Links between computability and dynamics in multidimensional symbolic dynamics

Second step: Sub-dynamics of multidimensional sofic and Applications to find local rules

floripadynsys : Workshop on Dynamics, Numeration and Tilings

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Recall of the previous episode
Subshifts defined by forbidden patterns

Definition: Subshift of forbidden patterns $\mathcal{F} \subset \mathcal{A}^*$

$$T(\mathcal{A}, d, \mathcal{F}) = \{ x \in \mathcal{A}^\mathbb{Z}^d : \text{patterns of } \mathcal{F} \text{ does not appear in } x \} \subseteq \mathcal{A}^\mathbb{Z}^d$$

Some classes of subshifts:

- **T fullshift (FS)** $\iff \mathcal{F} = \emptyset$ and $T = T(\mathcal{A}, d, \mathcal{F}) = \mathcal{A}^\mathbb{Z}^d$,

- **T subshift of finite type (SFT)** $\iff \exists \mathcal{F} \subset \mathcal{A}^*$ a finite set such that $T = T(\mathcal{A}, d, \mathcal{F})$,

- **T subshift sofic (Sofic)** $\iff \exists \mathcal{F} \subset \mathcal{A}^*$ a finite set and $\pi$ a morphism such that $T = \pi(T(\mathcal{A}, d, \mathcal{F}))$
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$T(A, d, \mathcal{F}) = \{x \in A^{\mathbb{Z}^d} : \text{patterns of } \mathcal{F} \text{ does not appear in } x\} \subseteq A^{\mathbb{Z}^d}$

Some classes of subshifts:

$T$ fullshift ($\mathcal{FS}$) $\iff \mathcal{F} = \emptyset$ and $T = T(A, d, \mathcal{F}) = A^{\mathbb{Z}^d}$

$T$ subshift of finite type ($\mathcal{SFT}$) $\iff \exists \mathcal{F} \subset A^*$ a finite set such that $T = T(A, d, \mathcal{F})$

$T$ subshift sofic ($\mathcal{Sofic}$) $\iff \exists \mathcal{F} \subset A^*$ a rational set such that $T = T(A, 1, \mathcal{F})$ (Weiss-73)
1D vs 2D SFT/sofic subshifts

1D SFT/sofic subshifts
- SFT/sofic has periodic configurations

2D SFT/sofic subshifts
- exists aperiodic SFT/sofic
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Theorem (Mozes 1989)

Given a substitution $s$, there exists a SFT $T(B, d, F)$ and a factor map $\pi : B \to A$ such that $\pi(T(B, d, F)) = T_s$.
Moreover $\pi$ is a conjugacy almost everywhere and $T(B, d, F)$ is substitutive.
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- \( \mathbf{T}(A, 1, \mathcal{F}) = \emptyset \) is decidable

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Recall

$q_0 \quad \rightarrow \quad 1, 1 \rightarrow q_1$
$q_1 \quad \rightarrow \quad 1, 1 \rightarrow q_2$
$q_2 \quad \rightarrow \quad 0/\#, 1, \leftarrow$

... # # # # # # # # # # 0 1 # # # # # # # # # ...

$q_2$
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\[
\begin{array}{c}
q_0 \quad 1, 1, \rightarrow \\
q_1 \\
q_2 \\
q_F \quad 1, 1, \rightarrow \\
0/\#, 1, \leftarrow \\
0/\#, 1, \rightarrow \\
1/\#, 1, \rightarrow \\
1/\#, 1, \leftarrow \\
1/\#, 1, \rightarrow \\
0/\#, 1, \leftarrow \\
\end{array}
\]

\[
\begin{array}{cccccccc}
\# & \# & \# & \# & \# & 1 & 1 & 1 & 0 & 1 & \# & \# & \# & \# & \#
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```
q0  0, #, 1, ←
    ↑
q1  0, #, 1, →
    ↓
q2  1, 1, ←
    ↓
qF  1, 1, →
```

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```
q_F 1 1, →
\u2212\u03c9, 1, ←
q_0 0/\#, 1, ←
\u2261, 1, ←
\u2261, 1, →
0/\#, 1, →
\u2261, 1, ←
0/\#, 1, ←
q_1
q_2
q_F
```

... # # # # # # # 1 1 1 1 1 # # # # # # ...
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A strip of level \( n \) allows to code space-time diagram of \( M \) of size \( 2^n \times 2^{2n} \), thus:

\[
M \text{ halts} \iff T_{\text{Calcul}}(M) = \emptyset
\]
Dynamical operations on subshifts
Factor operation: **Fact**

**Definition**

Let $T \subseteq \mathcal{A}^{\mathbb{Z}^d}$ be a subshift and $\pi : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{B}^{\mathbb{Z}^d}$ a morphism, $\text{Fact}_\pi(T) = \pi(T) \subseteq \mathcal{B}^{\mathbb{Z}^d}$ is $T$. 

Exemple: Consider:

$\mathcal{A} = \{/uni25FB, /uni220E, /uni220E\}$

$\Sigma = T(\mathcal{A}, 1, \{/uni220E/uni220E, /uni220E/uni220E, /uni220E/uni25FB, /uni25FB/uni220E\}) \subset \mathcal{A}^{\mathbb{Z}^2}$.


Thus $\text{SFT}_\text{Cl}_F(SFT)$.

By definition $\text{Cl}_F(SFT)$ is the class of sofic subshifts.
Factor operation: **Fact**

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\[
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\]

**Exemple:** Consider:

- $A = \{\square, ■, ■\}$
- $\Sigma = T(A, 1, \{■■, ■■, ■□, □■\}) \subset A^{\mathbb{Z}}$.
- $\pi : A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ morphism such that $\pi : \begin{cases} 
\square \rightarrow \square \\
■ \rightarrow ■ \\
□ \rightarrow □
\end{cases}$

So $\text{Fact}_\pi(\Sigma) = \{x \in \{\square, ■\}^{\mathbb{Z}} / \text{blocks of } ■ \text{ have even sizes}\} = T\{0, ■\}, \{□\text{■}^{2n+1}□ : n \in \mathbb{N}\}$

Thus $SFT \nsubseteq Cl_F(SFT)$
Factor operation: **Fact**

**Definition**

Let $T \subseteq \mathcal{A}^\mathbb{Z}$ be a subshift and $\pi : \mathcal{A}^\mathbb{Z} \rightarrow \mathcal{B}^\mathbb{Z}$ a morphism, 

$\text{Fact}_\pi (T) = \pi (T) \subseteq \mathcal{B}^\mathbb{Z}$ is $T$.

**Exemple :**  
Consider:

- $\mathcal{A} = \{\Box, \blacksquare, \blacksquare\}$
- $\Sigma = T(\mathcal{A}, 1, \{\blacksquare\blacksquare, \blacksquare\blacksquare, \Box\Box, \Box\Box\}) \subset \mathcal{A}^\mathbb{Z}$.
- $\pi : \mathcal{A}^\mathbb{Z} \rightarrow \mathcal{B}^\mathbb{Z}$ morphism such that $\pi : \Box \rightarrow \Box$, $\blacksquare \rightarrow \blacksquare$, $\Box \rightarrow \Box$.

So $\text{Fact}_\pi (\Sigma) = \{\chi \in \{\Box, \blacksquare\}^\mathbb{Z} / \text{blocks of } \blacksquare \text{ have even sizes}\} = T\{0, \blacksquare\}, \{\Box\Box^{2n+1} \Box : n \in \mathbb{N}\}$

Thus $\text{SFT} \not\in \mathcal{C}l_F(\text{SFT})$

By definition $\mathcal{C}l_F(\text{SFT})$ is the class of **sofic subshifts**.

**Théorème (Weiss-73)**

In dimension 1, a subshift is sofic if and only if the set of forbidden patterns is rational (i.e. described by a finite automaton)
Subshift realized by subaction of a sofic?

Let $\Sigma = T(\{a, b, \$, 1, \{ba, \beta a^n b^m \alpha : n \neq m, \alpha \neq a, \beta \neq b\})$. Consider the subshift

$$T = \{x \in (\{a, b, \$\})^\mathbb{Z} : \exists y \in \Sigma \text{ tel que } x(.j) = y \text{ such that } j \in \mathbb{Z}\}$$
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$$T = \{x \in (\{a, b, \$\})^{\mathbb{Z}^2} : \exists y \in \Sigma \text{ tel que } x(\ldots j) = y \text{ such that } j \in \mathbb{Z}\}$$

| $\$ | $a'$ | $a$ | $b$ | $b'$ | $\$ | $a$ | $a''$ | $a$ | $b$ | $b''$ | $b$ | $b$ | $b$ | $\$ | $\$ | $a''$ | $b''$ | $\$ | $\$ |
| $a$ | $a'$ | $a$ | $b$ | $b'$ | $\$ | $a$ | $a''$ | $a$ | $b$ | $b''$ | $b$ | $b$ | $b$ | $\$ | $\$ | $a''$ | $b''$ | $\$ | $\$ |
| $a$ | $a'$ | $a$ | $b$ | $b'$ | $\$ | $a$ | $a''$ | $a$ | $b$ | $b'$ | $b$ | $b$ | $b$ | $\$ | $\$ | $a''$ | $b''$ | $\$ | $\$ |
| $a$ | $a''$ | $b''$ | $b$ | $\$ | $a$ | $a''$ | $a$ | $a$ | $a$ | $a''$ | $b''$ | $b$ | $b$ | $b$ | $\$ | $\$ | $a''$ | $b''$ | $\$ | $\$ |
| $a$ | $a''$ | $a$ | $b$ | $b'$ | $\$ | $a$ | $a''$ | $a$ | $a$ | $a$ | $b$ | $b$ | $b'$ | $b$ | $b$ | $\$ | $\$ | $a''$ | $b''$ | $\$ | $\$ |
| $a$ | $a''$ | $a$ | $b$ | $b'$ | $\$ | $a$ | $a''$ | $a$ | $a$ | $a$ | $b$ | $b$ | $b'$ | $b$ | $b$ | $\$ | $\$ | $a''$ | $b''$ | $\$ | $\$ |
| $a$ | $a''$ | $b''$ | $b$ | $\$ | $a$ | $a''$ | $a$ | $a$ | $a$ | $a$ | $a''$ | $b''$ | $b$ | $b$ | $b$ | $\$ | $\$ | $a''$ | $b''$ | $\$ | $\$ |
| $a$ | $a''$ | $b''$ | $b$ | $\$ | $a$ | $a''$ | $a$ | $a$ | $a$ | $a''$ | $b''$ | $b$ | $b$ | $b$ | $\$ | $\$ | $a''$ | $b''$ | $\$ | $\$ |
| $a$ | $a''$ | $b''$ | $b$ | $\$ | $a$ | $a''$ | $a$ | $a$ | $a$ | $a''$ | $b''$ | $b$ | $b$ | $b$ | $\$ | $\$ | $a''$ | $b''$ | $\$ | $\$ |
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Let $\Sigma = T(\{a, b, \$, 1, \{ba, \beta a^n b^m \alpha : n \neq m, \alpha \neq a, \beta \neq b\})$. Consider the subshift

$$T = \{x \in (\{a, b, \$\}^\mathbb{Z})^2 : \exists y \in \Sigma \text{ tel que } x(\cdot, j) = y \text{ such that } j \in \mathbb{Z}\}$$
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Projective subaction: \( \text{SA} \)

Let \( T \subset \mathcal{A}^{\mathbb{Z}^d} \) be a subshift and \( G \) be a sublattice of \( \mathbb{Z}^d \), the \( G \)-action on \( T \) is not necessary a subshift. However if we restrict to a row one obtains a subshift.

**Definition**

Let \( G \) be a sublattice of \( \mathbb{Z}^d \) generated by \( u_1, u_2, \ldots, u_{d'} \) \((d' \leq d)\). Let \( T \subset \mathcal{A}^{\mathbb{Z}^d} \) be a subshift:

\[
\text{SA}_G (T) = \left\{ y \in \mathcal{A}^{\mathbb{Z}^{d'}} : \exists x \in T \text{ tel que } \forall i_1, \ldots, i_{d'} \in \mathbb{Z}^{d'}, \ y_{i_1, \ldots, i_{d'}} = x_{i_1 u_1 + \cdots + i_{d'} u_{d'}} \right\}.
\]

Let \( G = \{ (i, i) : i \in \mathbb{Z} \} \subset \mathbb{Z}^2 \).
Example of projective subaction

\[\mathcal{F} = \left\{ \begin{array}{c}
\alpha^i, \beta^i, \alpha^j, \\
\beta^i, \gamma^i, \beta^i, \\
\gamma^i, \beta^i, \gamma^i
\end{array} \right\}
\text{such that}
\begin{align*}
\alpha^i & \in \{a, b\}, \\
\beta^i & \in \{b, \gamma\}, \\
\gamma^i & \in \{a, b\}.
\end{align*}
\]

\[\epsilon T_F \subset A^{\mathbb{Z}^2}\]
Example of projective subaction

$$\mathcal{F} = \left\{ \begin{array}{c}
\alpha, \beta, a, \bar{a}, \bar{b}, \bar{\beta}, \\
\bar{\alpha}, \bar{\beta}, \bar{a}, \bar{b}, \gamma, \bar{\gamma}, b, \bar{b}, \gamma, \bar{\gamma}, \bar{\beta}, \bar{\alpha}
\end{array} \right\},
$$
such that

$$\bar{\alpha} \in \left\{ \begin{array}{c} a, b \\
\bar{a}, \bar{b}
\end{array} \right\},
\bar{\beta} \in \left\{ \begin{array}{c} \bar{a}, \bar{b} \\
\bar{\alpha}, \bar{\beta}
\end{array} \right\},
\bar{\gamma} \in \left\{ \begin{array}{c} \bar{\alpha}, \bar{\beta} \\
\bar{\gamma}, \bar{\delta}
\end{array} \right\}.$$
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Some classes of subshifts:

- **T fullshift (FS)** $\iff \mathcal{F} = \emptyset$ and $T = T(\mathcal{A}, d, \mathcal{F}) = \mathcal{A}^{\mathbb{Z}^d}$,

- **T subshift of finite type (SFT)** $\iff \exists \mathcal{F} \subset \mathcal{A}^*$ a finite set such that $T = T(\mathcal{A}, d, \mathcal{F})$,

- **T subshift sofic (Sofic)** $\iff \exists \mathcal{F} \subset \mathcal{A}^*$ a rational set such that $T = T(\mathcal{A}, 1, \mathcal{F})$.
Subshifts defined by forbidden patterns

Definition: Subshift of forbidden patterns \( \mathcal{F} \subset A^* \)

\[
T(A, d, \mathcal{F}) = \{ x \in A^{\mathbb{Z}^d} : \text{patterns of } \mathcal{F} \text{ does not appear in } x \} \subseteq A^{\mathbb{Z}^d}
\]

Some classes of subshifts:

- **\( T \) fullshift (FS) \( \iff \mathcal{F} = \emptyset \) and \( T = T(A, d, \mathcal{F}) = A^{\mathbb{Z}^d} \),

- **\( T \) subshift of finite type (SFT) \( \iff \exists \mathcal{F} \subset A^* \) a finite set such that \( T = T(A, d, \mathcal{F}) \),

- **\( T \) subshift sofic (Sofic) \( \iff d = 1\) \( \exists \mathcal{F} \subset A^* \) a rational set such that \( T = T(A, 1, \mathcal{F}) \),

- **\( T \) effective (RE) \( \iff \exists \mathcal{F} \subset A^* \) a recursively enumerable set such that \( T = T(A, d, \mathcal{F}) \)
Computability obstruction

Proposition

\[
\mathcal{C} \mathcal{L}_{SA}(\mathcal{RE}) = \mathcal{RE}
\]
In particular \( \mathcal{C} \mathcal{L}_{SA}(Sofic) = \mathcal{C} \mathcal{L}_{\text{Fact,SA}}(SFT) \subset \mathcal{RE} \)

Proof:

Let \( T = T(B, 2, F) \) be a subshift such that \( F \) is enumerated by a Turing machine and denote \( F_m \) the \( m \) first patterns enumerated. Consider \( \Sigma = \mathcal{SA}_G(T) \):

\[
u \text{ is a forbidden pattern of } \Sigma \iff \exists m \text{ such that all patterns of support } [-m, m]^2 \text{ which satisfy } F_m \text{ does not contain } u \text{ in the center.}
\]

- For \( u \in \mathcal{A}^n \), consider a Turing machine \( M_u \) which on the enter \( m \) enumerate patterns of support \([-m, m]^2\) which contains \( u \) and satisfies \( F_m \). The machine \( M_u \) halts, and forbid \( u \), if no pattern are produced.

- The Turing machine which enumerates forbidden patterns of \( \Sigma \) is constructed using \( (M_u)_{u \in \mathcal{A}^n} \) in parallel.
An important tool: Simulation of effective subshifts by SFT
Realisation of effective subshift by sofic

Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

If $T \subset \mathcal{A}^\mathbb{Z}$ is an effective subshift, there is a subshift of finite type $T_{\text{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \rightarrow \mathcal{A}$ such that

\[
\pi(T_{\text{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in T, \forall i \in \mathbb{Z}, \ x_{ie_1 + ie_2} = y \right\}.
\]

Moreover $h_{\text{top}}(T_{\text{Final}}) = 0$.

\[ y \in T \text{ iff a "superposition" of } y \text{ in one direction is in } \pi(T_{\text{Final}}). \]

**Corollary:**

- $\mathcal{C}_{\text{SA}}(\text{Sofic}) = \mathcal{RE}$.
- Every $d$-dimensional effective subshift is conjugate to the sub-action of a subshift of finite type.
Realisation of effective subshift by sofic

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**Idea of the proof:**

**Layer 1:**

$$T_{\text{Align}} = \left\{ \right\}$$

**Aim:**

We want to eliminate each $x$ which contains forbidden patterns of $\Sigma$. 

**An important tool:** realization of effective subshift
Realisation of effective subshift by sofic

Theorem \textit{(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)}

If $T \subset A^\mathbb{Z}$ is an effective subshift, there is a subshift of finite type $T_{\text{Final}} \subset B^\mathbb{Z}^2$ and a factor map $\pi : B \to A$ such that

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Idea of the proof: Layer 2:

\begin{itemize}
  \item Layer 1:
  \item Layer 2:
  \item Layer 3:
  \item Layer 4:
\end{itemize}
Realisation of effective subshift by sofic

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\pi(T_{\text{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in T, \forall i \in \mathbb{Z}, x_{\mathbb{Z}e_1 + ie_2} = y \right\}.
$$

**Idea of the proof:**

**Layer 2:**

![Diagram](attachment:image.png)

An important tool: realization of effective subshift

Four layers
Realisation of effective subshift by sofic

**Theorem** *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

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**Idea of the proof:**

**Layer 2:**

![Diagram](image-url)
Realisation of effective subshift by sofic

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**Idea of the proof:**

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**Layer 2:**

![Diagram of Layer 2](image)
Realisation of effective subshift by sofic

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Layer 2:
Realisation of effective subshift by sofic

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![Layer 2 Diagram]
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**Idea of the proof:**

**Layer 2:**

An important tool: realization of effective subshift

Four layers
Realisation of effective subshift by sofic

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**Idea of the proof:**

**Layer 3:** Enumeration of forbidden patterns

![Diagram](Attachment)
Realisation of effective subshift by sofic

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\]

**Idea of the proof:**

**Layer 3:** Responsability zones of \( \mathcal{M}_{\text{Forbid}} \)

\( \mathcal{M}_{\text{Forbid}} \) of a level \( n \) can ask at \( \mathcal{M}_{\text{Search}} \) of the same level or neighbor \( \mathcal{M}_{\text{Search}} \) of the same level.
Realisation of effective subshift by sofic

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**Idea of the proof:**

**Layer 3:** Responsability zones of $\mathcal{M}_{\text{Forbid}}$

$\mathcal{M}_{\text{Forbid}}$ of a level $n$ can ask at $\mathcal{M}_{\text{Search}}$ of the same level or neighbor $\mathcal{M}_{\text{Search}}$ of the same level.
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**Idea of the proof:**

**Layer 4:**

An important tool: realization of effective subshift

Four layers

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Realisation of effective subshift by sofic

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**Idea of the proof:**

**Layer 4:**

- Forbid
- Search
- \(M_{\text{Forbidden}}\)
- \(M_{\text{Search}}\)
- \(x \in \Sigma\)
- \(T_{\text{Grid}}\)
Realisation of effective subshift by sofic

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Idea of the proof:

Layer 4:

$M_{\text{Search}}$ holds:

- There is enough space to code address.
- The time taken to give back the information is $t(n) \leq 2^n \times O(n^2 2^n)$ which is "absorbed" by the exponential time of the clock ($2^{2^n}$).
Realisation of effective subshift by sofic

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\]

**Idea of the proof:**

Layer 4:

Communication between layers:

- condition **Request**: \( \mathcal{M}_F \) ask \( \mathcal{M}_\text{Search} \) the value of a box in the responsibility zone and wait the answer
- condition **Forbid**: exclude configuration when forbidden pattern are encountered

To obtain \( \Sigma \):

- operation **Fact** to keep only letters of \( \mathcal{A}_\Sigma \)
- operation **SA** to keep only an horizontal line
Perspectives around sub-dynamic

Optimality of the construction

- A so huge alphabet.
- A long range of dependance to detect forbidden patterns. Wait course 3!
- Construction very rigid: What happens if we impose some mixing properties? Wait course 3!
Perspectives around sub-dynamic

Optimality of the construction

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Sub-dynamic

In Hochman-09 there is a characterization of subaction of $d$-dimensional sofic with $d \geq 3$. What happens for $d = 2$?
Perspectives around sub-dynamic

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Sub-dynamic
In *Hochman-09* there is a characterization of subaction of $d$-dimensional sofic with $d \geq 3$. What happens for $d = 2$?

Projective sub-dynamic of SFT
We have $\mathcal{C}_{SA}(Sofic) = \mathcal{RE}$. Which information we have about $\mathcal{C}_{SA}(SFT)$? In this case we cannot use additional alphabet to make computation. *Wait course 3!*

An important tool: realization of effective subshift
Applications
Applications to find local rules

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**A framework for computability results :**

- Computability Obstructions on SFTs are usually also obstructions for effective shifts.
- Prove the obstruction is the only obstruction for effective shifts.
- Use the previous theorem to go back to SFTs.
Applications to find local rules

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**Applications:**

- characterization of the entropy of multidimensional SFTs *(Hochman-Meyerovitch-10)*,
- characterization of multidimensional $\mathbf{S}$-adic subshift with local rules *(Aubrun-Sablik-12)*,
- characterization of tilings which approximate discrete plane *(Fernique-Sablik-12)*,
- characterization of periods of multidimensional SFTs *(Jeandel-Vanier-13)*,
- characterization of the function with measure quasi-periodicity *(Ballier-Jeandel-10)*,
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- ...
Approximation of discrete plane
**$n \to d$ tilings**

Let $v_1, \ldots, v_n$ be pairwise non-colinear vectors of $\mathbb{R}^d$ with $n > d > 0$.

- A $n \to d$ tile is a parallelootope generated by $d$ of the $v_i$'s, there are $\binom{n}{d}$ tiles.
- A $n \to d$ tiling is a face-to-face tiling of $\mathbb{R}^d$ by $n \to d$ tiles.
- The set $\mathcal{X}_{n \to d}$ of all tilings of $\mathbb{R}^d$ by $n \to d$-tiles is the **full $n \to d$ tiling space**.
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Consider a $n \rightarrow d$ tiling.
Lift

Map an arbitrary vertex onto an arbitrary vector of $\mathbb{Z}^n$. 
Modify the $k^{\text{th}}$ entry when moving along the $k^{\text{th}}$ direction.
Lift

$n \rightarrow d$ vertices are mapped onto vertices of $[0, 1]^n$. 
The whole tiling is mapped onto a stepped surface of \( \mathbb{R}^n \): its lift.
Planar tilings

Definition

A \( n \rightarrow d \) tilings set \( T \subset \mathcal{X}_{n \rightarrow d} \) is a \textit{planar tiling space} if there are a \( d \)-dimensional vector subspace \( V \subset \mathbb{R}^n \), the \textit{slope} and a positive integer \( w \), the \textit{width}, such that all tiling \( t \in T \) can be lifted into the slice \( V + [0, w)^n \).

\[ \]

Approximation of discrete plane

Problematic

Approximation of discrete plane
Planar tilings

Definition

A $n \to d$ tilings set $T \subset X_{n \to d}$ is a **planar tiling space** if there are a $d$-dimensional vector subspace $V \subset \mathbb{R}^n$, the **slope** and a positive integer $w$, the **width**, such that all tiling $t \in T$ can be lifted into the slice $V + [0, w)^n$.

The $w = 1$ case corresponds to **strong planar tilings**.
Planar tilings

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Here $w = 2$, if $w \geq 2$, this corresponds to weak planar tilings.
Planar tilings

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A $n \to d$-tilings set $T \subset \mathcal{X}_{n \to d}$ is a **planar tiling space** if there are a $d$-dimensional vector subspace $V \subset \mathbb{R}^n$, the **slope** and a positive integer $w$, the **width**, such that all tiling $t \in T$ can be lifted into the slice $V + [0, w)^n$.

Here $w = 3$, if $w \geq 2$, this corresponds to **weak planar tilings**.
Local rules

- A $n \to d$-pattern of size $r$ of a tiling $t \in X_{n \to d}$ is a set of tiles lying inside a ball of radius $r > 0$. For $T \subset X_{n \to d}$ denote $\mathcal{P}_r(T)$ the set of $n \to d$-pattern of size $r$ of each tiling of $T$.
- The set of tilings of forbidden $n \to d$-patterns $\mathcal{F}$ is

$$T_{\mathcal{F}} = \{ t \in X_{n \to d} : \text{no patterns of } \mathcal{F} \text{ appears in } t \}$$

- $T$ is a set of tilings of finite type if there exists $\mathcal{F}$ finite such that $T = T_{\mathcal{F}}$.

$\mathcal{F} = \left\{ \begin{array}{c}
\begin{array}{c}
\text{pattern 1} \\
\text{pattern 2} \\
\text{pattern 3}
\end{array}
\end{array}\right\}$
Consider these decorated $3 \rightarrow 2$ tiles: \[\{\text{\includegraphics{tile1.png}}, \text{\includegraphics{tile2.png}}, \text{\includegraphics{tile3.png}}\}\], which can match only if the corresponding edges have the same color.
Consider these decorated $3 \rightarrow 2$ tiles: \[\{\text{\includegraphics[width=0.05\textwidth]{tile1.png}}\,\text{\includegraphics[width=0.05\textwidth]{tile2.png}}\,\text{\includegraphics[width=0.05\textwidth]{tile3.png}}\}\], which can match only if the corresponding edges have the same color.
Colored local rules

Consider these decorated \(3 \rightarrow 2\) tiles: \(\{\text{tile 1}, \text{tile 2}, \text{tile 3}\}\), which can match only if the corresponding edges have the same color.
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Colored local rules

Consider these decorated 3 → 2 tiles: \{\begin{tikzpicture}
\end{tikzpicture}\}, which can match only if the corresponding edges have the same color.

A set of tilings has colored local rules if it is possible to decorate tiles to obtain it.
Colored local rules

Consider these decorated $3 \rightarrow 2$ tiles: \[ \{ \] , which can match only if the corresponding edges have the same color.
Colored local rules

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Consider these decorated $3 \rightarrow 2$ tiles: \[\{ \text{tiles} \}, \] which can match only if the corresponding edges have the same color.
Colored local rules

Consider these decorated $3 \rightarrow 2$ tiles:\n\[
\begin{align*}
\text{\begin{tikzpicture}[scale=0.5]}
\end{tikzpicture}}
\end{align*}
\]
, which can match only if the corresponding edges have the same color. This allows only small fluctuations and obtain a weak planar set of tilings.
Historic of the problem

Which vector space admits local rules or colored local rules?

\( n\)-fold tiling: plane tiling of slope \( \mathbb{R}(u_1, ..., u_n) + \mathbb{R}(v_1, ..., v_n) \), where

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  u_k = \cos \left( \frac{2k\pi}{n} \right) \quad \text{and} \quad v_k = \sin \left( \frac{2k\pi}{n} \right)
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\(^{(1)}\): Penrose 1974  \(^{(2)}\): Burkov 1988  \(^{(3)}\): Le 1992  \(^{(4)}\): Socolar 1989  
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Main results

- A vector $v \in \mathbb{R}^n$ is *computable* if there exists a computable function $f : \mathbb{N} \rightarrow \mathbb{Q}^n$ such that $\|v - f(n)\|_\infty \leq 2^{-n}$ for all $n \in \mathbb{N}$.

- The vector space $V \subset \mathbb{R}^n$ of dimension $d$ is *computable* if there exists a set of $d$ computable vectors which generate $V$.

Theorem (Fernique & S.)

A $d$-dimensional vector space $V$ admits $n \rightarrow d$ weak colored local rules (of width 3) for $n > d$ if and only if it is computable.

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Notion of natural local rules

Local rules are said *natural* if there are verified by strong planar tiling.
Computability obstruction
Algorithm to obtain the slope

**Input:** Local rules of the planar tilings set \( T \subset \mathcal{X}_{n \to d} \), the width \( w \) and an integer \( m \) which corresponds to the precision.

**Algorithm:**

1. \( r_0 := 2wm \), \( r := r_0 \) and \( d := 1 \)
2. While \( d \geq \frac{1}{2m} \) do
   - enumerate \( \mathcal{P}_r(T) \), the set of all the diameter \( r \) patterns centered on 0 allowed by these local rules (this takes exponential but finite time in \( r \))
   - enumerate \( X^{r_0}_r \), the'
     \[
     d = \max_{W_1, W_2 \in X_r} \tilde{d}(W_1, W_2)
     \]
   - \( r := r + 1 \)
3. Output: an element of \( W \in X_r \)

**The algorithm halts:** For sufficiently large \( r \) all vector spaces of \( X_r \) are near of \( V \), if not by compacity one obtains one other slope for the \( n \to d \) tiling.

**The algorithm holds:** There exists \( W' \in X_r \) such that \( \tilde{d}(W', V) \leq \frac{w}{r_0} \), thus

\[
\tilde{d}(W, V) \leq \tilde{d}(W, W') + \tilde{d}(W', V) \leq \frac{1}{2m} + \frac{w}{r_0} \leq \frac{1}{m}
\]
Realization of computable 3 $\rightarrow$ 2 planar tilings with colored local rules
Stripes of $3 \rightarrow 2$ strong planar tiling

For $3 \rightarrow 2$ strong planar tilling, intertwined stripes encoding Sturmian words.
Stripes of 3 → 2 strong planar tiling

Parallel stripes encode Sturmian words with the same slope.
Quasi-Sturmian words

Define the *Sturmian word* $s_{\rho, \alpha} \in \{0, 1\}^\mathbb{Z}$ of slope $\alpha \in [0, 1]$ and intercept $\rho$ by

$$s_{\rho, \alpha}(n) = 0 \iff (\rho + n\alpha) \mod 1 \in [0, 1 - \alpha).$$

For $x, y \in \{0, 1\}^\mathbb{Z}$ define $d(x, y) := \sup_{p \leq q} ||x_p x_{p+1} \ldots x_q||_0 - ||y_p y_{p+1} \ldots y_q||_0$.

**Fact:** Sturmian words with equal slopes are at distance at most one.

$$w = 1$$

$$s_{\rho, \alpha}:\ldots 0010001000100010001000010001000100010\ldots$$
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\( x \in \{0, 1\}^\mathbb{Z} \) is a quasi-Sturmian of slope \( \alpha \) if \( d(x, s_{\rho,\alpha}) \leq 1 \).

\[
\begin{align*}
\text{Quasi-Sturmian:} & \quad \ldots 0010001000100010001000100010001000100101010\ldots \\
\text{Approximation of discrete plane} & \quad \\
\end{align*}
\]

\[
w = 2
\]
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**Fact:** Two words in \( \{0, 1\}^\mathbb{Z} \) are at distance at most one if and only if each can be obtained from the other by performing letter replacements \( 0 \rightarrow 1 \) or \( 1 \rightarrow 0 \), without two consecutive replacements of the same type.

\[w = 2\]

\[s_{\rho,\alpha}: \ldots 0010001000100010001000100010001000100010\ldots\]

Quasi-Sturmian: \[\ldots 0010101000100000001001000010001001010\ldots\]

Changes: \[\uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \ldots\]
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![Graphical representation of Quasi-Sturmian words and codes](image-url)
From strong planar $3 \to 2$ tilings to quasi-Sturmian subshifts

$$T \subset \left\{ x \in \{0, 1\}^{\mathbb{Z}^2} : \forall m \exists \rho \ x_{(\cdot, m)} = s_{\rho, \alpha} \right\} \subset \left\{ x \in \{0, 1\}^{\mathbb{Z}^2} : \exists \rho \forall m \ d(x_{(\cdot, m)}, s_{\rho, \alpha}) \leq 1 \right\}$$

Sturmian subshift

Quasi-sturmian subshift

Approximation of discrete plane

Colored local rules for computable $3 \to 2$ planar tilings
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- Sturmian subshift
- Quasi-sturmian subshift

Approximation of discrete plane
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Sturmian subshift

Quasi-sturmian subshift
Realisation of effective subshift by sofic

**Theorem** *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

If \( T \subset \mathcal{A}^\mathbb{Z} \) is an effective subshift, there is a subshift of finite type \( T_{\text{Final}} \subset \mathcal{B}^{\mathbb{Z}^2} \) and a factor map \( \pi : \mathcal{B} \to \mathcal{A} \) such that

\[
\pi(T_{\text{Final}}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in T, \forall i \in \mathbb{Z}, \ x_{ie_1 + ie_2} = y \right\}.
\]

Moreover \( h_{\text{top}}(T_{\text{Final}}) = 0 \).

\( y \in T \) iff a "superposition" of \( y \) in one direction is in \( \pi(T_{\text{Final}}) \).
Independent quasi-Sturmian subshifts of slope $\alpha$ is sofic

If $\alpha$ is computable then $\{s_{\alpha,\rho} \in \{0,1\}\mathbb{Z} : \rho \in \mathbb{R}\}$ is an effective subshift. So there exists an SFT $T_{\{0,1\} \times B, \mathcal{F}} \subset (\{0,1\} \times B)^{\mathbb{Z}^2}$ such that:

$$\pi_1(T_{\{0,1\} \times B, \mathcal{F}}) = \{x \in \{0,1\}\mathbb{Z}^2, \exists \rho \in \mathbb{R}, \forall m \in \mathbb{Z}, w_{\mathbb{Z}e_1+me_2} = s_{\alpha,\rho}\}.$$
Independent quasi-Sturmian subshifts of slope $\alpha$ is sofic

Consider the SFT $\tilde{Z}_\alpha \subset (\{0, 1\} \times \mathcal{B} \times \{0, 1\})^\mathbb{Z}^2$ such that:

$$x \in \tilde{Z}_\alpha \iff \begin{cases} \pi_{12}(x) \in T_{\mathcal{B}, \mathcal{F}}, \\ \pi_3(x_{m,n}) = 0 \text{ and } \pi_3(x_{m,n+1}) = 1 \Rightarrow \pi_1(x_{m,n}) = 0, \\ \pi_3(x_{m,n}) = 1 \text{ and } \pi_3(x_{m,n+1}) = 0 \Rightarrow \pi_1(x_{m,n}) = 1. \end{cases}$$

On each line we add an independent valid coding.

\[
\begin{array}{ccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
Independent quasi-Sturmian subshifts of slope $\alpha$ is sofic

Define $\pi(x)_{m,n} = \begin{cases} 
\pi_1(x_{m,n}) & \text{if } \pi_3(x_{m,n}) = \pi_3(x_{m,n+1}), \\
1 - \pi_1(x_{m,n}) & \text{otherwise}.
\end{cases}$

$$\pi(\tilde{Z}_\alpha) = Z_\alpha = \left\{ x \in \{0, 1\}^\mathbb{Z}, \forall m \in \mathbb{Z}, \; d(x_{\cdot, m}, s_{\alpha, 0}) \leq 1 \right\}$$

After the factor $\pi$, each line is an independent quasi-sturmian of slope $\alpha$. 
Transformation of tiles of $\tilde{\mathbb{Z}}_\alpha$ in $3 \to 2$-tiles

Each tiles of $\tilde{\mathbb{Z}}_\alpha$ can be viewed as a wang tile.
We construct a set $\tau_{\alpha}^v$ of $3 \to 2$ colored tiles in the following way:

Call $\tilde{v}_i$-ribbon of a $3 \to 2$ tiling a maximal sequence of tiles, with two consecutive tiles being adjacent along an edge $\tilde{v}_i$.
Then, $\tau_{\alpha}^v$ exactly forms the $3 \to 2$ tilings whose $\tilde{v}_3$-ribbons has slope $\alpha$. 
Width of the planar tiling built width local rules

In the same way we construct the set of tiles $\tau_{\beta}^{v_2}$ and $\tau_{\alpha/\beta}^{v_1}$ and we consider colored $3 \to 2$ tilings formed with $\tau_{1,\alpha,\beta} = \tau_{\alpha/\beta}^{v_1} \times \tau_{\beta}^{v_2} \times \tau_{\alpha}^{v_3}$.

These tilings are all planar tilings of slope orthogonal to $(1, \alpha, \beta)$. Moreover, the width of such a tiling is at most 3, since any two of its vertices can be connected by a path made of two ribbons.

**Theorem (Fernique & S.)**

A $d$-dimensional vector space $V$ admits $n \to d$ weak colored local rules (of width 3) for $n > d$ if and only if it is computable.
Given a slope, it is possible to substitute each tile of a strong planar tiling by a "meta" tile arbitrary large. Thus decorations can be encoded by “fluctuations” at the cost of an increase of 1 in the width.

**Theorem (Fernique & S.)**

A $d$-dimensional vector space $V$ admits $n \to d$ weak local rules (of width 4) for $n > d$ if and only if it is computable.
Perspectives around this application

Decorated local rules

The computable slopes have natural decorated rules (of with 3) but it is possible to have strong decorated local rules (i.e., width 1)?
Perspectives around this application

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The computable slopes have natural decorated rules (of width 3) but it is possible to have strong decorated local rules (i.e., width 1)?

Undecorated local rules
Decorations can be encoded by “fluctuations” at the cost of an increase of 1 in the thickness, but the rules are no more natural.
Perspectives around this application

**Decorated local rules**

The computable slopes have natural decorated rules (of width 3) but it is possible to have strong decorated local rules (i.e., width 1)?

**Undecorated local rules**

Decorations can be encoded by “fluctuations” at the cost of an increase of 1 in the thickness, but the rules are no more natural.

**Natural undecorated local rules**

Only algebraic slopes can have natural undecorated rules (Le ’95). Even fewer slopes can have strong undecorated rules (Levitov ’88). There is yet no complete characterization of these slopes.