Homogenization of thin structures and singular measures

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Motivation

- Dimension reduction. Shells, skeletons, rod structures $\longrightarrow$ surfaces and segments structures;
- Reduction of the number of parameters. Asymptotic problems with two small parameters (microscopic length scale of the medium and structure thickness) $\longrightarrow$ problems with only one parameter;
- Porous media with rough geometry.

Let $\mu(x)$ be a positive finite Borel measure on a standard n-dimensional torus $\mathbb{T}^n \equiv \mathbb{R}^n/\mathbb{Z}^n$ or in $\mathbb{R}^n$. We identify μ with the corresponding periodic measure in $\mathbb{R}^n$. Without loss of generality, we may assume that

$$\int_{\mathbb{T}^n} d\mu(x) = 1.$$

To clarify the idea of introducing Sobolev spaces with measure, consider a simple example. Let μ be a positive finite Borel measure in a smooth bounded domain G. Consider the variational problem

$$\inf_{\varphi \in C_0^\infty(G)} \int\limits_G \big(a(x) \nabla \varphi(x) \cdot \nabla \varphi(x) + \varphi^2(x) - 2 f(x) \varphi(x) \big) d\mu(x),$$

where $a(x)$ ia a continuous positive definite matrix in $\overline{G}$ and $f(x)$ is a continuous function in $\overline{G}$. Our goal is to introduce a Sobolev space with measure μ in such a way that the minimum is attained and a minimizer is found as a solution to the corresponding Euler equation.

Definition 1. We say that a function $u \in L_2(\mathbb{T}^n, \mu)$ belongs to the space $H^1(\mathbb{T}^n, \mu)$ if there exists a vector-function $z \in (L_2(\mathbb{T}^n, \mu))^n$ and a sequence $\varphi_k \in C^\infty(\mathbb{T}^n)$ such that

$$\varphi_k \longrightarrow u \quad \text{in } L_2(\mathbb{T}^n, \mu) \text{ as } k \to \infty,$$

$$\nabla\varphi_k \longrightarrow z \quad \text{in } (L_2(\mathbb{T}^n, \mu))^n \text{ as } k \to \infty.$$

The function $z(x)$ is called the *gradient* or μ-*gradient* of $u(x)$ and is denoted by $\nabla^\mu u$.

Similarly, we can define the spaces $H^1(\mathbb{R}^n, \mu)$, $H^1_{\rm loc}(\mathbb{R}^n, \mu)$ and also the space $H^1(G, \mu)$ for an arbitrary domain $G \subset \mathbb{R}^n$ and a (locally) finite Borel measure μ on G.

Generally speaking, the gradient of a function of class $H^1(\mathbb{T}^n, \mu)$ is not unique. In particular, the zero function may have a nontrivial gradient. We illustrate this with

Example 1*.* In the square $[-1/2, 1/2]^2$, we consider the segment $\{-1/4 \leq x_1 \leq 1/4, x_2 = 0\}$ and introduce

$$d\mu = 2\chi(x_1)\,dx_1 \times \delta(x_2), \tag{1}$$

where $\chi(t)$ is the characteristic function of the segment $[-\frac{1}{4}, \frac{1}{4}]$ and $\delta(t)$ is the Dirac mass at zero.

Example (cont.)

Let $\psi(x) \in C_0^\infty$ coincide with a function of the form $\theta(x_1)x_2$ in a small neighborhood of the segment. Then $\psi = 0$ in $L_2(\mathbb{T}^2, \mu)$. Choosing $\varphi_k(x) = \psi(x)$ for all k in the definition of μ-gradient, we find $z(x) = \nabla^\mu \psi(x) = (0, \theta(x_1))$. Thus, any vector-valued function of the form $(0, \theta(x_1))$ with smooth $\theta(s)$ serves as the μ-gradient of zero. In fact, this assertion is valid for any $\theta(s)$ in L_2.

Gradients of zero, Example of H^1 space

The gradients of zero form a closed subspace of $(L_2(\mathbb{T}^n,\mu))^n$, denote it $\Gamma_\mu(0)$. The set of the gradients of any $H^1(\mathbb{T}^n,\mu)$-function is the sum of its arbitrary gradient and $\Gamma_\mu(0)$.

Example 2 (Segment). Consider the space $H^1(\mathbb{T}^n,\mu)$ (or $H^1(\mathbb{R}^n,\mu)$) for 1D Lebesgue measure μ on the segment $I=\{x\in\mathbb{R}^n : 0\le x_1\le a, x_2=x_3=\cdots=x_n=0\}$.

Proposition 1. *The space $H^1(\mathbb{T}^n,\mu)$ consists of all Borel functions $u(x)$ such that $u(s,0,0,\ldots,0)\in H^1(0,a)$. Moreover, $\nabla^\mu u(x)=(u'_{x_1}(x_1,0),\psi_2(x_1),\ldots,\psi_n(x_1))$, where $u'_{x_1}\equiv \frac{d}{ds}u(s,0,0,\ldots,0)\Big|_{s=x_1}$, and $\psi_2,\ \psi_3,\ldots,\psi_n$ are arbitrary functions in $L_2(0,a)$.*

Example 3 *("Urchin").* Consider the segments $I_1,\ I_2,\ I_N$ starting at the origin and directed along vectors $v_1,\ v_2, \dots, v_N$. Let $\mu_1,\ \mu_2, \dots, \mu_N$ be the standard $1D$ Lebesgue measures on the segments $I_1, \dots, I_N$ respectively, and let $\lambda_1, \dots, \lambda_N$ be arbitrary positive numbers. We set

$$\mu = \sum_{j=1}^{N} \lambda_j \mu_j .$$

A function $u(x)$ belongs $H^1(\mathbb{T}^n, \mu)$ if and only if $u\big|_{I_j} \in H^1(I_j)$, and the values of the restricted functions at the origin coincide for all segments (recall that an H^1-function of a single variable is continuous).

Example 4 (Reinforced shells). Let $\Pi_0 = \{x \in \mathbb{T}^n \,:\, x_1 = 0\}$. We set

$$d\tilde{\mu}(x) = \delta(x_1) \times dx' + dx, \quad x' = (x_2, \dots, x_n).$$

A function $u(x) \in H^1(\mathbb{T}^2, \tilde{\mu})$ if and only if $u \in H^1(\mathbb{T}^n)$ and the trace $u(x)\big|_{\Pi_0} \in H^1(T^{n-1})$.

Remark 1*.* If the co-dimension of a plane $\Pi \subset \mathbb{R}^n$ is greater than one, then the trace of a $H^1(\mathbb{R}^n)$-function on Π is not well-defined. Therefore, if μ is the Lebesgue measure on Π and $d\tilde{\mu} = d\mu + dx$, then $H^1(\mathbb{T}^n, \tilde{\mu})$ is isomorphic to the direct sum of the spaces $H^1(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n, \mu)$.

We denote

$$H(\mathbb{R}^n, \mu) = \{(u, z) : u \in H^1(\mathbb{R}^n, \mu), z = \nabla^\mu u)\}.$$

Suppose that Radon measures μ_k weakly converges, as $k \to \infty$, to μ in $\mathbb{R}^n$.

Definition 2. We say that $g_k \in L_2(\mathbb{R}^n, \mu_k)$ *weakly converges* in $L_2(\mathbb{R}^n, \mu_k)$ to $g \in L_2(\mathbb{R}^n, \mu)$ as $k \to \infty$ if

- $\|g_k\|_{L_2(\mathbb{R}^n,\mu)} \leq C$;
- $\lim\limits_{k\to\infty} \int\limits_{\mathbb{R}^n} g_k(x)\varphi(x)d\mu_k(x) = \int\limits_{\mathbb{R}^n} g(x)\varphi(x)d\mu(x)$

  for all $\varphi \in C_0^\infty(\mathbb{R}^n)$.

Definition 3. A sequence $\{g_k\}$ *converges strongly* to $g(x) \in L_2(\mathbb{R}^n, \mu_k)$ if it weakly converges and

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} g_k(x) h_k(x) d\mu_k(x) = \int_{\mathbb{R}^n} g(x) h(x) d\mu(x)$$

for any sequence $\{h_k(x)\}$ weakly converging to $h(x) \in L_2(\mathbb{R}^n, \mu)$ in $L_2(\mathbb{R}^n, \mu_k)$.

Lemma 1. *Let $\{g_k\}$ weakly converge to $g(x)$ in $L_2(\mathbb{R}^n, \mu_k)$. Then $\{g_k\}$ converges strongly if and only if*

$$\lim_{k\to\infty} \|g_k\|_{L_2(\mathbb{R}^n, \mu_k)} = \|g\|_{L_2(\mathbb{R}^n, \mu)}.$$

Lemma 2. *Let* $\{\mu_k\}$ *converge weakly to* μ*. Then any bounded sequence* $\{g_k(x)\}$*,* $\|g_k\|_{L_2(\mathbb{R}^n,\mu_k)} \leq C$ *converges weakly along a subsequence in* $L^2(\mathbb{R}^n,\mu_k)$ *towards some function* $g(x) \in L_2(\mathbb{R}^n,\mu)$*.*

Definition 4. The space $L_2^{\mathrm{pot}}(\mathbb{R}^n, \mu)$ is the closure of the linear set $\{\nabla\varphi : \varphi \in C_0^\infty(\mathbb{R}^n)\}$ in the $(L_2(\mathbb{R}^n, \mu))^n$-norm.

Definition 5. The space $L_2^{\mathrm{pot}}(\mathbb{R}^n, \mu)$ of solenoidal vector-valued functions is the orthogonal complement to the space $L_2^{\mathrm{pot}}(\mathbb{R}^n, \mu)$ in $(L_2(\mathbb{R}^n, \mu))^n$.

Let $K(x) \geq 0$ be a C_0^∞ function such that $\int\limits_{\mathbb{R}^n} K(x)dx = 1$ and $K(-x) = K(x)$. For a Radon measure $\mu(x)$ in $\mathbb{R}^n$ or on $\mathbb{T}^n$ we set

$$d\mu^\delta(x) = \rho^\delta(x)dx, \quad \rho^\delta(x) = \delta^{-n} \int\limits_{\mathbb{R}^n} K\Big(\frac{x-y}{\delta}\Big) d\mu(y).$$

The measures μ^δ locally weakly converge in $\mathbb{R}^n$ to μ.

We also introduce

$$\varphi^{\delta}(x) = \delta^{-n} \int\limits_{\mathbb{R}^n} K\left(\frac{y}{\delta}\right)\varphi(x-y)dy.$$

Then

$$\int\limits_{\mathbb{R}^n} \varphi^{\delta}(x)d\mu(x) = \int\limits_{\mathbb{R}^n} \varphi(x)d\mu^{\delta}(x)$$

Smoothing operator agreed with the measure

Lemma 3. *For every* $v \in L_2(\mathbb{R}^n, \mu)$ *there is* $v_\delta \in L_2(\mathbb{R}^n, \mu)$ *such that*

$$\int\limits_{\mathbb{R}^n} v_\delta(x)\varphi(x)d\mu^\delta(x) = \int\limits_{\mathbb{R}^n} v(x)\varphi^\delta(x)d\mu(x)$$

for all $\varphi \in C_0(\mathbb{R}^n)$. *The family* $v_\delta(x)$ *strongly converges to* $v(x)$ *in* $L_2(\mathbb{R}^n, \mu^\delta)$ *as* $\delta \to 0$.

Divergence operator

Definition 6. Let $g \in L_2(\mathbb{R}^n, \mu)$ and $v \in (L_2(\mathbb{R}^n, \mu))^n$. We say that $g(x) = \mathrm{div}^\mu v(x)$ if

$$\int_{\mathbb{R}^n} g(x)\varphi(x)d\mu(x) = -\int_{\mathbb{R}^n} v(x) \cdot \nabla\varphi(x)d\mu(x)$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n)$.

Let $a(x) = \{a_{ij}(x)\}$ be a symmetric $n \times n$-matrix,

$$\Lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda^{-1}|\xi|^2, \quad \Lambda > 0, \quad \xi \in \mathbb{R}^n$$

μ-a.e. in $\mathbb{R}^n$. Suppose that $f \in L_2(\mathbb{R}^n, \mu)$ and $\lambda > 0$.

Definition 7. We say that a pair $(u, \nabla^\mu u)$ with $u \in H^1(\mathbb{R}^n, \mu)$, satisfies the equation

$$-\mathrm{div}^\mu(a(x)\nabla^\mu u(x)) + \lambda u(x) = f(x) \tag{2}$$

in $L_2(\mathbb{R}^n, \mu)$, if for any $v \in H^1(\mathbb{R}^n, \mu)$ and any of its gradient $\nabla^\mu v$ it holds:

$$\int\limits_{\mathbb{R}^n} a(x)\nabla^\mu u(x)\cdot\nabla^\mu v(x)d\mu(x)+\lambda\int\limits_{\mathbb{R}^n} u(x)v(x)d\mu(x) = \int\limits_{\mathbb{R}^n} f(x)v(x)d\mu($$

A function $u \in H^1(\mathbb{R}^n, \mu)$ is called a *solution* if the last identity holds for some of its gradients.

Lemma 4. *The above equation has a unique solution* $(u, \nabla^\mu u)$*,* $u \in H^1(\mathbb{R}^n, \mu)$*. Moreover, the choice of the* μ*-gradient of* u *is uniquely determined by the condition* $a(x)\nabla^\mu u(x) \in (\Gamma_\mu(0))^\perp$*.*

In the special case $a(x) = Id$ the integral identity reads

$$\int\limits_{\mathbb{R}^n} \nabla^\mu u(x) \cdot \nabla^\mu v(x) d\mu(x) + \lambda \int\limits_{\mathbb{R}^n} u(x) v(x) d\mu(x) = \int\limits_{\mathbb{R}^n} f(x) v(x) d\mu(x).$$

The expression $\mathrm{div}^\mu \nabla^\mu u$ is called the μ*-Laplacian* of u.

Tangential gradient

A gradient $\nabla^{\mu} u$ of a function $u \in H^1(\mathbb{R}^n, \mu)$ is *tangential* if it is orthogonal to $\Gamma_\mu(0)$. Thus tangential gradient of u is the orthogonal projection of an arbitrary μ-gradient of u on $(\Gamma_\mu(0))^{\perp}$.

Definition 8. A periodic measure μ is said to be *two-connected* or *ergodic* if any function $u \in H^1(\mathbb{T}^n, \mu)$ such that $\nabla^\mu u = 0$ is equal to a constant μ-a.e.

Lemma 5. *Let a measure μ be 2-connected. Then the set* $\{g(x) \in L_2(\mathbb{T}^n, \mu) : g(x) = \mathrm{div}^\mu v(x)\}$ *is dense in* $\left\{u \in L_2(\mathbb{T}^n) : \int\limits_{\mathbb{T}^n} u(x) d\mu(x) = 0)\right\}$*.*

Exercise 1*.* Let Q be an open connected subset of $\mathbb{T}^n$, and let $d\mu(x) = \chi_Q dx$. Then μ is 2-connected.

Let a matrix $a_{ij}(x)$ be symmetric and uniformly elliptic μ-a.e.

Lemma 6. *The set of solutions to the equation*

$$-\mathrm{div}^{\mu}(a(x)\nabla^{\mu}u(x)) + u(x) = f, \quad f \in L_2(\mathbb{R}^n, \mu),$$

is dense in $L_2(\mathbb{R}^n, \mu)$. We denoted it by $\mathcal{D}$.

For $u \in \mathcal{D}$ we set $Au = f - u$. Then the operator $(A+I)^{-1}$ maps a function $f \in L_2(\mathbb{R}^n, \mu)$ to the corresponding solution of the equation. This operator is nonnegative, bounded and symmetric. Therefore, A is self-adjoint. Its domain is denoted by $\mathcal{D}(A)$. The equation can be written in the operator form $Au + \lambda u = f$.

The equation $Au + \lambda u = f$ is an Euler equation of the variational problem

$$\inf_{u \in H^1(\mathbb{R}^n, \mu)} \left\{ \int_{\mathbb{R}^n} \Big(a(x) \nabla^\mu u(x) \cdot \nabla^\mu u(x) + \lambda u^2(x) \Big) d\mu(x) - \int_{\mathbb{R}^n} 2 f(x) u(x) d\mu($$

Proposition 2. *Let $f \in L_2(\mathbb{R}^n, \mu)$. Then for each $\lambda > 0$ the above variational problem has a unique minimum point $u \in H^1(\mathbb{R}^n, \mu)$. It solves the equation $Au + \lambda u = f$.*

Similarly, we can treat the variational problem for the functional

$$\inf\left\{\int_{\mathbb{R}^n}\Big(a(x)\nabla^\mu u(x)\cdot\nabla^\mu u(x)+c(x)u^2(x)\Big)d\mu(x)-\int_{\mathbb{R}^n}2f(x)u(x)d\mu(x)$$

where $c(x)$ satisfies the estimate $\Lambda \le c(x) \le \Lambda^{-1}$. The Euler equation reads

$$-\mathrm{div}^\mu(a(x)\nabla^\mu u(x)) + c(x)u(x) = f(x).$$

Let G be a bounded Lipschitz domain in $\mathbb{R}^n$, and let $\mu(dx)$ be a positive finite Borel measure on G.

Definition 9. We say that $u \in \overset{\circ}{H}{}^1(G,\mu)$, and $z \in (L_2(G,\mu))^n$ is the gradient of u if there is a sequence $\varphi_k \in C_0^\infty(G)$ such that

$$\varphi_k \longrightarrow u \quad \text{in } L_2(G,\mu) \text{ as } k \to \infty,$$

$$\nabla\varphi_k \longrightarrow z \quad \text{in } (L_2(G,\mu))^n \text{ as } k \to \infty.$$

Dirichlet problem

$$-\mathrm{div} a(x)\nabla^\mu u(x) + c(x)u(x) = f(x) \quad \text{in } L_2(G,\mu)$$

$$u|_{\partial G} = 0.$$

Definition 10. We say that $u \in \overset{\circ}{H}{}^1(G,\mu)$ is a solution to the Dirichlet problem if for any $v \in \overset{\circ}{H}{}^1(G,\mu)$

$$\int\limits_G a(x)\big(\nabla^\mu u(x)\cdot\nabla^\mu v(x)+c(x)u(x)v(x)\big)d\mu(x) = \int\limits_G f(x)v(x)d\mu(x).$$

The existence and the uniqueness of a solution can be established in the standard way.

Definition 11. We say that $u(x) \in H^1(\mathbb{R}^n, \mu)$, and $z(x) \in (L_2(\mathbb{R}^n, \mu))^n$ is a μ-gradient of $u(x)$ if

$$\int_{\mathbb{R}^n} u(x)g(x)d\mu(x) = -\int_{\mathbb{R}^n} z(x) \cdot v(x)d\mu(x),$$

for each $g(x)$ and $v(x)$ such that $g(x) = \mathrm{div}^\mu v(x)$.

Proposition 3. *The two definitions of $H^1(\mathbb{R}^n, \mu)$ are equivalent.*

For a measure μ in $\mathbb{R}^n$ consider the smoothed measure $\mu^\delta = K^\delta \star \mu$. Then μ^δ locally weakly converge to μ as $\delta \to 0$.

Lemma 7. *Let* $g(x) = \mathrm{div}^\mu v(x)$*. Then there are* $g_\delta \in L_2(\mathbb{R}^n, \mu^\delta)$ *and* $v_\delta \in (L_2(\mathbb{R}^n, \mu^\delta))^n$ *such that*

$$\mathrm{div}^{\mu^\delta} v_\delta = g_\delta$$

and

$$g_\delta \to g \quad \text{strongly in } L_2(\mathbb{R}^n, \mu^\delta) \text{ as } \delta \to 0,$$

$$v_\delta \to v \quad \text{strongly in } (L_2(\mathbb{R}^n, \mu^\delta))^n \text{ as } \delta \to 0.$$

Theorem 8. *Let* $\mu^\delta = K^\delta \star \mu$*, and let* $u_\delta \in H^1(\mathbb{R}^n, \mu^\delta)$*. Suppose that*

$$u_\delta \rightharpoonup u \quad \textit{weakly in } L_2(\mathbb{R}^n, \mu^\delta) \textit{ as } \delta \to 0,$$

$$\nabla^{\mu^\delta} u_\delta \rightharpoonup z \quad \textit{weakly in } (L_2(\mathbb{R}^n, \mu^\delta))^n \textit{ as } \delta \to 0.$$

Then $u \in H^1(\mathbb{R}^n, \mu)$ *and* $z = \nabla^\mu u$*.*

Consider the elliptic equation

$$-\mathrm{div}^{\mu} a(x) \nabla^{\mu} u + \lambda u = f \quad \text{in } L_2(\mathbb{R}^n, \mu),$$

and the family of approximating equations of the form

$$-\mathrm{div}^{\mu^\delta} a_\delta(x) \nabla^{\mu^\delta} u + \lambda u = f_\delta \quad \text{in } \ L_2(\mathbb{R}^n, \mu^\delta).$$

Theorem 9. *Suppose that*

$$\Lambda|\xi|^2 \le a(x)\xi{\cdot}\xi \le \Lambda^{-1}|\xi|^2, \quad \Lambda|\xi|^2 \le a_\delta(x)\xi{\cdot}\xi \le \Lambda^{-1}|\xi|^2 \quad \forall x, \xi \in \mathbb{R}^n$$

$a_\delta(x) \longrightarrow a(x)$ *strongly in* $L_2(\mathbb{R}^n, \mu^\delta)$*, and* $f_\delta(x) \longrightarrow f(x)$ *strongly in* $L_2(\mathbb{R}^n, \mu^\delta)$*. Then* $u_\delta(x)$ *strongly converges to* $u(x)$ *in* $L_2(\mathbb{R}^n, \mu^\delta)$ *as* $\delta \longrightarrow 0$*.*

Non-degenerate periodic measures

Let μ be a periodic 2-connected measure in $\mathbb{R}^n$. For every $\xi \in \mathbb{R}^n$ consider the variational problem

$$\widehat{A}\xi \cdot \xi = \min_{v \in L_2^{\mathrm{pot}}(\mathbb{T}^n)} \int_{\mathbb{T}^n} (\xi + v(x)) \cdot (\xi + v(x)) d\mu(x).$$

Then $A\xi \cdot \xi$ is a nonnegative quadratic form in $\mathbb{R}^n$. The matrix of this quadratic form, denoted by $\widehat{A}$, is called *effective*.

Definition 12. A periodic measure μ is *non-degenerate* if $\widehat{A}$ is positive definite.

The kernel of $\widehat{A}$ is denoted by $\mathcal{K}^\mu$.

For a periodic matrix $a(x)$ such that

$$\Lambda|\xi|^2 \le a(x)\xi\cdot\xi \le \Lambda^{-1}|\xi|^2 \qquad \mu - \text{a.e.}$$

define

$$\widehat{A}_a\xi\cdot\xi = \min_{v\in L_2^{\mathrm{pot}}(\mathbb{T}^n)} \int\limits_{\mathbb{T}^n} a(x)(\xi+v(x))\cdot(\xi+v(x))d\mu(x).$$

Proposition 4. *The kernel of $\widehat{A}_a$ coincides with the kernel of $\widehat{A}$.*

$\widehat{A}_a$ is called the *effective matrix* of the operator $-\mathrm{div}^\mu(a(x)\nabla^\mu\cdot)$.

The Euler equation of the above variational problem reads:

find $v_\xi(x) \in L_2^{\mathrm{pot}}(\mathbb{T}^n, \mu)$ such that $a(x)(\xi + v_\xi(x)) \in L_2^{\mathrm{sol}}(\mathbb{T}^n, \mu)$.

Denote by Π_{pot} the orthogonal projection in $(L_2(\mathbb{T}^n, \mu))^n$ on the subspace $L_2^{\mathrm{pot}}(\mathbb{T}^n, \mu)$. Then the Euler equation takes the form:

find $v_\xi(x) \in L_2^{\mathrm{pot}}(\mathbb{T}^n, \mu)$ such that $\Pi_{\mathrm{pot}}\big(a(x)v_\xi(x)\big) = -\Pi_{\mathrm{pot}}\big(a(x)\xi\big)$

It is now clear that the operator mapping $v \in L_2^{\mathrm{pot}}(\mathbb{T}^n, \mu)$ to $\Pi_{\mathrm{pot}}\big(a(x)v_\xi(x)\big)$ is coercive in $L_2^{\mathrm{pot}}(\mathbb{T}^n, \mu)$.

The effective matrix $\widehat{A}_a$ can be written in the form

$$\widehat{A}_a\xi = \int\limits_{\mathbb{T}^n} a(x)(v_\xi(x) + \xi))d\mu(x), \quad \xi \in \mathbb{R}^n.$$

Denote by $V(x)$ the matrix whose columns are formed by vector-functions $v_{e_1}(x), \dots, v_{e_n}(x)$ ($\{e_j\}$ are the coordinate vectors in $\mathbb{R}^n$). Then

$$\widehat{A}_a = \int\limits_{\mathbb{T}^n} a(x)(Id + V(x)))d\mu(x).$$

Proposition 5. *The kernel* $\mathcal{K}^\mu$ *of* $\widehat{A}$ *(or* $\widehat{A}_a$) *coincides with the set of constant potential vectors.*
A vector $\eta \in \mathbb{R}^n$ *belongs to* $(\mathcal{K}^\mu)^\perp$ *if and only if there is* $v \in L_2^{\mathrm{sol}}(\mathbb{T}^n, \mu)$ *such that*

$$\int_{\mathbb{T}^n} v(x) d\mu(x) = \eta.$$

Consider the modified cell problem

Find $v_\xi^+(x) \in L_2^{\mathrm{pot}}(\mathbb{T}^n, \mu)$ such that $a(x)(\Pi^{\mathrm{eff}}\xi + v_\xi^+(x)) \in L_2^{\mathrm{sol}}(\mathbb{T}^n,$

Π^{eff} is the orthogonal projection on $(\mathcal{K}^\mu)^\perp$.

Corollary 10. *The relation holds*:

$$a(x)(\xi + v_\xi(x)) = a(x)(\Pi^{\mathrm{eff}}\xi + v^+).$$

The effective matrix $\widehat{A}_a$ can be expressed by

$$\widehat{A}_a\xi = \int_{\mathbb{T}^n} a(x)(v_\xi^+(x) + \Pi^{\mathrm{eff}}\xi))d\mu(x), \quad \xi \in \mathbb{R}^n.$$

Let μ be a periodic measure in $\mathbb{R}^n$. For $\varepsilon > 0$ we set $\mu_\varepsilon(dx) = \varepsilon^n \mu\Big(\dfrac{dx}{\varepsilon}\Big)$, i.e.,

$$\mu_\varepsilon(B) = \varepsilon^n \mu(\varepsilon^{-1} B)$$

for any Borel set $B \subset \mathbb{R}^n$.

The measure μ_ε weakly converge to the measure $\mu(\square)dx$, $\square = [0,1)^n$, as $\varepsilon \to 0$. In particular, if $\mu(\square) = 1$ then μ_ε converges to the standard Lebesgue measure.

Let G be a Jordan domain in $\mathbb{R}^n$.

Definition 13. We say that $u_\varepsilon \in L_2(G, \mu_\varepsilon)$ *two-scale converge* in $L_2(G, \mu_\varepsilon)$ to $u(x, y) \in L_2(G \times \square, dx \times \mu(y))$, as $\varepsilon \longrightarrow 0$, if

$$\|u_\varepsilon\|_{L_2(G,\mu_\varepsilon)} \leq C, \quad \varepsilon > 0,$$

and

$$\int\limits_G u_\varepsilon(x)\phi(x)\psi(\frac{x}{\varepsilon})d\mu_\varepsilon(x) \underset{\varepsilon\to 0}{\longrightarrow} \int\limits_G \int\limits_\square u(x, y)\varphi(x)\psi(y)dxd\mu(y)$$

for any $\varphi \in C_0^\infty(G)$ and $\psi \in C_{\mathrm{per}}^\infty(\square)$.

Mean value property

Lemma 11. *Suppose that* $g(x,y) \in C(\overline{G}; C_{\mathrm{per}}(\square))$*. Then*

$$\lim_{\varepsilon\to 0} \int\limits_G g\Big(x, \frac{x}{\varepsilon}\Big) d\mu_\varepsilon(x) = \int\limits_{G\times\square} g(x,y)\,dx\,d\mu(y).$$

Properties of two-scale convergence

Proposition 6 (weak compactness of a bounded sequnce). *Suppose that*

$$\|u^\varepsilon\|_{L_2(G,\mu_\varepsilon)} \leq C.$$

Then, along a subsequence $\varepsilon_k \longrightarrow 0$, the functions u^ε two-scale converge in $L_2(G,\mu_\varepsilon)$ to some function $u(x,y) \in L_2(G \times \Box, dx \times \mu(y))$.

Proposition 7 (lower semi-continuity of the norm). *Suppose that $u^\varepsilon(x)$ two-scale converge in $L_2(G,\mu_\varepsilon)$ to a function $u(x,y)$. Then*

$$\liminf_{\varepsilon \longrightarrow 0} \left\|u^\varepsilon\right\|_{L_2(G,\mu_\varepsilon)} \geq \left\|u(x,y)\right\|_{L_2(G\times\Box,dx\times\mu(y))}.$$

Definition 14. We say that $u^\varepsilon(x) \in L_2(G, \mu_\varepsilon)$ *strongly two-scale converge* to $u(x, y) \in L_2(G \times \square, dx \times \mu(y))$ in $L_2(G, \mu_\varepsilon)$ if $u^\varepsilon(x)$ two-scale converge to $u(x, y)$ and

$$\int\limits_G u^\varepsilon(x) v^\varepsilon(x)\, d\mu_\varepsilon(x) \longrightarrow \int\limits_{G\times\square} u(x,y)v(x,y)dxd\mu(y) \quad \text{as } \varepsilon \to 0.$$

for any $v^\varepsilon(x)$ which two-scale converges in $L_2(G, \mu_\varepsilon)$ to $v(x, y)$.

Equivalent definition reads

Definition 15. We say that $u^\varepsilon(x) \in L_2(G, \mu_\varepsilon)$ *strongly two-scale converge* to $u(x, y) \in L_2(G \times \square, dx \times \mu(y))$ in $L_2(G, \mu_\varepsilon)$ if $u^\varepsilon(x)$ two-scale converge to $u(x, y)$ in $L_2(G, \mu_\varepsilon)$ and

$$\lim_{\varepsilon\to 0} \int\limits_G |u^\varepsilon(x)|^2 d\mu_\varepsilon(x) = \int\limits_{G\times\square} |u(x,y)|^2 dxd\mu(y).$$

Proposition 8. *Suppose that* $u^\varepsilon(x) \in H^1(G, \mu_\varepsilon)$ *and*

$$\left\|u^\varepsilon\right\|_{L_2(G,\mu_\varepsilon)} \leq C, \quad \lim_{\varepsilon \longrightarrow 0} \varepsilon \left\|\nabla^\mu u^\varepsilon(x)\right\|_{(L_2(G,\mu_\varepsilon))^n} = 0.$$

Then, along a subsequence, u^ε *two-scale converge in* $L_2(G, \mu_\varepsilon)$ *to some function* $u^0(x)$ *which does not depend of* y*.*

$\mathcal{K}^\mu$ denotes the kernel of $\widehat{A}$, and Π^{eff} the operator of orthogonal projection in $\mathbb{R}^n$ on $(\mathcal{K}^\mu)^\perp$. We set $\nabla^{\text{eff}} = \Pi^{\text{eff}}\nabla$ and $H^{\text{eff}}(G) = \{u \in L_2(G) : \Pi^{\text{eff}}\nabla u \in (L_2(G))^n\}$.

Theorem 12. *Suppose that*

$$\left\|u^\varepsilon\right\|_{L_2(G,\mu_\varepsilon)} \le C, \quad \left\|\nabla^{\mu_\varepsilon} u^\varepsilon\right\|_{(L_2(G,\mu_\varepsilon))^n} \le C.$$

Then, along a subsequence,

$$u^\varepsilon(x) \xrightarrow{2} u^0(x) \quad \textit{two-scale in } L_2(G,\mu_\varepsilon) \textit{ as } \varepsilon \to 0,$$

$$\nabla^\mu u^\varepsilon(x) \xrightarrow{2} \nabla^{\text{eff}} u^0(x) + u_1(x,y) \quad \textit{two-scale in } (L_2(G,\mu_\varepsilon)) \textit{ as } \varepsilon \to 0;$$

with $u^0 \in H^{\text{eff}}(G)$ *and* $u_1 \in L_2(G; L_2^{\text{pot}}(\square,\mu))$.

Theorem 13. *If*

$$\|u^\varepsilon\|_{L_2(G,\mu_\varepsilon)} \le C, \quad \varepsilon\|\nabla^{\mu_\varepsilon} u^\varepsilon\|_{(L_2(G,\mu_\varepsilon))^n} \le C,$$

then there is a subsequences $\varepsilon_k \to 0$ *and a function* $u^0(x,y) \in L_2(G; H^1_{\mathrm{per}}(\square,\mu))$ *such that*

$$u^\varepsilon(x) \xrightarrow{2} u^0(x,y) \quad \textit{two-scale in } L_2(G,\mu_\varepsilon),$$

$$\varepsilon\nabla^\mu u^\varepsilon(x) \xrightarrow{2} \nabla^\mu_y u^0(x,y) \quad \textit{two-scale in } (L_2(G,\mu_\varepsilon)).$$

Let μ be a periodic measure in $\mathbb{R}^n$, and let $\mu_\varepsilon = \varepsilon^n \mu\Big(\dfrac{dx}{\varepsilon}\Big)$. Consider an elliptic equation

$$-\mathrm{div}^{\mu_\varepsilon}\Big(a\Big(\frac{x}{\varepsilon}\Big)\nabla^{\mu_\varepsilon} u\Big) + c\Big(\frac{x}{\varepsilon}\Big)u = f^\varepsilon(x), \quad \text{in } L_2(\mathbb{R}^n, \mu_\varepsilon),$$

We assume that

$$\Lambda|\xi|^2 \le a(y)\xi\cdot\xi \le \Lambda^{-1}|\xi|^2, \qquad \xi \in \mathbb{R}^n$$

μ-a.e. We also assume that $0 < c_0 \le c(y) \le c_1$ μ-a.e. We set

$$\widehat{c} = \int_{\square} c(y)\,d\mu(y).$$

The equation

$$-\mathrm{div}\big(\widehat{A}_a \nabla u\big) + \widehat{c}u = f(x), \quad x \in \mathbb{R}^n,$$

is called *homogenized*. The solution to this equation is denoted by $u^0(x)$. Under our assumptions, this equation has a unique solution in $L_2(\mathbb{R}^n)$.

Theorem 14. *If $f^\varepsilon(x)$ converge strongly (weakly) in $L_2(\mathbb{R}^n, \mu_\varepsilon)$ to a function $f(x) \in L_2(\mathbb{R}^n)$, then*

$$u^\varepsilon(x) \longrightarrow u^0(x) \quad \textit{strongly (weakly) in } L_2(\mathbb{R}^n, \mu_\varepsilon) \textit{ as } \varepsilon \to 0,$$

Moreover (flux convergence),

$$a\Big(\frac{x}{\varepsilon}\Big)\nabla^{\mu_\varepsilon} u^\varepsilon \rightharpoonup \widehat{A}_a \nabla^{\mathrm{eff}} u^0(x) \quad \textit{weakly in } (L_2(\mathbb{R}^n, \mu_\varepsilon))^n \textit{ as } \varepsilon \to 0.$$

Convergence of energy

Proposition 9. *If f^ε converges to f strongly in $L^2(\mathbb{R}^n, \mu_\varepsilon)$, then the energy converges*:

$$\lim_{\varepsilon\to 0} \int\limits_{\mathbb{R}^n} a\left(\frac{x}{\varepsilon}\right)\nabla^{\mu_\varepsilon} u^\varepsilon(x)\cdot\nabla^{\mu_\varepsilon} u^\varepsilon(x) d\mu_\varepsilon(x) = \int\limits_{\mathbb{R}^n} \widehat{A}_a \nabla^{\text{eff}} u^0 \cdot \nabla^{\text{eff}} u^0 dx.$$

Let G be a Lipschitz bounded domain. Consider the Dirichlet problem

$$-\mathrm{div}^{\mu_\varepsilon}\Big(a\Big(\frac{x}{\varepsilon}\Big)\nabla^{\mu_\varepsilon}u^\varepsilon\Big)+c\Big(\frac{x}{\varepsilon}\Big)u^\varepsilon=f^\varepsilon(x)\quad \text{in}\ \ L_2(G,\mu_\varepsilon),$$

$$u^\varepsilon\in \overset{\circ}{H}{}^1(G,\mu_\varepsilon).$$

and homogenized Dirichlet problem

$$-\mathrm{div}\,(\widehat{A}_1\nabla^{\mathrm{eff}}u^0)+\widehat{c}u^0=f\quad \text{in}\ \ G,$$

$$u^0\in H_0^{\mathrm{eff}}(G).$$

Both problems are well-posed, their solutions are denoted u^ε and u^0.

Theorem 15. *If* $f^{\varepsilon}(x)$ *strongly* (*weakly*) *converges in* $L_2(G,\mu_\varepsilon)$ *to* $f(x)\in L_2(G)$, *then,*

$$u^{\varepsilon}(x)\longrightarrow u^{0}(x)\quad \text{strongly (weakly) in } L_2(G,\mu_\varepsilon) \text{ as } \varepsilon\to 0,$$

Moreover, the flux convergence holds:

$$a\Big(\frac{x}{\varepsilon}\Big)\nabla^{\mu_\varepsilon}u^{\varepsilon}\rightharpoonup \widehat{A}_1\nabla^{\text{eff}}u^{0}(x)\quad \text{weakly in } (L_2(G,\mu_\varepsilon))^n \text{ as } \varepsilon\to 0$$

and, in the case of the strong convergence of f^{ε}, *the energy convergence holds*:

$$\lim_{\varepsilon\to 0}\int\limits_G a\Big(\frac{x}{\varepsilon}\Big)\nabla^{\mu_\varepsilon}u^{\varepsilon}(x)\cdot\nabla^{\mu_\varepsilon}u^{\varepsilon}(x)d\mu_\varepsilon(x)=\int\limits_G \widehat{A}_1\nabla^{\text{eff}}u^{0}\cdot\nabla^{\text{eff}}u^{0}dx.$$

Comments

The developed technique can be successfully used in the study of homogenization problems for higher contrast singular and thin structures, for example, singular double porosity problems, the homogenization of parabolic problems in variable spaces, elasticity problems for thin frames, nonlinear operators in variable spaces, and many other problems. It has not only intrinsic interest for homogenization theory, but also significance in relation to close topics such as the central limit theorem, spectral problems, the commutativity of diagram under the limit passage with respect to the period size and the thickness of structure, and many other aspects.