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Homogenization of random singular structures and random measures

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Abstract. The paper deals with the homogenization of random statistically homogeneous singular structures and variational problems involving integration with respect to ergodic stationary random measures.

Introduction

In this paper we consider problems of homogenization of differential operators and variational problems in variable spaces L^2 involving integration with respect to rapidly oscillating stationary random measures. The homogenization of random singular structures such as networks and frames is an important special case.

We obtain our results by developing previous studies in the following two directions: homogenization of equations with statistically homogeneous random coefficients and of periodic singular structures and measures.

Periodic measures and periodic singular structures were studied in [3], [4], [10], [12]. For an arbitrary periodic Radon measure $\mu=\mu(dx)$ on $\mathbb{R}^n$, the Sobolev spaces $H^1(\mathbb{T}^n,\mu(dx))$, where $\mathbb{T}^n$ is the torus of periodicity, and $H^1(\mathbb{R}^n,\mu(dx))$, were defined by closing the set of smooth functions in a suitable norm. It was found that the gradients of functions are not defined unambiguously. It was proved that there is a so-called *tangential* subspace $\mathbb{T}(x)\subset\mathbb{R}^n$. The set of gradients was described in terms of this subspace, and the *tangential* gradient was defined. Several results were obtained concerning the existence and uniqueness of solutions in $L^2(\mathbb{R}^n,\mu(dx))$ of elliptic equations of the form

$$-\operatorname{div} a(x)\nabla u+u=f(x)$$

and of certain more general elliptic and evolution equations and systems.

Considering the family of measures $\mu^\varepsilon(dx)=\varepsilon^n\mu(\frac{dx}{\varepsilon})$ with a small positive parameter ε, the problem of homogenization involves the important property of 2-connectedness of μ. Recall (see, for example, [2]) that the measure $\mu(dx)$ is 2-*connected* if every periodic function that belongs to the Sobolev space $H^1(\mathbb{T}^n,d\mu)$ and has zero μ-gradient is equal to a constant μ-a. e. A measure is 2-connected if and only if the corresponding Markov process on the periodicity torus is ergodic. In what follows we replace the term “2-connected” by the term “connected” or “ergodic”. Under the assumption that the periodic measure is connected, certain results on homogenization were obtained, singular structures

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and the corresponding thin structures were studied, along with the problem of whether the diagram of passage to the limit (as the thickness of the structure and the size of the mesh of periodicity tend to zero) is commutative, and the technique of two-scale convergence was adapted (see [5], [9]).

We recall some classical results on the homogenization of differential operators with random coefficients (see [7], [25]).

Let $a(x,\omega)=\{a_{ij}(x,\omega)\}$, $i,j=1,2,\dots,n$, be an ergodic statistically homogeneous random function, and let

$$\Lambda|\xi|^2 \leqslant a(x,\omega)\xi\cdot\xi \leqslant \Lambda^{-1}|\xi|^2, \qquad \Lambda>0, \quad \xi\in\mathbb{R}^n.$$

The following assertion holds for the family of equations

$$-\operatorname{div} a\biggl(\frac{x}{\varepsilon},\omega\biggr)\nabla u^\varepsilon+\lambda u^\varepsilon=f(x), \qquad f\in L^2(\mathbb{R}^n), \quad \lambda>0.$$

Assertion. *As $\varepsilon\to 0$, the solution u^ε converges almost surely (a. s.) in $L^2(\mathbb{R}^n)$ to a solution of the elliptic equation*

$$-\operatorname{div}\hat a\nabla u^0+\lambda u^0=f(x)$$

with constant non-random coefficients. The matrix $\hat a$ depends only on $a(x,\omega)$ and not on f or λ.

The homogenization theorem also holds for random perforated domains (see [6]). Let $G=G(\omega)$ be a random set in $\mathbb{R}^n$ and assume that the characteristic function of $G(\omega)$ is an ergodic statistically homogeneous random field. Assume that $G(\omega)$ is open and connected and let $G^\varepsilon(\omega)=\varepsilon G(\omega)$ be the homothetic contraction of $G(\omega)$. Then the following relation holds for the solution of the Neumann problem

$$-\operatorname{div} a\biggl(\frac{x}{\varepsilon},\omega\biggr)\nabla u^\varepsilon+\lambda u^\varepsilon=f(x) \quad \text{in} \quad G^\varepsilon(\omega), \qquad f\in C_0^\infty(\mathbb{R}^n),$$
$$\frac{\partial u^\varepsilon}{\partial\nu^\varepsilon}\equiv a\biggl(\frac{x}{\varepsilon},\omega\biggr)\nabla u^\varepsilon\cdot n=0 \quad \text{on} \quad \partial G^\varepsilon(\omega),$$

where n is the normal to $\partial G^\varepsilon(\omega)$:

$$\lim_{\varepsilon\to 0}\int_{G^\varepsilon}|u^\varepsilon(x,\omega)-u^0(x)|^2\,dx=0$$

a. s., where u^0 is the solution of the homogenized equation

$$-\operatorname{div}\hat a\nabla u^0+\lambda u^0=\theta f, \qquad x\in\mathbb{R}^n,$$

and θ is the density of the random set $G=G(\omega)$.

The homogenization technique for equations with random coefficients enables us to study the asymptotic behaviour of stochastic processes (walks) in random media (in particular, the problem of the validity of the central limit theorem (CLT)). For every realization of the medium we consider the diffusion process ξ_t with the corresponding parabolic equation

$$\frac{\partial v}{\partial t}=\operatorname{div} a(x,\omega)\nabla v.$$

The following theorem holds. For almost all realizations of the medium and any initial condition $\xi_0 = x \in \mathbb{R}^n$, the variables $\frac{\xi_t - x}{\sqrt{t}}$ converge distributionwise as $t \to \infty$ to the centred normal variable with correlation matrix $\hat{a}$, where $\hat{a}$ is the averaged matrix of the corresponding elliptic operator.

For walks in a random set, the Neumann boundary condition corresponds to the reflection of ξ_t in $\partial G(\omega)$. For such walks the so-called central limit theorem in mean was proved in [6]. For certain classes of random sets a more exact (individual) CLT was proved in [14] (see also the end of § 9 below).

Our purpose is to generalize these results to singular random structures. Let μ_ω be an ergodic stationary random measure on $\mathbb{R}^n$, that is, a family of Radon measures such that for every bounded Borel set B the function $\mu_\omega(B + x)$ is an ergodic statistically homogeneous random field. We assign to μ_ω the equation

$$-\operatorname{div} a(x, \omega)\nabla u + \lambda u = f, \qquad f \in L^2(\mathbb{R}^n, \mu_\omega),$$

understood in the sense of the integral identity

$$\int_{\mathbb{R}^n} a(x, \omega)\nabla u \nabla\varphi \, d\mu_\omega(x) + \int_{\mathbb{R}^n} u\varphi \, d\mu_\omega(x) = \int_{\mathbb{R}^n} f\varphi \, d\mu_\omega(x) \quad \forall\varphi \in C_0^\infty(\mathbb{R}^n),$$

where the gradient ∇u is unambiguously defined by the condition that the flux $a\nabla u$ is tangential. For piecewise-smooth structures the tangential subspace can be defined in the standard way. For arbitrary measures it was defined and studied in [2], [10]. In § 10 we define and study the tangential subspace and tangential gradient with respect to a certain measure on the probability space connected with the random measure μ_ω.

For $\varepsilon > 0$ we put $\mu_\omega^\varepsilon(dx) = \varepsilon^n \mu_\omega(\frac{dx}{\varepsilon})$. We shall study the asymptotic behaviour of the solution of the equation

$$-\operatorname{div} a\left(\frac{x}{\varepsilon}, \omega\right)\nabla u^\varepsilon + \lambda u^\varepsilon = f^\varepsilon \quad \text{in} \quad L^2(\mathbb{R}^n, \mu_\omega^\varepsilon)$$

as $\varepsilon \to 0$. The homogenization theorem for this equation holds if the random measure μ_ω is connected. Let us consider this condition in more detail. The measures μ_ω are, in a sense, realizations of a certain measure in Ω, the so-called Palm measure (see [13] and § 1 below). Several results, including Birkhoff's ergodic theorem, were obtained for it in the theory of random measures. In analogy with the periodic case, the Palm measure is assumed to be *connected*, which is equivalent to the assumption that the corresponding Markov process is ergodic. The connectedness of the Palm measure is defined in terms of the corresponding Sobolev spaces.

If the measure is connected, then the homogenization theorem (see Theorem 6.1) holds for almost all ω: if f^ε converges to f in $L^2(\mathbb{R}^n, \mu_\omega^\varepsilon)$ as $\varepsilon \to 0$, then u^ε converges in $L^2(\mathbb{R}^n, \mu_\omega^\varepsilon)$ to the solution of the equation

$$\operatorname{div} \hat{a}\nabla u + \lambda u = f$$

with a constant non-negative matrix $\hat{a}$. For sufficiently regular f this convergence is equivalent to the relation

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} |u^\varepsilon(x) - u(x)|^2 \, d\mu_\omega^\varepsilon(x) = 0.$$

Given a random measure, we can consider the corresponding random walk in a random medium and the corresponding parabolic equation. For these parabolic equations we also obtain homogenization theorems, which can be used in the study of the properties of random walks in stationary random media. Some models of these walks are actively studied by specialists in probability theory (see, for example, [26], [27]). From the point of view of probabilistic interpretation, an important result of this paper is the proof of the CLT in mean (see Theorem 9.4).

With the exception of some conditional results (see § 9), we do not consider the individual CLT in this paper. This will be dealt with in a separate paper.

Typical examples of singular random structures are percolation models in $\mathbb{R}^n$ and Voronoi diagrams of stationary point processes in $\mathbb{R}^n$. For these structures, see § 7 and Example 1.8. In particular, for the percolation model, we solve Kesten's well-known problem on the existence of effective conductance (see § 7).

We have already mentioned the important role played in the asymptotic analysis of random measures by the Palm measure and the corresponding Sobolev spaces and spaces of divergence-free and potential vector functions. We prefer to prove homogenization theorems using the method of two-scale convergence. We introduce and develop a version of this convergence for random measures, which can also be used in other applications.

§ 1. Random measures and their properties

1.1. General definitions. Let $(\Omega, \mathcal{F}, \mathsf{P})$ be the standard probability space with dynamical system T_x, that is, a group of measurable maps $T_x\colon \Omega \longrightarrow \Omega$ such that

(i) $T_{x+y} = T_x \cdot T_y$, $x, y \in \mathbb{R}^n$, $T_0 = \mathrm{Id}$,

(ii) $\mathsf{P}(T_x^{-1}(A)) = \mathsf{P}(A)$ for all $x \in \mathbb{R}^n$, $A \in \mathcal{F}$,

(iii) $T_x(\omega)$ is a measurable map from $(\mathbb{R}^n \times \Omega, \mathcal{B} \times \mathcal{F})$ to $(\Omega, \mathcal{F})$, where $\mathcal{B}$ is the Borel σ-algebra.

In what follows we assume that Ω is a *compact metric space*, $\mathcal{F}$ is the Borel σ-algebra on Ω and T_x is a map from $\mathbb{R}^n \times \Omega$ to Ω continuous in this metric.

We also assume that T_x is *ergodic*, that is, the P-measure of every invariant set in $\mathcal{F}$ is either 0 or 1.

In many cases only a discrete group of transformations T_z, $z \in \mathbb{Z}^n$, of the probability space is given. Let us recall the standard procedure of passing from a discrete group to a dynamical system T_x, $x \in \mathbb{R}^n$.

Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be a torus with the Lebesgue measure dy, and let

$$\widetilde{\Omega} = \Omega \times \mathbb{T}^n, \qquad \widetilde{\omega} = \{\omega, y\}, \qquad \tilde{\mu} = \mu \times dy,$$
$$\widehat{T}(x)\widetilde{\omega} = \{T_{[x+y]}\omega,\ x + y - [x+y]\},$$

where $[x+y]$ is the integer part of $x+y$. It is well known that the dynamical system $\widehat{T}_x$ is ergodic if and only if the group T_z is ergodic.

Example 1.1 (the periodic case). Let $\Omega = \square$ be the unit cube, $\square = \{\omega \in \mathbb{R}^n,\ 0 \leqslant \omega_j \leqslant 1,\ j = 1, \dots, n\}$. There is a dynamical system $T(x)\omega = \omega + x \pmod 1$ on Ω with respect to which the Lebesgue measure is invariant and ergodic. The realization of $f(\omega) \in L^2(\square)$ has the form $f(x + \omega)$.

Example 1.2 (the quasi-periodic case). Let $\Omega = \square$ be the unit cube in $\mathbb{R}^m$ and let μ be the Lebesgue measure on this cube. For $x \in \mathbb{R}^n$ we put $T(x)\omega = \omega + \lambda x \pmod 1$, where $\lambda = (\lambda_{ij})$ is an $n \times m$ matrix. It is obvious that μ is invariant under $T(x)$. This dynamical system is ergodic if $\lambda_{ij} k_j \neq 0$ for all integer vectors $k \neq 0$. Hence, $L^2(\Omega) = L^2(\square)$ is the space of periodic functions of m variables, and the realizations have the form $f(\omega + \lambda x)$. These realizations are called *quasi-periodic functions* if $f(\omega)$ is continuous on $\square$.

Recall that the random field $f(x, \omega)$, $x \in \mathbb{R}^n$, $\omega \in \Omega$, is said to be *stationary* if there is a measurable function $\tilde{f} = \tilde{f}(\omega)$ on Ω such that $f(x,\omega) = \tilde{f}(T(x)\omega)$.

A *Radon measure* on $\mathbb{R}^n$ is defined to be a non-negative Borel measure that takes finite values on compact sets. We denote by $T_x\mu$, where $x \in \mathbb{R}^n$, the shift of μ, that is, $T_x\mu(B) = \mu(B + x)$ for all Borel sets $B \subset \mathbb{R}^n$.

A family μ_ω, $\omega \in \Omega$, of Radon measures on $\mathbb{R}^n$ is said to be *stationary* if

$$T_x\mu_\omega = \mu_{T_x\omega}, \qquad x \in \mathbb{R}^n$$

for all $\omega \in \Omega$. Since

$$\int_{\mathbb{R}^n} \varphi(y)\, d\mu_{T_x\omega}(y) = \int_{\mathbb{R}^n} \varphi(y - x)\mu_\omega(y) \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n),$$

we arrive at the following definition.

Definition 1.1. A family of Radon measures μ_ω on $\mathbb{R}^n$, $\omega \in \Omega$, is called a *stationary random measure* if for every $\varphi \in C_0^\infty(\mathbb{R}^n)$ the random function

$$F_\varphi(x,\omega) = \int_{\mathbb{R}^n} \varphi(y - x)\, d\mu_\omega(y) \tag{1.1}$$

is stationary and measurable, that is,

$$\int_{\mathbb{R}^n} \varphi(y - x)\, d\mu_\omega(y) = F_\varphi(T_x\omega), \tag{1.2}$$

where $F_\varphi(\omega) = \int_{\mathbb{R}^n} \varphi(y)\, d\mu_\omega(y)$.

We impose the following technical condition for the sake of simplicity:

$$F_\varphi \in L^\infty(\Omega, \mathsf{P}) \quad \forall \varphi \in C_0^\infty. \tag{1.3}$$

This conditions actually means that

$$\mu_\omega(A) \leqslant C(A) < \infty \quad \text{a. s. in the sense of } \mathsf{P} \ (\mathsf{P}\text{-a. s.}) \tag{1.4}$$

for all bounded Borel sets $A \subset \mathbb{R}^n$.

Let $\square = [0,1)^n$. The quantity

$$m = \mathsf{E}(\mu(\square)) = \int_\Omega \int_\square d\mu_\omega(x)\, d\mathsf{P}(\omega)$$

is called the *intensity of the random measure*. Here and below E stands for the mathematical expectation with respect to P. The condition (1.4) implies that m is finite. We also assume that the intensity of μ is positive, whence

$$0 < m < \infty. \tag{1.5}$$

Here are some examples of stationary random measures.

Example 1.3 (the periodic case: continuation of Example 1.1). Consider a periodic Radon measure on $\mathbb{R}^n$, $\mu(\,\cdot\,+z)=\mu(\,\cdot\,)$ for all $z\in\mathbb{Z}^n$. The corresponding random measure

$$\mu_\omega=\mu(\,\cdot\,+\omega),\qquad \omega\in[0,1)^n=\mathbb{T}^n,$$

is stationary, as can be verified directly.

Example 1.4. Let $\rho\in L^1(\Omega,\mathsf{P})$ and $d\mu_\omega(y)=\rho(T_y\omega)\,dy$. It is obvious that μ_ω is stationary since

$$\begin{aligned}\int_{\mathbb{R}^n}\varphi(y-x)\,d\mu_\omega(y)&=\int_{\mathbb{R}^n}\varphi(y-x)\rho(T_y\omega)\,dy\\&=\int_{\mathbb{R}^n}\varphi(y)\rho(T_{x+y}\omega)\,dy=\int_{\mathbb{R}^n}\varphi(y)\,d\mu_{T_x\omega}(y).\end{aligned}$$

In many cases, a random measure is a priori invariant only under a discrete group of transformations T_z, $z\in\mathbb{Z}^n$. A measure invariant under a dynamical system $\widehat{T}_x$, $x\in\mathbb{R}^n$, can be constructed in the usual way: for $\widehat{\omega}=\{\omega,y\}$ we put

$$\mu_{\widehat{\omega}}=\mu_\omega+y.$$

It is easy to verify that $\mu_{\widehat{\omega}}$ is invariant under the transformations $\widehat{T}_x$.

Example 1.5 (a random perforated structure, in particular, a dispersed ball structure). Random perforated media can also be described in terms of random measures. Let $\mathcal{U}$ be a measurable subset of Ω such that $\mathsf{P}(\mathcal{U})>0$ and $\mathsf{P}(\Omega\setminus\mathcal{U})>0$. Let us recall that a *stationary random set* in $\mathbb{R}^n$ is defined to be a realization of the event $\Omega\setminus\mathcal{U}$, that is, a subset of $\mathbb{R}^n$ of the form

$$Q(\omega)=\{x\in\mathbb{R}^n\colon T_x\omega\in\Omega\setminus\mathcal{U}\}.$$

In many cases the complement of a random set is a union of separate inclusions or grains (say, balls). Such random sets are called *perforated media*. The corresponding random measure is defined by the formula

$$d\mu_\omega(x)=\rho(T_x\omega)\,dx,\qquad \rho(\omega)=\mathbf{1}_{\{\Omega\setminus\mathcal{U}\}}.$$

The definition of this measure implies that it is stationary. It is clear that the homogenization theorem does not hold for random sets that have a. s. two or more unbounded connected components. Examples of "good" media are given by *dispersed ball structures* in $\mathbb{R}^3$, which are defined as random media such that almost all realizations of $\mathcal{U}$ are unions of disjoint open balls in $\mathbb{R}^3$ with the radii of these balls belonging to $[r_1,r_2]$, $0<r_1<r_2<\infty$.

Example 1.6. Consider the integer lattice $\mathbb{Z}^n$ and let Γ be the set of all edges (links) joining neighbouring points. We denote the edges by γ_j and use the natural numbering $j\in\mathbb{Z}^n$. We shall now construct a probability space with a dynamical system. We equip $\Omega=\big\{\{\eta_j\}_{j\in\mathbb{Z}^n}\big\}$ with the system of seminorms $\|\eta^1-\eta^2\|_j=|\eta^1_j-\eta^2_j|$, $j\in\mathbb{Z}^n$. The Borel σ-algebra $\mathcal{B}$ in Ω coincides with the cylindric σ-algebra. Let P_0 be a probabilty measure on the real line whose support is contained in a finite segment of the positive half-line. Let P be the measure on Ω defined as the direct product of the measures P_0 on every component η_j. Consider the dynamical system $T_z,\ z \in \mathbb{Z}^n$, on Ω given by

$$(T_z(\eta))_j = \eta_{j+z}.$$

We consider the random variables ξ_j: $\xi_j(\eta) = \eta_j$ on Ω. By definition, the ξ_j are independent identically distributed positive random variables, $\xi_j \in L^\infty$. Denoting by $l_j(dx)$ the standard Lebesgue measure on γ_j, we consider the random measure

$$\mu_\eta(dx) = \sum_{j\in\mathbb{Z}^n} \xi_j(\eta) l_j(dx).$$

This measure is stationary only with respect to the integer shifts

$$\mu_\omega(A + j) = \mu_{T_j\omega}(A)$$

for all Borel sets $A \subset \mathbb{R}^n$ and all $j \in \mathbb{Z}^n$. An equivalent model with continuous time can be constructed by the same method.

In a similar way we can construct random measures on more complicated periodic lattices and measures whose density is non-constant along the segments.

Example 1.7. If in the construction of Example 1.6 we define P_0 to be the sum

$$(1-p)\delta(s) + p\delta(s-1), \qquad 0 < p < 1,$$

then we obtain the well-known percolation model. For the sake of simplicity, we restrict ourselves to the two-dimensional case. It is a well-known fact of percolation theory (see [16], [18]) that there is a critical value $p_c = 0.5$ such that for $p > p_c$ with probability 1 there is precisely one infinite cluster consisting of edges to which the value 1 is assigned, whereas for $p < p_c$ with probability 1 there is no infinite cluster. We shall consider only the supercritical case when $p > p_c$ and retain the value 1 only at the edges that belong to the infinite cluster. Using the constructions of the preceding example, we obtain a random measure on the infinite cluster.

Example 1.8 (the Voronoi diagram). Let us define a point process in $\mathbb{R}^n$. Let $\mathcal{S}$ be a map from Ω to the set of at most countable locally finite subsets of $\mathbb{R}^n$.

Definition 1.2. We say that $\mathcal{S}$ is a *point process* in $\mathbb{R}^n$ if the following conditions hold:

(i) for every bounded Borel set $A \subset \mathbb{R}^n$ the cardinality (number of points) of $A \cap \mathcal{S}(\omega)$, which is denoted by $\#(A \cap \mathcal{S}(\omega))$, is a measurable random variable whose values are finite a. s.,

(ii) for every bounded Borel set A the random function

$$\xi_A(x,\omega) \overset{\text{def}}{=} \#((A+x) \cap \mathcal{S}), \qquad x \in \mathbb{R}^n,$$

is a statistically homogeneous measurable random function, that is,

$$\xi_A(x,\omega) = \#(A \cap \mathcal{S}(T_x\omega)), \qquad A \in \mathcal{B}(\mathbb{R}^n), \quad x \in \mathbb{R}^n.$$

In particular, for every set of Borel sets $A_1,\dots,A_k$ and every $x\in\mathbb{R}^n$ we have

$$\mathcal{L}\big(\#(A_1\cap\mathcal{S}),\dots,\#(A_k\cap\mathcal{S})\big)=\mathcal{L}\big(\#((A_1+x)\cap\mathcal{S}),\dots,\#((A_k+x)\cap\mathcal{S})\big),$$

where $\mathcal{L}$ stands for the distribution of a random variable.

The quantity $m=\mathsf{E}(\#([0,1]^n\cap\mathcal{S}))$ is called the *intensity of the point process.* We shall consider processes of finite positive intensity.

An important example of a point process in $\mathbb{R}^n$ is the so-called Poisson process, which is characterized by the following two properties:

(i) for some $\lambda>0$ and every bounded Borel set $A\subset\mathbb{R}^n$ the random variable $\#(A\cap\mathcal{S})$ has the Poisson distribution with parameter $\lambda|A|$,

(ii) for every set of disjoint Borel sets $A_1,\dots,A_k$ the random variables $\#(A_1\cap\mathcal{S}),\dots,\#(A_k\cap\mathcal{S})$ are independent.

A proof of the existence of such processes can be found, for example, in [13].

The definition of the Voronoi diagram of a point process can be made as follows. For an arbitrary point process in $\mathbb{R}^n$ we number the points of $\mathcal{S}(\omega)$ and denote them by $\zeta_1(\omega),\dots,\zeta_k(\omega),\dots$. It is well known (see [13]) that this numbering can be assumed to be measurable, so that the ζ_k are random vectors in $\mathbb{R}^n$. The Voronoi diagram is defined to be the set of convex sets defined for every $\zeta_i(\omega)$ by the equality

$$M_i(\omega)=\biggl\{x\in\mathbb{R}^n\colon\ \operatorname{dist}(x,\zeta_i)<\operatorname{dist}\biggl(x,\bigcup_{j\ne i}\zeta_j\biggr)\biggr\}.$$

In our assumptions the M_i are a. s. convex polyhedra (see [13]).

We denote the $(n-1)$-dimensional faces of M_i by Γ_{ij}, where j is chosen in such a way that Γ_{ij} is contained in the hyperplane equidistant from ζ_i and ζ_j. We put

$$d\mu_\omega(x)=\sum_{i,j=1}^{\infty}\mathbf{1}_{\Gamma_{ij}(\omega)}(x)l_{ij}(dx),$$

where $\mathbf{1}_{\Gamma_{ij}}$ is the characteristic function of the set Γ_{ij} and $l_{ij}(dx)$ is the standard Lebesgue measure on the corresponding hypersurface. It is easy to verify that $d\mu_\omega(x)$ is a stationary random measure of finite intensity.

Example 1.9 (the "box" structure). Consider the $(n-1)$-dimensional faces of the unit cube $\bigl[-\frac12,\frac12\bigr]^n$ and all their integer shifts. Let S_{ij} be the face that separates the cubes with centres i and j in $\mathbb{Z}^n$, $|i-j|=1$. By constructing a probability space, a dynamical system and a family of independent identically distributed positive random variables ξ_{ij} in the same way as in Example 1.6, we can consider on every S_{ij} the surface Lebesgue measure with density ξ_{ij}. The sum of these measures over $i,j\in\mathbb{Z}^n$ is a stationary random measure.

Example 1.10 (the composite structure). Consider the following line segments on the two-dimensional coordinate plane:

$$I_1=\{x\in\mathbb{R}^2\colon x_2=0,\ 0\leqslant x_1\leqslant1\},\qquad I_2=\{x\in\mathbb{R}^2\colon x_1=0,\ 0\leqslant x_2\leqslant1\}.$$

Let $I_1^i=I_1+i$ and $I_j^2=I_2+j$, $i,j\in\mathbb{Z}^2$, be their integer shifts. We put $Q^k=\bigl[-\frac14,\frac14\bigr]^2+k$, $k\in\mathbb{Z}^2$. Let η_1^i, η_2^j and η_3^k be three families of independent identically distributed non-negative random variables. We put

$$d\mu(x) = \sum_k \eta_3^k \mathbf{1}_{Q^k}(x)\, dx + \sum_i \eta_1^i \mathbf{1}_{I_1^i}(x)\, dx_1 + \sum_j \eta_2^j \mathbf{1}_{I_2^j}(x)\, dx_2.$$

This random measure is a sum of absolutely continuous and singular measures. It is stationary with respect to integer shifts.

1.2. Some results of Palm theory. For the convenience of the reader, we state several results of the theory of stationary random measures.

First we consider an example. Let ρ be a non-negative random variable with $\rho \in L^1(\Omega, d\mathsf{P})$. Consider the measures $d\mu_\omega(x) = \rho(T_x\omega)\, dx$ on $\mathbb{R}^n$ and the measure $d\boldsymbol{\mu}(\omega) = \rho(\omega)\, d\mathsf{P}(\omega)$ on Ω. Since the densities $\rho(T_x\omega)$ are realizations of $\rho(\omega)$ on the trajectories $T_x\omega$, it is natural to call μ_ω a *realization of* $\boldsymbol{\mu}$. The measure on Ω and its realizations on the trajectories $T_x\omega$ are related by *Campbell's formula*,

$$\int_\Omega \int_{\mathbb{R}^n} f(x, T_x\omega)\, d\mu_\omega(x)\, d\mathsf{P}(\omega) = \int_\Omega \int_{\mathbb{R}^n} f(x, \omega)\, d\boldsymbol{\mu}(\omega)\, dx. \tag{1.6}$$

Indeed,

$$\begin{aligned}\int_\Omega \int_{\mathbb{R}^n} f(x, T_x\omega)\rho(T_x\omega)\, dx\, d\mathsf{P}(\omega) &= \int_{\mathbb{R}^n} \left(\int_\Omega f(x, T_x\omega)\rho(T_x\omega)\, d\mathsf{P}(\omega) \right) dx \\ &= \int_{\mathbb{R}^n} \int_\Omega f(x, \omega)\rho(\omega)\, d\mathsf{P}(\omega)\, dx = \int_{\mathbb{R}^n} \int_\Omega f(x, \omega)\, d\boldsymbol{\mu}(\omega)\, dx.\end{aligned}$$

Here we have used Fubini's theorem and the fact that $d\mathsf{P}(\omega)$ is invariant under the maps T_x. The formula (1.6) holds for all $f \in L^1(\mathbb{R}^n \times \Omega, dx \times d\boldsymbol{\mu})$.

It turns out that for an arbitrary stationary random measure $d\mu_\omega(x)$ there is a measure $\boldsymbol{\mu}$ on Ω related to the $d\mu_\omega(x)$ by formula (1.6).

Definition 1.3. *The Palm measure* of the random measure μ_ω is the measure $\boldsymbol{\mu}$ on $(\Omega, \mathcal{F})$ defined by the formula

$$\boldsymbol{\mu}(A) = \int_\Omega \int_{\mathbb{R}^n} \mathbb{I}_\Box(x) \mathbb{I}_A(T_x\omega)\, d\mu_\omega(x)\, d\mathsf{P}(\omega), \tag{1.7}$$

where $\mathbb{I}$ is the characteristic function of a set and $\Box = [0, 1)^n$.

Lemma 1.1 [13]. *$\boldsymbol{\mu}$ is a finite Borel measure on Ω. The equality (1.6) holds for all functions $f = f(x, \omega)$ integrable with respect to $dx \times \boldsymbol{\mu}$ and all non-negative measurable functions $f = f(x, \omega)$.*

There is another way to construct $\boldsymbol{\mu}$ that involves the approximation of measures μ_ω by smoothed absolutely continuous measures defined using a smoothing operator. For every $\delta > 0$ we consider the operator $\mathcal{K}^\delta$ whose action on every Radon measure $d\mu(x)$ on $\mathbb{R}^n$ is defined by the formula

$$d\mu^\delta(x) = (\mathcal{K}^\delta\, d\mu)(x) = \tilde{\rho}^\delta(x)\, dx, \qquad \tilde{\rho}^\delta(x) = \delta^{-n} \int_{\mathbb{R}^n} K\left(\frac{x - y}{\delta}\right) d\mu(y), \tag{1.8}$$

where K is a positive C_0^∞-function, $\int_{\mathbb{R}^n} K(x)\,dx = 1$ and $K(-x) = K(x)$. It is easy to verify that $\mathcal{K}^\delta\mu$ is a Radon measure and μ^δ converges locally weakly to μ as $\delta \downarrow 0$, that is, for every continuous function φ with compact support in $\mathbb{R}^n$ we have

$$\lim_{\delta\to 0}\int_{\mathbb{R}^n}\varphi(x)\,d\mu^\delta(x) = \int_{\mathbb{R}^n}\varphi(x)\,d\mu(x).$$

Putting

$$d\mu_\omega^\delta(x) = \mathcal{K}^\delta\,d\mu_\omega(x) = \tilde{\rho}_\omega^\delta(x)\,dx, \qquad \tilde{\rho}_\omega^\delta(x) = \delta^{-n}\int_{\mathbb{R}^n} K\left(\frac{x-y}{\delta}\right)d\mu_\omega(y), \tag{1.9}$$

we obtain a family of absolutely continuous measures μ_ω^δ that converge a. s. locally weakly to μ_ω as $\delta \downarrow 0$.

By (1.2), the function $\tilde{\rho}_\omega^\delta(x)$ can be represented as follows:

$$\tilde{\rho}_\omega^\delta(x) = \rho^\delta(T_x\omega), \qquad \rho^\delta(\omega) = \delta^{-n}\int_{\mathbb{R}^n} K\left(\frac{y}{\delta}\right)d\mu_\omega(y).$$

Hence, the μ_ω^δ can be represented as $d\mu_\omega^\delta(x) = \rho^\delta(T_x\omega)\,dx$, and these measures are realizations of the measure $\boldsymbol{\mu}^\delta \overset{\text{def}}{=} \rho^\delta(\omega)\,d\mathsf{P}(\omega)$ in Ω.

We claim that the family of measures $\boldsymbol{\mu}^\delta(d\omega)$ converges weakly to $\boldsymbol{\mu}$ in Ω. First we observe that the family of measures $\boldsymbol{\mu}^\delta(d\omega)$ is weakly compact in Ω, since Ω is a compact metric space and $\int_\Omega \boldsymbol{\mu}^\delta(d\omega) = m$, which can be proved as follows. Let $t \to \infty$. Then

$$\begin{aligned}\int_\Omega \boldsymbol{\mu}^\delta(d\omega) &= \int_\Omega \rho^\delta(\omega)\,d\mathsf{P}(\omega) = t^{-n}\mathsf{E}\int_{t\square}\rho^\delta(T_x\omega)\,dx \\ &= t^{-n}\mathsf{E}\int_{t\square}\int_{\mathbb{R}^n}\delta^{-n}K\left(\frac{x-y}{\delta}\right)d\mu_\omega(y)\,dx \\ &= t^{-n}\mathsf{E}\int_{\mathbb{R}^n}d\mu_\omega(y)\int_{t\square}\delta^{-n}K\left(\frac{x-y}{\delta}\right)dx \longrightarrow m.\end{aligned}$$

Since the left-hand side of this formula does not depend on t, it is equal to m. Let us verify that every limit measure $\tilde{\boldsymbol{\mu}} = \lim_{\delta\to 0}\boldsymbol{\mu}^\delta$ coincides with $\boldsymbol{\mu}$. To do this, we consider the function $K = K(x)$ occurring in (1.9) and an arbitrary continuous function $f = f(\omega)$ on Ω. We have

$$\begin{aligned}\int_\Omega f(\omega)\,d\tilde{\boldsymbol{\mu}}(\omega) &= \lim_{\delta\downarrow 0}\int_\Omega f(\omega)\boldsymbol{\mu}^\delta(d\omega) = \lim_{\delta\downarrow 0}\int_{\mathbb{R}^n}K(x)\mathsf{E}\{f(\omega)\rho^\delta(\omega)\}\,dx \\ &= \lim_{\delta\downarrow 0}\int_{\mathbb{R}^n}K(x)\mathsf{E}\{f(T_x\omega)\rho^\delta(T_x\omega)\}\,dx = \lim_{\delta\downarrow 0}\mathsf{E}\int_{\mathbb{R}^n}K(x)f(T_x\omega)\,d\mu_\omega^\delta(x) \\ &= \mathsf{E}\int_{\mathbb{R}^n}K(x)f(T_x\omega)\,d\mu_\omega(x) = \int_{\mathbb{R}^n}\int_\Omega K(x)f(\omega)\,d\boldsymbol{\mu}(\omega)\,dx = \int_\Omega f(\omega)\,d\boldsymbol{\mu}(\omega).\end{aligned}$$

Here we have used Campbell's formula, (1.4) and Lebesgue's theorem. Since f is an arbitrary function, we have proved the desired coincidence of measures.

We define the smoothing operator for functions $\varphi \in C(\Omega)$ by the formula

$$\varphi^\delta(\omega) = \widetilde{\mathcal{K}}^\delta\varphi(\omega) = \delta^{-n}\int_{\mathbb{R}^n}K\left(\frac{y}{\delta}\right)\varphi(T_y\omega)\,dy. \tag{1.10}$$

Let us note that $\varphi^\delta \in C(\Omega)$ since the group T_x is continuous. By Campbell's formula, we have

$$\begin{aligned}\int_\Omega \varphi^\delta(\omega)\, d\boldsymbol{\mu}(\omega) &= \frac{1}{\delta^n}\int_\Omega\int_{\mathbb{R}^n} K\left(\frac{x}{\delta}\right)\varphi(T_x\omega)\,dx\,d\boldsymbol{\mu}(\omega)\\ &= \frac{1}{\delta^n}\int_\Omega\int_{\mathbb{R}^n} K\left(\frac{x}{\delta}\right)\varphi(T_{-x}\omega)\,dx\,d\boldsymbol{\mu}(\omega)\\ &= \frac{1}{\delta^n}\int_\Omega\int_{\mathbb{R}^n} K\left(\frac{x}{\delta}\right)\varphi(\omega)\,d\mu_\omega(x)\,d\mathsf{P}(\omega) = \int_\Omega \varphi(\omega)\boldsymbol{\mu}^\delta(d\omega). \quad (1.11)\end{aligned}$$

Theorem 1.1 (the ergodic theorem [13]). *Let the dynamical system T_x be ergodic and assume that the stationary random measure μ has finite intensity $m > 0$. Then*

$$\lim_{t\to\infty}\frac{1}{t^n|A|}\int_{tA} g(T_x\omega)\,d\mu_\omega(x) = \int_\Omega g(\omega)d\boldsymbol{\mu}(\omega) \quad \textit{a. s. with respect to } \mathsf{P} \qquad (1.12)$$

for all bounded Borel sets A, $|A| > 0$, and all $g(\omega) \in L^1(\Omega, \boldsymbol{\mu})$.

In the case when $\boldsymbol{\mu} = \mathsf{P}$, Theorem 1.1 coincides with the well-known theorem of Birkhoff.

The ergodic theorem stated above implies that the measures

$$\mu^\varepsilon = \varepsilon^n\, d\mu_\omega(\varepsilon^{-1}x)$$

converge weakly to a determinate measure $m\,dx$ a. s. in the sense of P as $\varepsilon \to 0$.

We have already mentioned that condition (1.5) implies that the Palm measure is a finite Borel measure on the compact metric space Ω. Not all these measures are Palm measures for stationary measures.

Example 1.11. Let Ω be a two-dimensional torus, let $T(x)$ be a one-dimensional dynamical system of shifts along an irrational cable, let I be a line segment on the torus and let μ be a one-dimensional Lebesgue measure concentrated on this segment. Then it can be shown that

(i) if I is parallel to the cable, then μ is not a Palm measure,

(ii) if I is orthogonal to the cable, then μ is a Palm measure.

§ 2. Sobolev spaces with measure

Let us recall the definition of the Sobolev space $H^1(\mathbb{R}^n, \mu_\omega)$.

Definition 2.1. We say that $z = z(x)$, $z \in (L^2(\mathbb{R}^n, \mu_\omega))^n$, is a gradient of $u(x)$ in $L^2(\mathbb{R}^n, \mu_\omega)$ if there is a sequence $u_k \in C_0^\infty(\mathbb{R}^n)$ such that

$$u_k \to u \quad \text{in} \quad L^2(\mathbb{R}^n, \mu_\omega), \qquad \nabla u_k \to z \quad \text{in} \quad (L^2(\mathbb{R}^n, \mu_\omega))^n$$

as $k \to \infty$. The *space* $H(\mathbb{R}^n, \mu_\omega)$ is defined to be the set of pairs (u, z), $z = \nabla u$, equipped with the natural norm

$$\left(\|u\|^2_{L^2(\mathbb{R}^n,\mu_\omega)} + \|z\|^2_{L^2(\mathbb{R}^n,\mu_\omega)}\right)^{1/2}.$$

This set is a closed subspace of $(L^2(\mathbb{R}^n, \mu_\omega))^{n+1}$.

Sometimes it is only the first component of the pair $(u, \nabla u)$ that is called an *element of the Sobolev space*. Let us note that a function can have more than one gradient. The gradient is defined to within the *set of gradients of the zero function*, which will be denoted by $\Gamma_{\mu_\omega}(0)$. This set is a closed subset of $(L^2(\mathbb{R}^n, \mu_\omega))^n$. The spaces $H^1_{\mathrm{loc}}(\mathbb{R}^n, \mu_\omega)$, as well as the spaces $H^1(G, \mu_\omega)$ and $H^1_0(G, \mu_\omega)$ for any Lipschitz domain G, can be defined likewise.

For a detailed description of Sobolev spaces, see [2]–[5] and [8]–[10].

Our next purpose is to describe some spaces of functions on Ω associated with the measure μ.

Lemma 2.1. *Let $v \in L^2(\Omega, \boldsymbol{\mu})$. Then* P*-almost all realizations $v(T_x\omega)$ belong to $L^2_{\mathrm{loc}}(\mathbb{R}^n, \mu_\omega)$. Moreover,*

$$\mathsf{E} \int_A v^2(T_x\omega)\, d\mu_\omega(x) = |A|\, \|v\|^2_{L^2(\Omega, \boldsymbol{\mu})} \tag{2.1}$$

for all $A \in \mathcal{B}(\mathbb{R}^n)$.

Proof. This follows immediately from Campbell's formula and Fubini's theorem.

A converse assertion also holds and can be stated as follows: if almost all realizations of the measurable function $v = v(\omega)$ belong to $L^2_{\mathrm{loc}}(\mathbb{R}^n, \mu_\omega)$ and

$$\mathsf{E} \int_{\square} v^2(T_x\omega)\, d\mu_\omega(x) < \infty,$$

then $v \in L^2(\Omega, \mu)$ and (2.1) holds.

Let us construct the space $H^1(\Omega, \boldsymbol{\mu})$. Consider the set of continuous functions on Ω such that the limits

$$(\partial_i u)(\omega) = \lim_{\delta \to 0} \frac{u(T_{\delta e_i}\omega) - u(\omega)}{\delta}$$

exist for all ω and $i = 1, 2, \ldots, n$ and are continuous on Ω. This set is dense in $L^2(\Omega, \boldsymbol{\mu})$, as can be verified by using the fact that $C(\Omega)$ is dense in $L^2(\Omega)$ and considering for an arbitrary $\varphi \in C(\Omega)$ the family of functions

$$\varphi^\delta(\omega) = \delta^{-n} \int_{\mathbb{R}^n} K\left(\frac{y}{\delta}\right) \varphi(T_y\omega)\, dy, \tag{2.2}$$

where K is a positive C_0^∞-function whose integral is equal to 1. The desired properties of φ^δ follow from the continuity of T_x. We denote this set of functions by $C^1(\Omega)$.

Definition 2.2. We say that $u \in L^2(\Omega, \boldsymbol{\mu})$ *belongs to* $H^1(\Omega, \boldsymbol{\mu})$ and $z \in (L^2(\Omega, \boldsymbol{\mu}))^n$ is a *gradient* of this function if there is a sequence $u_k \in C^1(\Omega)$ such that $u_k \to u$ in $L^2(\Omega, \boldsymbol{\mu})$ and $\partial_i u_k \to z_i$ in $L^2(\Omega, \boldsymbol{\mu})$, $i = 1, 2, \ldots, n$, as $k \to \infty$.

We denote the space of pairs (u, z) by $H(\Omega, \boldsymbol{\mu})$ and the set of their first components by $H^1(\Omega, \boldsymbol{\mu})$.

The gradient of $u \in H^1(\Omega, \boldsymbol{\mu})$ will be denoted by $z = \partial u$.

As in the case of periodic singular measures (see [4]), $u \in H^1(\Omega, \boldsymbol{\mu})$ can have more than one gradient. It is defined to within a gradient of zero. The set of gradients of zero will be denoted by $\Gamma_{\boldsymbol{\mu}}(0)$.

Lemma 2.2. *Let u be a function belonging to $H^1(\Omega, \boldsymbol{\mu})$ and let z be a gradient of this function. Then* P*-almost all realizations $u(T_x\omega)$ belong to $H^1_{\mathrm{loc}}(\mathbb{R}^n, \mu_\omega)$, and $z(T_x\omega)$ is a gradient of $u(T_x\omega)$ regarded as a function of x.*

Proof. Consider an approximating sequence of functions $u_k \in C^1(\Omega)$. Their realizations $u_k(T_x\omega)$ belong to $C^1_{\mathrm{b}}(\mathbb{R}^n)$ and

$$\frac{\partial}{\partial x_i} u_k(T_x\omega) = (\partial_i u_k)(T_x\omega).$$

Here we use standard notation: $C^1_{\mathrm{b}}(\mathbb{R}^n)$ stands for the set of functions continuously differentiable in $\mathbb{R}^n$ equipped with the norm

$$\|u\|_{C^1_{\mathrm{b}}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} (|u(x)| + |\nabla u(x)|).$$

By Campbell's formula (1.6), we have

$$\mathsf{E} \int_A (u_k(T_x\omega) - u(T_x\omega))^2 \, d\mu_\omega(x) = |A| \int_\Omega (u_k(\omega) - u(\omega))^2 \, d\boldsymbol{\mu}(\omega) \to 0$$

as $k \to \infty$. By choosing a subsequence we obtain the relation

$$u_k(T_x\omega) \to u(T_x\omega) \quad \text{in} \quad L^2_{\mathrm{loc}}(\mathbb{R}^n, \mu_\omega)$$

P-a. s. as $k \to \infty$. We likewise obtain that

$$\frac{\partial}{\partial x_i} u_k(T_x\omega) \to z(T_x\omega) \quad \text{in} \quad (L^2_{\mathrm{loc}}(\mathbb{R}^n, \mu_\omega))^n$$

P-a. s. as $k \to \infty$, which completes the proof of the lemma.

The space $L^2_{\mathrm{pot}}(\Omega, \boldsymbol{\mu})$ of potential vectors is defined to be the closure of the set $\{\partial u, \ u \in C^1(\Omega)\}$ in $(L^2(\Omega, \boldsymbol{\mu}))^n$. The space $L^2_{\mathrm{sol}}(\Omega, \boldsymbol{\mu})$ of divergence-free vectors is defined to be the orthogonal complement of $L^2_{\mathrm{pot}}(\Omega, \boldsymbol{\mu})$ in $(L^2(\Omega, \boldsymbol{\mu}))^n$, that is, $L^2_{\mathrm{sol}}(\Omega, \boldsymbol{\mu}) = (L^2_{\mathrm{pot}}(\Omega, \boldsymbol{\mu}))^\perp$.

Lemma 2.3. *Let $v \in L^2_{\mathrm{pot}}(\Omega, \boldsymbol{\mu})$. Then* P*-a. s. the realizations $v(T_x\omega)$ belong to the spaces $L^2_{\mathrm{pot,loc}}(\mathbb{R}^n, \mu_\omega)$ defined as the closures in $(L^2_{\mathrm{loc}}(\mathbb{R}^n, \mu_\omega))^n$ of the gradients of smooth functions on $\mathbb{R}^n$.*

The proof of this lemma is as simple as that of Lemma 2.2. The main point is that $L^2_{\mathrm{pot}}(\Omega, \mu)$ is the closure of the set of smooth vectors $\partial u, \ u \in C^1(\Omega)$, for which this property of realizations is obvious. On the other hand, the space $L^2_{\mathrm{sol}}(\Omega, \mu)$ is defined in another way and, generally speaking, the set of smooth vectors is not dense in it. This is the reason why the proof of the following theorem is not quite trivial.

Theorem 2.1. P*-almost all realizations of an arbitrary vector-valued function $w \in L^2_{\mathrm{sol}}(\Omega, \boldsymbol{\mu})$ are divergence-free in the sense of the measure μ_ω, that is,*

$$w(T_x\omega) \in L^2_{\mathrm{loc}}(\mathbb{R}^n, d\mu_\omega), \qquad \int_{\mathbb{R}^n} w(T_x\omega) \cdot \nabla\varphi(x) \, d\mu_\omega(x) = 0 \quad \forall \varphi \in C^\infty_0(\mathbb{R}^n). \tag{2.3}$$

Before we prove Theorem 2.1, let us describe an auxiliary construction. We shall deal with the smoothed measures $\boldsymbol{\mu}^\delta$ and the smoothing operator φ^δ, $\varphi \in C(\Omega)$, defined by formula (1.10).

Lemma 2.4. *For any $b \in L^2(\Omega, d\boldsymbol{\mu})$ there is a $b_\delta \in L^2(\Omega, d\boldsymbol{\mu}^\delta)$ such that*

$$\int_\Omega b_\delta \varphi \, d\boldsymbol{\mu}^\delta = \int_\Omega b\varphi^\delta \, d\boldsymbol{\mu} \quad \forall \varphi \in C(\Omega). \tag{2.4}$$

Proof. Regarding the right-hand side of (2.4) as a functional $\Phi(\varphi)$, $\varphi \in C(\Omega)$, we obtain the obvious estimate

$$|\Phi(\varphi)|^2 \leqslant c \int_\Omega (\varphi^\delta)^2 \, d\boldsymbol{\mu}, \qquad c = \int_\Omega b^2 \, d\boldsymbol{\mu}.$$

By Jensen's integral inequality, we have $(\varphi^\delta)^2 \leqslant (\varphi^2)^\delta$, whence

$$|\Phi(\varphi)|^2 \leqslant c \int_\Omega (\varphi^2)^\delta \, d\boldsymbol{\mu} = c \int_\Omega \varphi^2 \, d\boldsymbol{\mu}^\delta$$

by property (1.11) of $\boldsymbol{\mu}^\delta$. Therefore, b_δ exists by Riesz' theorem, and

$$\int_\Omega |b_\delta|^2 \, d\boldsymbol{\mu}^\delta \leqslant \int_\Omega |b|^2 \, d\boldsymbol{\mu}. \tag{2.5}$$

Since $\varphi^\delta \to \varphi$ uniformly on Ω, we have $b_\delta \rightharpoonup b$ in $L^2(\Omega, d\boldsymbol{\mu}^\delta)$. The estimate (2.5) implies that this convergence is strong: $b_\delta \to b$ in $L^2(\Omega, d\boldsymbol{\mu}^\delta)$.

We now take a $v \in L^2_{\mathrm{sol}}(\Omega, \boldsymbol{\mu})$ and observe that the corresponding vector v_δ is divergence-free with respect to $\boldsymbol{\mu}^\delta$. Indeed, for $\partial\varphi$, $\varphi \in C^1(\Omega)$, (2.4) implies that

$$\int_\Omega v_\delta \cdot \partial\varphi \, d\boldsymbol{\mu}^\delta = \int_\Omega v \cdot (\partial\varphi)^\delta \, d\boldsymbol{\mu} = \int_\Omega v \cdot \partial\varphi^\delta \, d\boldsymbol{\mu} = 0.$$

Using the fact that the measure $\boldsymbol{\mu}^\delta$ is absolutely continuous with respect to P, that is, $\boldsymbol{\mu}^\delta = \rho^\delta \mathsf{P}$, $\rho^\delta \in L^1(\Omega, d\mathsf{P})$, we obtain that the vector

$$p^\delta \equiv v_\delta \rho^\delta$$

is divergence-free: $\int_\Omega p^\delta \cdot \partial\varphi \, d\mathsf{P}(\omega) = 0$ for all $\varphi \in C^1(\Omega)$. By a well-known theorem (see [14], Ch. 7), this vector is divergence-free on P-almost all realizations:

$$\int_{\mathbb{R}^n} v_\delta(T_x\omega) \cdot \nabla\psi(x) \, d\mu^\delta_\omega(x) = 0 \quad \forall \psi \in C_0^\infty(\mathbb{R}^n). \tag{2.6}$$

For $a \in C(\Omega)$ we have

$$\begin{aligned} 0 &= \int_\Omega a(\omega) \biggl[\int_{\mathbb{R}^n} v_\delta(T_x\omega) \cdot \nabla\psi(x) \, d\mu^\delta_\omega(x) \biggr] d\mathsf{P}(\omega) \\ &= \int_{\mathbb{R}^n} \int_\Omega a(T_{-x}\omega) v_\delta(\omega) \cdot \nabla\psi(x) \rho^\delta(\omega) \, d\mathsf{P}(\omega) \, dx. \end{aligned}$$

Here we used the fact that P is invariant under the T_x and applied Campbell's formula to

$$f(x, \omega) = a(T_{-x}\omega)\nabla\psi(x) \cdot v_\delta(\omega).$$

Since $v_\delta \to v$ in $L^2(\Omega, \boldsymbol{\mu}^\delta)$, we have

$$\lim_{\delta\to 0}\int_\Omega a(T_{-x}\omega)v_\delta(\omega)\cdot\nabla\psi(x)\rho^\delta(\omega)\,d\mathsf{P}(\omega) = \int_\Omega a(T_{-x}\omega)\nabla\psi(x)\cdot v(\omega)\,d\boldsymbol{\mu}(\omega)$$

for fixed $x \in \mathbb{R}^n$. Therefore,

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n}\int_\Omega a(T_{-x}\omega)v(\omega)\cdot\nabla\psi(x)\,d\boldsymbol{\mu}(\omega)\,dx \\ &= \int_\Omega a(\omega)\Big[\int_{\mathbb{R}^n} v(T_x\omega)\cdot\nabla\psi(x)\,d\mu_\omega(x)\Big]\,d\mathsf{P}(\omega), \end{aligned}$$

again by Campbell's formula. Since a is any element of $C(\Omega)$, we obtain that

$$\int_{\mathbb{R}^n} v(T_x\omega)\cdot\nabla\psi(x)\,d\mu_\omega(x) = 0$$

P-a. s., as required.

Definition 2.3. We say that $g(\omega) = \operatorname{div}_\omega v(\omega)$, $g \in L^2(\Omega, \boldsymbol{\mu})$, $v \in (L^2(\Omega, \boldsymbol{\mu}))^n$, if

$$\int_\Omega g(\omega)u(\omega)\,d\boldsymbol{\mu}(\omega) = -\int_\Omega v(\omega)\cdot\partial u(\omega)\,d\boldsymbol{\mu}(\omega) \quad \forall u \in C^1(\Omega). \tag{2.7}$$

It is clear that this identity holds for any pair $(u, \partial u) \in H^1(\Omega, \mu)$.

The next theorem is a generalization of Theorem 2.1 and can be proved likewise.

Theorem 2.2. *If* $g(\omega) = \operatorname{div}_\omega v$, *then* P-*a. s.* $\operatorname{div} v(T_x\omega) = g(T_x\omega)$ *in the sense of* μ_ω, *that is,*

$$\int_{\mathbb{R}^n} v(T_x\omega)\cdot\nabla\varphi\,d\mu_\omega(x) = -\int_{\mathbb{R}^n} g(T_x\omega)\varphi\,d\mu_\omega(x), \qquad \varphi \in C_0^\infty(\mathbb{R}^n). \tag{2.8}$$

Remark 2.1. It is easy to verify that (2.8) holds for a more general class of test functions. In particular, it holds for an arbitrary $\varphi \in H^1(\mathbb{R}^n, \mu_\omega)$ with compact support.

Consider the auxiliary problem

$$\int_\Omega \partial u(\omega)\cdot\partial v(\omega)\,d\boldsymbol{\mu}(\omega) + \int_\Omega u(\omega)v(\omega)\,d\boldsymbol{\mu}(\omega) = \int_\Omega f(\omega)v(\omega)\,d\boldsymbol{\mu}(\omega), \quad v \in H^1(\Omega, \boldsymbol{\mu}), \tag{2.9}$$

where $f \in L^2(\Omega, \boldsymbol{\mu})$. Since the left-hand side of this equation is the scalar product of $(u, \partial u)$ and $(v, \partial v)$ in $H(\Omega, \boldsymbol{\mu})$ and the right-hand side is a continuous functional, Riesz' theorem implies that this equation has precisely one solution $(u, \nabla u) \in H(\Omega, \boldsymbol{\mu})$. Hence, it has precisely one solution in $H^1(\Omega, \boldsymbol{\mu})$.

Let us note that in equation (2.9) the gradient of a solution is defined unambiguously by the following condition: ∂u is orthogonal to the gradients of zero, that is, to the $z \in \Gamma_{\boldsymbol{\mu}}(0)$. It is obvious that this condition is necessary. The following assertion implies that it is sufficient.

Proposition 2.1. *Every* $u \in H^1(\Omega, \boldsymbol{\mu})$ *has precisly one gradient orthogonal to* $\Gamma_{\boldsymbol{\mu}}(0)$.

Proof. Let ∂u be a gradient of u. Since $\Gamma_{\boldsymbol{\mu}}(0)$ is a closed subspace of $(L^2(\Omega, \boldsymbol{\mu}))^n$, the equation

$$(z + \partial u, v)_{(L^2(\Omega,\boldsymbol{\mu}))^n} = 0 \quad \forall v \in \Gamma_{\boldsymbol{\mu}}(0)$$

has precisely one solution in $\Gamma_{\boldsymbol{\mu}}(0)$. The function $(z+\partial u)$ gives the desired gradient.

The main condition imposed on the Palm measure can be stated as follows.

Definition 2.4. The measure $\boldsymbol{\mu}$ is said to be *ergodic* or 2-*connected* if $u = \text{const}$ $\boldsymbol{\mu}$-a. s. for every $u \in H^1(\Omega, \boldsymbol{\mu})$ such that $z = \partial u$ is equal to zero.

The following approximation lemma holds for ergodic measures.

Lemma 2.5. *If* μ *is ergodic, then the set* $\{g(\omega)\}$ *of functions of the form* $g(\omega) = \operatorname{div}_\omega v(\omega)$ *is dense in* $\bigl\{u \in L^2(\Omega, \boldsymbol{\mu})\colon \int_\Omega u\, d\boldsymbol{\mu}(\omega) = 0\bigr\}$.

Proof. Assume the opposite. Then there is a $\zeta \in L^2(\Omega, \boldsymbol{\mu})$, $\zeta \neq 0$, such that $\int_\Omega \zeta(\omega)\, d\boldsymbol{\mu}(\omega) = 0$ and ζ is orthogonal to every vector of the form $\operatorname{div} v$. Consider problem (2.9) with $f = \zeta$. Putting $v = u$, we obtain that

$$\int_\Omega |\nabla u|^2\, d\boldsymbol{\mu}(\omega) + \int_\Omega u^2\, d\boldsymbol{\mu}(\omega) = \int_\Omega \zeta u\, d\boldsymbol{\mu}(\omega),$$

whence

$$\int_\Omega u^2\, d\boldsymbol{\mu}(\omega) \leqslant \int_\Omega \zeta^2\, d\boldsymbol{\mu}(\omega). \tag{2.10}$$

The relation (2.9) implies that

$$\int_\Omega (\zeta - u) v\, d\boldsymbol{\mu}(\omega) = \int_\Omega \nabla u \nabla v\, d\boldsymbol{\mu}(\omega),$$

whence $\zeta - u = \operatorname{div}_\omega(\nabla u)$. The choice of ζ implies that $\int_\Omega (\zeta - u)\zeta\, d\boldsymbol{\mu}(\omega) = 0$. Further, we have

$$0 \leqslant \int_\Omega (\zeta - u)^2\, d\boldsymbol{\mu}(\omega) = -\int_\Omega u(\zeta - u)\, d\boldsymbol{\mu}(\omega),$$

whence

$$\int_\Omega u^2\, d\boldsymbol{\mu}(\omega) \geqslant \int_\Omega u\zeta\, d\boldsymbol{\mu}(\omega) = \int_\Omega \zeta^2\, d\boldsymbol{\mu}(\omega).$$

Using (2.10), we obtain that $\int_\Omega u^2\, d\boldsymbol{\mu}(\omega) = \int_\Omega \zeta^2\, d\boldsymbol{\mu}(\omega)$. Hence, $\nabla u = 0$. Since $\boldsymbol{\mu}$ is ergodic, we have $u = \text{const}$. Therefore, $\zeta = \text{const}$, whence $\zeta = 0$.

§ 3. Constructing a closed Dirichlet form and a self-adjoint operator

Let $a = (a_{ij}(\omega))$ be a symmetric, $\mathcal{F}$-measurable and positive-definite matrix such that

$$\Lambda^{-1}|\xi|^2 \leqslant a_{ij}\xi_i\xi_j \leqslant \Lambda|\xi|^2, \qquad \xi \in \mathbb{R}^n, \quad \Lambda > 0,$$

$\boldsymbol{\mu}$-a.s. In this section we regard μ_ω as a Borel measure on $\mathbb{R}^n$ and construct a self-adjoint operator

$$A = -\operatorname{div} a(T_x\omega)\nabla,$$

that acts on $L^2(\mathbb{R}^n, \mu_\omega)$. We also construct a closed Dirichlet form corresponding to it.

Let $f \in L^2(\mathbb{R}^n, \mu_\omega)$. We say that $u \in H^1(\mathbb{R}^n, \mu_\omega)$ is a solution of the equation

$$Au + u = f$$

if the following integral identity holds:

$$\int_{\mathbb{R}^n} a(T_x\omega)\nabla u(x) \cdot \nabla\varphi(x)\, d\mu_\omega(x) + \int_{\mathbb{R}^n} u(x)\varphi(x)\, d\mu_\omega(x) \\ = \int_{\mathbb{R}^n} f(x)\varphi(x)\, d\mu_\omega(x) \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n). \tag{3.1}$$

By Riesz' theorem, there is precisely one solution $(u, \nabla u)$. The unique gradient in (3.1) is determined by the condition that the flux $a\nabla u$ is tangential, that is, $a\nabla u \perp \Gamma_{\mu_\omega}(0)$. We claim that the set of solutions is dense in $L^2(\mathbb{R}^n, \mu_\omega)$.

Lemma 3.1. *The set of solutions of the equation $Au+u=f$, $f \in L^2(\mathbb{R}^n, \mu_\omega)$, is dense in $L^2(\mathbb{R}^n, \mu_\omega)$.*

Proof. We denote the set of solutions by $\mathcal{D}(A)$ and assume that there is a non-zero function f_0 orthogonal to $\mathcal{D}(A)$. Let v be a solution of equation (3.1) with the right-hand side f_0 and let u be a solution of this equation with an arbitrary right-hand side f. By continuity, the integral identity (3.1) holds for test functions belonging to $H^1(\mathbb{R}^n, \mu_\omega)$. Using v as a test function in the equation for u and vice versa, and subtracting one of the integral identities thus obtained from the other, we obtain that $(f, v) = 0$ for all $f \in L^2(\mathbb{R}^n, \mu_\omega)$. Therefore, $v = 0$ and $\nabla v \in \Gamma_{\mu_\omega}(0)$. Hence, $(a(T_x\omega)\nabla v, \nabla v) = 0$, which contradicts our assumption.

Now for the $u \in \mathcal{D}(A)$ corresponding to the right-hand side f we put $Au = f - u$. The operator $(A+I)^{-1}$ assigns to $f \in L^2(\mathbb{R}^n, \mu_\omega)$ the corresponding solutions of problem (3.1). This operator is bounded, non-negative and symmetric. Therefore, A is a self-adjoint operator.

The solution of the equation $Au + u = f$ (or equation (3.1), which is equivalent to it) provides a minimum in the variational problem

$$\inf_{u \in H^1(\mathbb{R}^n, \mu_\omega)} \left\{ \int_{\mathbb{R}^n} (a(x)\nabla u(x) \cdot \nabla u + u^2(x))\, d\mu(x) - 2\int_{\mathbb{R}^n} f(x)u(x)\, d\mu_\omega(x) \right\}. \tag{3.2}$$

We can define A in a somewhat different way by constructing the corresponding Dirichlet quadratic form. We put

$$\mathcal{E}(u, u) = \min_{\nabla u \in \Gamma_\mu(u)} \int_{\mathbb{R}^n} a(x)\nabla u(x) \cdot \nabla u(x)\, d\mu_\omega(x), \tag{3.3}$$

where $\Gamma_\mu(u)$ is the set of all gradients of u. We claim that this quadratic form is closed. Let $\mathcal{E}_1(u, v) = \mathcal{E}(u, v) + (u, v)$ and let $u_k \in H^1(\mathbb{R}^n, \mu_\omega)$ be a sequence of functions such that $\mathcal{E}_1(u_k-u_m,u_k-u_m)\to 0$ as $k,m\to\infty$. Our purpose is to prove that there is a $u_0\in H^1(\mathbb{R}^n,\mu_\omega)$ such that $\mathcal{E}_1(u_k-u_0,u_k-u_0)\to 0$ as $k\to\infty$. Consider the following scalar product in $\widetilde{H}(\mathbb{R}^n,\mu_\omega)$:

$$((u,\nabla u),(v,\nabla v))_{\widetilde{H}}=\int_{\mathbb{R}^n}\bigl(u(x)v(x)+a(T_x\omega)\nabla u(x)\cdot\nabla v(x)\bigr)\,d\mu_\omega(x).$$

Let $(u,z(u))$ be the projection of $(u,\nabla u)$ to the orthogonal complement of the subspace $(0,\Gamma_\mu(0))$ (with respect to this scalar product). It is easy to verify that

$$\mathcal{E}_1(u,u)=\int_{\mathbb{R}^n}\bigl(u^2(x)+a(T_x\omega)z(u)\cdot z(u)\bigr)\,d\mu_\omega(x).$$

Now the existence of the desired u_0 follows from the fact that $H(\mathbb{R}^n,\mu_\omega)$ is closed in $(L^2(\mathbb{R}^n,\mu_\omega))^{n+1}$.

We can also prove that the form $\mathcal{E}(u,u)$ is closed using the following relaxation theorem:

$$\mathcal{E}(u,u)=\int_{\mathbb{R}^n}\hat{a}\nabla^t u\cdot\nabla^t u\,dx,$$

where $\nabla^t u$ is the tangential gradient of u,

$$\hat{a}(x)\xi\cdot\xi=\min_{\eta\in T^\perp(x)}a(T_x\omega)(\xi+\eta)\cdot(\xi+\eta)$$

and $T^\perp(x)$ is the orthogonal complement of the tangential space of μ_ω at x. This theorem immediately implies that the form is closed.

A detailed discussion of problems concerning the construction of tangential gradient, tangential space and relaxation can be found in § 10.

Now let us verify that this form is Markov.

Lemma 3.2. *$\mathcal{E}(u,u)$ is a closed Markov form.*

Proof. By definition, a form is said to be Markov if for any $\varepsilon>0$ there is a monotonic function $\phi_\varepsilon(t)$, $t\in\mathbb{R}$,

$$\phi_\varepsilon(t)=t\quad\text{for}\quad t\in[0,1],\qquad -\varepsilon\leqslant\phi_\varepsilon(t)\leqslant 1+\varepsilon,\quad \phi'(t)\leqslant 1,$$

such that $\phi_\varepsilon(u)\in\mathcal{D}(\mathcal{E})$ and $\mathcal{E}(\phi_\varepsilon(u),\phi_\varepsilon(u))\leqslant\mathcal{E}(u,u)$ for all $u\in\mathcal{D}(\mathcal{E})$.

For every $\varepsilon>0$ consider the function ϕ_ε such that

$$\phi_\varepsilon'\in C_0^\infty(\mathbb{R}),\quad \phi_\varepsilon'(t)=1\quad\text{if}\quad t\in[0,1],\qquad 0\leqslant\phi_\varepsilon'\leqslant 1,\quad \operatorname{supp}\phi_\varepsilon'\subset(-\varepsilon,1+\varepsilon).$$

The obvious relation $\nabla^t\phi_\varepsilon(u)=\phi_\varepsilon'(u)\nabla^t u$, which holds for all $u\in H^1(\mathbb{R}^n,\mu_\omega)$, and the relaxation theorem stated above imply that $\mathcal{E}(\phi_\varepsilon(u),\phi_\varepsilon(u))\leqslant\mathcal{E}(u,u)$.

Lemma 3.2 implies, in particular, that the maximum principle holds for the semigroup e^{-tA}.

Along with the problem for the functions defined on the whole space, we shall deal with the Dirichlet problem and other boundary-value problems in bounded domains G with Lipschitz boundaries. Let us state one of these problems.

Let S be a part of ∂G. We shall assume that S is a closed subset of ∂G with Lipschitz boundary.

Given $f \in L^2(G, \mu_\omega)$ and $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$, we consider the variational problem

$$\inf_{(\varphi-\varphi_0)\in C_0^\infty(\mathbb{R}^n\setminus S)} \int_G \big(a(T_x\omega)\nabla\varphi\cdot\nabla\varphi + \varphi^2(x) - 2f(x)\varphi(x)\big)\, d\mu_\omega(x). \tag{3.4}$$

The *space* $\widetilde{H}(G, S, \mu_\omega)$ is defined to be the closure of the set of vector-valued functions $\{(\varphi, \nabla\varphi)\colon \varphi \in C_0^\infty(\mathbb{R}^n \setminus S)\}$ in the norm of $(L^2(G, d\mu_\omega))^{n+1}$. We denote the set of their first components by $H^1(G, S, d\mu_\omega)$. In particular, we denote this space by $H_0^1(G, d\mu_\omega)$ if S is empty.

A standard verification shows that problem (3.4) has precisely one minimum on the set $\{v\colon (v - \varphi_0, \nabla v - \nabla\varphi_0) \in \widetilde{H}(G, S, \mu_\omega)\}$, and the function at which this minimum is attained satisfies the equation

$$\int_G a(T_x\omega)\nabla u(x)\cdot\nabla\varphi(x)\, d\mu_\omega(x) + \int_G u(x)\varphi(x)\, d\mu_\omega(x) = \int_G f(x)\varphi(x)\, d\mu_\omega(x) \tag{3.5}$$

for all $\varphi \in H^1(G, S, d\mu_\omega)$.

§ 4. A dual definition of Sobolev spaces and approximation problems

In [19] one can find another (dual) definition of Sobolev spaces, which suits our purposes.

Recall that for $a \in L^2(\mathbb{R}^n, d\mu_\omega)$ and $b \in (L^2(\mathbb{R}^n, d\mu_\omega))^n$ we say that $\operatorname{div} b = a$ in the sense of μ_ω if

$$\int_{\mathbb{R}^n} b\cdot\nabla\varphi\, d\mu_\omega = -\int_{\mathbb{R}} a\varphi\, d\mu_\omega \quad \forall\varphi \in C_0^\infty(\Omega).$$

Let us note that, by continuity, this equality remains valid if the pair $(\varphi, \nabla\varphi)$ is replaced by an arbitrary $(u, v) \in H(\mathbb{R}^n, \mu_\omega)$.

Here is another definition of $H^1(\mathbb{R}^n, \mu_\omega)$.

Definition 4.1. A pair (u, v), $u \in L^2(\mathbb{R}^n, \mu_\omega)$, $v \in (L^2(\mathbb{R}^n, \mu_\omega))^n$, is an element of $H^1(\mathbb{R}^n, \mu_\omega)$ and $v = \nabla u$ if

$$\int_{\mathbb{R}^n} ua\, d\mu_\omega = -\int_{\mathbb{R}^n} v\cdot b\, d\mu_\omega \quad \text{whenever } \operatorname{div} b = a. \tag{4.1}$$

We claim that Definitions 4.1 and 2.1 are equivalent. It is obvious that (4.1) holds for all $(u, v) = (u, \nabla u) \in H(\mathbb{R}^n, \mu_\omega)$. Therefore, all elements of $H(\mathbb{R}^n, \mu_\omega)$ satisfy Definition 4.1. It is easy to verify that the set of functions satisfying Definition 2.1 is a closed subspace of $(L^2(\mathbb{R}^n, \mu_\omega))^{n+1}$. Assume that it is bigger than $H(\mathbb{R}^n, \mu_\omega)$. Then there is a non-zero element (u_0, v_0) for which (4.1) holds and the equality

$$\int_{\mathbb{R}^n} u_0(x)u(x)\, d\mu_\omega(x) = -\int_{\mathbb{R}^n} v_0(x)\cdot\nabla u(x)\, d\mu_\omega(x)$$

holds for all $u \in H^1(\mathbb{R}^n, \mu_\omega)$. This means that $u_0 = \operatorname{div} v_0$ with respect to μ_ω. Taking (u_0, v_0) for the test function in (4.1), we obtain the equality

$$\int_{\mathbb{R}^n} u_0^2 \, d\mu_\omega(x) = -\int_{\mathbb{R}^n} v_0^2 \, d\mu_\omega(x),$$

which contradicts the fact that (u_0, v_0) is a non-trivial function.

We shall need an analogue of Lemma 2.4.

Lemma 4.1. *For any $p \in L^2(\mathbb{R}^n, \mu_\omega)$ there is a $p_\delta \in L^2(\mathbb{R}^n, \mu_\omega^\delta)$ such that*

$$\int_{\mathbb{R}^n} p_\delta \varphi \, d\mu_\omega^\delta = \int_{\mathbb{R}^n} p\varphi^\delta \, d\mu_\omega \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n), \tag{4.2}$$

$$p_\delta \to p \quad in \quad L^2(\mathbb{R}^n, \mu^\delta), \tag{4.3}$$

where φ^δ is the standard smoothing:

$$\varphi^\delta(x) = \delta^{-n} \int_{\mathbb{R}^n} \varphi(x-y) K(\delta^{-1} y) \, dy$$

with the kernel $K(y)$ as in (1.8).

Lemma 4.2. *Let $\operatorname{div} b = a$ in the sense of μ. Then there are families b_δ and a_δ such that $b_\delta \to b$, $a_\delta \to a$ strongly in $L^2(\mathbb{R}^n, \mu^\delta)$, and $\operatorname{div} b_\delta = a_\delta$ in the sense of μ^δ.*

Proof. Let b_δ and a_δ be as in Lemma 4.1. It follows from (4.2) that

$$\int_{\mathbb{R}^n} b_\delta \cdot \nabla\varphi \, d\mu^\delta = \int_{\mathbb{R}^n} b \cdot \nabla(\varphi)^\delta \, d\mu = -\int_{\mathbb{R}^n} a\, \varphi^\delta \, d\mu = -\int_{\mathbb{R}^n} a_\delta \varphi \, d\mu^\delta,$$

as required.

The results obtained above enable us to establish the following assertion.

Theorem 4.1. *Assume that the sequence $(u_\delta, \nabla u_\delta)$ belongs to $H^1(\mathbb{R}^n, d\mu^\delta)$ and we have the weak convergence*

$$u_\delta \rightharpoonup u, \qquad \nabla u_\delta \rightharpoonup v \quad in \quad L^2(\mathbb{R}^n, d\mu^\delta). \tag{4.4}$$

Then $u \in H^1(\mathbb{R}^n, d\mu)$ and $v = \nabla u$.

Proof. Let us use Definition 4.1 of $H^1(\mathbb{R}^n, d\mu)$. If $\operatorname{div} b = a$ in the sense of μ_ω and b_δ, a_δ are as in Lemma 4.2, then we have

$$\int_{\mathbb{R}^n} u_\delta a_\delta \, d\mu_\omega^\delta = -\int_{\mathbb{R}^n} \nabla u_\delta \cdot b_\delta \, d\mu_\omega^\delta.$$

Taking into account the properties of strong convergence, we obtain that

$$\int_{\mathbb{R}^n} ua \, d\mu = -\int_{\mathbb{R}^n} v \cdot b \, d\mu.$$

Hence, $(u, v) \in H^1(\mathbb{R}^n, d\mu)$.

Theorem 4.1 is closely connected with the fact that μ^δ is obtained by smoothing μ. For other methods of approximating μ by absolutely continuous measures, passage to the limit in the variable Sobolev space may be invalid. Passage to the limit in Sobolev spaces was studied in [5], [21].

Consider the problem of convergence of solutions of elliptic equations in $\mathbb{R}^n$. As before, μ is a Radon measure on $\mathbb{R}^n$ and μ^δ is a smoothing. Consider the equations

$$\operatorname{div} A_\delta \nabla u^\delta + u_\delta = f_\delta \quad \text{in} \quad L^2(\mathbb{R}^n, d\mu^\delta), \tag{4.5}$$

$$\operatorname{div} A \nabla u + u = f \quad \text{in} \quad L^2(\mathbb{R}^n, d\mu) \tag{4.6}$$

under the following assumptions:

(i) $\lambda \leqslant A_\delta \leqslant \lambda^{-1}$ μ^δ-a. e., $\lambda > 0$,
(ii) $\lambda \leqslant A \leqslant \lambda^{-1}$ μ-a. e.,
(iii) $A_\delta \to A$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^n, d\mu^\delta)$,
(iv) $f^\delta \to f$ strongly in $L^2(\mathbb{R}^n, d\mu^\delta)$.

Theorem 4.2. *Let $u_\delta(x)$ and $u(x)$ be solutions of problems (4.5) and (4.6). Then*

$$u_\delta \to u \quad \textit{strongly in} \quad L^2(\mathbb{R}^n, d\mu^\delta). \tag{4.7}$$

Proof. Since the sequence $(u_\delta, \nabla u_\delta)$ is bounded in $H^1(\mathbb{R}^n, d\mu^\delta)$, we can assume without loss of generality that $u_\delta \rightharpoonup \tilde{u}$ weakly as $\delta \to 0$. Theorem 4.1 implies that $(\tilde{u}, \nabla\tilde{u}) \in H^1(\mathbb{R}^n, d\mu)$.

We can pass to the limit in the integral identity

$$\int_{\mathbb{R}^n} [A_\delta \nabla u_\delta \cdot \nabla\varphi + u_\delta \varphi]\, d\mu^\delta = \int_{\mathbb{R}^n} f^\delta \varphi\, d\mu^\delta$$

using the obvious relation

$$A_\delta \nabla u_\delta \rightharpoonup A\nabla\tilde{u} \quad \text{in} \quad L^2(\mathbb{R}^n, d\mu^\delta).$$

Therefore, $(\tilde{u}, \nabla\tilde{u})$ is a solution of equation (4.6), whence $(\tilde{u}, \nabla\tilde{u}) = (u, \nabla u)$. The strong convergence (4.7) follows from the convergence of energies

$$\begin{aligned}\lim_{\delta\to 0} \int_{\mathbb{R}^n} [A_\delta \nabla u_\delta \cdot \nabla u_\delta + u_\delta^2]\, d\mu^\delta &= \lim_{\delta\to 0} \int_{\mathbb{R}^n} u_\delta f^\delta\, d\mu^\delta \\ &= \int_{\mathbb{R}^n} fu\, d\mu = \int_{\mathbb{R}^n} [A\nabla u \cdot \nabla u + u^2]\, d\mu\end{aligned}$$

and lower semicontinuity.

§ 5. Two-scale convergence in spaces with random measures

Let us introduce the concept of two-scale convergence associated with the random measures

$$d\mu_\omega^\varepsilon(x) = \varepsilon^n\, d\mu_\omega\biggl(\frac{x}{\varepsilon}\biggr).$$

Let G be a domain in $\mathbb{R}^n$. In particular, G may coincide with $\mathbb{R}^n$. Let $T_x\widetilde{\omega}$ be a typical trajectory. The term "typical" means that formula (1.12) holds for the trajectory $T_x\widetilde{\omega}$ for all $g \in C(\Omega)$. Let us note that trajectories are typical a. s., as follows from the fact that $C(\Omega)$ is a separable space.

Consider a family of functions $v^\varepsilon(x)$ bounded in $L^2(G,\mu^\varepsilon_{\widetilde{\omega}})$, that is,

$$\limsup_{\varepsilon\to 0}\int_G |v^\varepsilon(x)|^2\,d\mu^\varepsilon_{\widetilde{\omega}}(x)<\infty. \tag{5.1}$$

Our purpose is to make the definition of the weak two-scale limit

$$v^\varepsilon(x) \xrightharpoonup{2} v(x,\omega), \tag{5.2}$$

where $v \in L^2(G\times\Omega, dx\times d\boldsymbol{\mu}(\omega))$. Although the original sequence $v^\varepsilon(x)$ is connected with the concrete trajectory $T_x\widetilde{\omega}$, the two-sale limit will be defined on $G\times\Omega$.

Definition 5.1. The relation (5.2) means that (5.1) holds and

$$\lim_{\varepsilon\downarrow 0}\int_G v^\varepsilon(x)\varphi(x)b\bigl(T_{\frac{x}{\varepsilon}}\widetilde{\omega}\bigr)\,d\mu^\varepsilon_{\widetilde{\omega}}(x)=\int_G\int_\Omega v(x,\omega)\varphi(x)b(\omega)\,d\boldsymbol{\mu}(\omega)\,dx \tag{5.3}$$

for all $\varphi(x)\in C_0^\infty(G)$ and $b(\omega)\in C^1(\Omega)$.

Theorem 1.1 implies that

$$\begin{aligned}
\limsup_{\varepsilon\downarrow 0}&\left|\int_G v^\varepsilon(x)\varphi(x)b\bigl(T_{\frac{x}{\varepsilon}}\widetilde{\omega}\bigr)\,d\mu^\varepsilon_{\widetilde{\omega}}(x)\right|\\
&\leqslant \limsup_{\varepsilon\downarrow 0}\|v^\varepsilon\|_{L^2(G,\,\mu^\varepsilon_{\widetilde{\omega}})}\biggl(\int_G\varphi^2(x)b^2\bigl(T_{\frac{x}{\varepsilon}}\widetilde{\omega}\bigr)\,d\mu^\varepsilon_{\widetilde{\omega}}(x)\biggr)^{1/2}\\
&\leqslant C\,C(\widetilde{\omega})\lim_{\varepsilon\downarrow 0}\biggl(\int_G\varphi^2(x)b^2\bigl(T_{\frac{x}{\varepsilon}}\widetilde{\omega}\bigr)\,d\mu^\varepsilon_{\widetilde{\omega}}(x)\biggr)^{1/2}\\
&= C\,C(\widetilde{\omega})\biggl(\int_G\int_\Omega\varphi^2(x)b^2(\omega)\,d\boldsymbol{\mu}(\omega)\,dx\biggr)^{1/2}
\end{aligned}$$

for P-almost all $\widetilde{\omega}$. Using the standard diagonal procedure, we obtain (see [1], [4]) that for the family of function v^ε for which (5.1) holds, there is a subsequence $\varepsilon_k\to 0$ such that the limit on the left-hand side of (5.3) exists and is a linear functional continuous on $L^2(G\times\Omega, dx\times\boldsymbol{\mu})$. Hence, this limit can be represented in the form

$$\int_G\int_\Omega v(x,\omega)\varphi(x)b(\omega)\,d\boldsymbol{\mu}(\omega)\,dx$$

with $v(x,\omega)\in L^2(G\times\Omega,\,dx\times d\boldsymbol{\mu}(\omega))$.

The result thus obtained can be stated as follows.

Lemma 5.1. *Every family of functions v^ε such that (5.1) holds converges along a subsequence to some $v(x,\omega)\in L^2(G\times\Omega, dx\times d\boldsymbol{\mu}(\omega))$ in the sense of weak two-scale convergence.*

Lemma 5.1 holds for an arbitrary sequence of functions bounded in $L^2(G,d\mu_{\widetilde{\omega}}(x))$, but the trajectory $T_x\widetilde{\omega}$ is typical (not arbitrary). As usual, the term "typical" is connected with the validity of the assertion of the ergodic theorem. A point $\omega \in \Omega$ is said to be *typical* if (1.12) holds at this point for all $g(\omega) \in C(\Omega)$. Let $\widetilde{\Omega}$ be the set of typical points. Then $\widetilde{\Omega}$ is invariant, $\mathsf{P}(\widetilde{\Omega}) = 1$ and $\boldsymbol{\mu}(\widetilde{\Omega}) = m$. The last equality follows from Campbell's formula. The following important assertion holds: *every function $g \in L^1(\Omega, \boldsymbol{\mu})$ can be changed on a set of $\boldsymbol{\mu}$-measure zero in such a way as to be defined on $\widetilde{\Omega}$, and (1.12) holds for all $\widetilde{\omega} \in \widetilde{\Omega}$.*

Let us note that for every $g \in L^1(\Omega, \boldsymbol{\mu})$, the relation (1.12) holds for all $\omega \in \Omega_1$, where Ω_1 is an invariant set of full $\boldsymbol{\mu}$-measure. We can assume without loss of generality that $\Omega_1 \subset \widetilde{\Omega}$. We have to define g on $\widetilde{\Omega} \setminus \Omega_1$. Consider a sequence $g^k \in C(\Omega)$, $g^k \to g$ in $L^1(\Omega, \boldsymbol{\mu})$ as $k \to \infty$. Formula (1.12) implies that

$$\lim_{m,k\to\infty} \lim_{t\to\infty} \frac{1}{t|A|} \int_{tA} |g^k(T_x\omega) - g^m(T_x\omega)| \, d\mu_\omega(x) = 0$$

for $\omega \in \widetilde{\Omega} \setminus \Omega_1$. One can find a measurable function $g(x)$, $x \in \mathbb{R}^n$, such that

$$\lim_{k\to\infty} \lim_{t\to\infty} \frac{1}{t|A|} \int_{tA} |g^k(T_x\omega) - g(x)| \, d\mu_\omega(x) = 0,$$

$$\lim_{t\to\infty} \frac{1}{t|A|} \int_{tA} g(x) \, d\mu_\omega(x) = \int_\Omega g(\omega') \, d\boldsymbol{\mu}(\omega')$$

for $\omega \in \widetilde{\Omega} \setminus \Omega_1$. In the case when $\mu_\omega(dx) = dx$, this assertion was proved in [14], Ch. 7, § 5. That proof remains valid in the present case. It remains to define g on $T_x\omega$, $\omega \in \widetilde{\Omega} \setminus \Omega_1$, by the equality $g(T_x\omega) = g(x)$.

In what follows we identify functions belonging to $L^2(\Omega, \boldsymbol{\mu})$ with modifications of them for which (1.12) holds.

Lemma 5.2. *In Definition 5.1, the class of test functions can be extended by requiring that $b(\omega) \in L^2(\Omega, \boldsymbol{\mu})$.*

Proof. Let $b(\omega) \in L^2(\Omega, \boldsymbol{\mu})$ and assume that the sequence of functions $b^k(\omega) \in C^1(\Omega)$ approximates $b(\omega)$ in the norm of $L^2(\Omega, \boldsymbol{\mu})$. Putting $Q = \operatorname{supp} \varphi$, we have

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \left| \int_Q v^\varepsilon(x)\varphi(x)\big(b^k\big(T_{\frac{x}{\varepsilon}}\widetilde{\omega}\big) - b\big(T_{\frac{x}{\varepsilon}}\widetilde{\omega}\big)\big) \, d\mu^\varepsilon_{\widetilde{\omega}}(x) \right| \\
&\qquad \leqslant C_\varphi \limsup_{\varepsilon\to 0} \|v^\varepsilon\|_{L^2(Q,\,\mu^\varepsilon_{\widetilde{\omega}})} \limsup_{\varepsilon\to 0} \left(\int_Q \big(b^k\big(T_{\frac{x}{\varepsilon}}\widetilde{\omega}\big) - b\big(T_{\frac{x}{\varepsilon}}\widetilde{\omega}\big)\big)^2 \, d\mu^\varepsilon_{\widetilde{\omega}}(x) \right)^{1/2} \\
&\qquad \leqslant C\|b^k(\omega) - b(\omega)\|_{L^2(Q,\boldsymbol{\mu})}.
\end{aligned}$$

The last relation follows from the ergodic theorem. The desired assertion can be easily deduced from this inequality.

The next two lemmas deal with other properties of two-scale convergence.

Lemma 5.3. *Let*

$$\|v^\varepsilon\|_{L^2(G,\,\mu^\varepsilon_{\widetilde{\omega}})} \leqslant C(\widetilde{\omega}), \qquad \lim_{\varepsilon\to 0} \varepsilon\|\nabla v^\varepsilon\|_{(L^2(G,\,\mu^\varepsilon_{\widetilde{\omega}}))^n} = 0.$$

Then

$$v^\varepsilon(x) \xrightharpoonup{2} v^0(x) \tag{5.4}$$

along a subsequence, where $v^0(x)$ does not depend on ω.

Proof. Lemma 5.1 implies that $v^\varepsilon \xrightharpoonup{2} v^0(x,\omega)$. Assume that $\psi \in L^2(\Omega,\boldsymbol{\mu})$ and $b \in (L^2(\Omega,\boldsymbol{\mu}))^n$ are such that $\psi(\omega) = \operatorname{div} b(\omega)$, which means that

$$\int_\Omega \nabla^\omega \theta(\omega) b(\omega)\, d\boldsymbol{\mu}(\omega) = \int_\Omega \theta(\omega)\psi(\omega)\, d\boldsymbol{\mu}(\omega)$$

for all $\theta \in H^1(\Omega,\boldsymbol{\mu})$. Theorem 2.2 implies that

$$\int_G v(x)\varphi(x)\psi(T_{\frac{x}{\varepsilon}}\omega)\, d\mu^\varepsilon_\omega(x) = \int_G \varepsilon\nabla\bigl(v(x)\varphi(x)\bigr) b\bigl(T_{\frac{x}{\varepsilon}}\omega\bigr)\, d\mu^\varepsilon_\omega(x) \quad \mathsf{P}\text{-a.s.} \tag{5.5}$$

for $v \in H^1(G,\mu^\varepsilon_\omega)$ and $\varphi \in C_0^\infty(G)$. Substituting $v(x) = v^\varepsilon(x)$ in (5.5) with v^ε as in (5.4) and passing to the two-scale limit on both sides of this equality, we obtain that

$$\int_G \int_\Omega v^0(x,\omega)\psi(\omega)\varphi(x)\, d\boldsymbol{\mu}(\omega)\, dx = 0$$

for all ψ representable in the form $\psi(\omega) = \operatorname{div}_\omega b(\omega)$. Since φ is arbitrary, Lemma 2.5 implies that

$$\int_\Omega v^0(x,\omega)\psi(\omega)\, d\boldsymbol{\mu}(\omega) = 0$$

for all $\psi(\omega)$ with the zero mean value. Hence, $v^0 = v^0(x)$.

We associate with $\boldsymbol{\mu}$ the quadratic form

$$\tau(\xi) = \min_{v \in L^2_{\mathrm{pot}}(\Omega,\mu)} \int_\Omega |\xi + v|^2\, d\boldsymbol{\mu}(\omega), \quad \xi \in \mathbb{R}^n.$$

We say that $\boldsymbol{\mu}$ is *non-degenerate* if this form is non-degenerate: $\tau(\xi) \geqslant c_0|\xi|^2$, $c_0 > 0$. In the case of degeneracy the kernel of τ consists of those $\xi \in \mathbb{R}^n$ that are potential vectors $\xi \in L^2_{\mathrm{pot}}(\Omega,\mu)$. The form $\tau(\xi)$ is a special case of the averaged matrix.

Proposition 5.1. *The set of mean values $\int_\Omega v\, d\boldsymbol{\mu}(\omega)$, where $v \in L^2_{\mathrm{sol}}(\Omega,\boldsymbol{\mu})$, coincides with the orthogonal complement of the kernel of τ.*

This assertion is a special case of results on averaged matrices that will be obtained in § 6 below.

Lemma 5.4. *Let*

$$\|v^\varepsilon\|_{L^2(G,\,\mu^\varepsilon_{\widetilde\omega})} \leqslant C(\widetilde\omega), \qquad \|\nabla v^\varepsilon\|_{(L^2(G,\,\mu^\varepsilon_{\widetilde\omega}))^n} \leqslant C(\widetilde\omega).$$

Then

$$v^\varepsilon(x) \xrightharpoonup{2} v^0(x), \tag{5.6}$$

$$\nabla v^\varepsilon(x) \xrightharpoonup{2} \nabla^{\mathrm{eff}} v^0(x) + v_1(x,u), \tag{5.7}$$

$$\nabla^{\mathrm{eff}} v^0 \in L^2(\mathbb{R}^n), \qquad v_1 \in L^2(G; L^2_{\mathrm{pot}}(\Omega,\boldsymbol{\mu})), \tag{5.8}$$

along a subsequence, where $\nabla^{\mathrm{eff}} = Q\nabla$ and Q is the orthogonal projector to the orthogonal complement of the kernel of τ. In particular, $v^0 \in H^1(G)$ if $\boldsymbol{\mu}$ is non-degenerate.

Proof. We denote the two-scale limit of ∇v^ε by $w(x,\omega)$. Let $b(\omega)\in L^2_{\rm sol}(\Omega,\boldsymbol{\mu})$ and $\langle b\rangle\equiv\int_\Omega b(\omega)\,d\boldsymbol{\mu}(\omega)=\eta\neq 0$. For every $\varphi\in C_0^\infty(G)$ we have

$$\int_G \nabla v^\varepsilon(x)\cdot b\big(T_{\frac{x}{\varepsilon}}\omega\big)\varphi(x)\,d\mu^\varepsilon_\omega(x)=\int_G \nabla(v^\varepsilon(x)\varphi(x))\cdot b\big(T_{\frac{x}{\varepsilon}}\omega\big)\,d\mu^\varepsilon_\omega(x)$$
$$-\int_G v^\varepsilon(x)\nabla\varphi(x)\cdot b\big(T_{\frac{x}{\varepsilon}}\omega\big)\,d\mu^\varepsilon_\omega(x)=-\int_G v^\varepsilon(x)\nabla\varphi(x)\cdot b\big(T_{\frac{x}{\varepsilon}}\omega\big)\,d\mu^\varepsilon_\omega(x).$$

Passing to the two-scale limit, we obtain that

$$\int_\Omega\int_G w(x,\omega)\cdot b(\omega)\varphi(x)\,dx\,d\boldsymbol{\mu}(\omega)=-\int_\Omega\int_G v^0(x)\nabla\varphi(x)\cdot b(\omega)\,dx\,d\boldsymbol{\mu}(\omega)$$
$$=\eta\cdot\int_G v^0(x)\nabla\varphi(x)\,dx.$$

Therefore,

$$\nabla v^0(x)\cdot\eta=\int_\Omega w(x,\omega)\cdot b(\omega)\,d\boldsymbol{\mu}(\omega)\in L^2(G),$$

whence $\nabla^{\rm eff}v^0\in L^2(G)$. It remains to observe that

$$-\int_\Omega\int_G v^0\nabla\varphi(x)\cdot b(\omega)\,dx\,d\boldsymbol{\mu}(\omega)=-\int_\Omega\int_G v^0\nabla^{\rm eff}\varphi(x)\cdot b(\omega)\,dx\,d\boldsymbol{\mu}(\omega)$$
$$=\int_\Omega\int_G \nabla^{\rm eff}v^0\cdot b(\omega)\varphi(x)\,dx\,d\boldsymbol{\mu}(\omega)$$

for all $b\in L^2_{\rm sol}(\Omega,\boldsymbol{\mu})$, whence

$$\int_\Omega\int_G (w(x,\omega)-\nabla^{\rm eff}v^0)\cdot b(\omega)\varphi(x)\,dx\,d\boldsymbol{\mu}(\omega)=0,$$

and $(w(x,\omega)-\nabla^{\rm eff}v^0)\in L^2(G;L^2_{\rm pot}(\Omega,\boldsymbol{\mu}))$.

Corollary 5.1. *Let G be a Lipschitz domain and assume that the sequence of functions $v^\varepsilon\in H^1_0(G,\,\mu^\varepsilon_{\widetilde\omega})$ is such that*

$$\|v^\varepsilon\|_{L^2(G,\,\mu^\varepsilon_{\widetilde\omega})}\leqslant C(\widetilde\omega),\qquad \|\nabla v^\varepsilon\|_{(L^2(G,\,\mu^\varepsilon_{\widetilde\omega}))^n}\leqslant C(\widetilde\omega).\tag{5.9}$$

Then we have the following convergence along a subsequence:

$$v^\varepsilon\xrightharpoonup{2}v^0(x)\in H^{\rm eff}_0(G),\tag{5.10}$$
$$\nabla v^\varepsilon\xrightharpoonup{2}\nabla^{\rm eff}v^0(x)+v_1(x,\omega),\tag{5.11}$$

where $H^{\rm eff}_0(G)$ is the closure of $C_0^\infty(G)$ in the norm of $H^{\rm eff}(G)$ (see formula (6.13) below), $\nabla^{\rm eff}v^0\in(L^2(G))^n$ and $v_1(x,\omega)\in L^2(G;L^2_{\rm pot}(\Omega,\boldsymbol{\mu}))$. In particular, $v^0\in H^1_0(G)$ if μ_ω is non-degenerate.

If the estimates (5.9) hold for $v^\varepsilon\in H^1(G,S,\mu^\varepsilon_{\widetilde\omega})$ and μ_ω is non-degenerate, then we have the following convergence along a subsequence:

$$v^\varepsilon\xrightharpoonup{2}v^0(x)\in H^1(G,S),\tag{5.12}$$
$$\nabla v^\varepsilon\xrightharpoonup{2}\nabla v^0(x)+v_1(x,\omega),\tag{5.13}$$

where $v_1(x,\omega)\in L^2(G;L^2_{\mathrm{pot}}(\Omega,\boldsymbol{\mu}))$ *and* $H^1(G,S)$ *is the closure of* $C_0^\infty(\mathbb{R}^n\setminus S)$ *in the norm of* $H^1(G)$.

Proof. We extend the v^ε by zero to the exterior of G, retaining the same notation for the extensions. Since the assumptions of Lemma 5.4 hold for the extended functions, we have $v^0\in L^2(\mathbb{R}^n)$ and $\nabla^{\mathrm{eff}}v^0\in L^2(\mathbb{R}^n)$. It is obvious that $v^0=0$ in the exterior of G. By [17], Ch. X, § 2, Lemma 2.6, we have $v^0\in H_0^{\mathrm{eff}}(G)$, which completes the proof of the first assertion. The second assertion can be proved likewise.

The following lemma also deals with an important property of two-scale convergence.

Lemma 5.5. *Let*

$$\|v^\varepsilon\|_{L^2(G,\,\mu^\varepsilon_{\widetilde\omega})}\leqslant C(\widetilde\omega),\qquad \varepsilon\|\nabla v^\varepsilon\|_{(L^2(G,\,\mu^\varepsilon_{\widetilde\omega}))^n}\leqslant C(\widetilde\omega).$$

Then we have the following convergence along a subsequence:

$$v^\varepsilon \xrightarrow{2s} v(x,\omega), \tag{5.14}$$

$$\varepsilon\nabla v^\varepsilon \xrightarrow{2s} \nabla_\omega v(x,\omega). \tag{5.15}$$

Proof. For functions of the form $\psi(\omega)=\operatorname{div}_\omega b(\omega)$, Theorem 2.2 implies that $\psi\big(T_{\frac{x}{\varepsilon}}\omega\big)=\varepsilon\operatorname{div} b\big(T_{\frac{x}{\varepsilon}}\omega\big)$ with respect to μ^ε_ω, whence

$$\int_G v^\varepsilon(x)\varphi(x)\psi(T_{\frac{x}{\varepsilon}}\omega)\,d\mu^\varepsilon_\omega=\int_G \varepsilon\nabla v^\varepsilon(x)\varphi(x)b\big(T_{\frac{x}{\varepsilon}}\omega\big)\,d\mu^\varepsilon_\omega$$

for all $\varphi\in C_0^\infty(G)$. Passing to the two-scale limit and denoting the limit of $\varepsilon\nabla v^\varepsilon$ by $p(x,\omega)$, we obtain that

$$\int_{\mathbb{R}^n}\int_\Omega v(x,\omega)\varphi(x)\psi(\omega)\,d\boldsymbol{\mu}(\omega)\,dx=\int_{\mathbb{R}^n}\int_\Omega p(x,\omega)\varphi(x)b(\omega)\,d\boldsymbol{\mu}(\omega)\,dx.$$

Therefore,

$$\int_\Omega v(x,\omega)\psi(\omega)\,d\boldsymbol{\mu}(\omega)=\int_\Omega p(x,\omega)b(\omega)\,d\boldsymbol{\mu}(\omega),$$

whence $p(x,\omega)=\nabla_\omega v(x,\omega)$ for almost all x. Here we used the equivalence of the following two definitions of $\widetilde H(\Omega,\boldsymbol{\mu})$:

(i) $\widetilde H(\Omega,\boldsymbol{\mu})$ is the closure of the set $\{(v(\omega),\nabla_\omega v(\omega))\colon v\in C^1(\Omega)\}$ in the norm of $(L^2(\Omega,\boldsymbol{\mu}))^{n+1}$,

(ii) $(v(\omega),z(\omega))\in\widetilde H(\Omega,\boldsymbol{\mu})$ if the equality

$$\int_\Omega v(\omega)\psi(\omega)\,d\boldsymbol{\mu}(\omega)=\int_\Omega z(\omega)b(\omega)\,d\boldsymbol{\mu}(\omega) \tag{5.16}$$

holds for $\psi(\omega) = \operatorname{div}_\omega b(\omega)$. To prove that these definitions are equivalent, note that (5.16) obviously holds for every element of $\widetilde{H}(\Omega, \boldsymbol{\mu})$ in the sense of the first definition. Assume that the second definition gives a bigger space. Then there is a $(v_0(\omega), z_0(\omega))$ such that (5.16) holds and

$$\int_\Omega v_0(\omega)v(\omega)\, d\boldsymbol{\mu}(\omega) + \int_\Omega z_0(\omega)\nabla_\omega v(\omega)\, d\boldsymbol{\mu}(\omega) = 0.$$

By Definition 2.3, this means that $v_0 = \operatorname{div}_\omega z_0$. Substituting the test function (v_0, z_0) in (5.16), we obtain that

$$\int_\Omega v_0^2(\omega)\, d\boldsymbol{\mu}(\omega) = -\int_\Omega z_0^2(\omega)\, d\boldsymbol{\mu}(\omega),$$

which contradicts the fact that (v_0, z_0) is non-trivial.

Now let us define strong two-scale convergence in the variable space $L^2(\mathbb{R}^n, \mu_{\widetilde{\omega}}^\varepsilon)$.

Definition 5.2. We say that we have *strong two-scale convergence* of the family of functions $\{v^\varepsilon\}$, $v^\varepsilon \in L^2(G, \mu_{\widetilde{\omega}}^\varepsilon)$, to $v^0(x,\omega) \in L^2(G \times \Omega, dx\, d\boldsymbol{\mu}(\omega))$ as $\varepsilon \to 0$ if we have two-scale convergence of v^ε to v^0 and

$$\lim_{\varepsilon \to 0} \int_G v^\varepsilon(x)u^\varepsilon(x)\, d\mu_{\widetilde{\omega}}(x) = \int_G \int_\Omega v^0(x,\omega)u^0(x,\omega)\, dx\, d\boldsymbol{\mu}(\omega) \tag{5.17}$$

for every family u^ε for which we have two-scale convergence to u^0 in $L^2(G, \mu_{\widetilde{\omega}}^\varepsilon)$.

Strong two-scale convergence will be denoted by $v^\varepsilon \overset{2s}{\to} v^0$. An important consequence of strong two-scale convergence is the convergence of the norms

$$\lim_{\varepsilon \to 0} \int_G (v^\varepsilon(x))^2\, d\mu_{\widetilde{\omega}}(x) = \int_G \int_\Omega (v^0(x,\omega))^2\, dx\, d\boldsymbol{\mu}(\omega). \tag{5.18}$$

It is easy to verify that this relation can be used instead of (5.17) in the definition of strong two-scale convergence.

We can now extend the class of test functions used in (5.3). Indeed, if we have two-scale convergence of u^ε to u^0 in $L^2(G, \mu_{\widetilde{\omega}}(x))$, then (5.17) holds if we have the strong two-scale convergence of v^ε.

§ 6. Homogenization

We shall now state the asymptotic problems studied in this paper. To the random measure μ_ω we assign the family of measures

$$d\mu_\omega^\varepsilon(x) = \varepsilon^n\, d\mu_\omega\left(\frac{x}{\varepsilon}\right), \qquad \varepsilon > 0, \tag{6.1}$$

and consider the equation $A^\varepsilon u^\varepsilon + u^\varepsilon = f^\varepsilon$, $f^\varepsilon \in L^2(\mathbb{R}^n, \mu_\omega^\varepsilon)$, whose integral identity has the form

$$\int_{\mathbb{R}^n} a\big(T_{\frac{x}{\varepsilon}}\omega\big)\nabla u^\varepsilon(x)\nabla\varphi(x)\,d\mu_\omega^\varepsilon(x) + \int_{\mathbb{R}^n} u^\varepsilon(x)\varphi(x)\,d\mu_\omega^\varepsilon(x) = \int_{\mathbb{R}^n} f^\varepsilon(x)\varphi(x)\,d\mu_\omega^\varepsilon(x). \tag{6.2}$$

The function u^ε is a solution of the variational problem

$$\min\Big\{\int_{\mathbb{R}^n} a\big(T_{\frac{x}{\varepsilon}}\omega\big)\nabla u\cdot\nabla u\,d\mu_\omega^\varepsilon(x) + \int_{\mathbb{R}^n} u^2\,d\mu_\omega^\varepsilon(x) - 2\int_{\mathbb{R}^n} f^\varepsilon u\,d\mu_\omega^\varepsilon(x)\Big\}, \tag{6.3}$$

where the minimum is taken over all $u \in H^1(\mathbb{R}^n, \mu_\omega^\varepsilon)$. Instead of this minimum, we can consider the infimum over the set of all smooth finitary functions. We have already shown that problem (6.2) is well posed and has a. s. precisely one solution for every positive ε.

Since μ_ω^ε converges weakly to the Lebesgue measure, the effective equation is an ordinary equation.

Let us recall the definitions of strongly and weakly converging sequences of functions in variable spaces.

Definition 6.1. We say that a family v^ε, $\|v^\varepsilon\|_{L^2(\mathbb{R}^n,\mu^\varepsilon)} \leqslant C$, *converges weakly* in $L^2(\mathbb{R}^n, \mu^\varepsilon)$ to $v^0 \in L^2(\mathbb{R}^n, dx)$ if

$$\int_{\mathbb{R}^n} v^\varepsilon(x)\varphi(x)\,d\mu^\varepsilon(x) \underset{\varepsilon\downarrow 0}{\longrightarrow} \int_{\mathbb{R}^n} v^0(x)\varphi(x)\,dx \tag{6.4}$$

for all $\varphi \in C_0(\mathbb{R}^n)$. The family v^ε, $\|v^\varepsilon\|_{L^2(\mathbb{R}^n,\mu^\varepsilon)} \leqslant C$, *converges strongly* in $L^2(\mathbb{R}^n, \mu^\varepsilon)$ to $v^0 \in L^2(\mathbb{R}^n, dx)$ if

$$\int_{\mathbb{R}^n} v^\varepsilon(x)p^\varepsilon(x)\,d\mu^\varepsilon(x) \underset{\varepsilon\downarrow 0}{\longrightarrow} \int_{\mathbb{R}^n} v^0(x)p^0(x)\,dx \tag{6.5}$$

for every sequence p^ε converging weakly in $L^2(\mathbb{R}^n, \mu^\varepsilon)$ to p^0.

Strong and weak convergence in a bounded domain can be defined likewise.

Assuming that the f^ε in (6.2) converge strongly (weakly) to some $f \in L^2(\mathbb{R}^n)$, we shall study the asymptotic behaviour of u^ε as $\varepsilon \downarrow 0$. In other words, we shall homogenize equation (6.2).

In many cases it is sufficient to assume that $f \in C_0(\mathbb{R}^n)$ and does not depend on ε.

We formally define the averaged matrix a^{eff} by the equality

$$a^{\text{eff}}\xi\cdot\xi = \inf \int_\Omega a(\omega)(\nabla^{\boldsymbol{\mu}}\varphi(\omega)+\xi)\cdot(\nabla^{\boldsymbol{\mu}}\varphi(\omega)+\xi)\,d\boldsymbol{\mu}(\omega), \tag{6.6}$$

where $\xi\in\mathbb{R}^n$ is an arbitrary constant vector and the infimum is taken over all $\varphi\in C^1(\Omega)$. Here is an equivalent definition:

$$a^{\text{eff}}\xi\cdot\xi = \min \int_\Omega a(\omega)(v(\omega)+\xi)\cdot(v(\omega)+\xi)\,d\boldsymbol{\mu}(\omega), \qquad v\in L^2_{\text{pot}}(\Omega,\boldsymbol{\mu}). \tag{6.7}$$

The function $v^\xi(\omega)$ at which the minimum is attained is a solution of the problem

$$v^\xi\in L^2_{\text{pot}}(\Omega,\boldsymbol{\mu}), \qquad a(\omega)(\xi+v^\xi(\omega))\in L^2_{\text{sol}}(\Omega,\boldsymbol{\mu}). \tag{6.8}$$

This solution depends linearly on ξ. Now a^{eff} can be defined by the equality

$$a^{\text{eff}}\xi = \int_\Omega a(\omega)(\xi+v^\xi(\omega))\,d\boldsymbol{\mu}(\omega). \tag{6.9}$$

It is easy to verify that a^{eff} is a symmetric matrix.

The following matrix can be useful:

$$w(\omega) = (v_i^{e_j}(\omega)), \qquad i,j=1,\dots,n, \tag{6.10}$$

where the e_j comprise the standard basis of $\mathbb{R}^n$. By the ergodic theorem, the measures μ^ε_ω converge weakly P-a. s. to $m\,dx$ as $\varepsilon\downarrow 0$ on every compact subset of $\mathbb{R}^n$. In what follows we assume without loss of generality that $m=1$. Hence, μ^ε converges to the standard Lebesgue measure.

Now consider the formally homogenized problem

$$-\operatorname{div} a^{\text{eff}}\nabla u^0 + u^0 = f, \tag{6.11}$$

where $f\in L^2(\mathbb{R}^n)$. This equation can be written in the integral form

$$\int_{\mathbb{R}^n} a^{\text{eff}}\nabla u^0(x)\nabla\varphi(x)\,dx + \int_{\mathbb{R}^n} u^0(x)\varphi(x)\,dx = \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx \tag{6.12}$$

for all $\varphi\in C_0^\infty$. Since the matrix a^{eff} is non-negative, this problem has precisely one solution in $L^2(\mathbb{R}^n)$. If a^{eff} is positive definite, then we have an elliptic problem whose solution belongs to $H^1(\mathbb{R}^n)$. In the case when a^{eff} is partially degenerate, we can only assert that $a^{\text{eff}}\nabla u^0$ belongs to $L^2(\mathbb{R}^n)$, which implies that u^0 belongs to the function space

$$H^{\text{eff}} = \{u\in L^2(\mathbb{R}^n)\colon a^{\text{eff}}\nabla u\in (L^2(\mathbb{R}^n))^n\} \tag{6.13}$$

with the norm $\|u\|^2_{H^{\text{eff}}} = \|u\|^2_{L^2(\mathbb{R}^n)} + \|a^{\text{eff}}\nabla u\|^2_{L^2(\mathbb{R}^n)}$. It is obvious that neither H^{eff} nor the kernel of a^{eff} depend on the choice of the positive-definite matrix $a(\omega)$ in (6.2) (they depend only on μ_ω).

Let us note that μ_ω is non-degenerate if and only if a^{eff} is positive definite. Moreover, the following assertion holds.

Lemma 6.1. *The kernel of* $a^{\rm eff}$ *coincides with the set of constant potential vectors in* $L^2_{\rm pot}(\Omega,\boldsymbol{\mu})$*. In particular,* μ_ω *is non-degenerate if and only if there are no non-trivial constant functions in* $L^2_{\rm pot}(\Omega,\boldsymbol{\mu})$*. A vector* η *belongs to the orthogonal complement of the kernel of* $a^{\rm eff}$ *if and only if there is a* $v\in L^2_{\rm sol}(\Omega,\boldsymbol{\mu})$ *such that* $\int_\Omega v(\omega)\,d\boldsymbol{\mu}(\omega)=\eta$.

Proof. The first assertion follows immediately from formula (6.7). It is clear that the mean value of every divergence-free vector is orthogonal to every constant potential vector. On the other hand, the set of vectors of the form $a^{\rm eff}\xi$, $\xi\in\mathbb{R}^n$, coincides with the orthogonal complement of the kernel of $a^{\rm eff}$ since $a^{\rm eff}$ is symmetric. By (6.8) and (6.9), every $a^{\rm eff}\xi$ is the mean value of the divergence-free vector $a(\omega)(\xi+v^\xi)$. The lemma is proved.

Note that the integral identity (6.12) holds for all $\varphi\in H^{\rm eff}$.

We now state a theorem on homogenization in the whole space.

Theorem 6.1. *Assume that the* f^ε *in problem* (6.2) (*or problem* (6.3), *which is equivalent*) *converges strongly* (*weakly*) *in* $L^2(\mathbb{R}^n,\mu^\varepsilon_\omega)$ *to* $f\in L^2(\mathbb{R}^n)$ *as* $\varepsilon\to 0$*. Then the solutions* u^ε *of problem* (6.2) *converge strongly* (*weakly*) *in* $L^2(\mathbb{R}^n,\mu^\varepsilon_\omega)$ *to the solution* u^0 *of problem* (6.11) P-*a. s.*

The proof will be based on the technique of stochastic two-scale convergence developed in § 5.

Note that in the case when the solution of the limit problem is a continuous function that decreases sufficiently rapidly at infinity, the convergence in the assertion of Theorem 6.1 can be expressed in the usual way:

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}(u^\varepsilon(x)-u^0(x))^2\,d\mu^\varepsilon_\omega(x)=0.$$

This condition holds, for example, if the right-hand side of (6.2) is continuous, does not depend on ε and admits the estimate $f(x)\leqslant g(|x|)$, where $g(s)$ is a function non-negative and monotonic on $[0,\infty)$ such that $g(|x|)\in L^2(\mathbb{R}^n)$.

Similar results hold for boundary value problems. For example, in the Dirichlet problem

$$-\operatorname{div} a\bigl(T_{\frac{x}{\varepsilon}}\omega\bigr)\nabla u^\varepsilon(x)+u^\varepsilon(x)=f^\varepsilon(x),\qquad (u^\varepsilon-\varphi_0)\in H^1_0(G,d\mu_\omega),\tag{6.14}$$

the difference between u^0 and φ_0 belongs to $H^{\rm eff}_0(G)$, which is the closure of $C_0^\infty(\mathbb{R}^n)$ in the norm

$$\|\varphi\|^2=\int_G(\varphi^2(x)+a^{\rm eff}\nabla\varphi(x)\cdot\nabla\varphi(x))\,dx.$$

The integral identity

$$\int_G a^{\rm eff}\nabla u^0\cdot\nabla\varphi\,dx+\int_G u^0\varphi\,dx=\int_G f\varphi\,dx\tag{6.15}$$

holds for all $\varphi\in H^{\rm eff}_0(G)$. It is easy to verify that the solution of this problem is unique.

Theorem 6.2. *Assume that the f^ε in problem (6.14) converges strongly (weakly) in $L^2(G,\mu^\varepsilon_\omega)$ to $f\in L^2(G)$ as $\varepsilon\to 0$. Then the solution u^ε of problem (3.5) converges strongly (weakly) in $L^2(G,\mu^\varepsilon_\omega)$ to the solution u^0 of problem (6.15) P-a. s.*

Consider the mixed boundary-value problem. Let G be a domain with piecewise-smooth boundary and let S be a closed subset of ∂G with Lipschitz boundary. Recall that u^ε is called a solution of the mixed boundary-value problem

$$\begin{gathered} -\operatorname{div} a\bigl(T_{\frac{x}{\varepsilon}}\omega\bigr)\nabla u^\varepsilon(x)+u^\varepsilon(x)=f^\varepsilon(x),\\ u^\varepsilon|_S=\varphi_0,\qquad \frac{\partial}{\partial\nu_a}u^\varepsilon|_{(\partial G\setminus S)}=0, \end{gathered} \tag{6.16}$$

if $(u^\varepsilon-\varphi_0)\in H^1(G,S,\mu^\varepsilon_\omega)$ and

$$\int_{\mathbb{R}^n} a\bigl(T_{\frac{x}{\varepsilon}}\omega\bigr)\nabla u^\varepsilon(x)\cdot\nabla\varphi(x)\,d\mu^\varepsilon_\omega(x)+\int_{\mathbb{R}^n} u^\varepsilon(x)\varphi(x)\,d\mu^\varepsilon_\omega(x)=\int_{\mathbb{R}^n} f^\varepsilon(x)\varphi(x)\,d\mu^\varepsilon_\omega(x) \tag{6.17}$$

for all $\varphi\in H^1(G,S,\mu^\varepsilon_\omega)$. When dealing with the mixed boundary-value problem, we shall assume for the sake of simplicity that μ_ω is non-degenerate. Then $u^0-\varphi_0$ belongs to $H(G,S)$, which is the closure of $C_0^\infty(\mathbb{R}^n\setminus S)$ in the norm of $H^1(G)$. The integral identity (6.15) holds for u^0 for all $\varphi\in H(G,S)$.

Theorem 6.3. *Assume that μ_ω is non-degenerate and that the f^ε converge strongly (weakly) in $L^2(G,\mu^\varepsilon_\omega)$ to $f\in L^2(G)$ as $\varepsilon\to 0$. Then the solution u^ε of problem (6.16) converges strongly (weakly) P-a. s. in $L^2(G,\mu^\varepsilon_\omega)$ to a u^0 such that $(u^0-\varphi_0)\in H(G,S)$ and (6.15) holds for all $\varphi\in H(G,S)$.*

In all the homogenization problems mentioned above, we have convergence of fluxes and energies.

Theorem 6.4. *Let the assumptions of Theorem 6.1 hold. Then the following relations hold P-a. s.:*

$$a\bigl(T_{\frac{x}{\varepsilon}}\omega\bigr)\nabla u^\varepsilon \rightharpoonup a^{\mathrm{eff}}\nabla u^0 \quad \textit{weakly in}\quad L^2(\mathbb{R}^n,\mu^\varepsilon_\omega) \tag{6.18}$$

and, if f^ε converges strongly to f,

$$\int_{\mathbb{R}^n} a\bigl(T_{\frac{x}{\varepsilon}}\omega\bigr)\nabla u^\varepsilon\cdot\nabla u^\varepsilon\,d\mu^\varepsilon_\omega(x)\to\int_{\mathbb{R}^n} a^{\mathrm{eff}}\nabla u^0\cdot\nabla u^0\,dx. \tag{6.19}$$

In a similar way, the following relations hold for the Dirichlet problem and the mixed boundary-value problem:

$$a\bigl(T_{\frac{x}{\varepsilon}}\omega\bigr)\nabla u^\varepsilon \rightharpoonup a^{\mathrm{eff}}\nabla u^0 \quad \textit{weakly in}\quad L^2(G,\mu^\varepsilon_\omega) \tag{6.20}$$

and, in the case when the right-hand sides converge strongly,

$$\int_G a\bigl(T_{\frac{x}{\varepsilon}}\omega\bigr)\nabla u^\varepsilon\cdot\nabla u^\varepsilon\,d\mu^\varepsilon_\omega(x)\to\int_G a^{\mathrm{eff}}\nabla u^0\cdot\nabla u^0\,dx. \tag{6.21}$$

Proof of Theorem 6.1. The elementary a priori estimate

$$\|u^\varepsilon\|_{L^2(\mathbb{R}^n,\mu^\varepsilon)} + \|\nabla u^\varepsilon\|_{L^2(\mathbb{R}^n,\mu^\varepsilon)} \leqslant C$$

and Lemma 5.4 imply that

$$\begin{gathered} u^\varepsilon \xrightharpoonup{2} u^0(x), \\ \nabla u^\varepsilon \xrightharpoonup{2} \nabla^{\mathrm{eff}} u^0(x) + u_1(x,\omega), \qquad u_1 \in L^2\big(\mathbb{R}^n; L^2_{\mathrm{pot}}(\Omega,\boldsymbol{\mu})\big), \\ a\big(T_{\frac{x}{\varepsilon}}\big)\nabla u^\varepsilon \xrightharpoonup{2} a(\omega)\big(\nabla^{\mathrm{eff}} u^0 + u_1\big) \end{gathered} \tag{6.22}$$

for all typical realizations. Let us substitute the test function $\varepsilon\varphi(x)v\big(T_{\frac{x}{\varepsilon}}\omega\big)$ with $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $v(\omega) \in C^1(\Omega)$ in equation (6.2):

$$\begin{gathered} \int_{\mathbb{R}^n} a\big(T_{\frac{x}{\varepsilon}}\omega\big)\nabla u^\varepsilon \varphi(x) \cdot \nabla_\omega v\big(T_{\frac{x}{\varepsilon}}\omega\big)\, d\mu^\varepsilon + \varepsilon \int_{\mathbb{R}^n} a\big(T_{\frac{x}{\varepsilon}}\omega\big)\nabla u^\varepsilon \nabla\varphi(x) v\big(T_{\frac{x}{\varepsilon}}\omega\big)\, d\mu^\varepsilon \\ + \varepsilon \int_{\mathbb{R}^n} u^\varepsilon \varphi(x) v\big(T_{\frac{x}{\varepsilon}}\omega\big)\, d\mu^\varepsilon = \varepsilon \int_{\mathbb{R}^n} f^\varepsilon(x)\varphi(x)v\big(T_{\frac{x}{\varepsilon}}\omega\big)\, d\mu^\varepsilon. \end{gathered}$$

Passing to the two-scale limit, we obtain that

$$\int_{\mathbb{R}^n}\int_\Omega a(\omega)(\nabla u^0(x) + u_1(x,\omega)) \cdot \nabla_\omega v(\omega)\varphi(x)\, d\boldsymbol{\mu}(\omega)\, dx = 0.$$

Therefore,

$$\int_\Omega a(\omega)(\nabla u^0(x) + u_1(x,\omega))\nabla_\omega v(\omega)\, d\boldsymbol{\mu}(\omega) = 0$$

for almost all x. Hence, $u_1(x,\omega)$ is a solution of the auxiliary problem (6.8) for $\xi = \nabla^{\mathrm{eff}} u^0(x)$, that is, $u_1(x,\omega) = w(\omega)\nabla^{\mathrm{eff}} u^0(x)$. Hence,

$$\int_\Omega a(\omega)(\nabla u^0(x) + u_1(x,\omega))\mathsf{P}(d\omega) = a^{\mathrm{eff}}\nabla u^0(x).$$

Combining this equality with (6.22), we obtain that

$$a\big(T_{\frac{x}{\varepsilon}}\omega\big)\nabla u_\varepsilon \rightharpoonup a^{\mathrm{eff}}\nabla u^0 \quad \text{in} \quad L^2\big(\mathbb{R}^n, d\mu_\omega^\varepsilon\big)$$

(convergence of fluxes). Passing to the limit in (6.2), we obtain that $u^0(x)$ satisfies the equation

$$\int_{\mathbb{R}^n} a^{\mathrm{eff}}\nabla u^0(x) \cdot \nabla\varphi(x)\, dx + \int_{\mathbb{R}^n} u^0(x)\varphi(x)\, dx = \int_{\mathbb{R}^n} f(x)\varphi(x)\, dx. \tag{6.23}$$

This equation has precisely one solution, and the solutions u^ε of equation (6.2) (or of the variational problem (6.3)) converge weakly with respect to μ^ε to the unique solution $u^0(x)$ of problem (6.23), which is a determinate function.

We claim that u^ε converges strongly to u^0 in $L^2(\mathbb{R}^n,\mu^\varepsilon)$ if f^ε converges strongly to f. Our arguments are based on the weak convergence proved above. Consider the following auxiliary problem:

$$-\operatorname{div} a\big(T_{\frac{x}{\varepsilon}}\omega\big)\nabla v^\varepsilon(x) + v^\varepsilon(x) = u^\varepsilon(x), \qquad v^\varepsilon \in H^1\big(\mathbb{R}^n,\mu_\omega^\varepsilon\big). \tag{6.24}$$

The solution v^ε of this problem converges weakly to the solution of the homogenized problem

$$-\operatorname{div} a^{\mathrm{eff}} \nabla v(x) + v(x) = u^0(x), \qquad v \in H^{\mathrm{eff}}(\mathbb{R}^n).$$

Using v^ε as a test function in the original equation, using u^ε as a test function in equation (6.24) and subtracting one of the integral identities thus obtained from the other, we obtain the equality

$$\int_{\mathbb{R}^n} (u^\varepsilon(x))^2 \, d\mu_\omega^\varepsilon(x) = \int_{\mathbb{R}^n} v^\varepsilon(x) f(x) \, d\mu_\omega^\varepsilon(x).$$

Considering the limit functions, we likewise obtain that

$$\int_{\mathbb{R}^n} (u^0(x))^2 \, dx = \int_{\mathbb{R}^n} v(x) f(x) \, dx.$$

Hence,

$$\lim_{\varepsilon\to 0} \|u^\varepsilon\|^2_{L^2(\mathbb{R}^n,\mu_\omega^\varepsilon)} = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} v^\varepsilon(x) f(x) \, d\mu_\omega^\varepsilon(x) = \int_{\mathbb{R}^n} v(x) f(x) \, dx = \int_{\mathbb{R}^n} (u^0(x))^2 \, dx,$$

which completes the proof of strong convergence.

The assertions of Theorem 6.3 concerning the weak and strong convergence of solutions of boundary-value problems can be proved likewise.

Proof of Theorem 6.4. First we shall prove the theorem for the problem stated in the whole space. Since the u^ε converge to u^0 and the $\nabla_{\mu_\omega^\varepsilon} u^\varepsilon$ are bounded in $L^2(\mathbb{R}^n, \mu_\omega^\varepsilon)$, Lemma 5.4 implies that we have the weak two-scale convergence

$$a(T_{\frac{x}{\varepsilon}}\omega) \nabla_{\mu_\omega^\varepsilon} u^\varepsilon \overset{2}{\rightharpoonup} a(\omega)\big(\nabla^{\mathrm{eff}} u^0 + v_1(x,\omega)\big), \qquad \varepsilon \to 0,$$

where $v_1(x,\omega) \in L^2(\mathbb{R}^n; L^2_{\mathrm{pot}}(\Omega, \boldsymbol{\mu}))$. In particular, we have the weak convergence

$$a\big(T_{\frac{x}{\varepsilon}}\omega\big) \nabla_{\mu_\omega^\varepsilon} u^\varepsilon \rightharpoonup \int_\Omega a(\omega)\big(\nabla^{\mathrm{eff}} u^0 + v_1(x,\omega)\big) \, d\boldsymbol{\mu}(\omega)$$

in $L^2(\mathbb{R}^n, \mu_\omega^\varepsilon)$. It was shown in the proof of Theorem 6.1 that $v_1(x,\omega)$ has the form $v_1(x,\omega) = w(\omega)\nabla u^0(x)$, where $w(\omega)$ is the solution of the auxiliary problem (6.8), (6.10). Therefore,

$$\int_\Omega a(\omega)\big(\nabla^{\mathrm{eff}} u^0 + v_1(x,\omega)\big) Q(d\omega) = a^{\mathrm{eff}} \nabla u^0,$$

as was to be shown.

The convergence of fluxes in boundary-value problems (with homogeneous or non-homogeneous boundary conditions) can be verified likewise.

Let us prove the convergence of energies. Since the pair $(u^\varepsilon, \nabla u^\varepsilon)$ belongs to $H^1(\mathbb{R}^n, \mu_\omega^\varepsilon)$, it can be used as a test function in (6.2), which implies that

$$\int_{\mathbb{R}^n} a\big(T_{\frac{x}{\varepsilon}}\omega\big) \nabla u^\varepsilon \cdot \nabla u^\varepsilon \, d\mu_\omega^\varepsilon(x) + \int_{\mathbb{R}^n} |u^\varepsilon|^2 \, d\mu_\omega^\varepsilon(x) = \int_{\mathbb{R}^n} f^\varepsilon u^\varepsilon \, d\mu_\omega^\varepsilon(x).$$

Passing to the limit in this equality and using the strong convergence of u^ε to u^0 in $L^2(\mathbb{R}^n,\mu_\omega^\varepsilon)$, we obtain that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} a\bigl(T_{\frac{x}{\varepsilon}}\omega\bigr)\nabla u^\varepsilon\cdot\nabla u^\varepsilon\,d\mu_\omega^\varepsilon(x)=-\int_{\mathbb{R}^n}|u^0|^2\,dx+\int_{\mathbb{R}^n}fu^0\,dx$$

P-a. s. On the other hand, using $u^0(x)$ as a test function in (6.12), we obtain the equality

$$\int_{\mathbb{R}^n} a^{\mathrm{eff}}\nabla u^0\cdot\nabla u^0\,dx=-\int_{\mathbb{R}^n}|u^0|^2\,dx+\int_{\mathbb{R}^n}fu^0\,dx.$$

The last two relations imply that the energies converge in the case when the equation is defined on the whole space.

For boundary-value problems, the strong convergence of solutions and the convergence of energies can be verified in the same way as was done for problems in the whole space.

To conclude this section, we state a theorem on the (two-scale) convergence of arbitrary solutions which can be useful in the study of properties of the solutions of boundary-value problems.

Theorem 6.5. *Let the functions $u^\varepsilon(x)$ be solutions of the equation*

$$-\operatorname{div} a\bigl(T_{\frac{x}{\varepsilon}}\omega\bigr)\nabla u^\varepsilon+u^\varepsilon=f^\varepsilon(x)$$

in the domain $G\subset\mathbb{R}^n$ in the sense of the integral identity

$$\int_G\bigl(a\bigl(T_{\frac{x}{\varepsilon}}\omega\bigr)\nabla u^\varepsilon\cdot\nabla\varphi(x)+u^\varepsilon\varphi\bigr)\,d\mu_\omega^\varepsilon(x)=\int_G f^\varepsilon(x)\varphi(x)\,d\mu_\omega^\varepsilon(x)\quad\forall\varphi\in C_0^\infty(G).$$

Further, let $f^\varepsilon\to f$ strongly in $L^2(G,\mu_\omega^\varepsilon)$ and let $\|u^\varepsilon\|_{L^2(G,\,\mu_\omega^\varepsilon)}\leqslant C$. Then for every typical realization, every two-scale limit u^0 of u^ε depends only on the slow variable x and satisfies the equation

$$\operatorname{div}\bigl(a^{\mathrm{eff}}\nabla u^0\bigr)+u^0=f.$$

We have the following convergence of gradients and fluxes:

$$\nabla u^\varepsilon\xrightharpoonup{2}\nabla^{\mathrm{eff}}u^0+w(\omega)\nabla^{\mathrm{eff}}u^0,$$
$$a\bigl(T_{\frac{x}{\varepsilon}}\omega\bigr)\nabla u^\varepsilon\xrightharpoonup{2}a(\omega)\nabla^{\mathrm{eff}}u^0+a(\omega)w(\omega)\nabla^{\mathrm{eff}}u^0(x),$$

where $w(\omega)$ is the matrix defined in (6.8), (6.10).

Theorem 6.5 can be proved by the methods used in the proofs of the preceding assertions in this section.

§ 7. Kesten's problem

Let us apply the technique developed above to the study of effective diffusion on an infinite cluster in the two-dimensional percolation model. Consider the standard percolation model in $\mathbb{R}^2$, in which to every node of $\mathbb{Z}^2$ the values 1 and 0 are assigned with probabilities p and $1-p$, respectively, and the corresponding random variables are independent. Neighbouring nodes to which the value 1 is assigned are joined by edges. It is well known (see, for example, [18]) that for $p > p_c \sim 0.59$, the set of edges thus obtained has a. s. precisely one unbounded connected component, which is called an *infinite cluster*.

We shall study an important example of a singular random measure defined as follows. The support of this measure coincides with the union of the edges belonging to the infinite cluster. On each of these edges the random measure μ_ω is defined to be the standard one-dimensional Lebesgue measure $\mathcal{H}^1(\gamma_k)$, where γ_k is the corresponding edge. This definition implies that μ_ω is a random locally finite Borel measure of finite intensity stationary with respect to integer shifts. The construction of the probability space and of the random ergodic dynamical system in this example are well known (see [18]). According to [18], the density of the infinite cluster is positive for all $p > p_c$. Therefore, the intensity of μ_ω is positive if $p > p_c$.

Lemma 7.1. *The measure μ_ω is ergodic.*

Proof. Consider an arbitrary $u(\omega) \in H^1(\Omega, \boldsymbol{\mu})$ such that $\nabla u(\omega) = 0$. By Lemma 2.2, we have $u \in H^1(\mathbb{R}^2, \mu_\omega)$ P-a. s. and $\nabla_{\mu_\omega} u(T_x\omega) = 0$. According to [3] and [12] the function $u(T_x\omega)$ belongs P-a. s. to $\mathcal{H}^1(\gamma_k)$ on every segment of the infinite cluster and has a continuous modification. It is easy to verify that the tangential gradient of this function coincides on every segment with the derivative along the segment and vanishes under our assumptions. Therefore, $u(T_x\omega)$ is equal to a constant on the segments of the infinite cluster P-a. s. Since the infinite cluster is connected, this function is equal to a constant on the whole cluster P-a. s. Combining this with the ergodicity of T_x and using Campbell's formula, we obtain that $u(\omega) = \text{const}$ $\boldsymbol{\mu}$-a. s.

A property of the channels in the corresponding percolation model (see [18]) implies that μ_ω is non-degenerate if $p > p_c$. This property, in a form adapted for our purposes, can be described as follows.

For an arbitrary sequence $x_0, x_1, \dots, x_k$ of elements of $\mathbb{Z}^2$ such that $|x_{i+1} - x_i| = 1$ for all $i = 1, 2, \dots, k$, a *path* is defined to be a sequence of edges $\gamma_0, \gamma_1, \dots, \gamma_k$, $\gamma_i = [x_i, x_{i+1}]$.

Theorem 7.1. *Let $p > p_c$. Then there are constants $c(p) > 0$ and $c_1(p) > 0$ such that P-a. s. for all sufficiently large N, there are at least $c(p)N$ disjoint paths (channels) in the square $[0, N]^2$ joining its upper and lower sides and such that the edges contained in these paths belong to the infinite cluster. Moreover, one can assume that the length of each channel does not exceed $c_1(p)N$.*

Let us verify that μ_ω is non-degenerate. First we observe that the averaged matrix is isotropic, since the original model is stochastically invariant under rotation through angles that are multiples of $\pi/2$. Therefore, $a^{\text{eff}} = \kappa^{\text{eff}} I$, where I is the identity matrix, and it is sufficient to verify that $\kappa^{\text{eff}} > 0$.

Consider the following Dirichlet problem with non-homogeneous boundary conditions in $G=[0,1]^2$:

$$\begin{aligned} -\operatorname{div} \nabla_{\mu_\omega^\varepsilon} u^\varepsilon + u^\varepsilon &= 0, \\ (u^\varepsilon - x_1) &\in H_0^1(G, \mu_\omega^\varepsilon). \end{aligned} \tag{7.1}$$

If $\kappa^{\mathrm{eff}} = 0$, then the energy of the limit problem vanishes, and the theorem of convergence of energies for the Dirichlet problem implies that

$$\lim_{\varepsilon\to 0} \int_G |\nabla u^\varepsilon|^2 \, d\mu_\omega^\varepsilon(x) = 0.$$

Therefore, it is sufficient to verify that $\int_G |\nabla u^\varepsilon|^2 \, d\mu_\omega^\varepsilon(x) > C > 0$ a. s. with respect to P. By Theorem 7.1, for P-almost all ω and all sufficiently small $\varepsilon > 0$, one can find in G at least $c(p)/\varepsilon$ disjoint channels that join the upper and lower sides of the square, and the length of every polygonal arc that forms a channel is at most $c_1(p)$. On each of these arcs we take the natural parametrization defined by the arc-length. The restriction of u^ε to each of these polygonal arcs is an H^1-function of the coordinate thus defined. The tangential gradient $\nabla^t u^\varepsilon$ is parallel to the corresponding segments of this polygonal arc and coincides with the derivative of u^ε along it. Denoting the number of edges in the jth channel by N_j and the above coordinate by t, we obtain that

$$1 = \left(\int_0^{\varepsilon N_j} \frac{du^\varepsilon}{dt} \, dt \right)^2 \leqslant \int_0^{\varepsilon N_j} dt \int_0^{\varepsilon N_j} \left| \frac{du^\varepsilon}{dt} \right|^2 dt.$$

Hence,

$$\int_0^{\varepsilon N_j} \left| \frac{du^\varepsilon}{dt} \right|^2 dt \geqslant \frac{1}{c_1(p)} \,.$$

Adding these inequalities for all channels, we obtain that

$$\sum_j \int_0^{\varepsilon N_j} \left| \frac{du^\varepsilon}{dt} \right|^2 dt \geqslant \frac{c(p)}{\varepsilon c_1(p)} \,.$$

The definition of μ^ε implies that $d\mu^\varepsilon(x) = m\varepsilon \, dt$ on every channel, where m is the density of the infinite cluster. Therefore,

$$\int_G |\nabla u^\varepsilon|^2 \, d\mu^\varepsilon(x) \geqslant \sum_j \int_0^{\varepsilon N_j} \left| \frac{du^\varepsilon}{dt} \right|^2 d\mu^\varepsilon(x) = m\varepsilon \sum_j \int_0^{\varepsilon N_j} \left| \frac{du^\varepsilon}{dt} \right|^2 dt \geqslant \frac{mc(p)}{c_1(p)} \,.$$

This implies that the energy is uniformly positive. Hence, the effective matrix is positive definite.

Now consider Kesten's problem as stated. Assume that $p > p_c$ in the percolation model considered above. Let S be the union of the upper and lower sides of the square $G = [0,1]^2$ and consider the following problem in G:

$$\begin{gathered} -\operatorname{div} \nabla_{\mu_\omega^\varepsilon} u^\varepsilon = 0, \\ (u^\varepsilon - \psi) \in H^1(G, S, \mu_\omega^\varepsilon), \qquad \frac{\partial}{\partial x_1} u^\varepsilon \Big|_{\partial G\setminus S} = 0, \end{gathered} \tag{7.2}$$

where ψ is an arbitrary C_0^∞-function (in particular, we can assume that ψ coincides with a linear function in the neighbourhood of G). Problem (7.2) can have more than one solution since the intersection of the infinite cluster with the square $\frac{1}{\varepsilon}G$ is, generally speaking, disconnected. Consider a family of solutions u^ε, $\varepsilon > 0$, such that

$$\|u^\varepsilon\|_{L^\infty(G)} \leqslant C. \tag{7.3}$$

It is easy to verify that such solutions exist by adding a small positive potential to the right-hand side of (7.2). For fixed $\varepsilon > 0$ and arbitrary $\lambda > 0$, the maximum principle holds for the solution of the problem

$$\begin{gathered} -\operatorname{div} \nabla_{\mu_\omega^\varepsilon} u_\lambda^\varepsilon + \lambda u_\lambda^\varepsilon = 0, \\ (u_\lambda^\varepsilon - \psi) \in H^1(G, S, \mu_\omega^\varepsilon), \qquad \frac{\partial}{\partial x_1} u_\lambda^\varepsilon \Big|_{\partial G \setminus S} = 0, \end{gathered}$$

whence

$$\|u_\lambda^\varepsilon\|_{L^\infty(G)} \leqslant \|\psi\|_{L^\infty}.$$

Passing to the limit as $\lambda \to 0$ in the integral identity corresponding to this problem, we establish that every weak limit point of the family u_λ^ε is a solution of problem (7.2). Note that the tangential gradient of the solution of problem (7.2) is defined unambiguously.

Now let us write equation (7.2) in the form

$$-\operatorname{div} \nabla_{\mu_\omega^\varepsilon} u^\varepsilon + u^\varepsilon = u^\varepsilon.$$

Under the condition (7.3), we can pass to the weak limit along a subsequence on the right-hand side of this equation. By Theorem 6.3, the limit function satisfies the equation

$$\begin{gathered} \kappa^{\text{eff}} \Delta u^0 = 0, \\ (u^0 - \psi) \in H^1(G, S), \qquad \frac{\partial}{\partial x_1} u^0 \big|_{\partial G \setminus S} = 0, \end{gathered}$$

whose solution is unique since κ^{eff} is positive. Hence, every family of solutions such that condition (7.3) holds converges weakly in $L^2(G, \mu_\omega^\varepsilon)$ to the solution of the last problem. By the theorem on the convergence of fluxes, we have

$$\nabla_{\mu_\omega^\varepsilon} u^\varepsilon \rightharpoonup \kappa^{\text{eff}} \nabla u^0$$

weakly in $L^2(G, \mu_\omega^\varepsilon)$. To prove the convergence of energies, we use $(u^\varepsilon - \psi)$ as a test function in the integral identity corresponding to problem (7.2) and pass to the limit as $\varepsilon \to 0$. This yields

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_G |\nabla_{\mu_\omega^\varepsilon} u^\varepsilon|^2 \, d\mu_\omega^\varepsilon(x) &= \lim_{\varepsilon \to 0} \int_G \nabla_{\mu_\omega^\varepsilon} u^\varepsilon \cdot \nabla \psi \, d\mu_\omega^\varepsilon(x) \\ &= \int_G \kappa^{\text{eff}} \nabla u^0 \cdot \nabla \psi \, dx = \kappa^{\text{eff}} \int_G |\nabla u^0|^2 \, dx. \end{aligned}$$

Here we have used the convergence of fluxes and the integral identity corresponding to the limit problem.

Thus we obtain the following method of determining the effective conductance of the infinite cluster. We define $\kappa(N)$ to be the minimal energy in the variational problems

$$\kappa(N)=\inf_{\substack{u\in C^\infty([0,N]^2)\\ u(0,x_2)=0,\ u(N,x_2)=N}}\frac{1}{N^2}\int_{[0,N]^2}|\nabla u(x)|^2\,d\mu_\omega(x).$$

We have proved that

$$\lim_{N\to\infty}\kappa(N)=\kappa^{\mathrm{eff}}$$

P-a. s.

Remark 7.1. The results obtained in this paper for two-dimensional percolation models remain valid in higher dimensions (bear in mind that the property of channels was proved in all dimensions). We also note that Kesten's problem for the continuous percolation model was considered in [6].

§ 8. An example

In this section we consider in more detail the example of random measures absolutely continuous with respect to the Lebesgue measure. Let $d\mu_\omega(x)=\rho(T_x\omega)\,dx$, where $\rho(\omega)>0,\ \ \rho\in L^1(\Omega,\mathsf{P})$. In this case $d\boldsymbol{\mu}(\omega)=\rho(\omega)\,d\mathsf{P}(\omega)$.

Assume that $\frac{1}{\rho}\in L^1(\Omega,\mathsf{P})$. We claim that μ_ω is ergodic and non-degenerate.

Let $\nabla^{\boldsymbol{\mu}}_\omega u(\omega)=0$. Consider a sequence of $C^1(\Omega)$-functions $u_m(\omega)$ such that

$$\int_\Omega|u_m(\omega)-u(\omega)|^2\,d\boldsymbol{\mu}(\omega)\longrightarrow 0,\qquad \int_\Omega|\nabla u_m(\omega)|^2\,d\boldsymbol{\mu}(\omega)\longrightarrow 0$$

as $m\to\infty$. For $k>0$ we put

$$u_m^k(\omega)=\begin{cases}u_m(\omega) & \text{if}\quad -k\leqslant u_m(\omega)\leqslant k,\\ k & \text{if}\quad u_m(\omega)\geqslant k,\\ -k & \text{if}\quad u_m(\omega)\leqslant -k,\end{cases}$$

and

$$u^k(\omega)=\begin{cases}u(\omega) & \text{if}\quad -k\leqslant u(\omega)\leqslant k,\\ k & \text{if}\quad u(\omega)\geqslant k,\\ -k & \text{if}\quad u(\omega)\leqslant -k.\end{cases}$$

We have

$$\int_\Omega\bigl|u_m^k(\omega)-u^k(\omega)\bigr|^2\,d\boldsymbol{\mu}(\omega)\leqslant\int_\Omega|u_m(\omega)-u(\omega)|^2\,d\boldsymbol{\mu}(\omega)\longrightarrow 0$$

as $m\to\infty$. Hence,

$$\lim_{m\to\infty}\mathsf{E}\int_G\bigl|u_m^k(T_x\omega)-u^k(T_x\omega)\bigr|^2\rho(T_x\omega)\,dx=0$$

for all bounded Borel sets G. Therefore, the relation

$$\lim_{m\to\infty}\int_G\bigl|u_m^k(T_x\omega)-u^k(T_x\omega)\bigr|^2\rho(T_x\omega)\,dx=0$$

holds along a subsequence P-a.s. Since the functions $u_m^k(T_x\omega)$ are uniformly bounded, we have

$$\lim_{m\to\infty}\int_G \big|u_m^k(T_x\omega)-u^k(T_x\omega)\big|^2\,dx=0$$

P-a.s. Passing once again to a subsequence, we obtain that

$$\lim_{m\to\infty}\int_G |\nabla u_m(T_x\omega)|^2\rho(T_x\omega)\,dx=0.$$

Therefore,

$$\begin{aligned}\int_G \big|\nabla u_m^k(T_x\omega)\big|\,dx &\leqslant \int_G |\nabla u_m(T_x\omega)|\,dx\\ &\leqslant \biggl(\int_G |\nabla u_m(T_x\omega)|^2\rho(T_x\omega)\,dx\biggr)^{1/2}\biggl(\int_G \rho^{-1}(T_x\omega)\,dx\biggr)^{1/2}\longrightarrow 0\end{aligned}$$

as $m\to\infty$. Hence, $u^k(T_x\omega)$ is constant P-a.s. on G. Since G is an arbitrary set, $u^k(T_x\omega)$ is constant P-a.s. on $\mathbb{R}^n$. Since T_x is ergodic, this constant does not depend on $\omega\in\Omega$. Hence, $u(\omega)$ is equal to a constant P-a.s. and $\boldsymbol{\mu}$-a.s., which proves that $d\mu_\omega(x)$ is ergodic.

To show that $d\boldsymbol{\mu}(\omega)=\rho(\omega)\,d\mathsf{P}(\omega)$ is non-degenerate, we assume that there is a sequence of functions $u_m\in C^1(\Omega)$ such that

$$\lim_{m\to\infty}\int_\Omega |\nabla_\omega u_m(\omega)-\xi|^2\rho(\omega)\,d\mathsf{P}(\omega)=0 \tag{8.1}$$

for some $\xi\in\mathbb{R}^n\setminus\{0\}$. Then

$$\begin{aligned}&\int_\Omega |\nabla_\omega u_m(\omega)-\xi|d\mathsf{P}(\omega)\\ &\qquad\leqslant \biggl(\int_\Omega |\nabla_\omega u_m(\omega)-\xi|^2\rho(\omega)\,d\mathsf{P}(\omega)\biggr)^{1/2}\biggl(\int_\Omega \rho^{-1}(\omega)\,d\mathsf{P}(\omega)\biggr)^{1/2}\longrightarrow 0\end{aligned}$$

as $m\to\infty$. On the other hand, $\int_\Omega \nabla_\omega u_m(\omega)\,d\mathsf{P}(\omega)=0$, whence

$$|\xi|=\biggl|\int_\Omega \bigl(\nabla_\omega u_m(\omega)-\xi\bigr)\,d\mathsf{P}(\omega)\biggr|\leqslant\int_\Omega |\nabla_\omega u_m(\omega)-\xi|\,d\mathsf{P}(\omega).$$

Hence, (8.1) cannot hold and $\boldsymbol{\mu}$ is non-degenerate.

Now consider the more general case of an absolutely continuous measure when the density $\rho(\omega)\in L^1(\Omega)$ can vanish on a set of positive measure.

Assume that there is a measurable set $\mathcal{U}\subset\Omega$ such that

(i) $\rho=0$ in $\mathcal{U}$,

(ii) $\mathbf{1}_{\Omega\setminus\mathcal{U}}(\omega)\rho^{-1}(\omega)\in L^1(\Omega)$, where $\mathbf{1}_{\Omega\setminus\mathcal{U}}$ is the characteristic function of the set $\Omega\setminus\mathcal{U}$,

(iii) the set $Q(\omega)=\{x\in\mathbb{R}^n\colon T_x\omega\in\Omega\setminus\mathcal{U}\}$ is open and connected P-a.s.

Then the measure $\boldsymbol{\mu}$ is connected. In the special case when $\rho(\omega)=\mathbf{1}_{\Omega\setminus\mathcal{U}}$, we are dealing with homogenization in a perforated domain. The proof of connectedness is similar to the proof of ergodicity given at the beginning of this section.

Note that, generally speaking, we do not assert that μ_ω is non-degenerate.

§ 9. Parabolic problems. The central limit theorem

In this section we consider the homogenization problem for parabolic equations in variable stochastically homogeneous spaces and prove the central limit theorem in mean.

We study the asymptotic behaviour of solutions of the Cauchy problem

$$\begin{gathered}\partial_t u^\varepsilon(x,t) = \operatorname{div}_{\mu_\omega} a\bigl(T_{\frac{x}{\varepsilon}}\omega\bigr)\nabla u^\varepsilon(x,t),\\ (x,t)\in\mathbb{R}^n\times(0,\infty), \qquad u^\varepsilon(x,0,\omega)=f^\varepsilon(x,\omega)\end{gathered} \tag{9.1}$$

as $\varepsilon\to 0$. We assume that the functions f^ε belong P-a. s. to the spaces $L^2(\mathbb{R}^n,\mu_\omega^\varepsilon)$. In particular, f can be a determinate continuous function that does not depend on ε and decays sufficiently rapidly at infinity.

Theorem 9.1. *Assume that f^ε a. s. converges strongly as $\varepsilon\to 0$ in the variable space $L^2(\mathbb{R}^n,\mu_\omega^\varepsilon)$ to a function $f(x)\in L^2(\mathbb{R}^n)$. Then the solution of problem (9.1) converges strongly in $L^2(\mathbb{R}^n,d\mu_\omega^\varepsilon)$ a. s. for every $t\geqslant 0$ to the solution u^0 of the homogenized problem*

$$\begin{gathered}\partial_t u^0(x,t)=\operatorname{div}(a^{\mathrm{eff}}\nabla u^0(x,t)),\\ (x,t)\in\mathbb{R}^n\times(0,\infty), \qquad u^0(x,0)=f(x),\end{gathered} \tag{9.2}$$

where a^{eff} is defined by formula (6.6) and coincides with the averaged matrix of the elliptic problem.

We can deduce the desired assertions from similar assertions on the homogenization of elliptic operators using a theorem of Trotter–Kato type [20] on the convergence of semigroups in variable spaces. Before stating this theorem we recall a definition. Let μ^h and μ be Radon measures on $\mathbb{R}^n$ and let $\mu^h\rightharpoonup\mu$. If B_h and B are operators on $L^2(\mathbb{R}^n,d\mu^h)$ and $L^2(\mathbb{R}^n,d\mu)$, respectively, and $\|B_h\|,\|B\|\leqslant M_0$, then we say that $B_h\to B$ if

$$B_h f_h\to Bf \quad \text{in} \quad L^2(\mathbb{R}^n,d\mu^h)$$

whenever $f_h\to f$ in $L^2(\mathbb{R}^n,d\mu^h)$.

Theorem 9.2 [20], [24]. *Let A and A^h be non-negative self-adjoint operators in $L^2(\mathbb{R}^n,d\mu)$ and $L^2(\mathbb{R}^n,d\mu^h)$, respectively, and assume that $(A^h+\lambda)^{-1}\to(A+\lambda)^{-1}$ strongly for all $\lambda>0$. Then $e^{-A^h t}\to e^{-At}$ strongly for all $t\geqslant 0$, and we have weak convergence*

$$(A^h+\lambda)^{-1}f_h\rightharpoonup(A+\lambda)^{-1}f \quad \textit{in} \quad L^2(\mathbb{R}^n,d\mu^h)$$

whenever $f_h\rightharpoonup f$ in $L^2(\mathbb{R}^n,d\mu^h)$.

We shall need another theorem on passage to the limit in parabolic equations. This theorem will be applied to equation (9.1) with a fixed ε. Let us state it for $\varepsilon=1$.

Consider the two Cauchy problems

$$\begin{gathered}\frac{\partial u}{\partial t}=\operatorname{div}_{\mu_\omega} a(T_x\omega)\nabla u, \qquad \frac{\partial u^\delta}{\partial t}=\operatorname{div}_{\mu_\omega^\delta} a^\delta(T_x\omega)\nabla u^\delta,\\ u|_{t=0}=u^\delta|_{t=0}=\varphi\in C_0^\infty,\end{gathered}$$

where the μ_ω^δ are smoothed measures and

$$a^\delta(T_x\omega) \to a(T_x\omega) \quad \text{in} \quad L^2(\mathbb{R}^n, d\mu_\omega^\delta) \quad \text{strongly}$$

as $\delta \to 0$.

Theorem 9.3.

$$u^\delta(x,t,\omega) \to u(x,t,\omega) \quad \textit{in} \quad L^2(\mathbb{R}^n, d\mu_\omega^\delta) \quad \textit{strongly}$$

P-*a. s. as* $\delta \to 0$.

Proof. This theorem follows immediately from Theorems 4.2 and 9.2.

Now let us proceed to the central limit theorem. Consider the equation

$$\begin{gathered} \partial_t u^\varepsilon(x,t,\omega) = \operatorname{div}_{\mu_\omega} a\big(T_{\frac{x}{\varepsilon}}\omega\big)\nabla u^\varepsilon(x,t) \quad \text{in} \quad \mathbb{R}^n \times (0,+\infty), \\ u^\varepsilon(x,0) = \phi^\varepsilon(x) = \varepsilon^{-n}\phi\left(\frac{x}{\varepsilon}\right), \end{gathered} \tag{9.3}$$

where $\phi(y)$ is a non-negative C_0^∞-function such that $\int_{\mathbb{R}^n}\phi(y)dy = 1$. With this initial condition it is sufficient to solve problem (9.3) for $\varepsilon = 1$ and put $u^\varepsilon(x,t,\omega) = \varepsilon^{-n}u^1\big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega\big)$.

By the maximum principle, $u^\varepsilon(x,t,\omega)$ is non-negative. We claim that u^ε converges for all $t > 0$ to the fundamental solution $G(x,t)$ of the homogenized equation in the following sense (as $\varepsilon \to 0$):

$$\lim_{\varepsilon\to 0} \mathsf{E}\bigg(\int_{\mathbb{R}^n}\big\{u^\varepsilon(x,t,\omega) - \gamma(\omega)G(x,t)\big\}\varphi(x)\,d\mu_\omega(x)\bigg)^2 = 0 \tag{9.4}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, where

$$\gamma(\omega) = \int_{\mathbb{R}^n}\phi(x)\,d\mu_\omega(x).$$

Definition 9.1. We say that the CLT holds *in mean* if (9.4) holds for all $\varphi \in C_0^\infty(\mathbb{R}^n)$.

We shall assume only the following hypothesis.

(H1) The ellipticity condition $\rho^\delta(T_x\omega) > 0$ holds for the *smoothed measures* $\mu_\omega^\delta = \rho^\delta(T_x\omega)\,d\mathsf{P}$ *for all* $\delta > 0$.

Note that for a stationary random measure of positive intensity, this hypothesis imposes no restrictions on the measure since in formula (1.9) (which defines the smoothed measure μ_ω^δ) the kernel $K(y)$, instead of being finitary, can decay fairly rapidly (say, exponentially).

Theorem 9.4. *Let* (H1) *hold. Then so does* (9.4).

Proof. Consider the auxiliary problem

$$\begin{gathered} \partial_s v^\varepsilon(x,s) + \operatorname{div}_{\mu_\omega} a(T_{\frac{x}{\varepsilon}}\omega)\nabla v^\varepsilon(x,s) = 0 \quad \text{in} \quad \mathbb{R}^n \times (-\infty,t), \\ v^\varepsilon(x,t) = \varphi(x) \end{gathered} \tag{9.5}$$

with reversed time. Let φ coincide with the test function ϕ in (9.3). It is obvious that $v^\varepsilon(s) = e^{(t-s)A^\varepsilon}\varphi$, where e^{-tA^ε} is the semigroup generated by A^ε. Since A^ε is self-adjoint, we have

$$\begin{aligned}\int_{\mathbb{R}^n} u^\varepsilon(x,t)\varphi(x)\,d\mu_\omega^\varepsilon(x) &= \int_{\mathbb{R}^n} e^{-tA^\varepsilon}\phi^\varepsilon(x)\varphi(x)\,d\mu_\omega^\varepsilon(x)\\ &= \int_{\mathbb{R}^n} e^{-tA^\varepsilon}\varphi(x)\phi^\varepsilon(x)\,d\mu_\omega^\varepsilon(x) = \int_{\mathbb{R}^n} v^\varepsilon(x,0)\phi^\varepsilon(x)\,d\mu_\omega^\varepsilon(x).\end{aligned}$$

We can write this relation in the equivalent form

$$\int_{\mathbb{R}^n} u^\varepsilon(x,t)\varphi(x)\,d\mu_\omega^\varepsilon(x) = \int_{\mathbb{R}^n} V^\varepsilon(x,t)\phi^\varepsilon(x)\,d\mu_\omega^\varepsilon(x),$$

where V^ε is the solution of the problem

$$\begin{gathered}\partial_t V^\varepsilon(x,t) = \operatorname{div}_{\mu_\omega} a(T_{\frac{x}{\varepsilon}}\omega)\nabla V^\varepsilon(x,t) \quad \text{in} \quad \mathbb{R}^n\times(0,\infty),\\ V^\varepsilon(x,0) = \varphi(x).\end{gathered} \tag{9.6}$$

By Theorem 9.1, V^ε converges for all $t \geqslant 0$ to the solution V^0 of the problem

$$\begin{gathered}\partial_t V^0(x,t) = \operatorname{div}\bigl(a^{\text{eff}}\nabla V^0(x,t)\bigr),\\ (x,t)\in\mathbb{R}^n\times(0,\infty), \qquad V^0(x,0) = \varphi(x).\end{gathered}$$

We denote by $V^\varepsilon(x,y,t,\omega)$ the solution of the problem

$$\begin{gathered}\partial_t V^\varepsilon(x,y,t,\omega) = \operatorname{div}_{\mu_\omega} a(T_{\frac{x}{\varepsilon}}\omega)\nabla V^\varepsilon(x,y,t,\omega) \quad \text{in} \quad \mathbb{R}^n\times(0,\infty),\\ V^\varepsilon(x,y,0,\omega) = \varphi(x-y),\end{gathered} \tag{9.7}$$

where the differentiation is with respect to x and the parameter $y\in\mathbb{R}^n$ determines the shift of the argument in the initial condition. First we assume that the solution $V^\varepsilon(x,y,t)$ is continuous in x. Properties of μ_ω^ε and a^ε imply that

$$V^\varepsilon(x,0,t,\omega) = V^\varepsilon\bigl(x-y,-y,t,T_{\frac{y}{\varepsilon}}\omega\bigr)$$

for all $y\in\mathbb{R}^n$. Putting $y=x$, we obtain that

$$V^\varepsilon(x,0,t,\omega) = V^\varepsilon\bigl(0,-x,t,T_{\frac{x}{\varepsilon}}\omega\bigr).$$

By Lemma 3.2, the maximum principle holds for the solution of problem (9.6). Therefore,

$$|V^\varepsilon(0,x',t,\omega) - V^\varepsilon(0,x'',t,\omega)| \leqslant \|\nabla\varphi\|_{L^\infty}\,|x'-x''| \tag{9.8}$$

for all x', $x''\in\mathbb{R}^n$, that is, the Lipschitz condition with respect to x holds for $V^\varepsilon(0,x,t,\omega)$ with the constant $\|\nabla\varphi\|_{L^\infty}$ for all $\varepsilon>0$, $t>0$ and $\omega\in\Omega$.

Using Campbell's formula, we obtain that

$$\begin{aligned}
&\mathsf{E}\int_{\mathbb{R}^n}\big(V^\varepsilon(x,t,\omega)-V^0(x,t)\big)^2\,d\mu^\varepsilon_\omega(x)\\
&\qquad=\mathsf{E}\int_{\mathbb{R}^n}\big(V^\varepsilon\big(0,-x,t,T_{\frac{x}{\varepsilon}}\omega\big)-V^0(x,t)\big)^2\,d\mu^\varepsilon_\omega(x)\\
&\qquad=\int_\Omega\int_{\mathbb{R}^n}\big(V^\varepsilon(0,-x,t,\omega)-V^0(x,t)\big)^2\,dx\,d\boldsymbol{\mu}(\omega)\\
&\qquad=\int_{\mathbb{R}^n}\bigg(\int_\Omega(V^\varepsilon(0,-x,t,\omega)-V^0(x,t))^2\,d\boldsymbol{\mu}(\omega)\bigg)\,dx.
\end{aligned}\tag{9.9}$$

Combining this with Theorem 9.1, we obtain the equality

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\bigg(\int_\Omega\big(V^\varepsilon(0,-x,t,\omega)-V^0(x,t)\big)^2\,d\boldsymbol{\mu}(\omega)\bigg)\,dx=0.$$

The inequality (9.8) implies that $\int_\Omega(V^\varepsilon(0,-x,t,\omega)-V^0(x,t))^2\,d\boldsymbol{\mu}(\omega)$ is a Lipschitz function of x with Lipschitz constant not depending on ε or t. Hence, this integral tends to zero uniformly in x as $\varepsilon\to 0$.

Now let us discard the assumption of continuity. Consider the relation (9.9) for the smoothed measures $\mu^\varepsilon_{\omega,\delta}$ with fixed $\varepsilon>0$. Passing to the limit as $\delta\to 0$, we obtain that

$$\mathsf{E}\int_{\mathbb{R}^n}\big(V^\varepsilon(x,t,\omega)-V^0(x,t)^\varepsilon\big)^2\,d\mu^\varepsilon_\omega=\int_{\mathbb{R}^n}\int_\Omega\Phi_\varepsilon(x,t,\omega)\,d\mu\,dx,$$

where $\Phi_\varepsilon(x,t,\omega)$ is the weak (in $L^2(\mathbb{R}^n\times\Omega)$) limit of the sequence

$$\big(V^\varepsilon_\delta(0,-x,t,\omega)-V^0(x,t)\big)^2.$$

It is clear that

$$|\Phi_\varepsilon(x',t,\omega)-\Phi_\varepsilon(x,t,\omega)|\leqslant C|x'-x|.$$

The following relations hold a. s.:

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\varphi(x)G(x,t)\lambda(\omega)\,d\mu^\varepsilon_\omega(x)&=\int_{\mathbb{R}^n}\varphi(x)G(x,t)\lambda(\omega)\,dx\\
&=\lambda(\omega)u^0(0,t)=\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}u^0(x,t)\phi^\varepsilon(x)\,d\mu^\varepsilon_\omega(x).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\mathsf{E}\bigg(\int_{\mathbb{R}^n}\varphi(x)\big(u^\varepsilon(x,t)-\lambda(\omega)G(x,t)\big)\,d\mu^\varepsilon_\omega(x)\bigg)^2\\
&\qquad=\limsup_{\varepsilon\to 0}\mathsf{E}\bigg(\int_{\mathbb{R}^n}\big(V^\varepsilon(x,t)-V^0(x,t)\big)\phi^\varepsilon(x)\,d\mu^\varepsilon_\omega(x)\bigg)^2\\
&\qquad\leqslant C\limsup_{\varepsilon\to 0}\mathsf{E}\int_{\mathbb{R}^n}\big(V^\varepsilon\big(0,-x,t,T_{\frac{x}{\varepsilon}}\omega\big)-V^0(x,t)\big)^2\phi^\varepsilon(x)\,d\mu^\varepsilon_\omega(x)\\
&\qquad=C\limsup_{\varepsilon\to 0}\int_\Omega\int_{\mathbb{R}^n}\big(V^\varepsilon(0,-x,t,\omega)-V^0(x,t)\big)^2\phi^\varepsilon(x)\,dx\,d\boldsymbol{\mu}(\omega)\\
&\qquad=C\limsup_{\varepsilon\to 0}\int_{\mathbb{R}^n}\int_\Omega\big(V^\varepsilon(0,-x,t,\omega)-V^0(x,t)\big)^2\,d\boldsymbol{\mu}(\omega)\phi^\varepsilon(x)\,dx.
\end{aligned}$$

We have shown above that the inner integral on the right-hand side of the last equality tends to zero uniformly in x. Therefore, the whole expression on the right-hand side tends to zero, which completes the proof of the theorem.

Corollary 9.1. *The assertion of Theorem 9.4 holds for every stationary random measure of positive intensity.*

We make several remarks on the "individual" central limit theorem, which states that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\{u^\varepsilon(x,t,\omega)-\gamma(\omega)G(x,t)\}\phi(x)\,d\mu_\omega^\varepsilon=0\quad\forall\phi\in C_0^\infty(\mathbb{R}^n)\tag{9.10}$$

for P-almost all ω. We assume without loss of generality that $\gamma(\omega)=1$. In this case we have

$$\int_{\mathbb{R}^n}u^\varepsilon(x,t,\omega)\,d\mu_\omega^\varepsilon=\int_{\mathbb{R}^n}\phi(x)\,d\mu_\omega=1.$$

We shall try to pass to the limit directly in equation (9.3). Consider the following conditions on u^ε:

$$\limsup_{\varepsilon\to 0}\int_{\mathbb{R}^n}|u^\varepsilon(x,t,\omega)|^2\,d\mu_\omega^\varepsilon<\infty\quad\forall t>0,\tag{9.11}$$

$$\limsup_{\varepsilon\to 0}\int_{|x|\geqslant\delta}u^\varepsilon(x,t,\omega)\,d\mu_\omega^\varepsilon\leqslant C_\delta(t),\qquad\lim_{t\to 0}C_\delta(t)=0\quad\forall\delta>0.\tag{9.12}$$

Theorem 9.5. *If the conditions (9.11) and (9.12) hold, then so does (9.10) and, moreover,*

$$u^\varepsilon(x,t,\omega)\rightharpoonup G(x,t)\quad\textit{in}\quad L^2(\mathbb{R}^n,d\mu_\omega^\varepsilon)\quad\forall t>0,$$

where $G(x,t)$ is the fundamental solution of the homogenized equation.

Proof. The sequence $u^\varepsilon(x,t_0,\omega)$, $t_0>0$, is bounded in $L^2(\mathbb{R}^n,d\mu_\omega^\varepsilon)$. We assume without loss of generality that this sequence is weakly convergent. By Theorem 9.1, the sequence $u^\varepsilon(x,t,\omega)$ converges weakly for all $t\geqslant t_0$. Hence, we can assume that

$$u^\varepsilon(x,t,\omega)\rightharpoonup u^0(x,t)\quad\text{in}\quad L^2(\mathbb{R}^n,d\mu_\omega^\varepsilon)\quad\forall t>0,$$

where $u^0(x,t)$ is the solution of the homogenized equation. It remains to verify that $u^0(x,t)=\delta(0)$, that is,

$$\lim_{t\to 0}\int_{\mathbb{R}^n}u^0(x,t)\psi(x)=\psi(0)\quad\forall\psi\in C_0^\infty(\mathbb{R}^n).$$

Let us use (9.12). It is obvious that

$$u^\varepsilon(x,t)\,d\mu_\omega^\varepsilon\rightharpoonup u^0(x,t)\,dx$$

in the sense of the weak convergence of measures. Then (9.12) implies that

$$\int_{\mathbb{R}^n}u^0(x,t)\,dx=\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}u^\varepsilon(x,t,\omega)\,d\mu_\omega^\varepsilon=1,\qquad\int_{|x|\geqslant\delta}u^0(x,t)\,dx\leqslant C_\delta(t).$$

Hence, the desired equality holds: $\lim_{t\to 0}u(x,t)=\delta(0)$, that is, $u(x,t)=G(x,t)$.

It is rather difficult to verify conditions (9.11) and (9.12). This verification involves the so-called Gaussian estimates for the transition probabilities or, in terms of differential equations, estimates of Nash–Aronson type for the fundamental solution. In [23] (see also [14], Ch. 8) conditions (9.11) and (9.12) were verified in the following cases:

(i) walks in the exterior of a dispersion random ball structure,

(ii) walks on the infinite cluster in the continuous percolation model on the plane.

It would be of interest to verify these conditions for other models, in particular, for walks on the infinite cluster in the network model considered above.

§ 10. Construction of the tangential gradient

In this section we study the structure of gradients of zero and the structure of the tangential subspace of the measures μ_ω on $\mathbb{R}^n$ and the measure $\boldsymbol{\mu}$ on Ω. Let us recall that the gradients of zero are denoted by $\Gamma_{\mu_\omega}(0)$ and $\Gamma_{\boldsymbol{\mu}}(0)$, respectively. First we shall prove several assertions concerning these subspaces.

Lemma 10.1. (i) *Let $\xi(\omega) \in L^\infty(\Omega,\boldsymbol{\mu})$ and $g(\omega) \in \Gamma_{\boldsymbol{\mu}}(0)$. Then $\xi g \in \Gamma_{\boldsymbol{\mu}}(0)$. If $\zeta(x) \in L^\infty(\mathbb{R}^n,\mu_\omega)$ and $h(x) \in \Gamma_{\mu_\omega}(0)$, then $\zeta h \in \Gamma_{\mu_\omega}(0)$.*

(ii) *The set $\Gamma_{\boldsymbol{\mu}}(0) \cap L^\infty(\Omega,\boldsymbol{\mu})$ is dense in $\Gamma_{\boldsymbol{\mu}}(0)$. The set $\Gamma_{\mu_\omega}(0) \cap L^\infty(\mathbb{R}^n,\mu_\omega)$ is dense in $\Gamma_{\mu_\omega}(0)$.*

Proof. (i) It is sufficient to prove the first assertion for functions $\xi \in C^1(\Omega)$. Let ψ_k be a sequence of functions of class $C^1(\Omega)$ such that

$$\psi_k \to 0 \quad \text{in} \quad L^2(\Omega,\boldsymbol{\mu}), \qquad \nabla_\omega \psi_k \to g \quad \text{in} \quad (L^2(\Omega,\boldsymbol{\mu}))^n.$$

We have

$$\xi\psi_k \to 0 \quad \text{in} \quad L^2(\Omega,\boldsymbol{\mu}), \qquad \nabla_\omega(\xi\psi_k) \to \xi g \quad \text{in} \quad (L^2(\Omega,\boldsymbol{\mu}))^n,$$

whence $\xi g \in \Gamma_{\boldsymbol{\mu}}(0)$.

(ii) For an arbitrary $\xi \in \Gamma_{\boldsymbol{\mu}}(0)$ we define g_k to be the characteristic function of the set $\{\omega\colon |\xi(\omega)| \leqslant k\}$. Then $g_k\xi \in \Gamma_{\boldsymbol{\mu}}(0) \cap L^\infty(\Omega,\boldsymbol{\mu})$ and $g_k\xi \to \xi$ as $k \to \infty$. The other assertions can be proved likewise.

We denote by $\Pi_{\boldsymbol{\mu}}$ and Π_{μ_ω} the orthogonal projectors in $(L^2(\Omega,\boldsymbol{\mu}))^n$ and $(L^2(\mathbb{R}^n,\mu_\omega))^n$ to $\Gamma_{\boldsymbol{\mu}}(0)$ and $\Gamma_{\mu_\omega}(0)$, respectively.

Lemma 10.2. *Let the functions $\xi(\omega)$, $g(\omega)$, $\zeta(x)$ and $h(x)$ be as in Lemma 10.1. Then $\Pi_{\boldsymbol{\mu}}(\xi g) = \xi\Pi_{\boldsymbol{\mu}}(g)$ and $\Pi_{\mu_\omega}(\zeta h) = \zeta\Pi_{\mu_\omega}(h)$.*

Proof. For every $\psi(\omega) \in \Gamma_{\boldsymbol{\mu}}(0)$, Lemma 10.1 implies that

$$0 = \bigl(g - \Pi_{\boldsymbol{\mu}}(g), \xi\psi\bigr) = \bigl(\xi g - \xi\Pi_{\boldsymbol{\mu}}(g), \psi\bigr),$$

where the scalar product is taken in $L^2(\Omega,\boldsymbol{\mu})$. Since $\xi\Pi_{\boldsymbol{\mu}}(g) \in \Gamma_{\boldsymbol{\mu}}(0)$, we have $\xi\Pi_{\boldsymbol{\mu}}(g) = \Pi_{\boldsymbol{\mu}}(\xi g)$.

Lemma 10.3. *There is an $\mathcal{F}$-measurable subspace $\mathcal{L}(\omega) \subset \mathbb{R}^n$ such that*

$$\Gamma_{\boldsymbol{\mu}}(0) = \{g \in (L^2(\Omega,\boldsymbol{\mu}))^n\colon g(\omega) \in \mathcal{L}(\omega)\}.$$

Proof. Let $e_1, e_2, \dots, e_n$ be the standard basis of $\mathbb{R}^n$. Put

$$\psi_j(\omega) = \Pi_{\boldsymbol{\mu}}(e_j)(\omega), \qquad \mathcal{L}(\omega) = \operatorname{span}\{\psi_j(\omega),\ j = 1, 2, \dots, n\}.$$

Put $\Theta = \{g \in (L^2(\Omega, \boldsymbol{\mu}))^n \colon g(\omega) \in \mathcal{L}(\omega)\}$. We claim that $\Gamma_{\boldsymbol{\mu}}(0) = \Theta$. For $g \in \Gamma_{\boldsymbol{\mu}}(0) \cap (L^\infty(\Omega, \boldsymbol{\mu}))^n$ we have

$$g = g_1 e_1 + g_2 e_2 + \dots + g_n e_n = \Pi_{\boldsymbol{\mu}}(g) = g_1 \psi_1 + g_2 \psi_2 + \dots + g_n \psi_n.$$

Since $\Gamma_{\boldsymbol{\mu}}(0) \cap (L^\infty(\Omega, \boldsymbol{\mu}))^n$ is dense in $\Gamma_{\boldsymbol{\mu}}(0)$ and Θ is closed, we have $\Gamma_{\boldsymbol{\mu}}(0) \subset \Theta$.

We shall now prove the reverse inclusion. Since $\psi_1, \dots, \psi_n \in \Gamma_{\boldsymbol{\mu}}(0)$, Lemma 10.1 implies that $\sum_{j=1}^n \xi_j(\omega)\psi_j(\omega) \in \Gamma_{\boldsymbol{\mu}}(0)$ for all $\xi_j \in L^\infty(\Omega, \boldsymbol{\mu})$. We claim that the set of such elements is dense in Θ. Indeed, if for some $g \in \Theta$ we have

$$0 = \left(g, \sum_{j=1}^n \xi_j \psi_j\right) = \sum_{j=1}^n (\xi_j, g\psi_j)$$

for all $\xi_j \in L^\infty(\Omega, \boldsymbol{\mu})$, then $g(\omega)\psi_j(\omega) = 0$ $\ \boldsymbol{\mu}$-a. s., and the definition of Θ implies that $g = 0$.

In the same vein, we construct the sets $\mathcal{L}_{\mu_\omega}(x)$:

$$\kappa_j = \Pi_{\mu_\omega}(e_j), \qquad \mathcal{L}_{\mu_\omega}(x) = \operatorname{span}\{\kappa_j,\ j = 1, 2, \dots, n\}.$$

Theorem 10.1. *The subspace $\Gamma_{\mu_\omega}(0)$ can be represented as*

$$\Gamma_{\mu_\omega}(0) = \{h \in L^2(\mathbb{R}^n, \mu_\omega) \colon h(x) \in \mathcal{L}_{\mu_\omega}(x)\}. \tag{10.1}$$

The equality

$$\mathcal{L}_{\mu_\omega}(x) = \mathcal{L}_{\boldsymbol{\mu}}(T_x\omega) \tag{10.2}$$

holds.

Proof. We prove (10.1) following the scheme of the proof of Lemma 10.3. To prove (10.2), we observe that $L_{\mu_\omega}(x)$ is measurable with respect to (ω, x) in the sense of the σ-algebra $\mathcal{F} \times \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra. Since the measures $d\mu_{T_y\omega}(x)$ and $\mu_\omega(dx+y)$ coincide, we have $\mathcal{L}_{\mu_{T_y\omega}}(x) = \mathcal{L}_{\mu_\omega}(x-y)$. Therefore, one can find a measurable subspace $\mathcal{L}_1(\omega) \subset \mathbb{R}^n$ such that $L_{\mu_\omega}(x) = \mathcal{L}_1(T_x\omega)$.

It is easy to verify that $\mathcal{L}(\omega) \subset \mathcal{L}_1(\omega)$. Indeed, if $(g(\omega), \nabla_{\boldsymbol{\mu}} g(\omega)) \in \widetilde{H}(\Omega, \boldsymbol{\mu})$, then P-a. s. $g(T_x\omega) \in L^2_{\mathrm{loc}}(\mathbb{R}^n, \mu_\omega)$ and $\nabla_{\boldsymbol{\mu}}(T_x\omega)$ is a gradient of $g(T_x\omega)$.

To prove the reverse inclusion, we need the following proposition.

Proposition 10.1. *Let $b(\omega) \in (L^2(\Omega, \boldsymbol{\mu}))^n$ and $g(\omega) \in L^2(\Omega, \boldsymbol{\mu})$. Assume that $b(T_x\omega)$ is P-a. s. a μ_ω-gradient of $g(T_x\omega)$. Then $b(\Omega) = \nabla_{\boldsymbol{\mu}} g(\omega)$.*

Proof. We shall use (5.16). For $r(\omega) \in L^2(\Omega, \boldsymbol{\mu})$ and $d(\omega) \in (L^2(\Omega, \boldsymbol{\mu}))^n$ such that $r(\omega) = \operatorname{div}_\omega d(\omega)$, Birkhoff's theorem implies that the following relations hold

P-a. s.:

$$\begin{aligned}\int_\Omega r(\omega)g(\omega)\,d\boldsymbol{\mu}(\omega) &= \lim_{T\to\infty}\frac{1}{T^n}\int_{(0,T)^n} r(T_x)g(T_x\omega)\,d\mu_\omega(x)\\ &= -\lim_{T\to\infty}\frac{1}{T^n}\int_{(0,T)^n} d(T_x\omega)\,b(T_x\omega)d\mu_\omega(x)\\ &= -\int_\Omega d(\omega)\,b(\omega)\,d\boldsymbol{\mu}(\omega).\end{aligned}$$

The proof of the second equality follows the scheme of the proof of Theorem 2.1.

Now assume that $\mathcal{L}_1(\omega)$ is bigger than $\mathcal{L}(\omega)$ by a set of positive measure $\boldsymbol{\mu}$. Let $s(\omega)$ be a unit vector belonging to $\mathcal{L}_1(\omega)$, orthogonal to $\mathcal{L}(\omega)$ and measurable on this set of positive measure $\boldsymbol{\mu}$. We put $s(\omega)$ equal to zero at other values of ω. By the first assertion of Theorem 10.1, $s(T_x\omega)$ is a gradient of zero in $L^2(\mathbb{R}^n,\mu_\omega)$ P-a. s. By Proposition 10.1 $s(\omega)$ is a gradient of zero in $L^2(\Omega,\boldsymbol{\mu})$, which contradicts the assumption.

Definition 10.1. The *tangential space* $\mathcal{T}_{\boldsymbol{\mu}}(\omega)$ is defined to be the orthogonal complement of $\mathcal{L}(\omega)$ in $\mathbb{R}^n$. In a similar way, $\mathcal{T}_{\mu_\omega}(x)$ is defined to be the orthogonal complement of $\mathcal{L}_{\mu_\omega}(x)$ in $\mathbb{R}^n$.

By Theorem 10.1, we have $\mathcal{T}_{\mu_\omega}(x)=\mathcal{T}_{\boldsymbol{\mu}}(T_x\omega)$. Let $u(\omega)$ be an arbitrary element of $H^1(\Omega,\boldsymbol{\mu})$. We denote by $\partial^t_{\boldsymbol{\mu}}u(\omega)$ the orthogonal projection (in $\mathbb{R}^n$) of $\partial_{\boldsymbol{\mu}}u(\omega)$ to $\mathcal{T}_{\mu_\omega}(x)$. The function $\partial^t_{\boldsymbol{\mu}}u(\omega)$ is called a *tangential gradient of* u. Let us note that every gradient of every $u(\omega)\in H^1(\Omega,\boldsymbol{\mu})$ can be unambiguously represented in the form

$$\partial_{\boldsymbol{\mu}}u(\omega)=\partial^t_{\boldsymbol{\mu}}u(\omega)+g(\omega), \tag{10.3}$$

where $g(\omega)\in\mathcal{L}(\omega)$.

The tangential gradient of an arbitrary $u(x)\in H^1(\mathbb{R}^n,\mu_\omega)$ can be defined likewise. We denote it by $\nabla^t u$.

Now consider the following relaxation problem. Let $a(\omega)$ be a positive-definite symmetric matrix and let $(u(\omega),\nabla u(\omega))\in H(\Omega,\boldsymbol{\mu})$ be a solution of the problem

$$\operatorname{div}(a(\omega)\nabla u(\omega))+u(\omega)=f(\omega),\qquad f\in L^2(\Omega,\boldsymbol{\mu}). \tag{10.4}$$

As mentioned above, the gradient $a(\omega)\nabla u(\omega)$ is then defined unambiguously by the condition of the orthogonality of ∂u to $\Gamma_{\boldsymbol{\mu}}(0)$ (generally speaking, this gradient is not tangential).

We wish to find a matrix $a^r(\omega)$ such that, for any $f\in L^2(\Omega,\boldsymbol{\mu})$, the solution of the problem

$$\operatorname{div}(a^r(\omega)\partial u(\omega))+u(\omega)=f(\omega)$$

has the form $(u,\partial^t u)$, where $u(\omega)$ coincides with the solution of problem (10.4) but the corresponding gradient is tangential. We solve this local problem using pointwise projection to the tangential subspace. Namely, we have to solve the following problem: given an $\eta\in\mathbb{R}^n$, find a $\zeta\in\mathcal{T}_{\boldsymbol{\mu}}(\omega)$ such that $a(\omega)(\eta+\zeta(\omega))\in\mathcal{T}_{\boldsymbol{\mu}}(\omega)$. It is easy to verify that the matrix $a^r(\omega)$ of the linear map $\eta\to a(\omega)(\eta+\zeta(\omega))=a^r(\omega)\eta$ is symmetric and $a^r(\omega)\eta=0$ for $\eta\in\mathcal{L}(\omega)$. Let Π^t be the orthogonal projector to $\mathcal{T}_{\boldsymbol{\mu}}(\omega)$ in $\mathbb{R}^n$. We have $a^r(\omega)\Pi^t\eta=a^r(\omega)\eta$. Since $a^r(\omega)\eta=a(\omega)\eta$ for all $\eta \in \mathcal{T}_{\boldsymbol{\mu}}(\omega)$ and the flux $a\nabla u$ in problem (10.4) is orthogonal to $\Gamma_{\boldsymbol{\mu}}(0)$, we have $a(\omega)\nabla u(\omega) \in \mathcal{T}_{\boldsymbol{\mu}}(\omega)$ $\boldsymbol{\mu}$-a. s. and

$$a(\omega)\nabla u(\omega) = a^r(\omega)\nabla u(\omega) = a^r(\omega)\Pi^t\partial u(\omega),$$

which solves the relaxation problem.

The relaxation problem for the measures μ_ω on $\mathbb{R}^n$ can be solved likewise. Using Theorem 10.1, we obtain that $a^r_{\mu_\omega}(x) = a^r(T_x\omega)$ for μ_ω-almost all x P-a. s.

Theorem 10.2. *Let $a(\omega)$ be a positive-definite symmetric matrix. Then P-a. s. for any $f \in L^2(\mathbb{R}^n, \mu_\omega)$, the solutions $(u, \nabla u)$, $(v, \nabla v)$ of the equations*

$$-\operatorname{div}(a(T_x\omega)\nabla u(x)) + u(x) = f(x),$$
$$-\operatorname{div}(a^r(T_x\omega)\nabla v(x)) + v(x) = f(x)$$

have equal first components, and ∇v is the tangential gradient of u, that is, $u(x) = v(x)$ and $\nabla v = \nabla^t u$.

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