null}(Q) = \mathrm{Span}(\{1\})$. Thus Eq. (4) $\Longrightarrow U_0$ does not depend on y.

U_1 must satisfy

$$QU_1 = -F(y,x).\nabla U_0(z,x) \tag{7}$$

Q is not invertible, we know that $\mathrm{Null}(Q) = \mathrm{Span}(\{1\})$.
$\mathrm{Null}(Q^*)$ has dimension 1 and is generated by the invariant probability $\mathbb{P}$.
By Fredholm alternative, the Poisson equation $QU = g$ has a solution U if g satisfies $g \perp \mathrm{Null}(Q^*)$, i.e. $\int g d\mathbb{P} = 0$, i.e. $\mathbb{E}[g(Y(0))] = 0$.

Since the r.h.s. of Eq. (7) is centered, this equation has a solution U_1

$$U_1(z,y,x) = -Q^{-1}F(y,x).\nabla U_0(z,x)$$

$$U_1(z,y,x) = -Q^{-1}[F(y,x)].\nabla U_0(z,x) \tag{8}$$

up to an additive constant, where $-Q^{-1} = \int_0^\infty dz e^{zQ}$.

Substitute (8) into (6): $\frac{\partial U_0}{\partial z} = QU_2 + F.\nabla U_1$ and take the expectation w.r.t $\mathbb{P}$. We get that U_0 must satisfy

$$\frac{\partial U_0}{\partial z} = \mathbb{E}\left[F.\nabla(-Q^{-1}F.\nabla U_0)\right]$$

This is the solvability condition for (6) and this is the limit Kolmogorov equation for the process X^ε:

$$\frac{\partial U_0}{\partial z} = \mathcal{L}U_0$$

with the limit generator

$$\mathcal{L} = \int_0^\infty \mathbb{E}\left[F.\nabla(e^{zQ}F.\nabla)\right] dz$$

Using the probabilistic representation of the semi-group e^{zQ} we get

$$\mathcal{L} = \int_0^\infty \mathbb{E}[F(Y(0),x).\nabla F(Y(z),x).\nabla]dz$$

Rigorous proof: The generator

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon^2} Q + \frac{1}{\varepsilon} F(y,x).\nabla$$

of $(X^\varepsilon(z), Y(\frac{z}{\varepsilon^2}))$ is such that

$$f(Y(\frac{z}{\varepsilon^2}), X^\varepsilon(z)) - f(Y(\frac{s}{\varepsilon^2}), X^\varepsilon(s)) - \int_s^z \mathcal{L}^\varepsilon f(Y(\frac{u}{\varepsilon^2}), X^\varepsilon(u)) du$$

is a martingale for any test function f.

$\Longrightarrow$ Convergence of martingale problems.

cf G. Papanicolaou, Asymptotic analysis of stochastic equations, MAA Stud. in Math. **18** (1978), 111-179.

H. J. Kushner, *Approximation and weak convergence methods for random processes* (MIT Press, Cambridge, 1984).

Convergence of martingale problems

Assume for a while: $\forall f \in \mathcal{C}_b^\infty$, *there exists* f^ε *such that:*

$$\sup_{x\in K, y\in S} |f^\varepsilon(y,x) - f(x)| \stackrel{\varepsilon\to 0}{\longrightarrow} 0, \qquad \sup_{x\in K, y\in S} |\mathcal{L}^\varepsilon f^\varepsilon(y,x) - \mathcal{L}f(x)| \stackrel{\varepsilon\to 0}{\longrightarrow} 0.$$

Assume tightness (*in* D, *ask Lenya*) and extract $\varepsilon_p \to 0$ such that $X^{\varepsilon_p} \to X$.

Take $z_1 < ... < z_n < s < z$ and $h_1, ..., h_n \in \mathcal{C}_b^\infty$:

$$\mathbb{E}\Big[\Big(f^\varepsilon(Y(\frac{z}{\varepsilon^2}), X^\varepsilon(z)) - f^\varepsilon(Y(\frac{s}{\varepsilon^2}), X^\varepsilon(s)) - \int_s^z \mathcal{L}^\varepsilon f^\varepsilon(Y(\frac{u}{\varepsilon^2}), X^\varepsilon(u))du\Big) h_1(X^\varepsilon(z_1))...h_n(X^\varepsilon(z_n))\Big] = 0$$

Take $\varepsilon_p \to 0$ so that $X^{\varepsilon_p} \to X$:

$$\mathbb{E}\Big[\Big(f(X(z)) - f(X(s)) - \int_s^z \mathcal{L}f(X(u))du\Big) h_1(X(z_1))...h_n(X(z_n))\Big] = 0$$

X is solution of the martingale problem associated to $\mathcal{L}$.

Perturbed test function method

Proposition: $\forall f \in \mathcal{C}_b^\infty$, *there exists a family* f^ε *such that:*

$$\sup_{x\in K, y\in S} |f^\varepsilon(y,x) - f(x)| \stackrel{\varepsilon\to 0}{\longrightarrow} 0, \qquad \sup_{x\in K, y\in S} |\mathcal{L}^\varepsilon f^\varepsilon(y,x) - \mathcal{L}f(x)| \stackrel{\varepsilon\to 0}{\longrightarrow} 0.$$

Proof: Define $f^\varepsilon(y,x) = f(x) + \varepsilon f_1(y,x) + \varepsilon^2 f_2(y,x)$.
Applying $\mathcal{L}^\varepsilon = \frac{1}{\varepsilon^2}Q + \frac{1}{\varepsilon}F(y,x).\nabla$ to f^ε, one gets:

$$\mathcal{L}^\varepsilon f^\varepsilon = \frac{1}{\varepsilon}\left(Qf_1 + F(y,x).\nabla f(x)\right) + \left(Qf_2 + F.\nabla f_1(y,x)\right) + O(\varepsilon).$$

Define the corrections f_j as follows:

1. $$f_1(y,x) = -Q^{-1}\left(F(y,x).\nabla f(x)\right).$$

Q has an inverse on the subspace of centered functions.

$$f_1(y,x) = \int_0^\infty du \mathbb{E}[F(Y(u),x).\nabla f(x) | Y(0) = y].$$

2. $$f_2(y,x) = -Q^{-1}\left(F.\nabla f_1(y,x) - \mathbb{E}[F.\nabla f_1(y,x)]\right).$$

It remains: $\mathcal{L}^\varepsilon f^\varepsilon = \mathbb{E}[F.\nabla f_1(y,x)] + O(\varepsilon)$.

One-dimensional case

$$\frac{dX^\varepsilon}{dz} = \frac{1}{\varepsilon} F\left(Y(\frac{z}{\varepsilon^2}), X^\varepsilon(z)\right), \qquad X^\varepsilon(z=0) = x_0 \in \mathbb{R}$$

Then $X^\varepsilon \overset{\varepsilon \to 0}{\longrightarrow} X$ where X is the diffusion process with generator

$$\mathcal{L} = a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}$$

with

$$\begin{aligned} a(x) &= \int_0^\infty du \mathbb{E}\left[F(Y(0),x)F(Y(u),x)\right] \\ b(x) &= \int_0^\infty du \mathbb{E}\left[F(Y(0),x)\partial_x F(Y(u),x)\right] \end{aligned}$$

The limit process can be identified as the solution of the **stochastic differential equation**

$$dX = b(X)dz + \sqrt{2a(X)}dW_z$$

where W is a Brownian motion.

Limit theorems - Random vs. periodic

$$\frac{dX^\varepsilon}{dz}(z) = \frac{1}{\varepsilon} F\Big(Y\big(\frac{z}{\varepsilon^2}\big), X^\varepsilon(z), \frac{z}{\varepsilon^{2+c}}\Big), \qquad X^\varepsilon(0) = x_0 \in \mathbb{R}^d.$$

$F(y, x, \phi)$ is periodic with respect to ϕ.

Case 1. Slow phase: $-2 < c < 0$ and $\mathbb{E}[F(Y(0), x, \phi)] = 0$.

Case 2. Fast phase: $c = 0$ and $\langle \mathbb{E}[F(Y(0), x, \phi)] \rangle_\phi = 0$.

Case 3. Ultra-fast phase: $c > 0$ and $\langle \mathbb{E}[F(Y(0), x, \phi)] \rangle_\phi = 0$.

The processes $(X^\varepsilon(z))_{z \geq 0}$ converge to X with generator $\mathcal{L}_j$:

$$\mathcal{L}_1 f(x) = \left\langle \int_0^\infty du \mathbb{E}\left[F(Y(0), x, .).\nabla \left(F(Y(u), x, .).\nabla f(x)\right)\right] \right\rangle_\phi ,$$

$$\mathcal{L}_2 f(x) = \int_0^\infty du \left\langle \mathbb{E}\left[F(Y(0), x, .).\nabla \left(F(Y(u), x, . + u).\nabla f(x)\right)\right] \right\rangle_\phi ,$$

$$\mathcal{L}_3 f(x) = \int_0^\infty du \mathbb{E}\left[\langle F(Y(0), x, .) \rangle_\phi .\nabla \Big(\langle F(Y(u), x, .) \rangle_\phi .\nabla f(x) \Big) \right].$$

The averaging theorem revisited

Consider the random differential equation

$$\frac{dX^\varepsilon}{dz} = F\left(Y\left(\frac{z}{\varepsilon}\right), X^\varepsilon(z)\right), \quad X^\varepsilon(0) = x_0$$

where we do not assume that $F(y,x)$ is centered. We denote its mean by

$$\bar{F}(x) = \mathbb{E}[F(Y(0),x)]$$

Then $(Y(z/\varepsilon), X^\varepsilon(z))$ is a Markov process with generator

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon}Q + F(y,x)\cdot\nabla$$

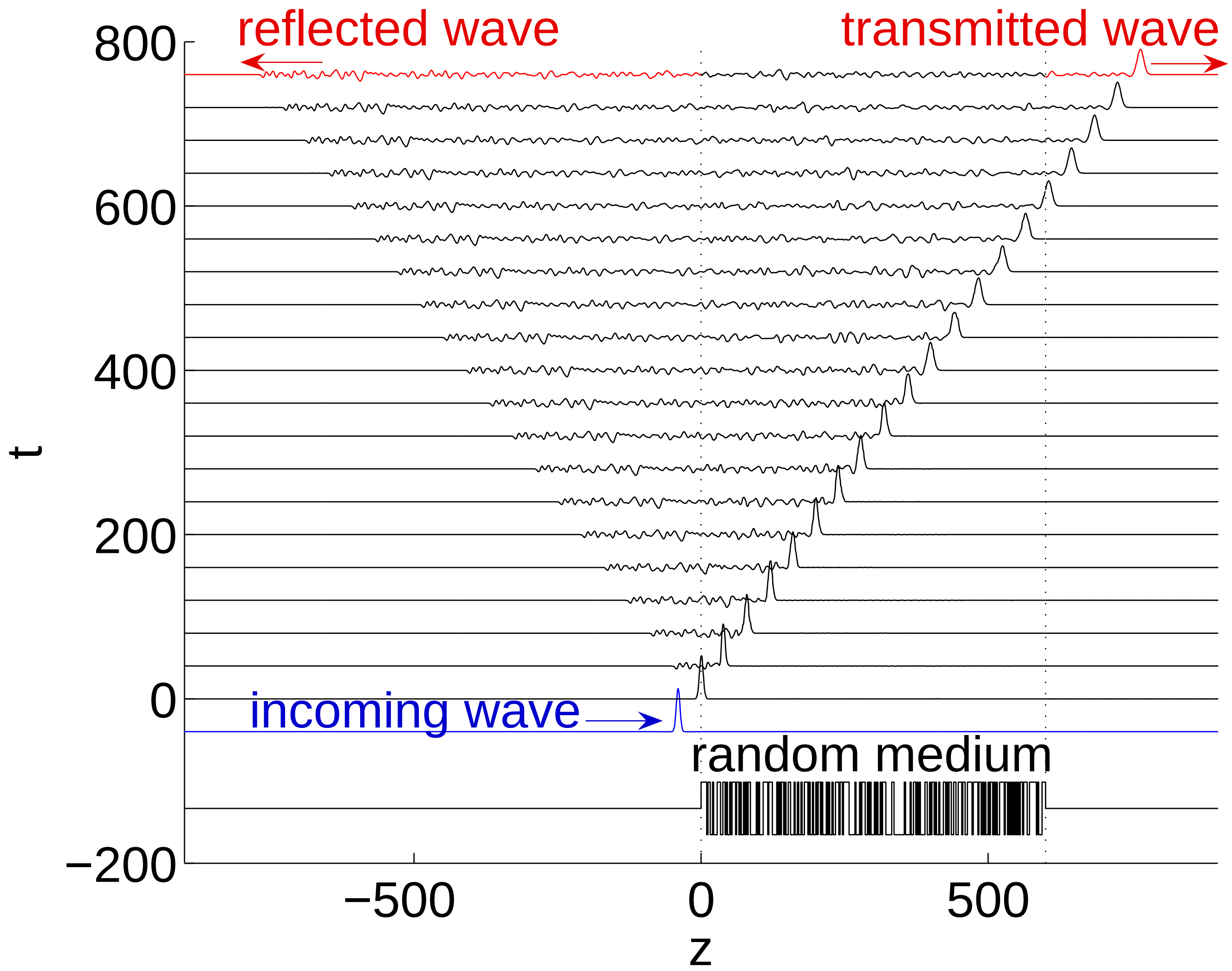
Let $f(x)$ be a test function. Define $f^\varepsilon(y,x) = f(x) + \varepsilon f_1(y,x)$ where f_1 solves the Poisson equation

$$Qf_1(y,x) + \left[F(y,x)\cdot\nabla f(x) - \bar{F}(x)\cdot\nabla f(x)\right] = 0$$

We get $\mathcal{L}^\varepsilon f^\varepsilon(y,x) = \bar{F}(x)\cdot\nabla f(x) + O(\varepsilon)$. Therefore the processes $X^\varepsilon(z)$ converge to the solution of the martingale problem associated with the generator $\mathcal{L}f(x) = \bar{F}(x)\cdot\nabla f(x)$. The solution is the deterministic process $\bar{X}(z)$

$$\frac{d\bar{X}}{dz} = \bar{F}(\bar{X}(z)), \quad \bar{X}(0) = x_0.$$

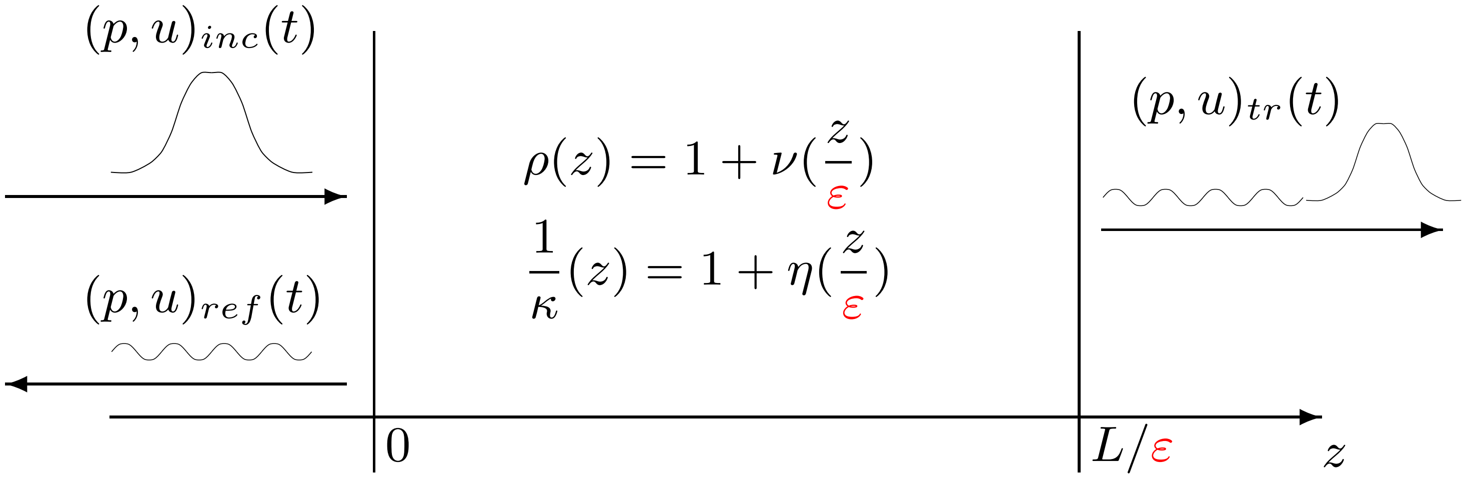
Long distance propagation



Long distance propagation $l_c \ll \lambda \ll L$

Acoustic equations for pressure p and velocity u:

$$\frac{\partial p}{\partial t} + \kappa(z)\frac{\partial u}{\partial z} = 0$$
$$\rho(z)\frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = 0$$



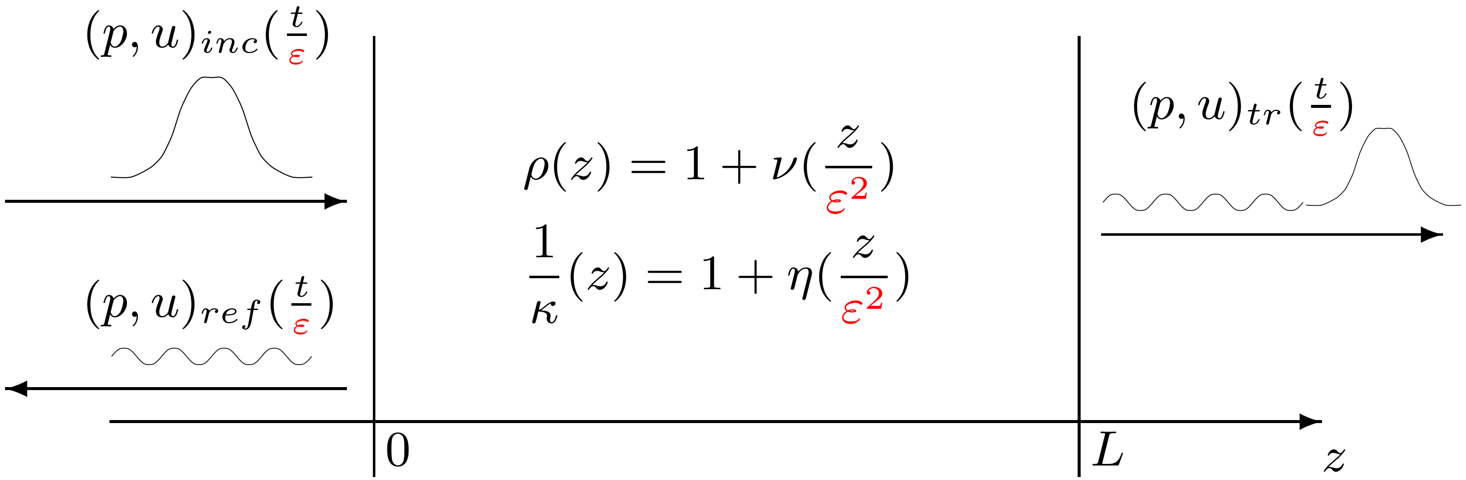
IC: right-going pulse incoming from the left homogeneous half-space.

Long distance propagation $l_c \ll \lambda \ll L$

Acoustic equations for pressure p and velocity u:

$$\frac{\partial p}{\partial t} + \kappa(z)\frac{\partial u}{\partial z} = 0$$

$$\rho(z)\frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = 0$$



IC: right-going pulse incoming from the left homogeneous half-space.