Links between computability and dynamics

in multidimensional symbolic dynamics

Second step: Sub-dynamics of multidimensional sofic and Applications to find local rules

floripadynsys : Workshop on Dynamics, Numeration and Tilings

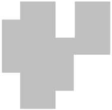
M. Sablik

LIP, ENS de Lyon
LATP, Université Aix-Marseille

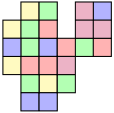
November 2013

Recall of the previous episode

Subshifts defined by forbidden patterns



support $\mathbb{U} \subset \mathbb{Z}^d$ finite



pattern $u \in \mathcal{A}^{\mathbb{U}}$

Definition: Subshift of forbidden parterns $\mathcal{F} \subset \mathcal{A}^*$

$$\mathbf{T}(\mathcal{A}, d, \mathcal{F}) = \{x \in \mathcal{A}^{\mathbb{Z}^d} \ : \ \text{patterns of } \mathcal{F} \text{ does not appear in } x\} \subseteq \mathcal{A}^{\mathbb{Z}^d}$$

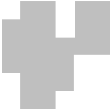
Some classes of subshifts:

$\mathbf{T}$ *fullshift* $(\mathcal{FS})$ $\iff$ $\mathcal{F} = \varnothing$ and $\mathbf{T} = \mathbf{T}(\mathcal{A}, d, \mathcal{F}) = \mathcal{A}^{\mathbb{Z}^d}$,

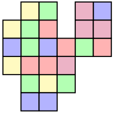
$\mathbf{T}$ *subshift of finite type* $(\mathcal{SFT})$ $\iff$ $\exists \mathcal{F} \subset \mathcal{A}^*$ a finite set such that $\mathbf{T} = \mathbf{T}(\mathcal{A}, d, \mathcal{F})$

$\mathbf{T}$ *subshift sofic* $(\mathcal{S}ofic)$ $\iff$ $\exists \mathcal{F} \subset \mathcal{A}^*$ a finite set and π a morphism such that $\mathbf{T} = \pi(\mathbf{T}(\mathcal{A}, d, \mathcal{F}))$

Subshifts defined by forbidden patterns



support $\mathbb{U} \subset \mathbb{Z}^d$ finite



pattern $u \in \mathcal{A}^{\mathbb{U}}$

Definition: Subshift of forbidden parterns $\mathcal{F} \subset \mathcal{A}^*$

$$\mathbf{T}(\mathcal{A}, d, \mathcal{F}) = \{x \in \mathcal{A}^{\mathbb{Z}^d} \ : \ \text{patterns of } \mathcal{F} \text{ does not appear in } x\} \subseteq \mathcal{A}^{\mathbb{Z}^d}$$

Some classes of subshifts:

$\mathbf{T}$ *fullshift* $(\mathcal{FS})$ $\iff$ $\mathcal{F} = \emptyset$ and $\mathbf{T} = \mathbf{T}(\mathcal{A}, d, \mathcal{F}) = \mathcal{A}^{\mathbb{Z}^d}$,

$\mathbf{T}$ *subshift of finite type* $(\mathcal{SFT})$ $\iff$ $\exists \mathcal{F} \subset \mathcal{A}^*$ a finite set such that $\mathbf{T} = \mathbf{T}(\mathcal{A}, d, \mathcal{F})$

$\mathbf{T}$ *subshift sofic* $(\mathcal{S}ofic)$ $\underset{d=1}{\iff}$ $\exists \mathcal{F} \subset \mathcal{A}^*$ a rational set such that $\mathbf{T} = \mathbf{T}(\mathcal{A}, 1, \mathcal{F})$ *(Weiss-73)*

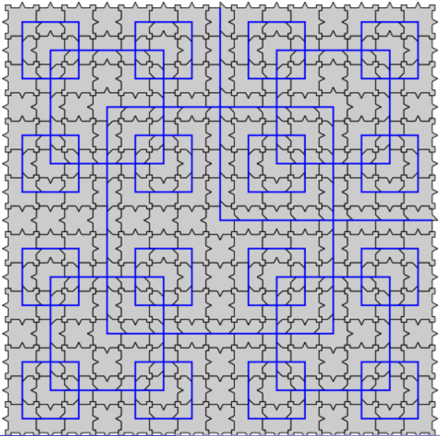
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations

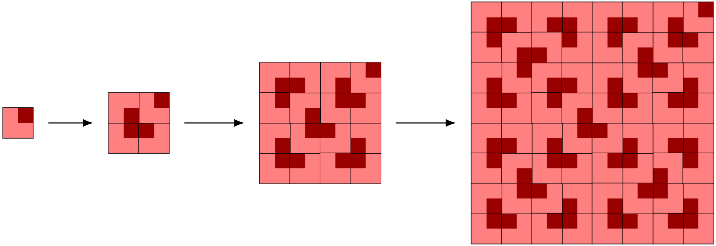
2D SFT/sofic subshifts

- exists aperiodic SFT/sofic



1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts	**2D SFT/sofic subshifts**
SFT/sofic has periodic configurations	exists aperiodic SFT/sofic
Substitutive subshift are not sofic	Substitutive subshift are sofic



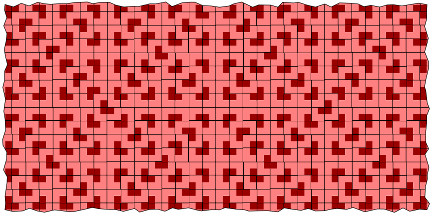
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic



Theorem *(Mozes 1989)*

Given a substition s, there exists a SFT $\mathbf{T}(\mathcal{B}, d, \mathcal{F})$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that $\pi(\mathbf{T}(\mathcal{B}, d, \mathcal{F})) = \mathbf{T}_s$.
Moreover π is a conjugacy almost everywhere and $\mathbf{T}(\mathcal{B}, d, \mathcal{F})$ is substitutive.

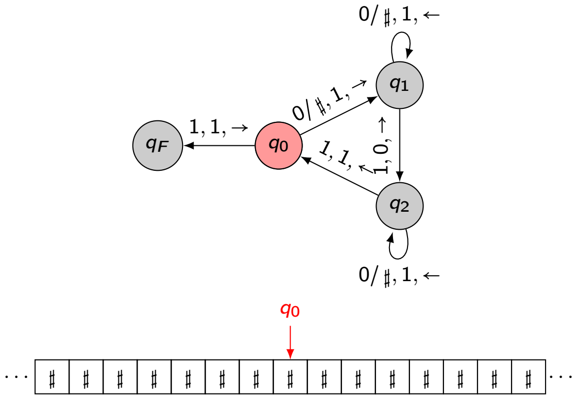
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A}, 1, \mathcal{F}) = \varnothing$? is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \varnothing$? is undecidable



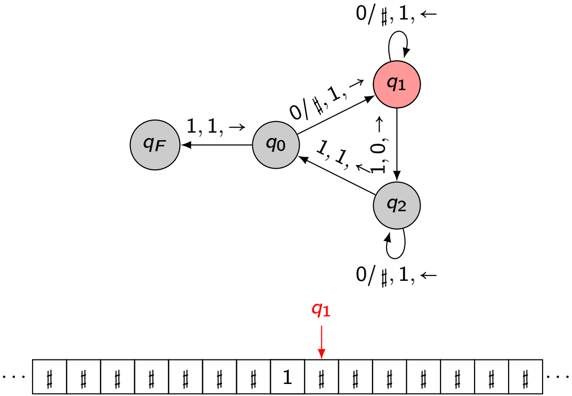
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A}, 1, \mathcal{F}) = \varnothing?$ is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \varnothing?$ is undecidable



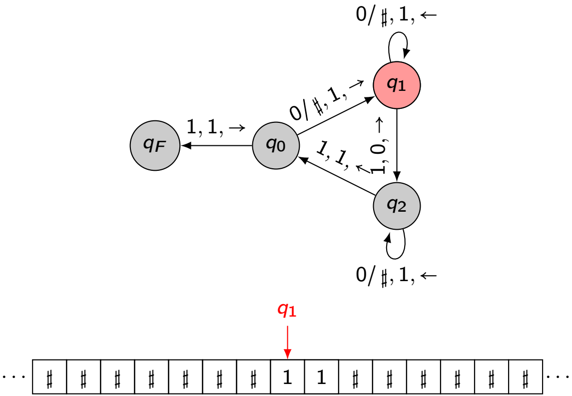
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A}, 1, \mathcal{F}) = \emptyset$? is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \emptyset$? is undecidable



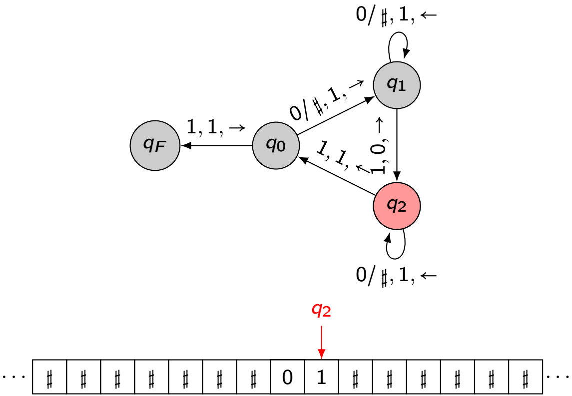
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A}, 1, \mathcal{F}) = \varnothing$? is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \varnothing$? is undecidable



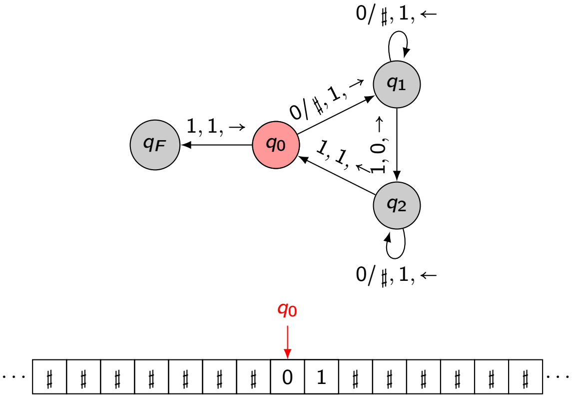
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A}, 1, \mathcal{F}) = \varnothing?$ is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \varnothing?$ is undecidable



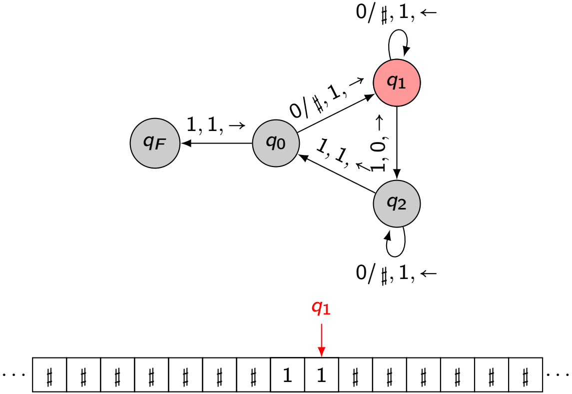
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A},1,\mathcal{F}) = \varnothing$? is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A},2,\mathcal{F}) = \varnothing$? is undecidable



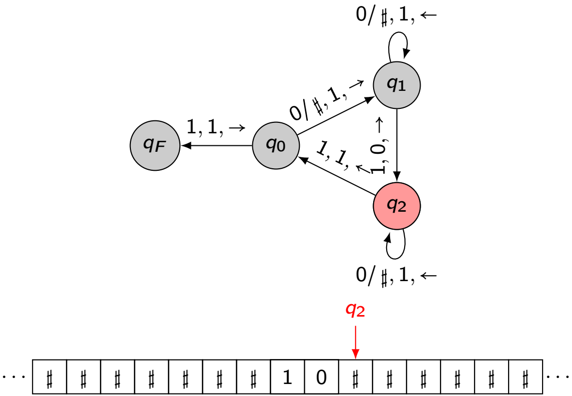
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A}, 1, \mathcal{F}) = \varnothing?$ is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \varnothing?$ is undecidable



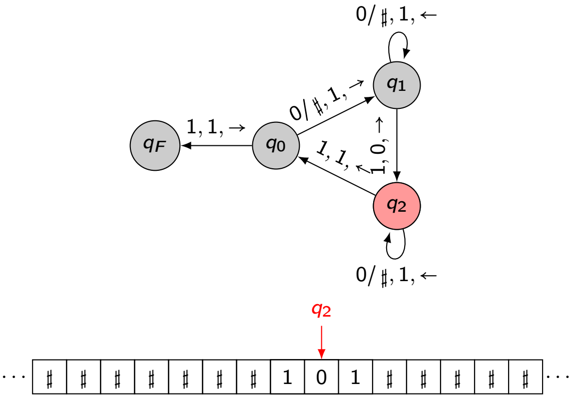
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A}, 1, \mathcal{F}) = \varnothing?$ is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \varnothing?$ is undecidable



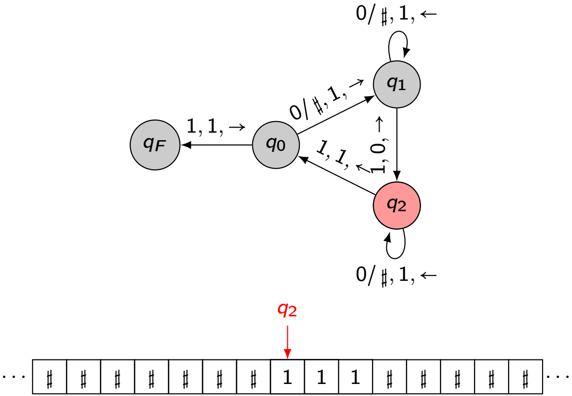
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A}, 1, \mathcal{F}) = \varnothing?$ is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \varnothing?$ is undecidable



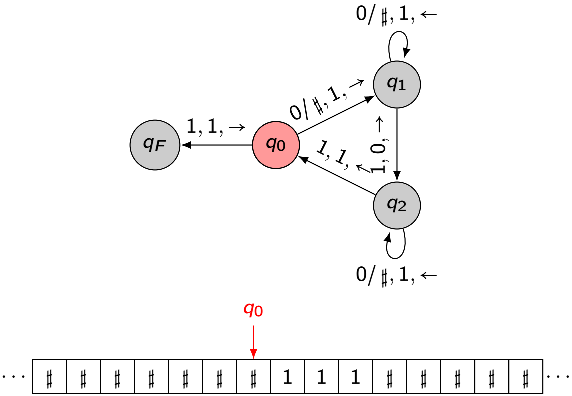
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A}, 1, \mathcal{F}) = \varnothing?$ is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \varnothing?$ is undecidable



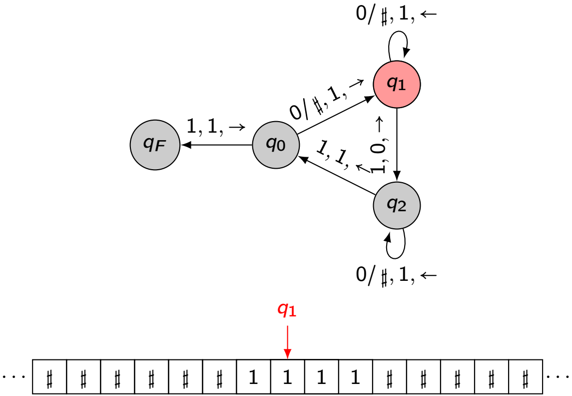
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A}, 1, \mathcal{F}) = \varnothing$? is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \varnothing$? is undecidable



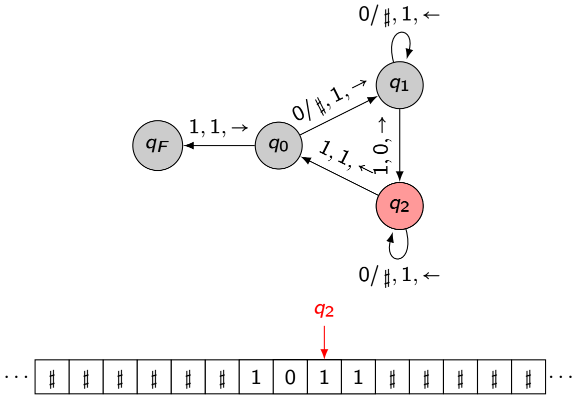
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A}, 1, \mathcal{F}) = \varnothing?$ is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \varnothing?$ is undecidable



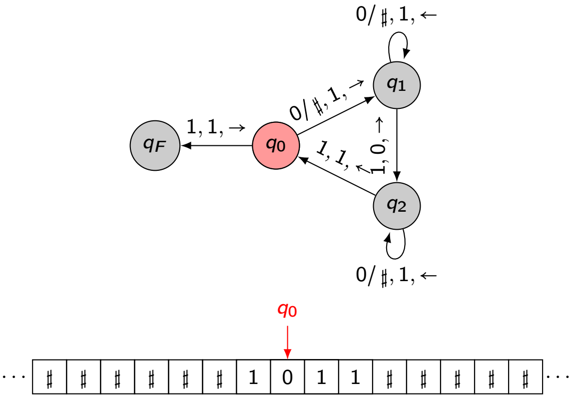
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A}, 1, \mathcal{F}) = \varnothing?$ is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \varnothing?$ is undecidable



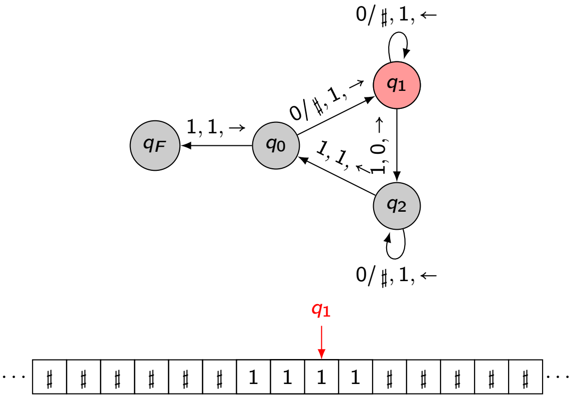
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A}, 1, \mathcal{F}) = \varnothing$? is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \varnothing$? is undecidable



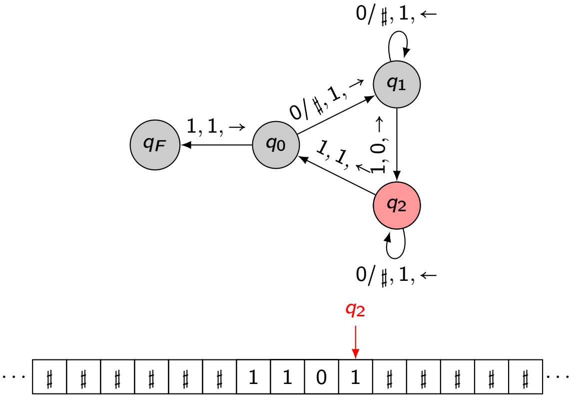
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A}, 1, \mathcal{F}) = \varnothing?$ is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \varnothing?$ is undecidable



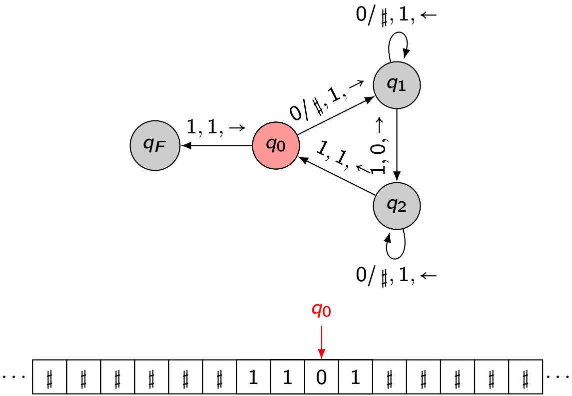
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A}, 1, \mathcal{F}) = \varnothing$? is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \varnothing$? is undecidable



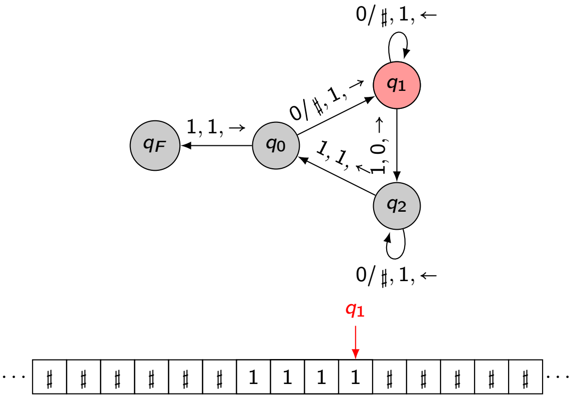
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A}, 1, \mathcal{F}) = \varnothing$? is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \varnothing$? is undecidable



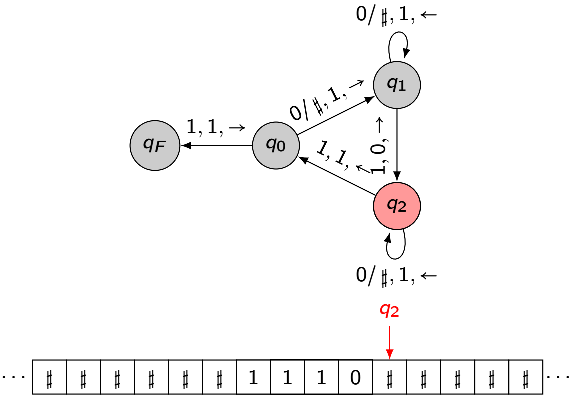
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A}, 1, \mathcal{F}) = \varnothing$? is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \varnothing$? is undecidable



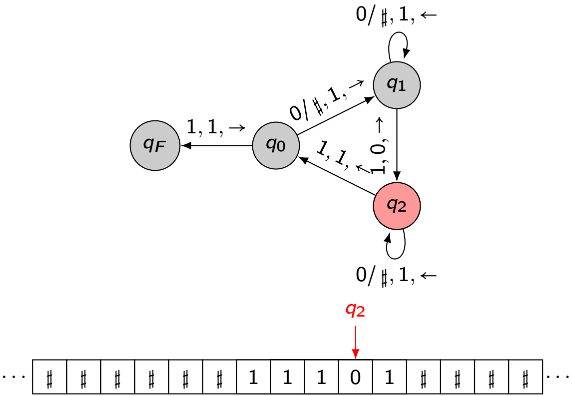
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A}, 1, \mathcal{F}) = \varnothing?$ is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \varnothing?$ is undecidable



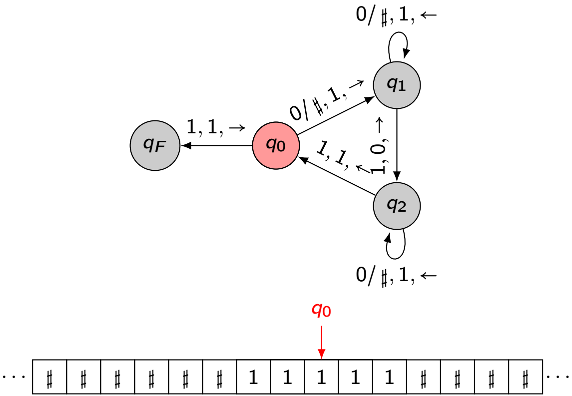
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A}, 1, \mathcal{F}) = \varnothing?$ is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \varnothing?$ is undecidable



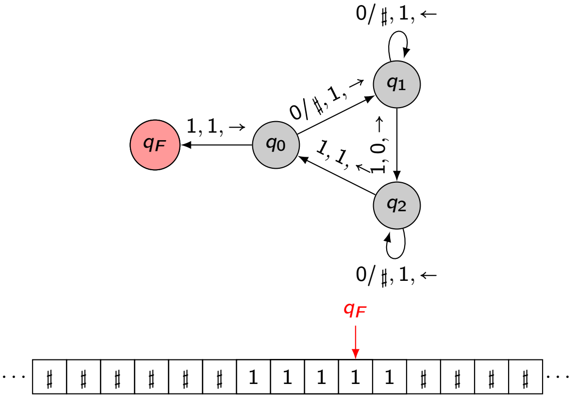
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A},1,\mathcal{F}) = \varnothing?$ is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A},2,\mathcal{F}) = \varnothing?$ is undecidable



1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A}, 1, \mathcal{F}) = \varnothing?$ is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \varnothing?$ is undecidable

···	♯	♯	♯	♯	♯	♯	1	1, q_1	1	1	♯	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	♯	♯, q_0	1	1	1	♯	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	♯	♯	1, q_2	1	1	♯	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	♯	♯	1	0, q_2	1	♯	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	♯	♯	1	0	♯, q_2	♯	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	♯	♯	1	1, q_1	♯	♯	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	♯	♯	0, q_0	1	♯	♯	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	♯	♯	0	1, q_2	♯	♯	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	♯	♯	1, q_1	1	♯	♯	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	♯	♯	1	♯, q_1	♯	♯	♯	♯	♯	♯	···
···	♯	♯	♯	♯	♯	♯	♯	♯, q_0	♯	♯	♯	♯	♯	♯	♯	···

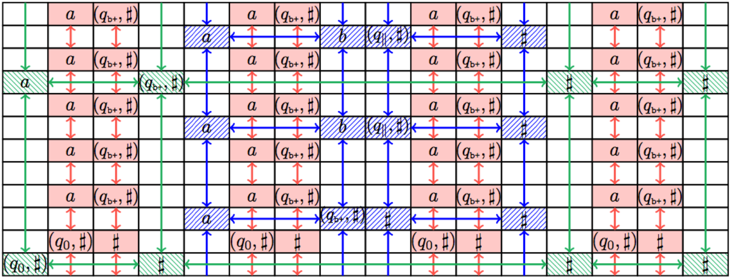
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts

- SFT/sofic has periodic configurations
- Substitutive subshift are not sofic
- $\mathbf{T}(\mathcal{A}, 1, \mathcal{F}) = \emptyset?$ is decidable

2D SFT/sofic subshifts

- exists aperiodic SFT/sofic
- Substitutive subshift are sofic
- $\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \emptyset?$ is undecidable



A strip of level n allows to code space-time diagram of $\mathcal{M}$ of size $2^n \times 2^{2^n}$, thus:

$$\mathcal{M} \text{ halts} \iff \mathbf{T}_{\texttt{Calcul}(\mathcal{M})} = \emptyset$$

Dynamical operations on subshifts

Factor operation: **Fact**

Definition

Let $\mathbf{T} \subseteq \mathcal{A}^{\mathbb{Z}^{\boldsymbol{d}}}$ be a subshift and $\pi : \mathcal{A}^{\mathbb{Z}^{\boldsymbol{d}}} \to \mathcal{B}^{\mathbb{Z}^{\boldsymbol{d}}}$ a morphism,
$\mathbf{Fact}_{\pi}(\mathbf{T}) = \pi(\mathbf{T}) \subseteq \mathcal{B}^{\mathbb{Z}^{\boldsymbol{d}}}$ is $\mathbf{T}$.

Factor operation: **Fact**

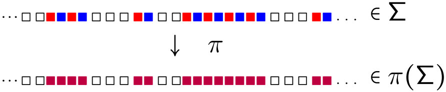
Definition

Let $\mathbf{T} \subseteq \mathcal{A}^{\mathbb{Z}^d}$ be a subshift and $\pi : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{B}^{\mathbb{Z}^d}$ a morphism,
$\mathbf{Fact}_\pi(\mathbf{T}) = \pi(\mathbf{T}) \subseteq \mathcal{B}^{\mathbb{Z}^d}$ is $\mathbf{T}$.

Exemple :

Consider:

- $\mathcal{A} = \{□, 🟥, 🟦\}$
- $\Sigma = \mathbf{T}(\mathcal{A}, 1, \{🟥🟥, 🟦🟦, 🟥□, □🟦\}) \subset \mathcal{A}^{\mathbb{Z}}$.
- $\pi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ morphism such that $\pi : \begin{cases} □ \to □ \\ 🟥 \to ■ \\ 🟦 \to ■ \end{cases}$



So $\mathbf{Fact}_\pi(\Sigma) = \{x \in \{□, ■\}^{\mathbb{Z}} /$ blocks of ■ have even sizes$\} = \mathbf{T}_{\{0,■\},\{□■^{2n+1}□ : n \in \mathbb{N}\}}$

Thus $SFT \subsetneq Cl_F(SFT)$

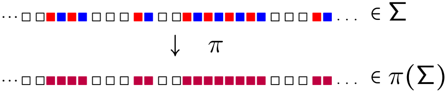
Factor operation: **Fact**

Definition

Let $\mathbf{T} \subseteq \mathcal{A}^{\mathbb{Z}^d}$ be a subshift and $\pi : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{B}^{\mathbb{Z}^d}$ a morphism,
$\mathbf{Fact}_\pi(\mathbf{T}) = \pi(\mathbf{T}) \subseteq \mathcal{B}^{\mathbb{Z}^d}$ is $\mathbf{T}$.

Exemple :
Consider:

- $\mathcal{A} = \{\square, {\color{red}\blacksquare}, {\color{blue}\blacksquare}\}$
- $\Sigma = \mathbf{T}(\mathcal{A}, 1, \{{\color{red}\blacksquare\blacksquare}, {\color{blue}\blacksquare\blacksquare}, {\color{red}\blacksquare}\square, \square{\color{blue}\blacksquare}\}) \subset \mathcal{A}^{\mathbb{Z}}$.
- $\pi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ morphism such that $\pi : \begin{cases} \square \to \square \\ {\color{red}\blacksquare} \to {\color{purple}\blacksquare} \\ {\color{blue}\blacksquare} \to {\color{purple}\blacksquare} \end{cases}$



So $\mathbf{Fact}_\pi(\Sigma) = \{x \in \{\square, {\color{purple}\blacksquare}\}^{\mathbb{Z}} /$ blocks of ${\color{purple}\blacksquare}$ have even sizes$\} = \mathbf{T}_{\{0,{\color{purple}\blacksquare}\},\{\square{\color{purple}\blacksquare}^{2n+1}\square : n \in \mathbb{N}\}}$

Thus $SFT \subsetneq \mathcal{C}l_F(SFT)$

By definition $\mathcal{C}l_F(SFT)$ is the class of *sofic subshifts* .

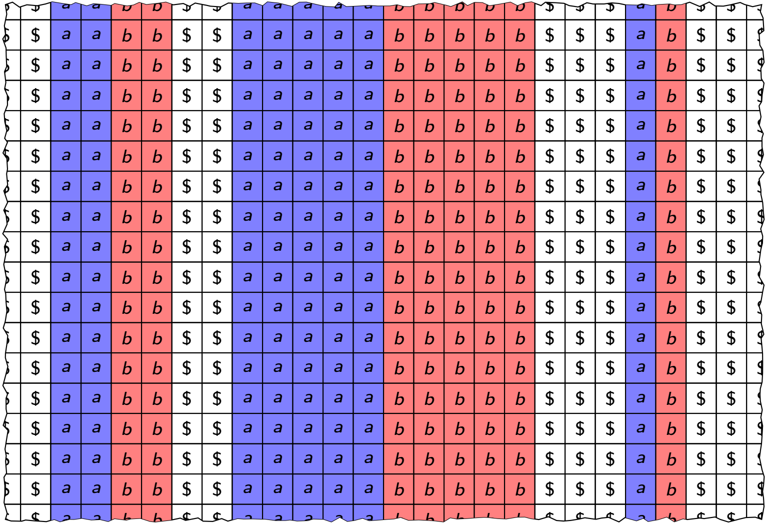
Théorème *(Weiss-73)*

In dimension 1, a subshift is sofic if and only if the set of forbidden pattens is rational (i.e. described by a finite automaton)

Subshift realized by subaction of a sofic?

Let $\Sigma = \mathbf{T}(\{a, b, \$\}, 1, \{ba, \beta a^n b^m \alpha : n \neq m, \alpha \neq a, \beta \neq b\})$. Consider the subshift

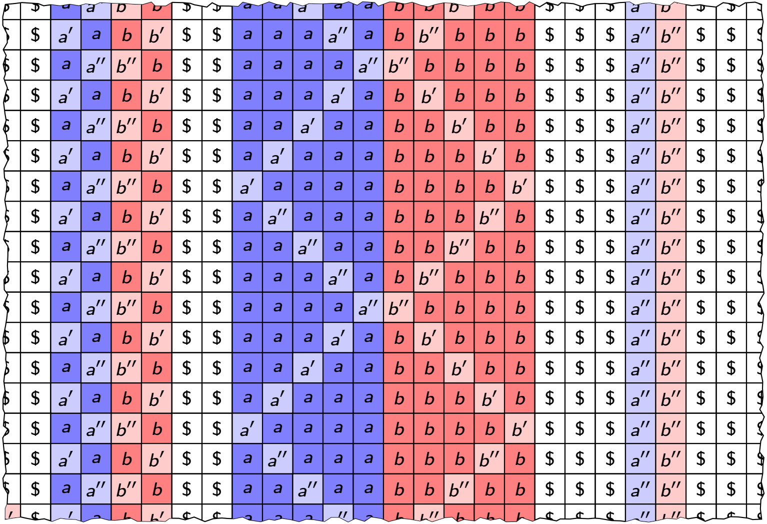
$$\mathbf{T} = \{x \in (\{a, b, \$\}^{\mathbb{Z}^2} : \exists y \in \Sigma \text{ tel que } x_{(.,j)} = y \text{ such that } j \in \mathbb{Z}\}$$



Subshift realized by subaction of a sofic?

Let $\Sigma = \mathbf{T}(\{a, b, \$\}, 1, \{ba, \beta a^n b^m \alpha : n \neq m, \alpha \neq a, \beta \neq b\})$. Consider the subshift

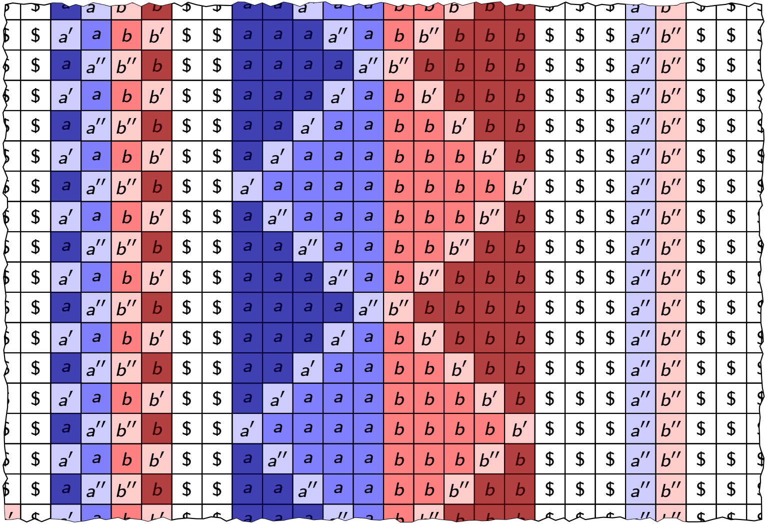
$$\mathbf{T} = \{x \in (\{a, b, \$\}^{\mathbb{Z}^2} : \exists y \in \Sigma \text{ tel que } x_{(.,j)} = y \text{ such that } j \in \mathbb{Z}\}$$



Subshift realized by subaction of a sofic?

Let $\Sigma = \mathbf{T}(\{a, b, \$\}, 1, \{ba, \beta a^n b^m \alpha : n \neq m, \alpha \neq a, \beta \neq b\})$. Consider the subshift

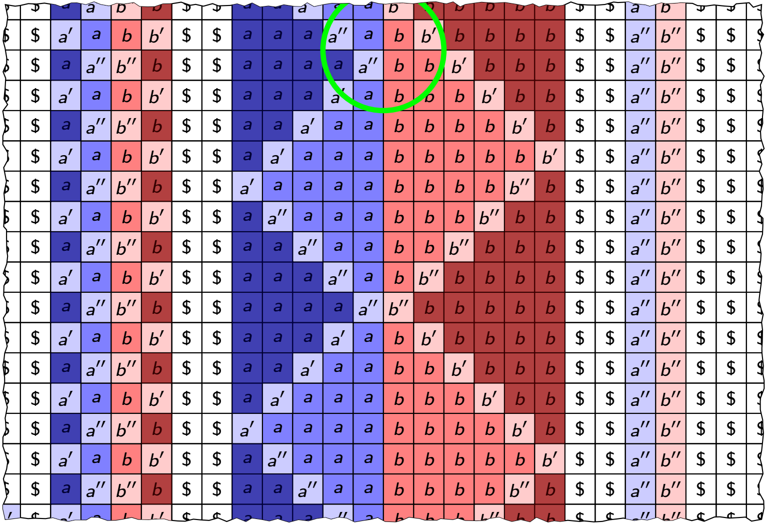
$$\mathbf{T} = \{x \in (\{a, b, \$\}^{\mathbb{Z}^2} : \exists y \in \Sigma \text{ tel que } x_{(.,j)} = y \text{ such that } j \in \mathbb{Z}\}$$



Subshift realized by subaction of a sofic?

Let $\Sigma = \mathbf{T}(\{a, b, \$\}, 1, \{ba, \beta a^n b^m \alpha : n \neq m, \alpha \neq a, \beta \neq b\})$. Consider the subshift

$$\mathbf{T} = \{x \in (\{a, b, \$\}^{\mathbb{Z}^2} : \exists y \in \Sigma \text{ tel que } x_{(.,j)} = y \text{ such that } j \in \mathbb{Z}\}$$



Projective subaction: **SA**

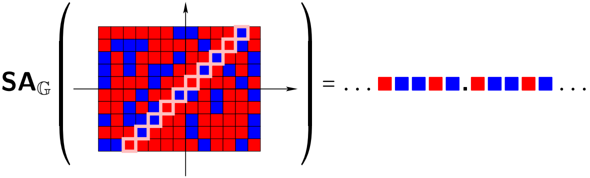
Let $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}^d}$ be a subshift and $\mathbb{G}$ be a sublattice of $\mathbb{Z}^d$, the $\mathbb{G}$-action on $\mathbf{T}$ is not necessary a subshift. However if we restrict to a row one obtains a subshift.

Definition

Let $\mathbb{G}$ be a sublattice of $\mathbb{Z}^d$ generated by $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_{d'}}$ $(d' \leq d)$. Let $\mathbf{T} \subseteq \mathcal{A}^{\mathbb{Z}^d}$ be a subshift :

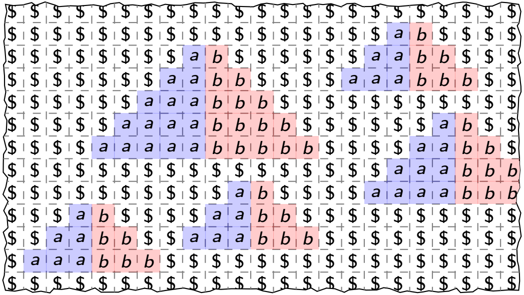
$$\mathbf{SA}_{\mathbb{G}}(\mathbf{T}) = \left\{ \begin{array}{c} y \in \mathcal{A}^{\mathbb{Z}^{d'}} \ : \ \exists x \in \mathbf{T} \text{ tel que } \forall i_1, \ldots, i_{d'} \in \mathbb{Z}^{d'}, \\ y_{i_1,\ldots,i_{d'}} = x_{i_1\mathbf{u_1}+\cdots+i_{d'}\mathbf{u_{d'}}} \end{array} \right\}.$$

Let $\mathbb{G} = \{(i,i) : i \in \mathbb{Z}\} \subset \mathbb{Z}^2$.



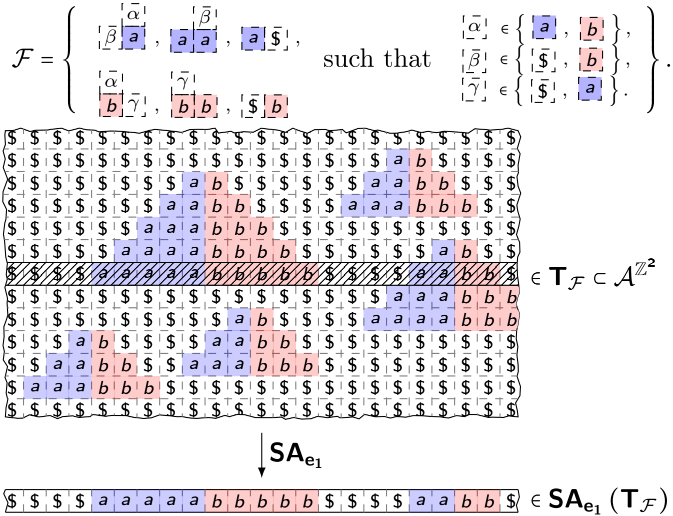
Example of projective subaction

$$\mathcal{F} = \left\{ \begin{array}{l} \begin{smallmatrix} & \alpha \\ \beta & a \end{smallmatrix}\ ,\ \begin{smallmatrix} & \beta \\ a & a \end{smallmatrix}\ ,\ \begin{smallmatrix} a & \$ \end{smallmatrix}\ , \\ \begin{smallmatrix} \alpha & \\ b & \gamma \end{smallmatrix}\ ,\ \begin{smallmatrix} \gamma & \\ b & b \end{smallmatrix}\ ,\ \begin{smallmatrix} \$ & b \end{smallmatrix} \end{array} \quad \text{such that} \quad \begin{array}{l} \alpha \in \{ a , b \}, \\ \beta \in \{ \$, b \}, \\ \gamma \in \{ \$, a \}. \end{array} \right\}.$$



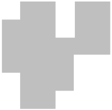
$\in \mathbf{T}_{\mathcal{F}} \subset \mathcal{A}^{\mathbb{Z}^2}$

Example of projective subaction

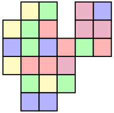


Thus $\mathcal{C}l_{\mathbf{SA}}(\mathcal{S}ofic) \neq \mathcal{S}ofic$.

Subshifts defined by forbidden patterns



support $\mathbb{U} \subset \mathbb{Z}^d$ finite



pattern $u \in \mathcal{A}^{\mathbb{U}}$

Definition: Subshift of forbidden parterns $\mathcal{F} \subset \mathcal{A}^*$

$$\mathbf{T}(\mathcal{A}, d, \mathcal{F}) = \{x \in \mathcal{A}^{\mathbb{Z}^d} : \text{ patterns of } \mathcal{F} \text{ does not appear in } x\} \subseteq \mathcal{A}^{\mathbb{Z}^d}$$

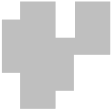
Some classes of subshifts:

$\mathbf{T}$ *fullshift* $(\mathcal{FS})$ $\iff$ $\mathcal{F} = \varnothing$ and $\mathbf{T} = \mathbf{T}(\mathcal{A}, d, \mathcal{F}) = \mathcal{A}^{\mathbb{Z}^d}$,

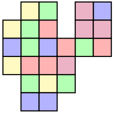
$\mathbf{T}$ *subshift of finite type* $(\mathcal{SFT})$ $\iff$ $\exists \mathcal{F} \subset \mathcal{A}^*$ a finite set such that $\mathbf{T} = \mathbf{T}(\mathcal{A}, d, \mathcal{F})$

$\mathbf{T}$ *subshift sofic* $(\mathcal{S}ofic)$ $\iff$ $\exists \mathcal{F} \subset \mathcal{A}^*$ a finite set and π a morphism such that $\mathbf{T} = \pi(\mathbf{T}(\mathcal{A}, d, \mathcal{F}))$

Subshifts defined by forbidden patterns



support $\mathbb{U} \subset \mathbb{Z}^d$ finite



pattern $u \in \mathcal{A}^{\mathbb{U}}$

Definition: Subshift of forbidden parterns $\mathcal{F} \subset \mathcal{A}^*$

$$\mathbf{T}(\mathcal{A}, d, \mathcal{F}) = \{x \in \mathcal{A}^{\mathbb{Z}^d} \ : \ \text{patterns of } \mathcal{F} \text{ does not appear in } x\} \subseteq \mathcal{A}^{\mathbb{Z}^d}$$

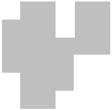
Some classes of subshifts:

$\mathbf{T}$ *fullshift* $(\mathcal{FS})$ $\iff$ $\mathcal{F} = \varnothing$ and $\mathbf{T} = \mathbf{T}(\mathcal{A}, d, \mathcal{F}) = \mathcal{A}^{\mathbb{Z}^d}$,

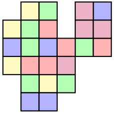
$\mathbf{T}$ *subshift of finite type* $(\mathcal{SFT})$ $\iff$ $\exists \mathcal{F} \subset \mathcal{A}^*$ a finite set such that $\mathbf{T} = \mathbf{T}(\mathcal{A}, d, \mathcal{F})$

$\mathbf{T}$ *subshift sofic* $(\mathcal{S}ofic)$ $\underset{d=1}{\iff}$ $\exists \mathcal{F} \subset \mathcal{A}^*$ a rational set such that $\mathbf{T} = \mathbf{T}(\mathcal{A}, 1, \mathcal{F})$

Subshifts defined by forbidden patterns



support $\mathbb{U} \subset \mathbb{Z}^d$ finite



pattern $u \in \mathcal{A}^{\mathbb{U}}$

Definition: Subshift of forbidden parterns $\mathcal{F} \subset \mathcal{A}^*$

$$\mathbf{T}(\mathcal{A}, d, \mathcal{F}) = \{x \in \mathcal{A}^{\mathbb{Z}^d} \ : \ \text{patterns of } \mathcal{F} \text{ does not appear in } x\} \subseteq \mathcal{A}^{\mathbb{Z}^d}$$

Some classes of subshifts:

$\mathbf{T}$ *fullshift* $(\mathcal{FS})$ $\iff$ $\mathcal{F} = \emptyset$ and $\mathbf{T} = \mathbf{T}(\mathcal{A}, d, \mathcal{F}) = \mathcal{A}^{\mathbb{Z}^d}$,

$\mathbf{T}$ *subshift of finite type* $(\mathcal{SFT})$ $\iff$ $\exists \mathcal{F} \subset \mathcal{A}^*$ a finite set such that $\mathbf{T} = \mathbf{T}(\mathcal{A}, d, \mathcal{F})$

$\mathbf{T}$ *subshift sofic* $(\mathcal{S}ofic)$ $\underset{d=1}{\iff}$ $\exists \mathcal{F} \subset \mathcal{A}^*$ a rational set such that $\mathbf{T} = \mathbf{T}(\mathcal{A}, 1, \mathcal{F})$

$\mathbf{T}$ *effective* $(\mathcal{RE})$ $\iff$ $\exists \mathcal{F} \subset \mathcal{A}^*$ a recursively enumerable set such that $\mathbf{T} = \mathbf{T}(\mathcal{A}, d, \mathcal{F})$

Computability obstruction

Proposition

$\mathcal{C}l_{\mathbf{SA}}(\mathcal{RE}) = \mathcal{RE}$
In particular $\mathcal{C}l_{\mathbf{SA}}(\mathcal{S}ofic) = \mathcal{C}l_{\mathbf{Fact},\mathbf{SA}}(\mathcal{SFT}) \subset \mathcal{RE}$

Proof:
Let $\mathbf{T} = \mathbf{T}(\mathcal{B}, 2, \mathcal{F})$ be a subshift such that $\mathcal{F}$ is enumerated by a Turing machine and denote $\mathcal{F}_m$ the m first patterns enumerated. Consider $\Sigma = \mathbf{SA}_{\mathbb{G}}(\mathbf{T})$:

u is a forbidden pattern of Σ $\iff$ $\exists m$ such that all patterns of support $[-m, m]^2$ which satisfy $\mathcal{F}_m$ does not contain u in the center.

- For $u \in \mathcal{A}^n$, consider a Turing machine $\mathcal{M}_u$ which on the enter m enumerate patterns of support $[-m, m]^2$ which contains u and satisfies $\mathcal{F}_m$. The machine $\mathcal{M}_u$ halts, and forbid u, if no pattern are produced.
- The Turing machine which enumerates forbidden patterns of Σ is constructed using $(\mathcal{M}_u)_{u \in \mathcal{A}^n}$ in parallel.

An important tool:
Simulation of effective subshifts by SFT

Realisation of effective subshift by sofic

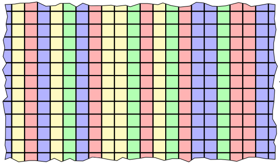
Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\texttt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\texttt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z},\ x_{\mathbb{Z}\mathbf{e_1}+i\mathbf{e_2}} = y \right\}.$$

Moreover $h_{\mathrm{top}}(\mathbf{T}_{\texttt{Final}}) = 0$.

$y \in \mathbf{T}$ iff a "superposition" of y in one direction is in $\pi(\mathbf{T}_{\texttt{Final}})$.



$\leftarrow y \in \mathbf{T}$

Corollary:

- $\mathcal{C}l_{\mathbf{SA}}(\mathcal{S}ofic) = \mathcal{RE}$.
- Every d-dimensional effective subshift is conjugate to the sub-action of a subshift of finite type.

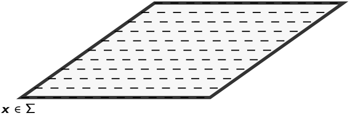
Realisation of effective subshift by sofic

Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

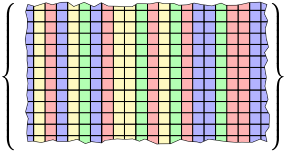
If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\mathtt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\mathtt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z}, \ x_{\mathbb{Z}\mathbf{e_1}+i\mathbf{e_2}} = y \right\}.$$

Idea of the proof:



Layer 1:

$\mathbf{T}_{\mathbf{Align}} =$ 

Aim:

We want to eliminate each x which contains forbidden patterns of Σ.

Realisation of effective subshift by sofic

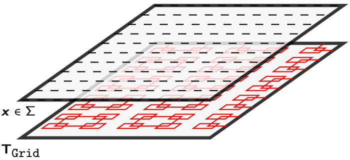
Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\texttt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\texttt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z},\ x_{\mathbb{Z}\mathbf{e_1}+i\mathbf{e_2}} = y \right\}.$$

Idea of the proof:

Layer 2:



Realisation of effective subshift by sofic

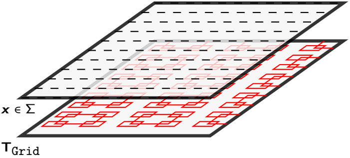
Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

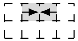
If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\texttt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\texttt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z},\ x_{\mathbb{Z}\mathbf{e_1}+i\mathbf{e_2}} = y \right\}.$$

Idea of the proof:

Layer 2:





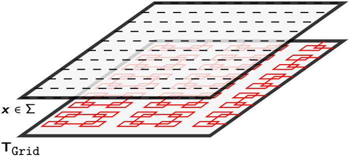
Realisation of effective subshift by sofic

Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

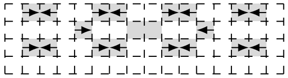
If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\texttt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\texttt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z},\ x_{\mathbb{Z}\mathbf{e_1}+i\mathbf{e_2}} = y \right\}.$$

Idea of the proof:



Layer 2:



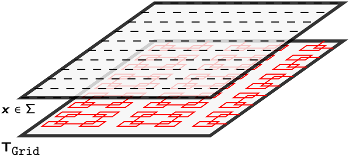
Realisation of effective subshift by sofic

Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

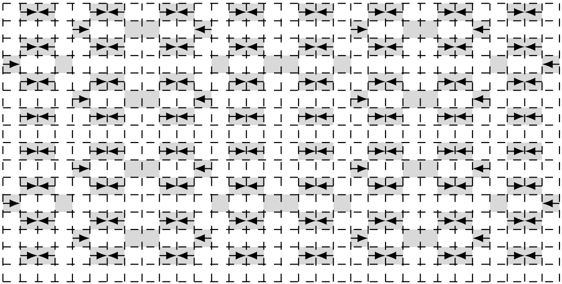
If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\texttt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\texttt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z},\ x_{\mathbb{Z}\mathbf{e_1}+i\mathbf{e_2}} = y \right\}.$$

Idea of the proof:



Layer 2:



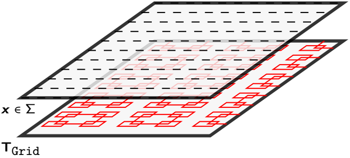
Realisation of effective subshift by sofic

Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

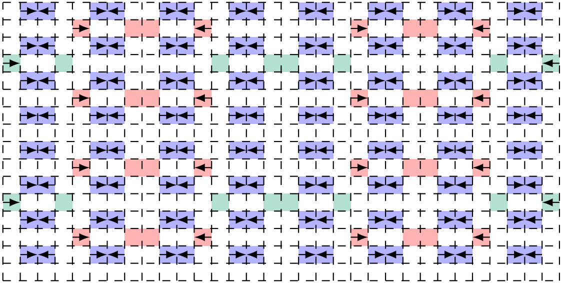
If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\texttt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\texttt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z},\ x_{\mathbb{Z}\mathbf{e_1}+i\mathbf{e_2}} = y \right\}.$$

Idea of the proof:



Layer 2:



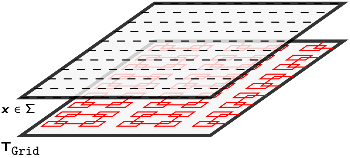
Realisation of effective subshift by sofic

Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

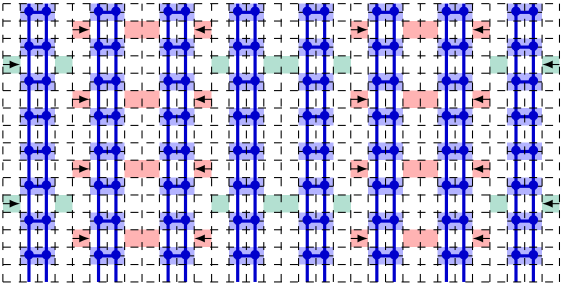
If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\texttt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\texttt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z},\ x_{\mathbb{Z}\mathbf{e_1}+i\mathbf{e_2}} = y \right\}.$$

Idea of the proof:



Layer 2:



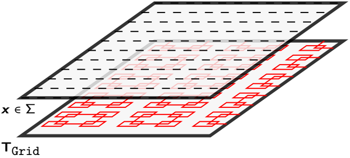
Realisation of effective subshift by sofic

Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

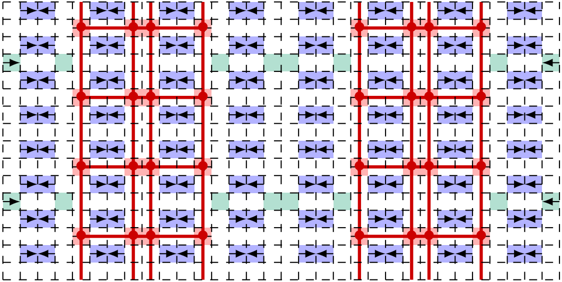
If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\texttt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\texttt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z},\ x_{\mathbb{Z}\mathbf{e_1}+i\mathbf{e_2}} = y \right\}.$$

Idea of the proof:



Layer 2:



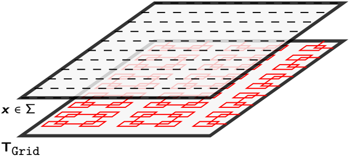
Realisation of effective subshift by sofic

Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

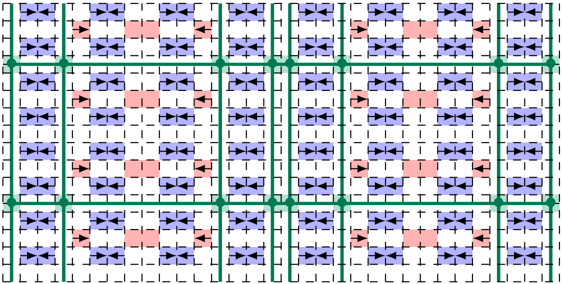
If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\texttt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\texttt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z},\ x_{\mathbb{Z}\mathbf{e_1}+i\mathbf{e_2}} = y \right\}.$$

Idea of the proof:



Layer 2:



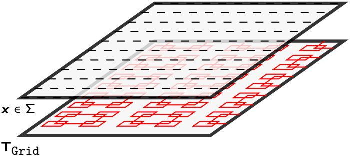
Realisation of effective subshift by sofic

Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

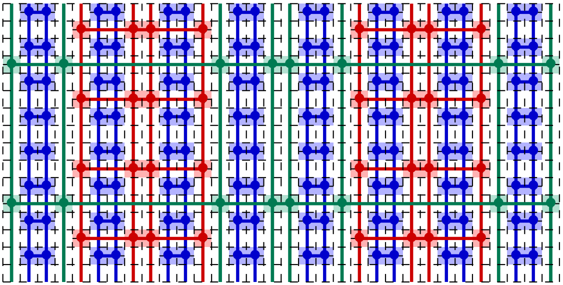
If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\texttt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\texttt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z},\ x_{\mathbb{Z}\mathbf{e_1}+i\mathbf{e_2}} = y \right\}.$$

Idea of the proof:



Layer 2:



Realisation of effective subshift by sofic

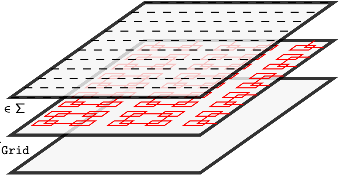
Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

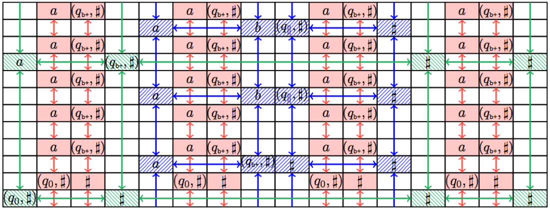
If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\texttt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\texttt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z},\ x_{\mathbb{Z}\mathbf{e_1}+i\mathbf{e_2}} = y \right\}.$$

Idea of the proof:

Layer 3: Enumeration of forbidden patterns





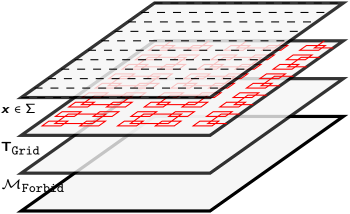
Realisation of effective subshift by sofic

Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

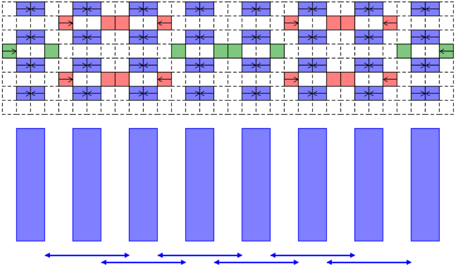
If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\texttt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\texttt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z},\ x_{\mathbb{Z}\mathbf{e_1}+i\mathbf{e_2}} = y \right\}.$$

Idea of the proof:



Layer 3: Responsability zones of $\mathcal{M}_{\texttt{Forbid}}$



$\mathcal{M}_{\texttt{Forbid}}$ of a level n can ask at $\mathcal{M}_{\texttt{Search}}$ of the same level or neighbor $\mathcal{M}_{\texttt{Search}}$ of the same level.

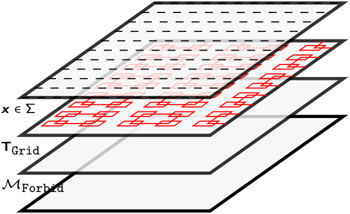
Realisation of effective subshift by sofic

Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

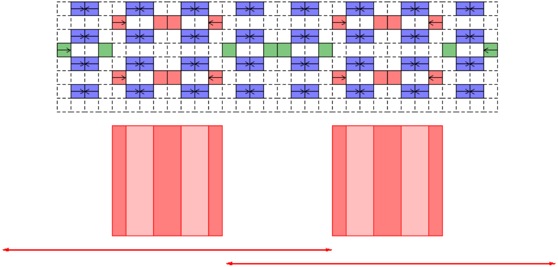
If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\texttt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\texttt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z},\ x_{\mathbb{Z}\mathbf{e_1}+i\mathbf{e_2}} = y \right\}.$$

Idea of the proof:



Layer 3: Responsability zones of $\mathcal{M}_{\texttt{Forbid}}$



$\mathcal{M}_{\texttt{Forbid}}$ of a level n can ask at $\mathcal{M}_{\texttt{Search}}$ of the same level or neighbor $\mathcal{M}_{\texttt{Search}}$ of the same level.

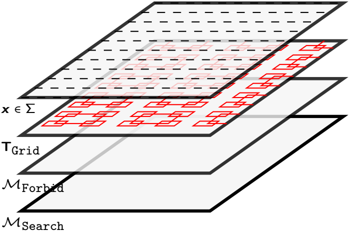
Realisation of effective subshift by sofic

Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

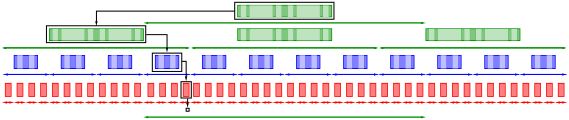
If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\texttt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\texttt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z},\ x_{\mathbb{Z}\mathbf{e_1}+i\mathbf{e_2}} = y \right\}.$$

Idea of the proof:



Layer 4:



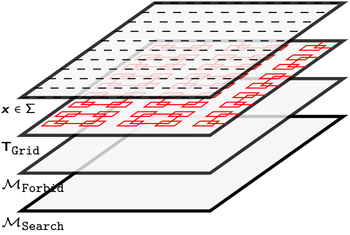
Realisation of effective subshift by sofic

Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

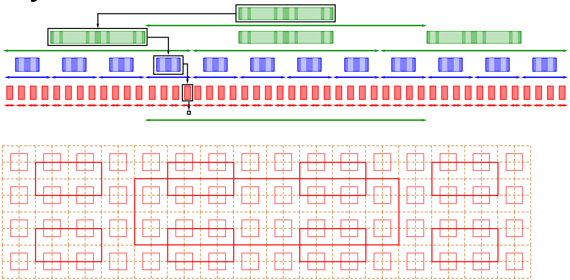
If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\texttt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\texttt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z},\ x_{\mathbb{Z}\mathbf{e_1}+i\mathbf{e_2}} = y \right\}.$$

Idea of the proof:



Layer 4:



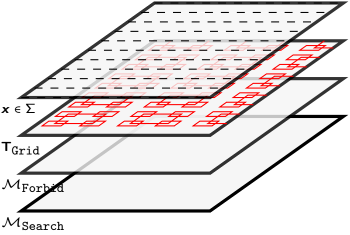
Realisation of effective subshift by sofic

Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

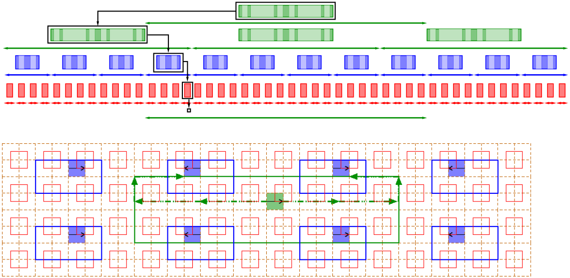
If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\texttt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\texttt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z},\ x_{\mathbb{Z}\mathbf{e_1}+i\mathbf{e_2}} = y \right\}.$$

Idea of the proof:



Layer 4:



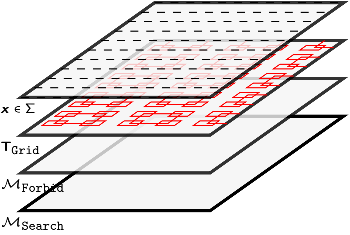
Realisation of effective subshift by sofic

Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

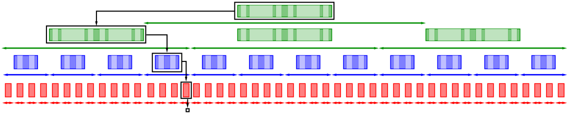
If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\mathtt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\mathtt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z},\ x_{\mathbb{Z}\mathbf{e_1}+i\mathbf{e_2}} = y \right\}.$$

Idea of the proof:



Layer 4:



$\mathcal{M}_{\mathtt{Search}}$ holds:

- There is enough space to code address.
- The time taken to give back the information is $t(n) \leq 2^n \times \mathcal{O}(n^2 2^n)$ which is "absorbed" by the exponential time of the clock (2^{2^n}).

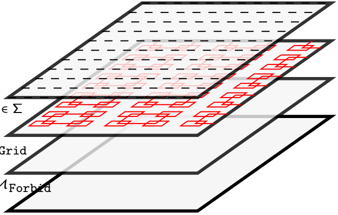
Realisation of effective subshift by sofic

Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\mathtt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\mathtt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z},\ x_{\mathbb{Z}\mathbf{e_1}+i\mathbf{e_2}} = y \right\}.$$

Idea of the proof:



Layer 4:

Communication between layers:

- condition **Request** : $\mathcal{M}_F$ ask $\mathcal{M}_{\text{Search}}$ the value of a box in the responsability zone and wait the answer
- condition **Forbid** : exclude configuration when forbidden pattern are encounter a

To obtain Σ :

- operation **Fact** to keep only letters of $\mathcal{A}_\Sigma$
- operation **SA** to keep only an horizontal line

Perspectives around sub-dynamic

Optimality of the construction

- A so huge alphabet.
- A long range of dependance to detect forbidden patterns. Wait course 3!
- Construction very rigid: What happens if we impose some mixing properties? Wait course 3!

Perspectives around sub-dynamic

Optimality of the construction

- A so huge alphabet.
- A long range of dependance to detect forbidden patterns. Wait course 3!
- Construction very rigid: What happens if we impose some mixing properties? Wait course 3!

Sub-dynamic

In *Hochman-09* there is a characterization of subaction of d-dimensional sofic with $d \geq 3$. What happens for $d = 2$?

Perspectives around sub-dynamic

Optimality of the construction

- A so huge alphabet.
- A long range of dependance to detect forbidden patterns. Wait course 3!
- Construction very rigid: What happens if we impose some mixing properties? Wait course 3!

Sub-dynamic

In *Hochman-09* there is a characterization of subaction of d-dimensional sofic with $d \geq 3$. What happens for $d = 2$?

Projective sub-dynamic of SFT

We have $\mathcal{C}l_{\mathbf{SA}}(\mathcal{S}ofic) = \mathcal{RE}$. Which information we have about $\mathcal{C}l_{\mathbf{SA}}(\mathcal{SFT})$? In this case we cannot use additional alphabet to make computation.
Wait course 3!

Applications

Applications to find local rules

Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\texttt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\texttt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z},\ x_{\mathbb{Z}\mathbf{e_1}+i\mathbf{e_2}} = y \right\}.$$

A framework for computability results :

- Computability Obstructions on SFTs are usually also obstructions for effective shifts.
- Prove the obstruction is the only obstruction for effective shifts.
- Use the previous theorem to go back to SFTs.

Applications to find local rules

Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\mathtt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\mathtt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z},\ x_{\mathbb{Z}\mathbf{e_1}+i\mathbf{e_2}} = y \right\}.$$

Applications:

- characterization of the entropy of multidimensional SFTs *(Hochman-Meyerovitch-10)*,
- characterization of multidimensional **S**-adic subshift with local rules *(Aubrun-Sablik-12)*,
- characterization of tilings which approximate discrete plane *(Fernique-Sablik-12)*,
- characterization of periods of multidimensional SFTs *(Jeandel-Vanier-13)*,
- characterization of the function with measure quasi-periodicity *(Ballier-Jeandel-10)*,
- ...

Applications to find local rules

Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\texttt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\texttt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z},\ x_{\mathbb{Z}\mathbf{e_1} + i\mathbf{e_2}} = y \right\}.$$

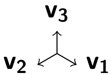
Applications:

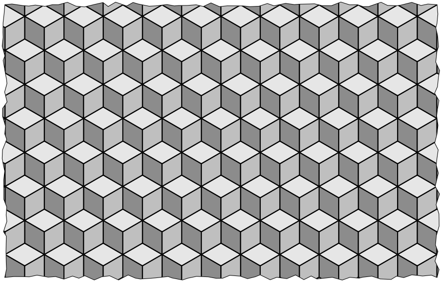
- characterization of the entropy of multidimensional SFTs *(Hochman-Meyerovitch-10)*,
- characterization of multidimensional **S**-adic subshift with local rules *(Aubrun-Sablik-12)*,
- characterization of tilings which approximate discrete plane *(Fernique-Sablik-12)*,
- characterization of periods of multidimensional SFTs *(Jeandel-Vanier-13)*,
- characterization of the function with measure quasi-periodicity *(Ballier-Jeandel-10)*,
- ...

Approximation of discrete plane

$n \to d$ tilings

Let $\mathbf{v_1}, \ldots, \mathbf{v_n}$ be pairwise non-colinear vectors of $\mathbb{R}^d$ with $n > d > 0$.

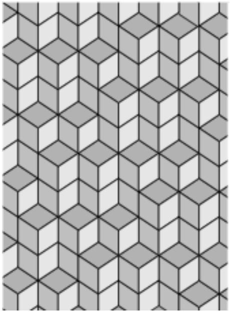




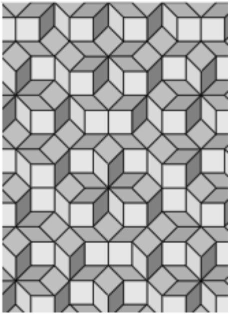
- A $n \to d$ *tile* is a parallelotope generated by d of the $\mathbf{v_i}$'s, there are $\binom{n}{d}$ tiles.
- A $n \to d$ *tiling* is a face-to-face tiling of $\mathbb{R}^d$ by $n \to d$ tiles.
- The set $\mathcal{X}_{n \to d}$ of all tilings of $\mathbb{R}^d$ by $n \to d$-tiles is the *full $n \to d$ tiling space*.

$n \to d$ tilings

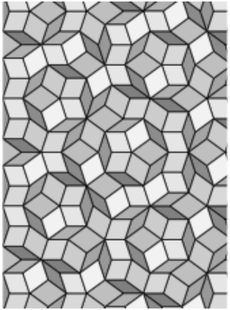
Let $\mathbf{v_1}, \ldots, \mathbf{v_n}$ be pairwise non-colinear vectors of $\mathbb{R}^d$ with $n > d > 0$.



$3 \to 2$



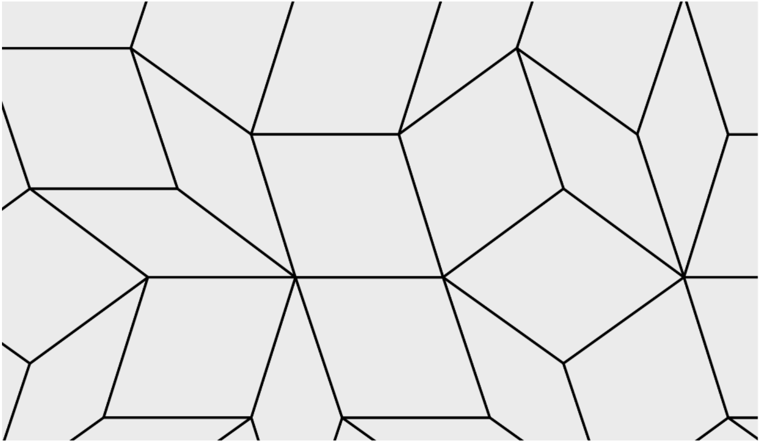
$4 \to 2$



$5 \to 2$

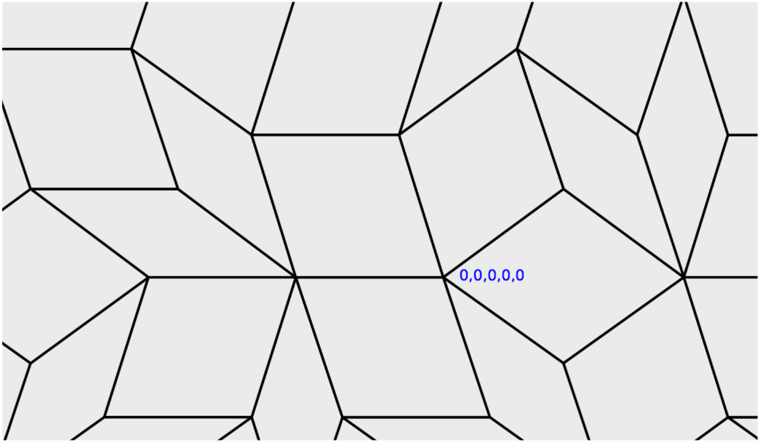
- A *$n \to d$ tile* is a parallelotope generated by d of the $\mathbf{v_i}$'s, there are $\binom{n}{d}$ tiles.
- A *$n \to d$ tiling* is a face-to-face tiling of $\mathbb{R}^d$ by $n \to d$ tiles.
- The set $\mathcal{X}_{n \to d}$ of all tilings of $\mathbb{R}^d$ by $n \to d$-tiles is the *full $n \to d$ tiling space*.

Lift



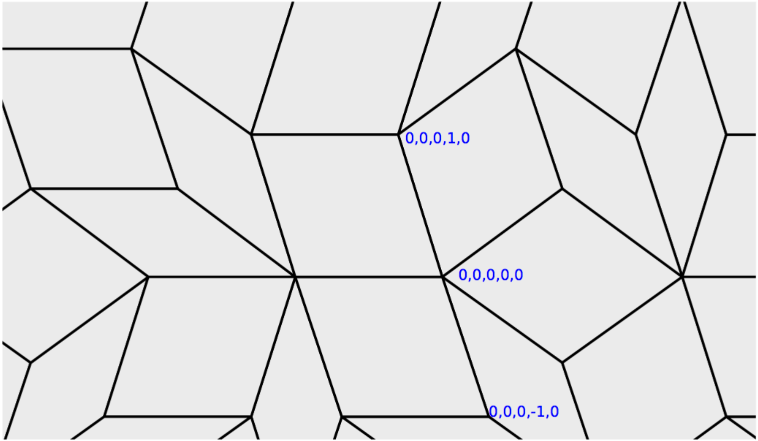
Consider a $n \to d$ tiling.

Lift



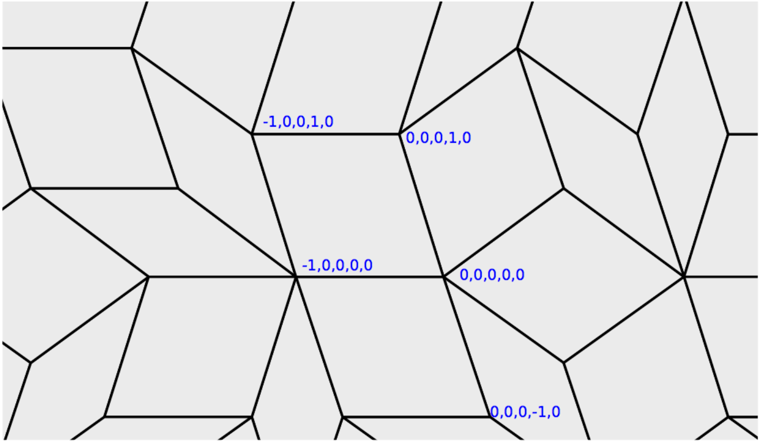
Map an arbitrary vertex onto an arbitrary vector of $\mathbb{Z}^n$.

Lift



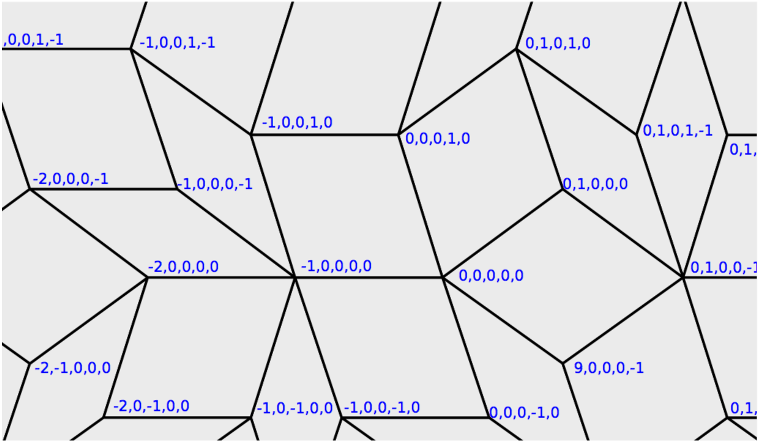
Modify the k^{th} entry when moving along the k^{th} direction.

Lift



$n \to d$ vertices are mapped onto vertices of $[0,1]^n$.

Lift

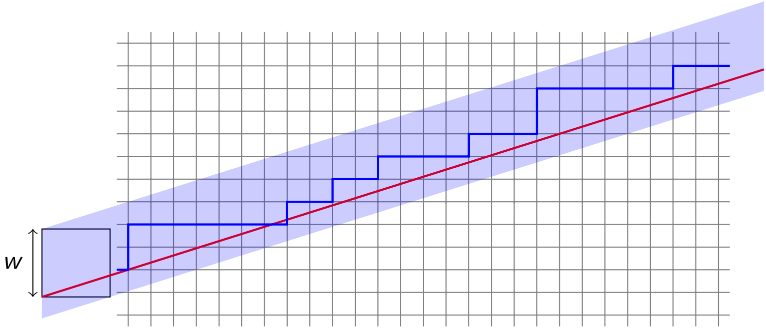


The whole tiling is mapped onto a stepped surface of $\mathbb{R}^n$: its *lift*.

Planar tilings

Definition

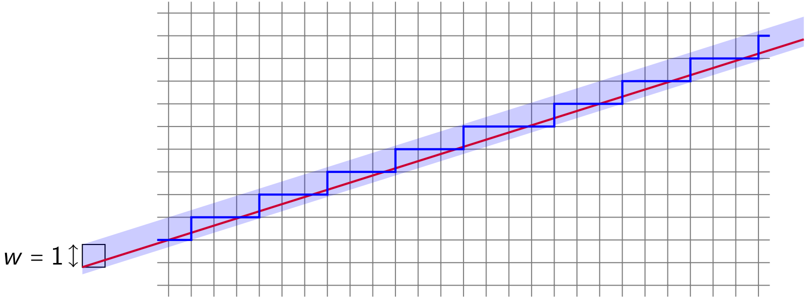
A $n \to d$ tilings set $\mathbf{T} \subset \mathcal{X}_{n \to d}$ is a *planar tiling space* if there are a d-dimensional vector subspace $V \subset \mathbb{R}^n$, the *slope* and a positive integer w, the *width*, such that all tiling $t \in \mathbf{T}$ can be lifted into the slice $V + [0, w)^n$.



Planar tilings

Definition

A $n \to d$ tilings set $\mathbf{T} \subset \mathcal{X}_{n \to d}$ is a *planar tiling space* if there are a d-dimensional vector subspace $V \subset \mathbb{R}^n$, the *slope* and a positive integer w, the *width*, such that all tiling $t \in \mathbf{T}$ can be lifted into the slice $V + [0, w)^n$.

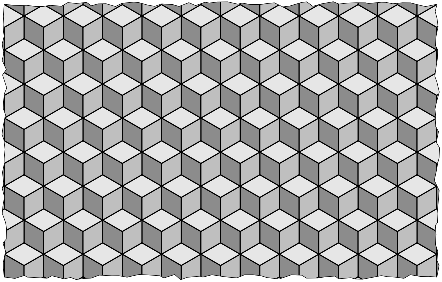


The $w = 1$ case corresponds to *strong planar tilings*.

Planar tilings

Definition

A $n \to d$-tilings set $\mathbf{T} \subset \mathcal{X}_{n \to d}$ is a *planar tiling space* if there are a d-dimensional vector subspace $V \subset \mathbb{R}^n$, the *slope* and a positive integer w, the *width*, such that all tiling $t \in \mathbf{T}$ can be lifted into the slice $V + [0, w)^n$.

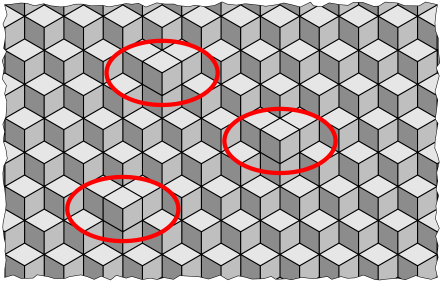


The $w = 1$ case corresponds to *strong planar tilings*.

Planar tilings

Definition

A $n \to d$-tilings set $\mathbf{T} \subset \mathcal{X}_{n \to d}$ is a *planar tiling space* if there are a d-dimensional vector subspace $V \subset \mathbb{R}^n$, the *slope* and a positive integer w, the *width*, such that all tiling $t \in \mathbf{T}$ can be lifted into the slice $V + [0, w)^n$.

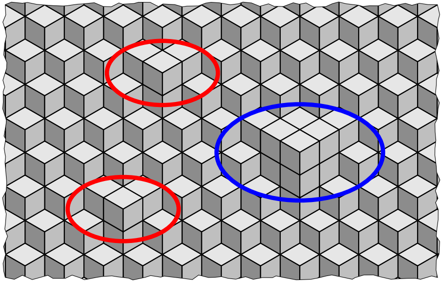


Here $w = 2$, if $w \geq 2$, this corresponds to *weak planar tilings*.

Planar tilings

Definition

A $n \to d$-tilings set $\mathbf{T} \subset \mathcal{X}_{n\to d}$ is a *planar tiling space* if there are a d-dimensional vector subspace $V \subset \mathbb{R}^n$, the *slope* and a positive integer w, the *width*, such that all tiling $t \in \mathbf{T}$ can be lifted into the slice $V + [0, w)^n$.



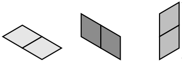
Here $w = 3$, if $w \geq 2$, this corresponds to *weak planar tilings*.

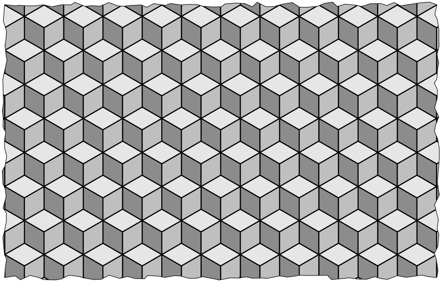
Local rules

- A *$n \to d$-pattern of size r* of a tiling $t \in \mathcal{X}_{n\to d}$ is a set of tiles lying inside a ball of radius $r > 0$. For $\mathbf{T} \subset \mathcal{X}_{n\to d}$ denote $\mathcal{P}_r(\mathbf{T})$ the set of $n \to d$-pattern of size r of each tiling of $\mathbf{T}$.
- The set of tilings of forbidden $n \to d$-patterns $\mathcal{F}$ is

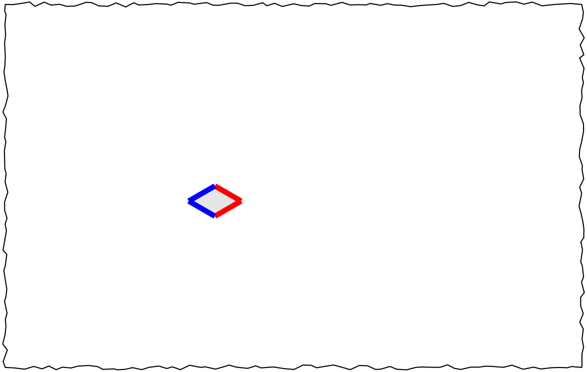
$$\mathbf{T}_{\mathcal{F}} = \{t \in \mathcal{X}_{n\to d} : \text{no patterns of } \mathcal{F} \text{ appears in } t\}$$

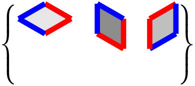
- $\mathbf{T}$ is a *set of tilings of finite type* if there exists $\mathcal{F}$ finite such that $\mathbf{T} = \mathbf{T}_{\mathcal{F}}$.

$\mathcal{F} = \{$  $\}$

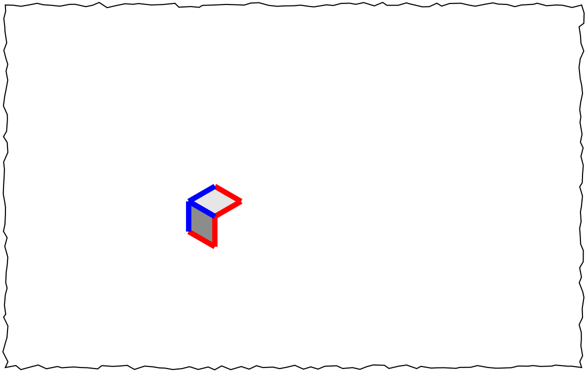


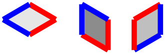
Colored local rules



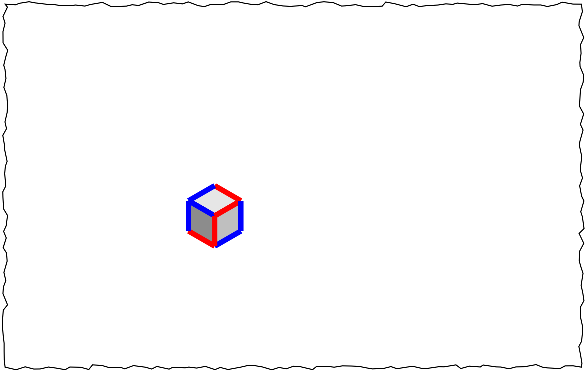
Consider these decorated $3 \to 2$ tiles: , which can match only if the corresponding edges have the same color.

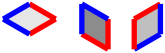
Colored local rules



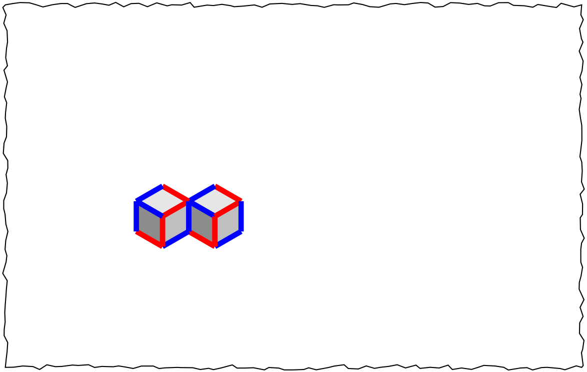
Consider these decorated $3 \to 2$ tiles: $\{$  $\}$, which can match only if the corresponding edges have the same color.

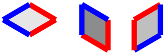
Colored local rules



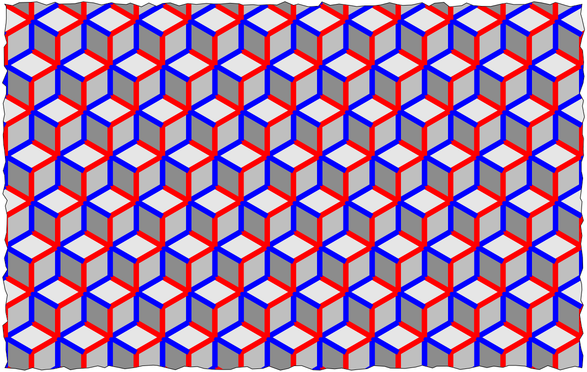
Consider these decorated $3 \to 2$ tiles: $\Big\{$  $\Big\}$, which can match only if the corresponding edges have the same color.

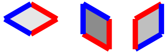
Colored local rules



Consider these decorated $3 \rightarrow 2$ tiles: $\left\{ \right.$  $\left. \right\}$, which can match only if the corresponding edges have the same color.

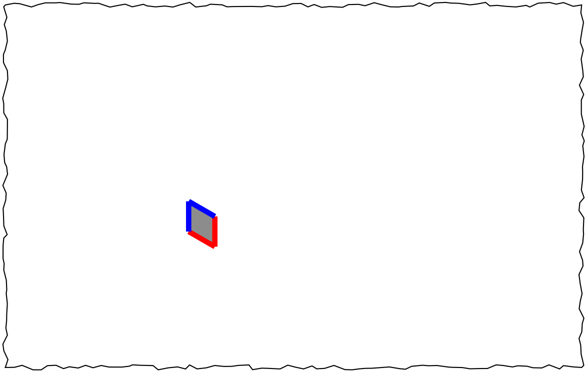
Colored local rules

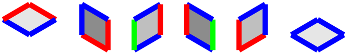


Consider these decorated $3 \to 2$ tiles: $\{$  $\}$, which can match only if the corresponding edges have the same color.

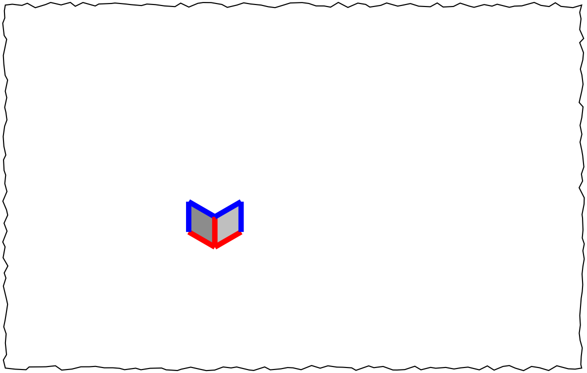
A set of tilings has *colored local rules* if it is possible to decorate tiles to obtain it.

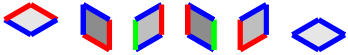
Colored local rules



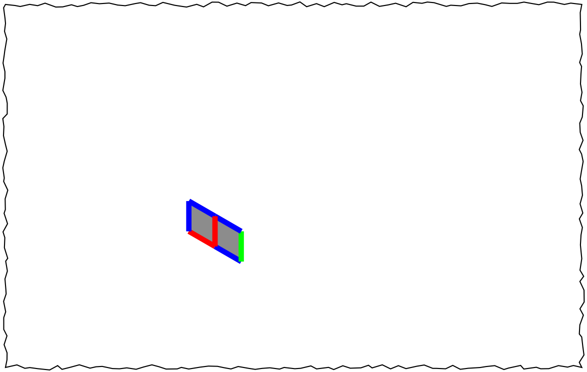
Consider these decorated $3 \to 2$ tiles: $\{$  $\}$, which can match only if the corresponding edges have the same color.

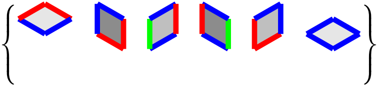
Colored local rules



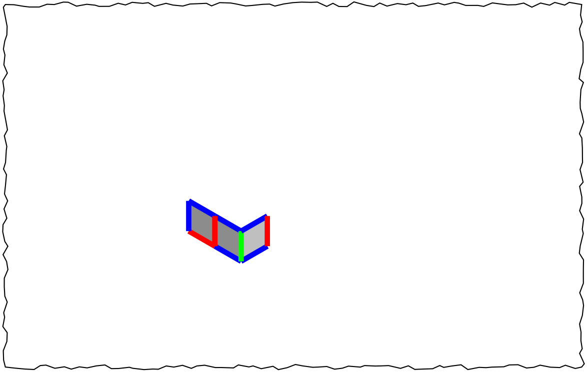
Consider these decorated $3 \to 2$ tiles: $\Bigg\{$  $\Bigg\}$, which can match only if the corresponding edges have the same color.

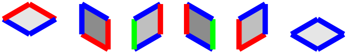
Colored local rules



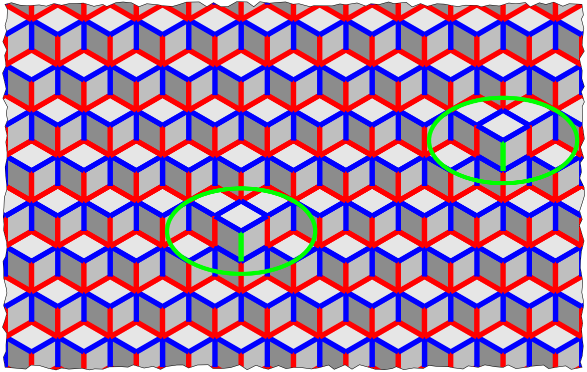
Consider these decorated $3 \to 2$ tiles: , which can match only if the corresponding edges have the same color.

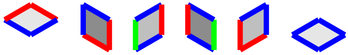
Colored local rules



Consider these decorated $3 \to 2$ tiles: $\Bigg\{$  $\Bigg\}$, which can match only if the corresponding edges have the same color.

Colored local rules



Consider these decorated 3 → 2 tiles: $\Big\{$  $\Big\}$, which can match only if the corresponding edges have the same color.
This allows only small fluctuations and obtain a weak planar set of tilings

Historic of the problem

Which vector space admits local rules or colored local rules?

n-fold tiling: plane tiling of slope $\mathbb{R}(u_1, ..., u_n) + \mathbb{R}(v_1, ..., v_n)$, where

$$u_k = \cos\left(\frac{2k\pi}{n}\right) \text{ and } v_k = \sin\left(\frac{2k\pi}{n}\right)$$

Slope of the Tiling	undecorated rules	decorated rules
$5, 10$-fold	stong	strong[(1)]
8-fold	none[(2)]	strong[(3)]
12-fold	none[(3)]	strong[(4)]
n-fold (with 4 not divide n)	weak[(5)]	strong?
quadratic slope in $\mathbb{R}^4$	weak[(6)]	strong[(7)]
non algebraic	none[(8)]	?

(1): *Penrose 1974* (2): *Burkov 1988* (3): *Le 1992* (4): *Socolar 1989*
(5): *Socolar 1990* (6): *Levitov 1988* (7): *Le & al. 1992* (8): *Le 1997*

Main results

- A vector $\mathbf{v} \in \mathbb{R}^n$ is *computable* if there exists a computable function $f : \mathbb{N} \longrightarrow \mathbb{Q}^n$ such that $\|\mathbf{v} - f(n)\|_\infty \leq 2^{-n}$ for all $n \in \mathbb{N}$.
- The vector space $V \subset \mathbb{R}^n$ of dimension d is *computable* if there exists a set of d computable vectors which generate V.

Main results

- A vector $\mathbf{v} \in \mathbb{R}^n$ is *computable* if there exists a computable function $f : \mathbb{N} \longrightarrow \mathbb{Q}^n$ such that $\|\mathbf{v} - f(n)\|_\infty \leq 2^{-n}$ for all $n \in \mathbb{N}$.
- The vector space $V \subset \mathbb{R}^n$ of dimension d is *computable* if there exists a set of d computable vectors which generate V.

Theorem *(Fernique & S.)*

A d-dimensional vector space V admits $n \to d$ weak colored local rules (of width 3) for $n > d$ if and only if it is computable.

Theorem *(Fernique & S.)*

A d-dimensional vector space V admits $n \to d$ weak local rules (of width 4) for $n > d$ if and only if it is computable.

Historic of the problem

Which vector space admits local rules or colored local rules?

n-fold tiling: plane tiling of slope $\mathbb{R}(u_1, ..., u_n) + \mathbb{R}(v_1, ..., v_n)$, where

$$u_k = \cos\left(\frac{2k\pi}{n}\right) \text{ and } v_k = \sin\left(\frac{2k\pi}{n}\right)$$

Slope of the Tiling	undecorated rules	decorated rules
$5, 10$-fold	stong	strong(1)
8-fold	none(2)	strong(3)
12-fold	none(3)	strong(4)
n-fold (with 4 not divide n)	weak(5)	strong?
quadratic slope in $\mathbb{R}^4$	weak(6)	strong(7)
non algebraic	none(8)	?
computable	weak	weak

(1): *Penrose 1974* (2): *Burkov 1988* (3): *Le 1992* (4): *Socolar 1989*
(5): *Socolar 1990* (6): *Levitov 1988* (7): *Le & al. 1992* (8): *Le 1997*

Historic of the problem

Which vector space admits local rules or colored local rules?

n-fold tiling: plane tiling of slope $\mathbb{R}(u_1, \ldots, u_n) + \mathbb{R}(v_1, \ldots, v_n)$, where

$$u_k = \cos\left(\frac{2k\pi}{n}\right) \text{ and } v_k = \sin\left(\frac{2k\pi}{n}\right)$$

Slope of the Tiling	undecorated rules	decorated rules
5, 10-fold	stong	strong[1]
8-fold	none[2]	strong[3]
12-fold	none[3]	strong[4]
n-fold (with 4 not divide n)	weak[5]	strong?
quadratic slope in $\mathbb{R}^4$	weak[6]	strong[7]
non algebraic	none[8]	?
computable	weak (non natural)	weak

Notion of natural local rules

Local rules are said *natural* if there are verified by strong planar tiling.

Computability obstruction

Algorithm to obtain the slope

Input: Local rules of the planar tilings set $\mathbf{T} \subset \mathcal{X}_{n \to d}$, the width w and an integer m which corresponds to the precision.

Algorithm:

- $r_0 := 2wm$, $r := r_0$ and $d := 1$
- While $d \geq \frac{1}{2m}$ do
  - enumerate $\mathcal{P}_r(\mathbf{T})$, the set of all the diameter r patterns centered on 0 allowed by these local rules (this takes exponential but finite time in r)
  - enumerate $\mathbb{X}^r_{r_0}$, the'
  $$d = \max_{W_1, W_2 \in \mathbb{X}_r} \widetilde{d}(W_1, W_2)$$
  - r:=r+1
- Output: an element of $W \in \mathbb{X}_r$

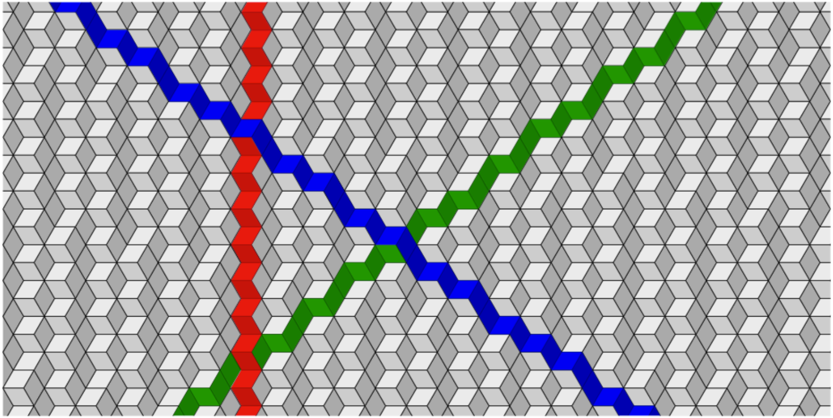
The algorithm halts: For sufficiently large r all vector spaces of $\mathbb{X}_r$ are near of V, if not by compacity one obtains one other slope for the $n \to d$ tiling.

The algorithm holds: There exists $W' \in \mathbb{X}_r$ such that $\widetilde{d}(W', V) \leq \frac{w}{r_0}$, thus

$$\widetilde{d}(W, V) \leq \widetilde{d}(W, W') + \widetilde{d}(W', V) \leq \frac{1}{2m} + \frac{w}{r_0} \leq \frac{1}{m}$$

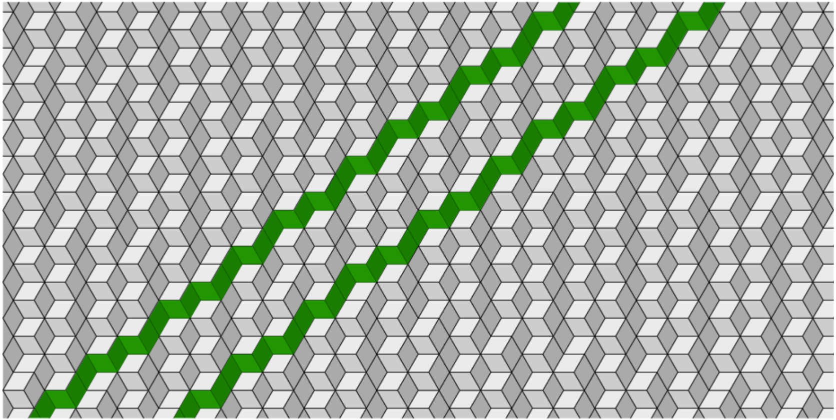
Realization of computable $3 \to 2$ planar tilings with colored local rules

Stripes of 3 → 2 strong planar tilling



For 3 → 2 strong planar tilling, intertwined stripes encoding Sturmian words.

Stripes of 3 → 2 strong planar tilling



Parallel stripes encode Sturmian words with the same slope.

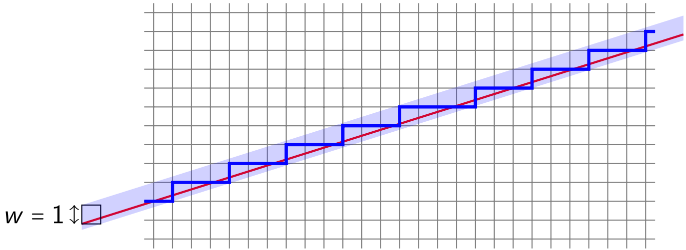
Quasi-Sturmian words

▶Define the *Sturmian word* $s_{\rho,\alpha} \in \{0,1\}^{\mathbb{Z}}$ of slope $\alpha \in [0,1]$ and intercept ρ by

$$s_{\rho,\alpha}(n) = 0 \iff (\rho + n\alpha) \mod 1 \in [0, 1-\alpha).$$

▶For $x, y \in \{0,1\}^{\mathbb{Z}}$ define $d(x,y) := \sup_{p \le q} \left| |x_p x_{p+1} \ldots x_q|_0 - |y_p y_{p+1} \ldots y_q|_0 \right|$.

Fact: *Sturmian words with equal slopes are at distance at most one.*



$s_{\rho,\alpha}$: $\ldots 0010001000100010000100010001000100010 \ldots$

Quasi-Sturmian words

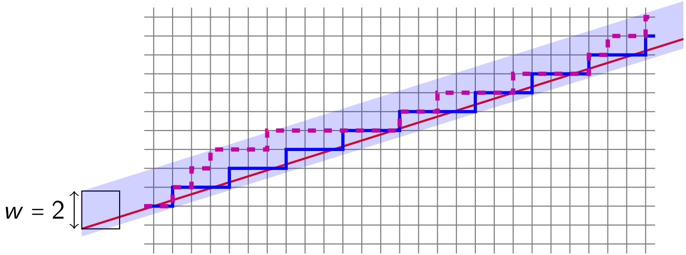
- Define the *Sturmian word* $s_{\rho,\alpha} \in \{0,1\}^{\mathbb{Z}}$ of slope $\alpha \in [0,1]$ and intercept ρ by

$$s_{\rho,\alpha}(n) = 0 \iff (\rho + n\alpha) \mod 1 \in [0, 1-\alpha).$$

- For $x, y \in \{0,1\}^{\mathbb{Z}}$ define $d(x,y) := \sup_{p \le q} \left| |x_p x_{p+1} \dots x_q|_0 - |y_p y_{p+1} \dots y_q|_0 \right|$.

Fact: *Sturmian words with equal slopes are at distance at most one.*

- $x \in \{0,1\}^{\mathbb{Z}}$ is *a quasi-Sturmian of slope* α if $d(x, s_{\rho,\alpha}) \le 1$.



$s_{\rho,\alpha}$: $\dots 00100010001000100010000100010001000100010 \dots$

Quasi-Sturmian: $\dots 00101010001000000010010000000001001001010 \dots$

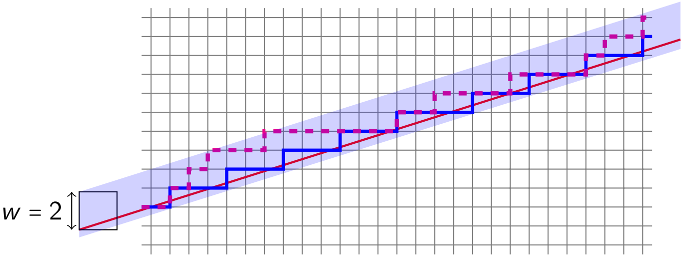
Quasi-Sturmian words

▶Define the *Sturmian word* $s_{\rho,\alpha} \in \{0,1\}^{\mathbb{Z}}$ of slope $\alpha \in [0,1]$ and intercept ρ by

$$s_{\rho,\alpha}(n) = 0 \iff (\rho + n\alpha) \mod 1 \in [0, 1-\alpha).$$

▶$x \in \{0,1\}^{\mathbb{Z}}$ is *a quasi-Sturmian of slope α* if $d(x, s_{\rho,\alpha}) \leq 1$.

Fact: *Two words in $\{0,1\}^{\mathbb{Z}}$ are at distance at most one if and only if each can be obtained from the other by performing letter replacements $0 \to 1$ or $1 \to 0$, without two consecutive replacements of the same type.*



$s_{\rho,\alpha}$: ...0010001000100010001000010001000010...
Quasi-Sturmian: ...0010101000100000001001000010001010...
Changes: ⇑ ⇑ ⇑ ⇑ ⇑⇑ ⇑⇑ ⇑ ...

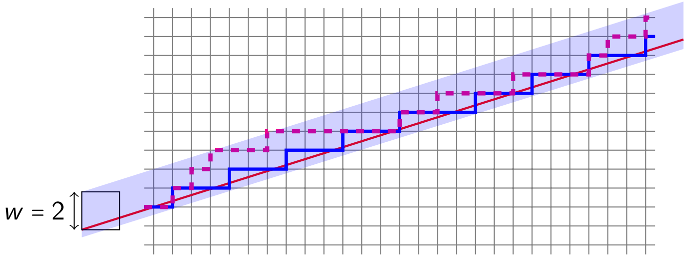
Quasi-Sturmian words

▶Define the *Sturmian word* $s_{\rho,\alpha} \in \{0,1\}^{\mathbb{Z}}$ of slope $\alpha \in [0,1]$ and intercept ρ by

$$s_{\rho,\alpha}(n) = 0 \;\Leftrightarrow\; (\rho + n\alpha) \mod 1 \in [0, 1-\alpha).$$

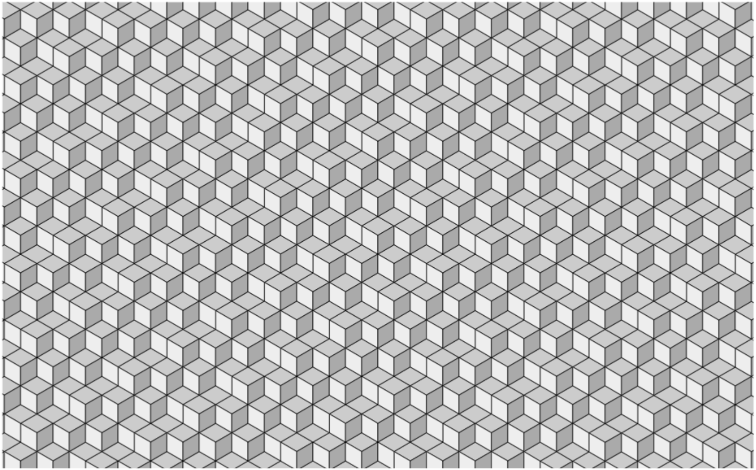
▶$x \in \{0,1\}^{\mathbb{Z}}$ is *a quasi-Sturmian of slope α* if $d(x, s_{\rho,\alpha}) \leq 1$.

Fact: *Two words in $\{0,1\}^{\mathbb{Z}}$ are at distance at most one if and only if each can be obtained from the other by performing letter replacements $0 \to 1$ or $1 \to 0$, without two consecutive replacements of the same type.*



$s_{\rho,\alpha}$: ...001000100010001000100010001000100010...
Quasi-Sturmian: ...001010100010000000100100001000100101 0...
Coding: ...000001111111111000000011000100010011 1...

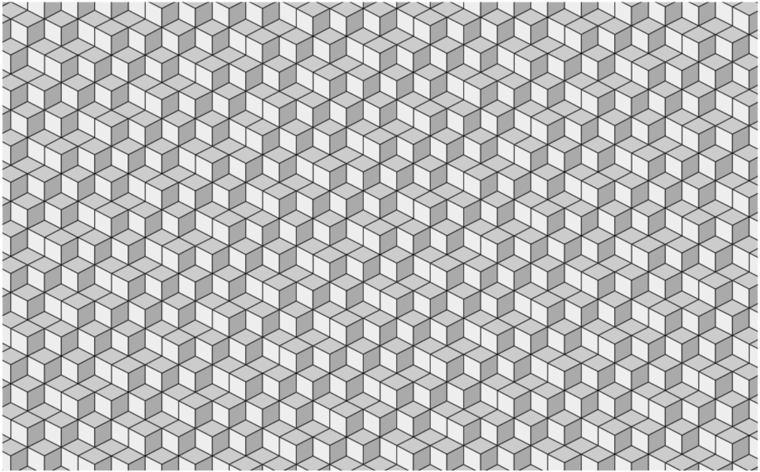
From strong planar 3 → 2 tilings to quasi-Sturmian subshifts



$\mathbf{T} \subset \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : \forall m \exists \rho \; x_{(.,m)} = s_{\rho,\alpha} \right\} \subset \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : \exists \rho \forall m \; d(x_{(.,m)}, s_{\rho,\alpha}) \leq 1 \right.$

Sturmian subshift — Quasi-sturmian subshift

From strong planar 3 → 2 tilings to quasi-Sturmian subshifts

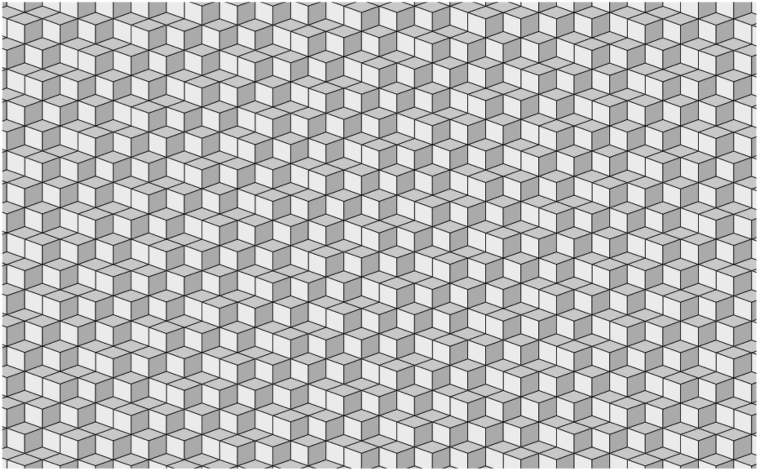


$\mathbf{T} \quad \subset \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : \forall m \exists \rho \; x_{(.,m)} = s_{\rho,\alpha} \right\} \subset \quad \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : \exists \rho \forall m \; d(x_{(.,m)}, s_{\rho,\alpha}) \leq 1 \right.$

Sturmian subshift

Quasi-sturmian subshift

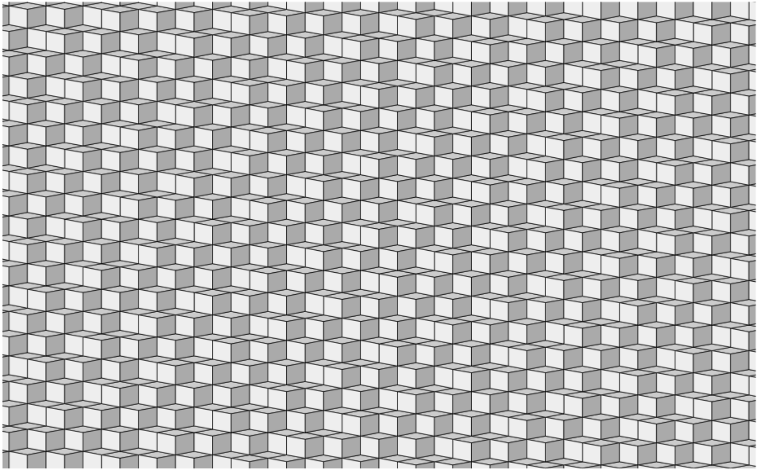
From strong planar 3 → 2 tilings to quasi-Sturmian subshifts



$$\mathbf{T} \subset \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : \forall m \exists \rho \; x_{(.,m)} = s_{\rho,\alpha} \right\} \subset \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : \exists \rho \forall m \; d(x_{(.,m)}, s_{\rho,\alpha}) \leq 1 \right.$$

Sturmian subshift — Quasi-sturmian subshift

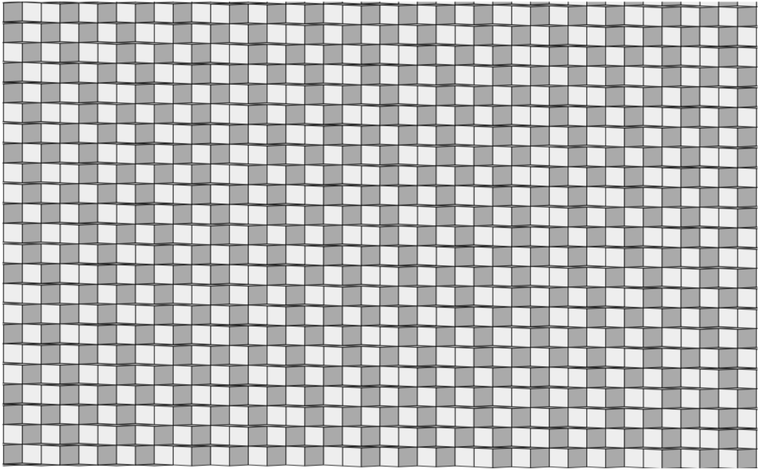
From strong planar 3 → 2 tilings to quasi-Sturmian subshifts



$\mathbf{T} \quad \subset \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : \forall m \exists \rho \; x_{(.,m)} = s_{\rho,\alpha} \right\} \subset \quad \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : \exists \rho \forall m \; d(x_{(.,m)}, s_{\rho,\alpha}) \leq 1 \right.$

Sturmian subshift — Quasi-sturmian subshift

From strong planar 3 → 2 tilings to quasi-Sturmian subshifts

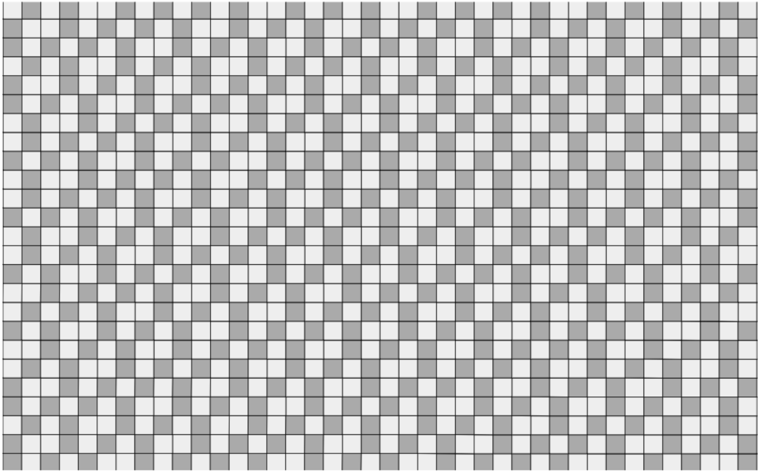


T $\subset \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : \forall m \exists \rho \; x_{(.,m)} = s_{\rho,\alpha} \right\} \subset \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : \exists \rho \forall m \; d(x_{(.,m)}, s_{\rho,\alpha}) \leq 1 \right.$

Sturmian subshift

Quasi-sturmian subshift

From strong planar 3 → 2 tilings to quasi-Sturmian subshifts



T $\subset \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : \forall m \exists \rho \; x_{(.,m)} = s_{\rho,\alpha} \right\} \subset \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : \exists \rho \forall m \; d(x_{(.,m)}, s_{\rho,\alpha}) \leq 1 \right.$

Sturmian subshift — Quasi-sturmian subshift

Realisation of effective subshift by sofic

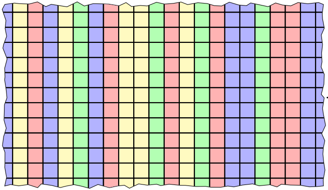
Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathbf{T}_{\mathtt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

$$\pi(\mathbf{T}_{\mathtt{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathbf{T}, \forall i \in \mathbb{Z},\ x_{\mathbb{Z}\mathbf{e_1}+i\mathbf{e_2}} = y \right\}.$$

Moreover $h_{\mathrm{top}}(\mathbf{T}_{\mathtt{Final}}) = 0$.

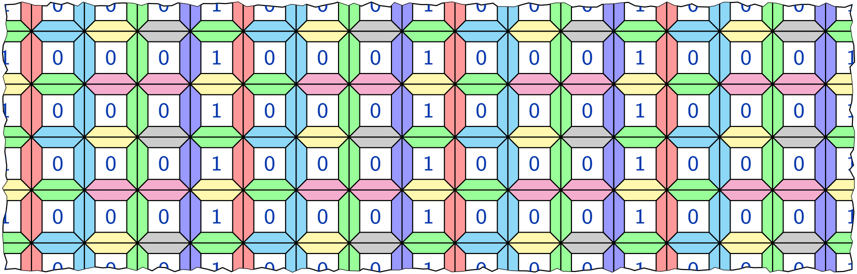
$y \in \mathbf{T}$ iff a "superposition" of y in one direction is in $\pi(\mathbf{T}_{\mathtt{Final}})$.



Independent quasi-Sturmian subshifts of slope α is sofic

If α is computable then $\left\{s_{\alpha,\rho} \in \{0,1\}^{\mathbb{Z}} : \rho \in \mathbb{R}\right\}$ is an effective subshift. So there exists an SFT $\mathbf{T}_{\{0,1\}\times\mathcal{B},\mathcal{F}} \subset (\{0,1\} \times \mathcal{B})^{\mathbb{Z}^2}$ such that:

$$\pi_1(\mathbf{T}_{\{0,1\}\times\mathcal{B},\mathcal{F}}) = \left\{x \in \{0,1\}^{\mathbb{Z}^2},\ \exists \rho \in \mathbb{R},\ \forall m \in \mathbb{Z},\ w_{\mathbb{Z}\mathbf{e_1}+m\mathbf{e_2}} = s_{\alpha,\rho}\right\}.$$

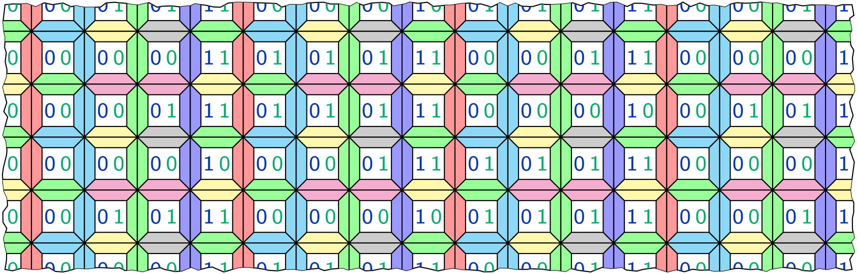


Each line is the same sturmian word $s_{\alpha,\rho}$

Independent quasi-Sturmian subshifts of slope α is sofic

Consider the SFT $\widetilde{Z}_\alpha \subset (\{0,1\} \times \mathcal{B} \times \{0,1\})^{\mathbb{Z}^2}$ such that:

$$x \in \widetilde{Z}_\alpha \iff \begin{cases} \pi_{12}(x) \in \mathbf{T}_{\mathcal{B},\mathcal{F}}, \\ \pi_3(x_{m,n}) = 0 \text{ and } \pi_3(x_{m,n+1}) = 1 \;\Rightarrow\; \pi_1(x_{m,n}) = 0, \\ \pi_3(x_{m,n}) = 1 \text{ and } \pi_3(x_{m,n+1}) = 0 \;\Rightarrow\; \pi_1(x_{m,n}) = 1. \end{cases}$$

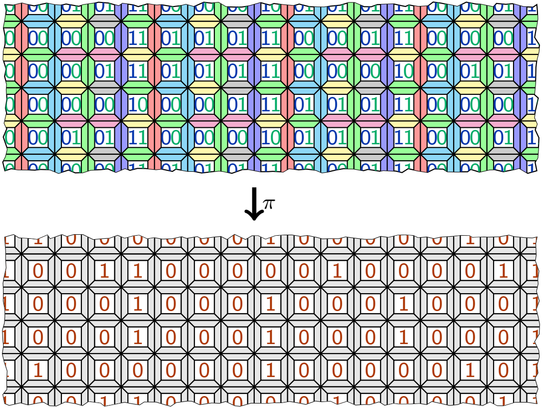


On each line we add an independent valid coding.

Independent quasi-Sturmian subshifts of slope α is sofic

Define $\pi(x)_{m,n} = \begin{cases} \pi_1(x_{m,n}) & \text{if } \pi_3(x_{m,n}) = \pi_3(x_{m,n+1}), \\ 1 - \pi_1(x_{m,n}) & \text{otherwise.} \end{cases}$

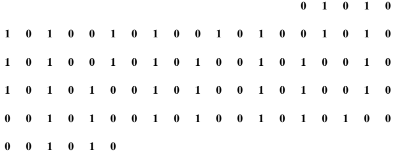
$$\pi(\widetilde{Z}_\alpha) = Z_\alpha = \left\{ x \in \{0,1\}^{\mathbb{Z}^2},\ \forall m \in \mathbb{Z},\ d(x_{(\cdot,m)}, s_{\alpha,0}) \le 1 \right\}$$

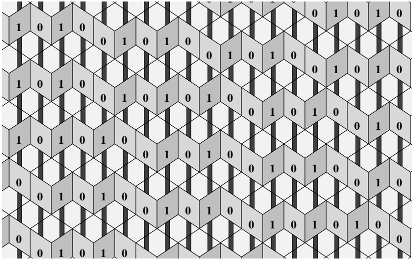


After the factor π, each line is an independent quasi-sturmian of slope α.

Transformation of tiles of $\widetilde{Z}_\alpha$ in $3 \to 2$-tiles

Each tiles of $\widetilde{Z}_\alpha$ can be viewed as a wang tile.
We construct a set $\tau_\alpha^{\mathbf{v}_3}$ of $3 \to 2$ colored tiles in the following way:

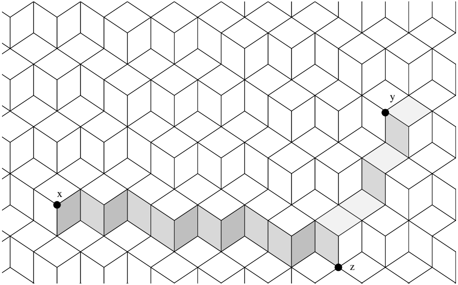




Call *$\vec{v}_i$-ribbon* of a $3 \to 2$ tiling a maximal sequence of tiles, with two consecutive tiles being adjacent along an edge $\vec{v}_i$.
Then, $\tau_\alpha^{\mathbf{v}_3}$ exactly forms the $3 \to 2$ tilings whose $\vec{v}_3$-ribbons has slope α.

Width of the planar tiling built width local rules

In the same way we construct the set of tiles $\tau_{\beta}^{\mathbf{v_2}}$ and $\tau_{\alpha/\beta}^{\mathbf{v_1}}$ and we consider colored $3 \to 2$ tilings formed with $\tau_{1,\alpha,\beta} = \tau_{\alpha/\beta}^{\mathbf{v_1}} \times \tau_{\beta}^{\mathbf{v_2}} \times \tau_{\alpha}^{\mathbf{v_3}}$.



These tilings are all planar tilings of slope orthogonal to $(1, \alpha, \beta)$. Moreover, the width of such a tiling is at most 3, since any two of its vertices can be connected by a path made of two ribbons.

Theorem *(Fernique & S.)*

A d-dimensional vector space V admits $n \to d$ weak colored local rules (of width 3) for $n > d$ if and only if it is computable.

How delete the colors?

Given a slope, it is possible to substitute each tile of a strong planar tiling by a "meta" tile arbitrary large. Thus decorations can be encoded by “fluctuations” at the cost of an increase of 1 in the width.

Theorem *(Fernique & S.)*

A d-dimensional vector space V admits $n \to d$ weak local rules (of width 4) for $n > d$ if and only if it is computable.

Perspectives around this application

Decorated local rules

The computable slopes have natural decorated rules (of with 3) but it is possible to have strong decorated local rules (i.e., width 1)?

Perspectives around this application

Decorated local rules

The computable slopes have natural decorated rules (of with 3) but it is possible to have strong decorated local rules (i.e., width 1)?

Undecorated local rules

Decorations can be encoded by "fluctuations" at the cost of an increase of 1 in the thickness, but the rules are no more natural.

Perspectives around this application

Decorated local rules

The computable slopes have natural decorated rules (of with 3) but it is possible to have strong decorated local rules (i.e., width 1)?

Undecorated local rules

Decorations can be encoded by "fluctuations" at the cost of an increase of 1 in the thickness, but the rules are no more natural.

Natural undecorated local rules

Only algebraic slopes can have natural undecorated rules *(Le '95)*. Even fewer slopes can have strong undecorated rules *(Levitov '88)*. There is yet no complete characterization of these slopes.