

Links between computability and dynamics
in multidimensional symbolic dynamics

First step: Aperiodicity and Undecidability

floripadynsys : Workshop on Dynamics, Numeration and Tilings

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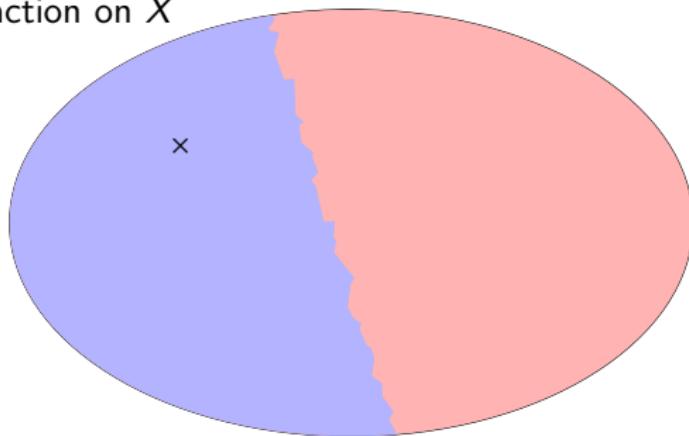
November 2013

Some motivations

Coding of dynamical systems

Given a dynamical system and a partition, it is possibly to code the trajectory.

F is a \mathbb{Z} -action on X

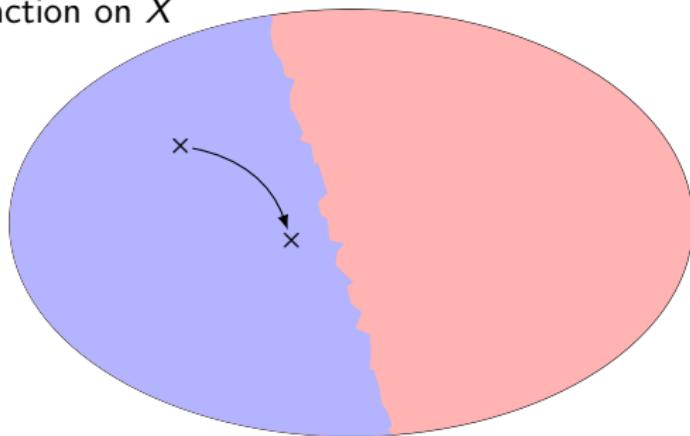


Coding: ...

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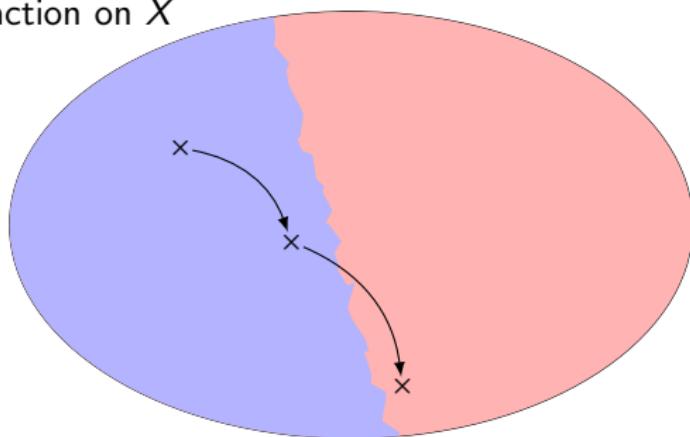


Coding: \dots

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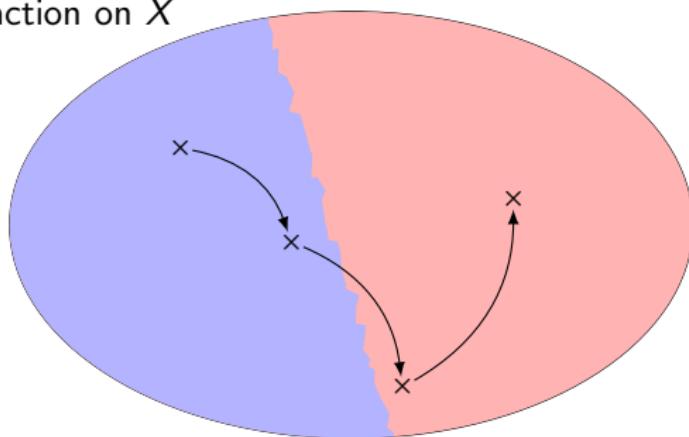
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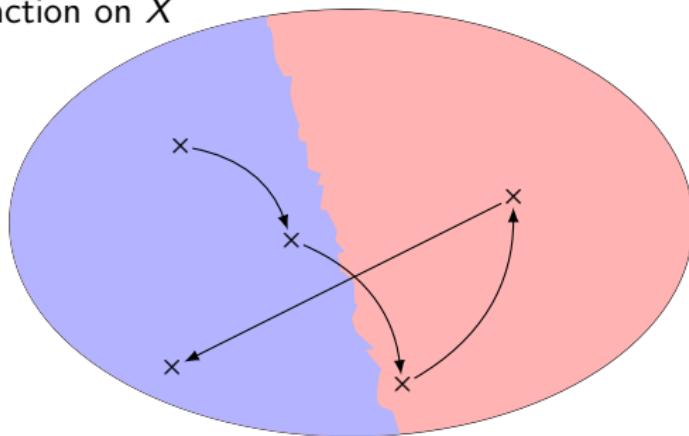
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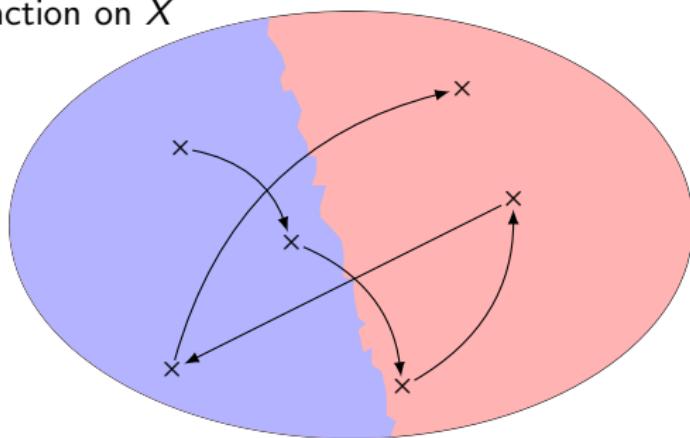
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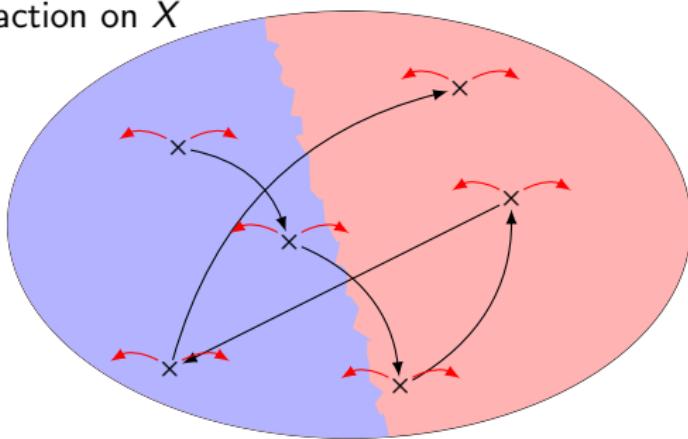
Coding:

$\dots \square \square \square \square \square \square \dots \in \mathcal{A}^{\mathbb{Z}}$

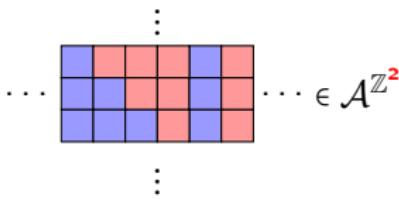
Coding of dynamical systems

Given a dynamical system and a partition, it is possible to code the trajectory.

F is a \mathbb{Z}^2 -action on X

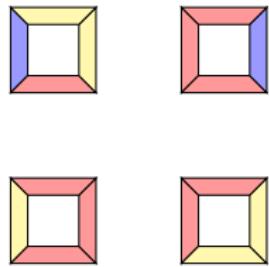


Coding:

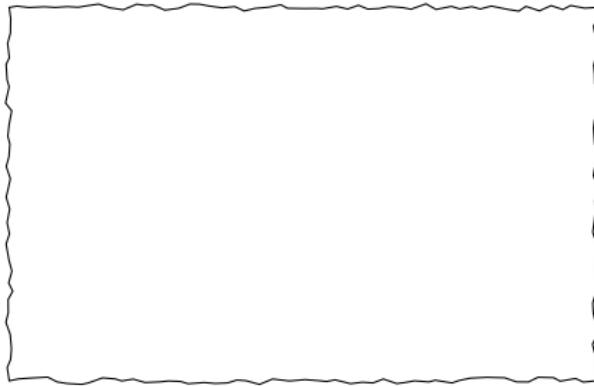


Wang's tilings

Tiles set



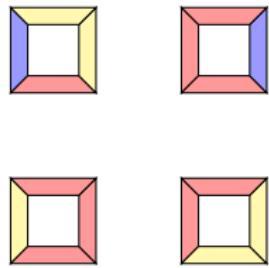
A tiling associated



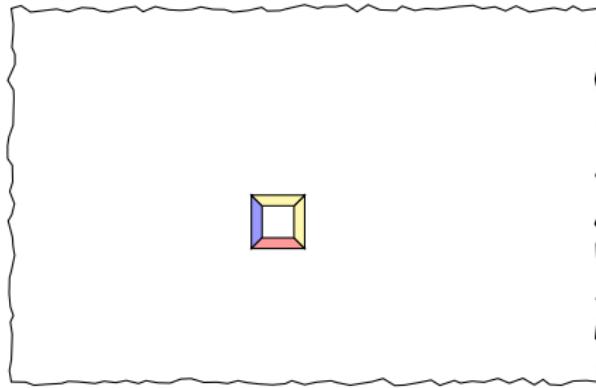
This tiles set can tile the plane?

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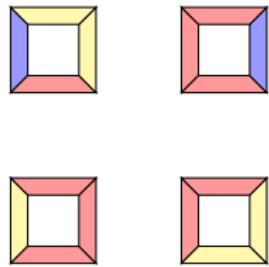
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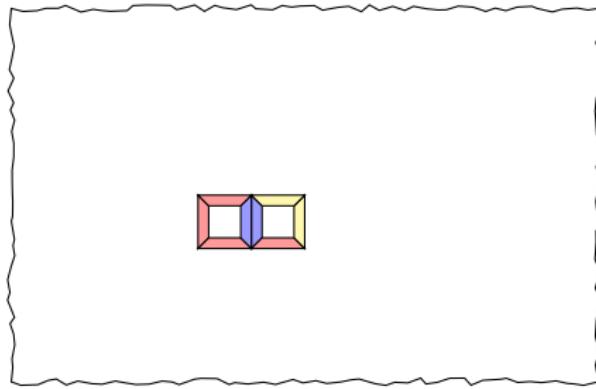
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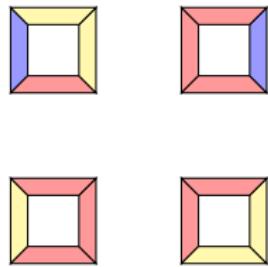
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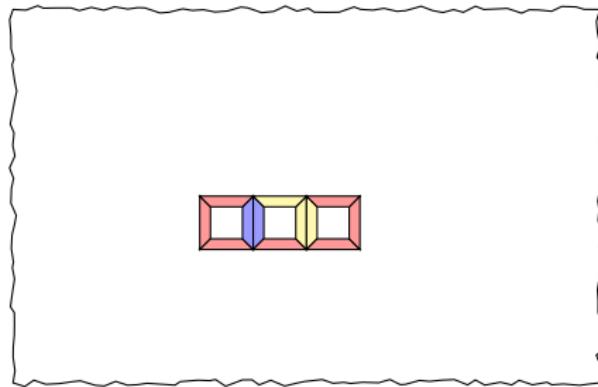
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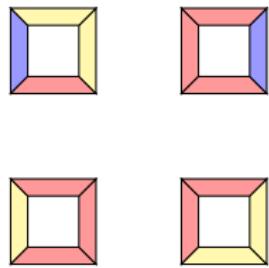
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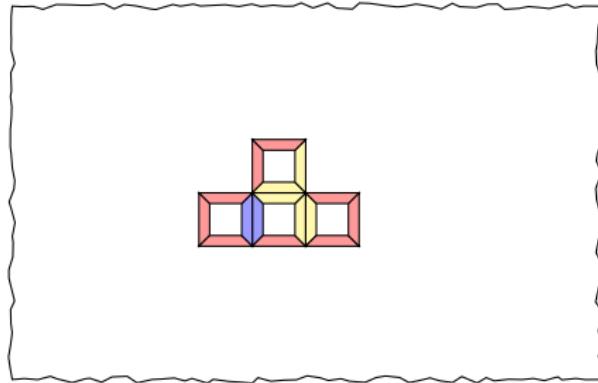
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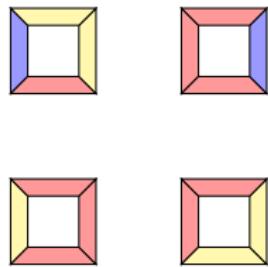
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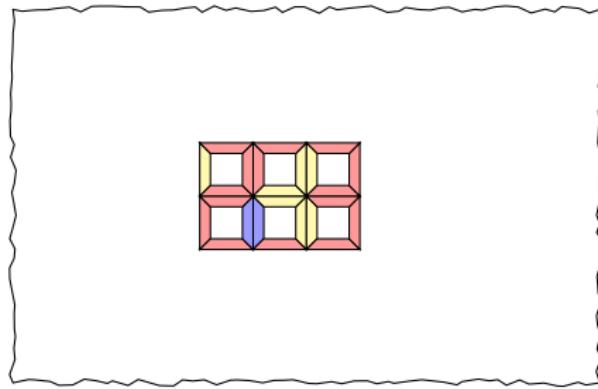
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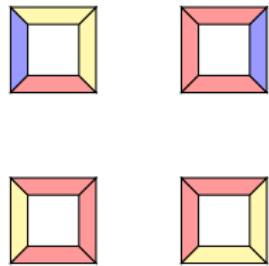
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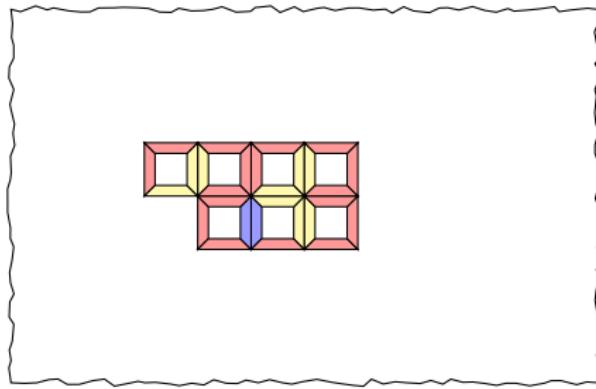
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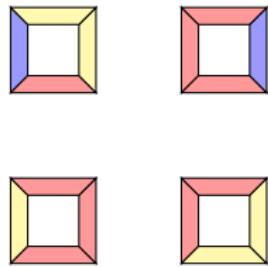
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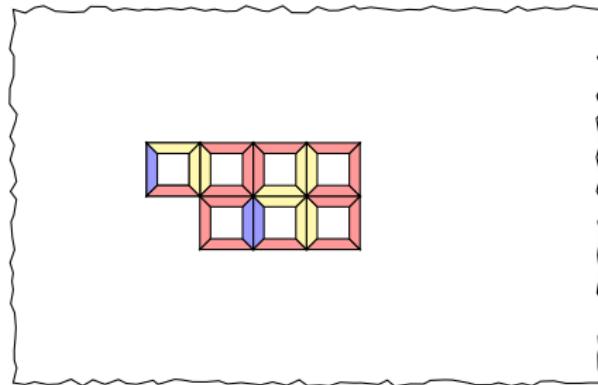
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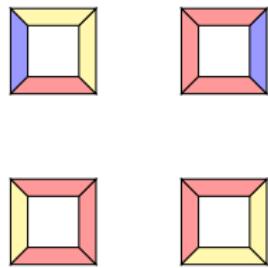
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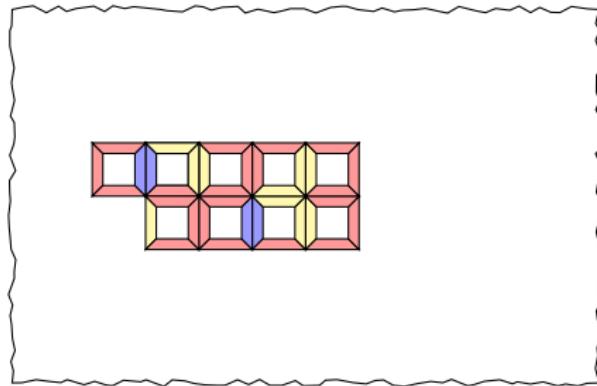
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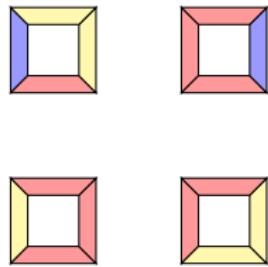
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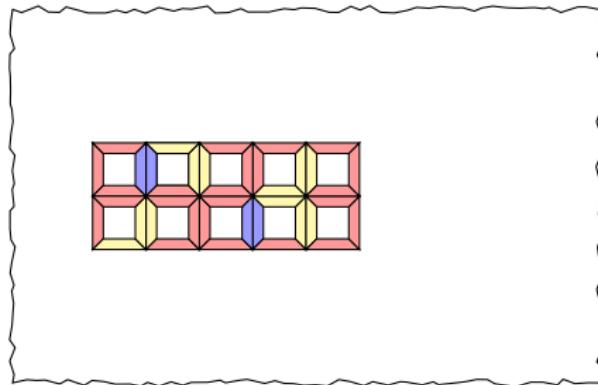
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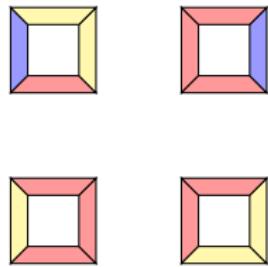
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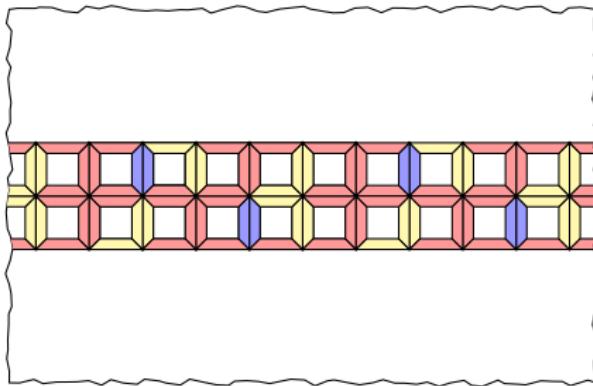
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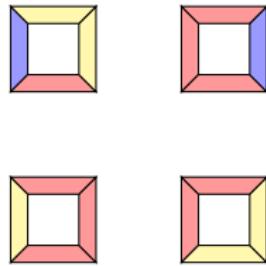
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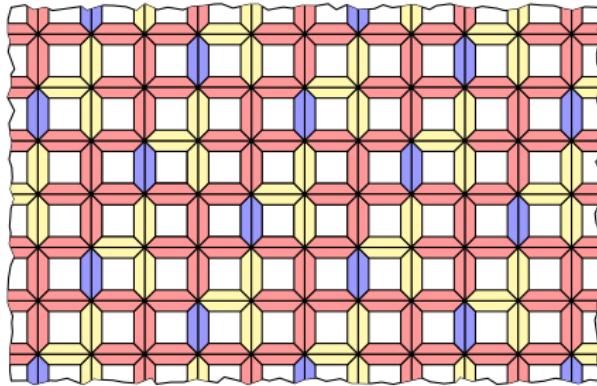
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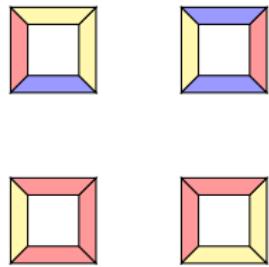


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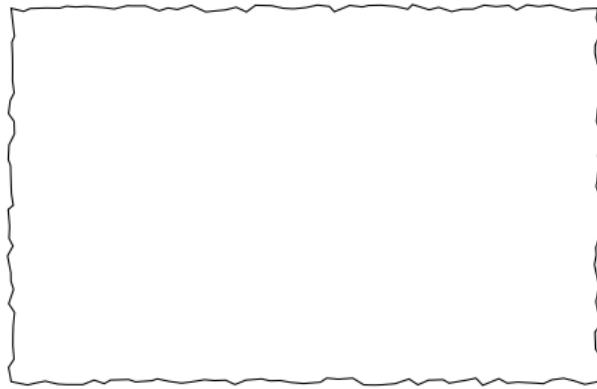
If the tiles set tiles the plane periodically, it is easy to find a configuration!

Wang's tilings

Tiles set



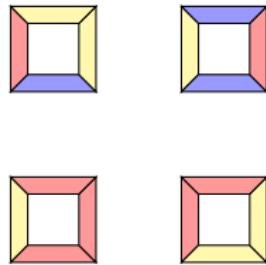
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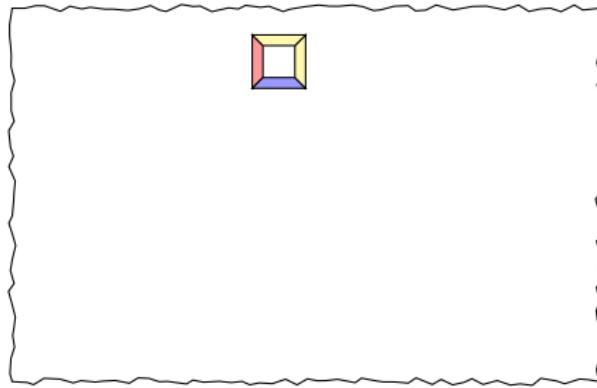
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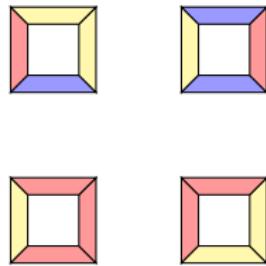
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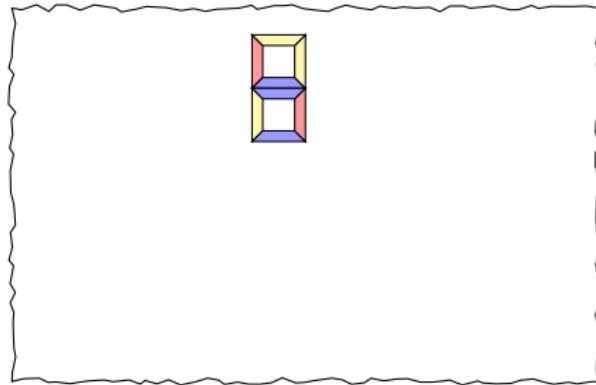
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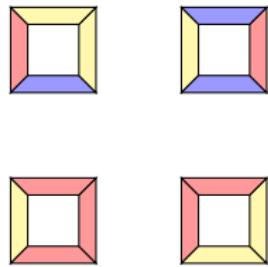
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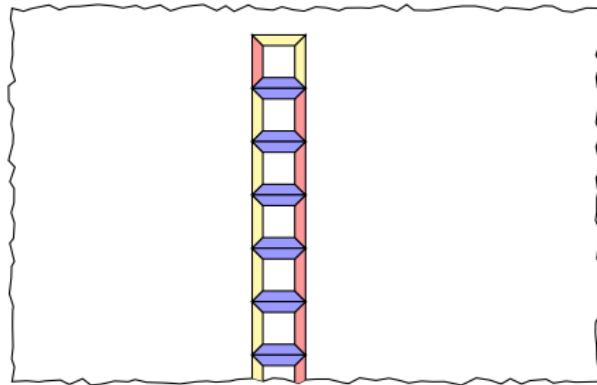
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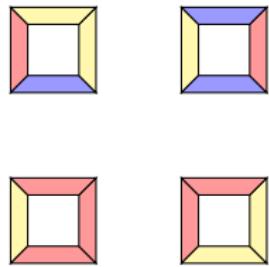
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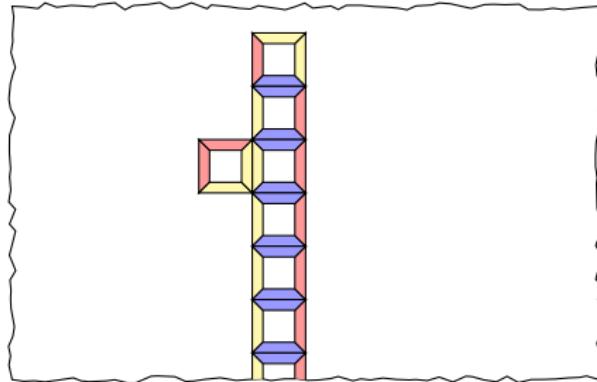
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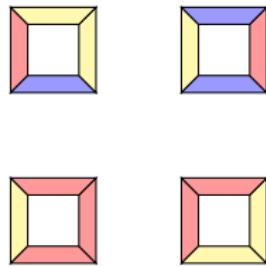
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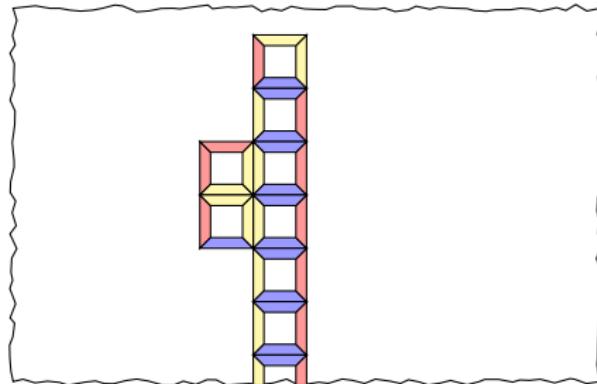
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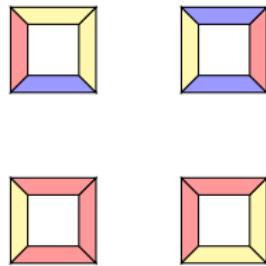
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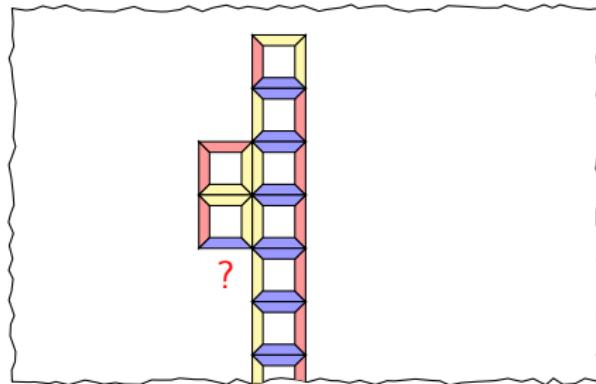
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This tiles set can tile the plane?

Theorem (*Berger-66, Robinson-71, Mozes-89, Kari-96...*)

- There exist tile sets which produces only aperiodic tilings.
- The domino problem is undecidable.

Outline of this course

- **Course 1:** Aperiodicity and decidability
 - How construct aperiodic tilings?
 - It is possible to tile the plane?
- **Course 2:** Sub-Action and Projective sub-action
 - Caractherization of Projective subaction for a sofic
 - Application to find local rules: cut and project tilings
- **Course 3:** Algorithmic optimizations in multidimensional symbolic dynamics
 - Entropy
 - Different classes of effective subshift

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Goal of the course

How construct local rules which force "complex" tilings?

Some elements of symbolic dynamics

Configuration and patterns

Let \mathcal{A} be a finite alphabet.

► $x : \mathbb{Z}^d \rightarrow \mathcal{A} \in \mathcal{A}^{\mathbb{Z}^d}$ is a *configuration*.

$$x = \begin{matrix} & \textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} & \textcircled{5} & \textcircled{6} & \textcircled{7} \\ \textcircled{1} & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \textcircled{2} & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \textcircled{3} & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \textcircled{4} & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \textcircled{5} & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \textcircled{6} & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \textcircled{7} & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \textcircled{8} & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \textcircled{9} & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \textcircled{10} & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \textcircled{11} & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \textcircled{12} & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \textcircled{13} & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \textcircled{14} & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \textcircled{15} & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \textcircled{16} & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{matrix} \in \{0, 1\}^{\mathbb{Z}^2}$$

► Let $\mathbb{U} \subset \mathbb{Z}^d$ be a finite set. A *pattern* is an element $p \in \mathcal{A}^{\mathbb{U}}$.



Support: $\mathbb{U} \subset \mathbb{Z}^d$ finite

0	1	1	1
0	1	0	1
1	0	1	1
0	0	0	1
0	0	0	
1	1		

$p \notin x$

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	0 1	1 0
0 1 0		0 1
1 0 1 0 1 0		
0 1 0 1		
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1 0		

$p \sqsubset x$

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► Let $\mathbb{U} \subset \mathbb{Z}^d$ be a finite set. A *pattern* is an element $p \in \mathcal{A}^{\mathbb{U}}$.

► Define the *language of support* \mathbb{U} :

$$\mathcal{L}_{\mathbb{U}}(x) = \{p \in \mathcal{A}^{\mathbb{U}} : \text{there exists } x \in \mathbf{T} \text{ such that } p \sqsubset x\}.$$

$$\mathcal{L}_1(\{x\}) = \{ \boxed{0}, \boxed{1} \}, \quad \mathcal{L}_2(\{x\}) = \left\{ \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}, \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}} \right\}, \quad \mathcal{L}_3(\{x\}) = \left\{ \boxed{\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}}, \boxed{\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}} \right\}$$

Subshift as dynamical system

- $\mathcal{A}^{\mathbb{Z}^d}$: *set of configurations*
 - ▶ compact
 - ▶ metrisable $d(x, y) \leq 2^{-n}$ where $n = \max \left\{ n \in \mathbb{N} : x_{[-n, n]^d} = y_{[-n, n]^d} \right\}$

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- Dynamic on $\mathcal{A}^{\mathbb{Z}^d}$ given by the shift, \mathbb{Z}^d -action defined for all $\mathbf{j} \in \mathbb{Z}^d$ by:

$$\begin{aligned}\sigma^{\mathbf{j}} : \quad \mathcal{A}^{\mathbb{Z}^d} &\longrightarrow \mathcal{A}^{\mathbb{Z}^d} \\ x = (x_i)_{i \in \mathbb{Z}^d} &\longmapsto \sigma^{\mathbf{j}}(x) = (x_{i+\mathbf{j}})_{i \in \mathbb{Z}^d}.\end{aligned}$$

$$\sigma^{(10, -3)} \left(\begin{array}{c|ccc} & & & \\ & & & \\ \hline \cdots & \text{red} & \text{blue} & \text{red} & \cdots \\ \cdots & \text{red} & \text{blue} & \text{red} & \cdots \\ & & & & \\ & & & & \end{array} \right) = \begin{array}{c|ccc} & & & \\ & & & \\ \hline \cdots & \text{red} & \text{blue} & \text{red} & \cdots \\ \cdots & \text{red} & \text{blue} & \text{red} & \cdots \\ & & & & \\ & & & & \end{array}$$

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Definition

A subshift is a closed shift-invariant subset of $\mathcal{A}^{\mathbb{Z}^d}$.

Let $x = \dots \square \square \square \blacksquare \square \square \square \dots \in \{ \square, \blacksquare \}^{\mathbb{Z}}$.

The orbit $\mathcal{O}(x) = \{ \sigma^n(x) : n \in \mathbb{Z} \} = \left\{ \dots \square \square \square \overset{-n}{\downarrow} \blacksquare \square \square \square \dots \right\}$ is shift-invariant.

But $\lim_{n \rightarrow \infty} \sigma^n(x) = \dots \square \square \square \square \square \square \dots$ so $\mathcal{O}(x)$ is not closed.

$\overline{\mathcal{O}(x)} = \{ \sigma^n(x) : n \in \mathbb{Z} \} \cup \{ \dots \square \square \square \square \square \square \dots \}$ is a subshift.

Morphism

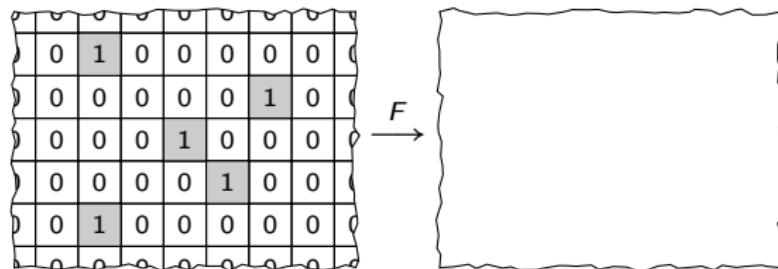
Definition

Let $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}^d}$ and $\mathbf{T}' \subset \mathcal{B}^{\mathbb{Z}^d}$. A function $F : \mathbf{T} \rightarrow \mathbf{T}'$ is a *morphism* if it is continuous and commutes with the shift (i.e. $F \circ \sigma_{\mathbf{T}}^i = \sigma_{\mathbf{T}'}^i \circ F \ \forall i \in \mathbb{Z}^d$).

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Let $\mathbb{U} = \begin{array}{|c|c|}\hline & \square \\ \square & \square \\ \hline \end{array}$ and $\bar{F}\left(\begin{array}{|c|c|}\hline c & \\ \hline a & b \\ \hline \end{array}\right) = a + b + c \pmod{2}$



Morphism

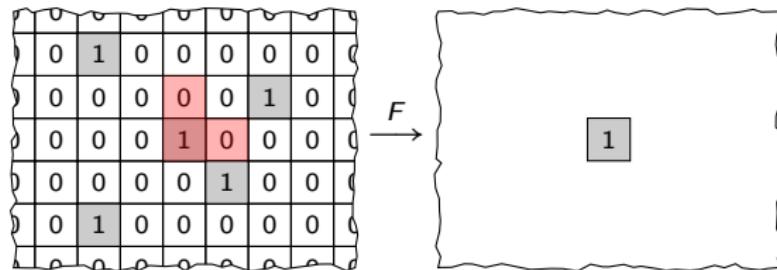
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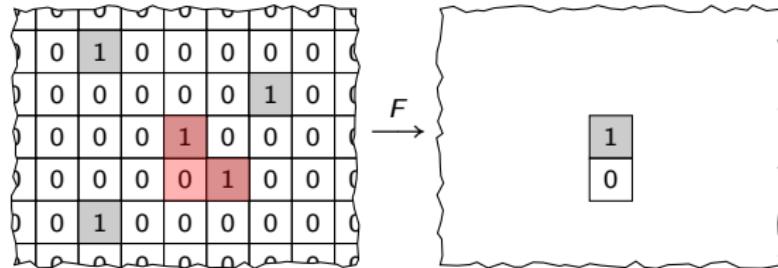
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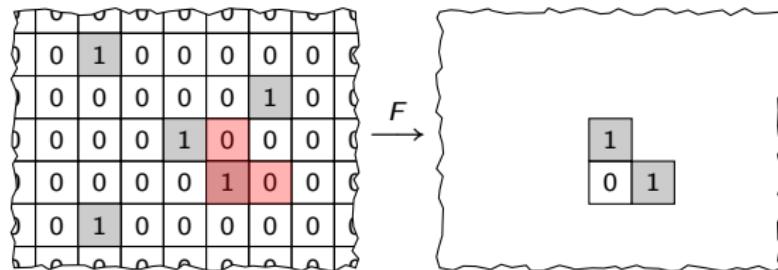
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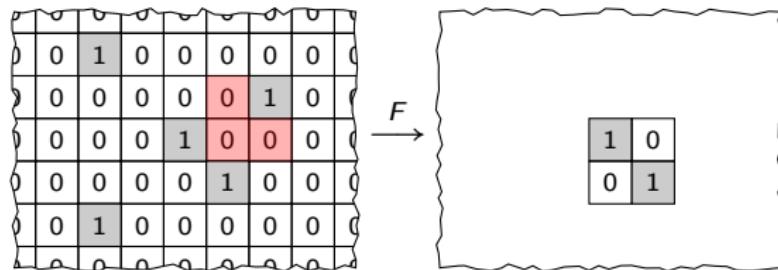
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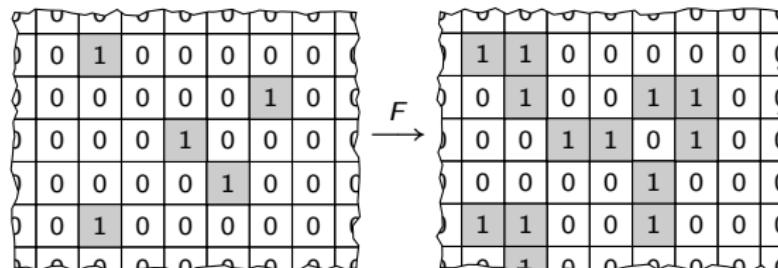
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Definition

- If $\pi : \mathbf{T} \rightarrow \mathbf{T}'$ is surjectif then π is a *factor*.
- If $\varphi : \mathbf{T} \rightarrow \mathbf{T}'$ is a bijective morphism then φ is a *conjugacy*.

Subshifts of finite type Sofic subshifts

Combinatory definition of subshifts

Definition

Let \mathcal{F} be a set of patterns. Define the *d-dimensional subshift of forbidden patterns \mathcal{F} on the alphabet \mathcal{A}* by:

$$\mathbf{T}(\mathcal{A}, d, \mathcal{F}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} : p \notin x \text{ for all } p \in \mathcal{F} \right\} = \bigcap_{p \in \mathcal{F}, i \in \mathbb{Z}^d} \mathcal{A}^{\mathbb{Z}^d} \setminus \sigma^{-i}([p]).$$

where $[p] = \{x \in \mathcal{A}^{\mathbb{Z}^d} : x_{\mathbb{U}} = p\}$ for $p \in \mathcal{A}^{\mathbb{U}}$.

Proposition: Combinatory definition of subshifts

$\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}^d}$ is a subshift if and only if there exists a set of forbidden patterns \mathcal{F} such that $\mathbf{T} = \mathbf{T}(\mathcal{A}, d, \mathcal{F})$.

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\Leftarrow $\mathbf{T}(\mathcal{A}, d, \mathcal{F})$ is a subshift.

\Rightarrow Let \mathbf{T} be a subshift, define $\mathcal{F} = \mathcal{A}^* \setminus \mathcal{L}(\mathbf{T})$.

- If $x \in \mathbf{T}$, every pattern of x is a pattern of $\mathcal{L}(\mathbf{T})$ so $x \in \mathbf{T}(\mathcal{A}, d, \mathcal{F})$.
- If $x \in \mathbf{T}(\mathcal{A}, d, \mathcal{F})$ then $\forall n \in \mathbb{N}$, $x_{[-n,n]^d} \in \mathcal{L}(\mathbf{T})$ that is to say $\exists y^n \in \mathbf{T}$ such that $x_{[-n,n]^d} = y^n_{[-n,n]^d}$. Thus $\lim_{n \rightarrow \infty} y^n = x \in \mathbf{T}$ since \mathbf{T} is closed.

Subshift of finite type

Definition

\mathbf{T} is a *subshift of finite type* if there exists \mathcal{F} a **finite** set of forbidden patterns such that

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The chessboard: Let $\mathcal{A} = \{0, 1\}$ and $\mathcal{F} = \left\{ \begin{array}{|c|c|} \hline i & \\ \hline i & \\ \hline \end{array} \right|, \begin{array}{|c|c|} \hline i & i \\ \hline \end{array} : i \in \mathcal{A} \right\}$. Consider

$$\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \left\{ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline & \text{v} & \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline \hline \end{array} \right|, \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline & \text{v} & \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline \hline \end{array} \right| \right\}$$

Subshift of finite type

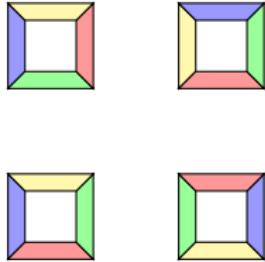
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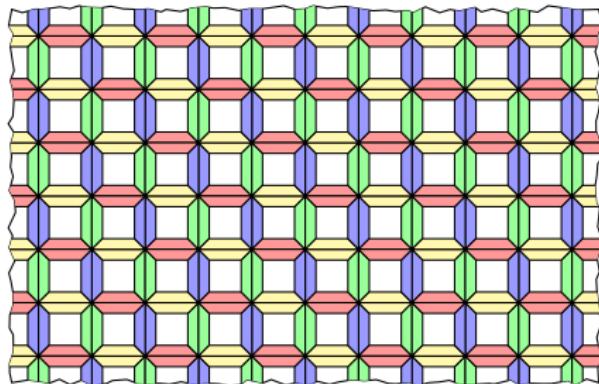
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Wang Tiles:

Tiles \mathcal{W} :



a tiling associated:



$$\mathbf{T}_{\mathcal{W}} = \left\{ (w_n)_{n \in \mathbb{Z}^2} \in \mathcal{W}^{\mathbb{Z}^2} : w_{i,j} = w_{i+1,j} \text{ et } \underline{w_{i,j}} = \overline{w_{i,j-1}} \right\}$$

Subshift of finite type

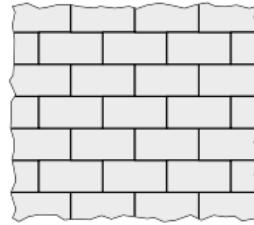
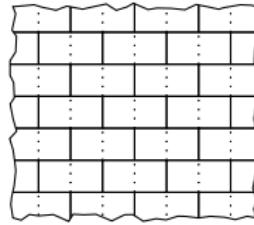
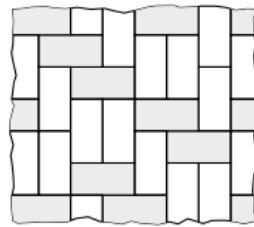
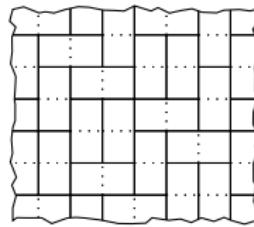
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Domino:

$$T_D = \left\{ \begin{array}{c} \square \\ \dots \\ \square \end{array} \right\}$$



Subshift of finite type

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Algebraic subshift of finite type:

1	0	1	0	0	1	0	1	1	0	1	1	0	1	1	1	0	0	1
1	0	0	1	1	1	0	0	1	0	0	1	0	0	1	0	0	1	0
0	1	1	1	0	1	0	0	0	1	1	1	0	0	0	1	1	0	0
1	1	0	1	0	0	1	1	1	1	0	1	0	0	0	0	0	0	1
0	1	0	0	1	1	1	0	1	0	1	1	0	0	0	0	0	0	0
1	1	0	0	0	1	0	1	1	0	0	1	0	0	0	0	0	0	0
1	0	1	1	1	1	0	0	1	0	0	0	1	1	1	1	0	1	0
0	1	1	1	1	1	0	1	1	1	0	1	0	1	0	1	0	1	0

$$\mathbf{T} = \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : \forall (i,j) \in \mathbb{Z}^2, x_{(i,j)} + x_{(i+1,j)} + x_{i,j+1} = 0 \pmod{2} \right\}$$

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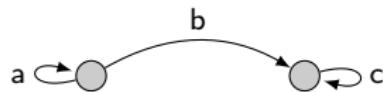
$$\mathbf{T} = \mathbf{T}(\mathcal{A}, d, \mathcal{F}) \subset \mathcal{A}^{\mathbb{Z}^d}$$

Proposition

If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}^d}$ is conjugate to a SFT, then \mathbf{T} is a SFT.

Subshift of finite type (case of dimension 1)

- SFT associated to the oriented graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$



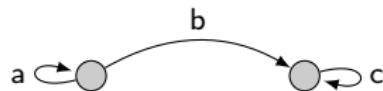
$$\begin{aligned}\mathbf{T}_{\mathcal{G}} &= \{e \in \mathcal{E}^{\mathbb{Z}} : \forall i \in \mathbb{Z}, t(e_i) = i(e_{i+1})\} \\ &= \mathbf{T}(\{a, b, c\}, 1, \{ba, ca, cb\})\end{aligned}$$

- For a subshift $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}}$, define the *Rauzy graph of order n*

$$\mathcal{G}_{\mathbf{T}}^n = \begin{cases} \mathcal{V}_{\mathbf{T}}^n = \mathcal{L}_{n-1}(\mathbf{T}) \\ ((u_0 \dots u_{n-2}, v_0 \dots v_{n-2}) \in \mathcal{E}_{\mathbf{T}}^n) \text{ ssi } u_0 \dots u_{n-2} v_{n-2} = u_0 v_0 \dots v_{n-2} \in \mathcal{L}_n(\mathbf{T}) \end{cases}$$

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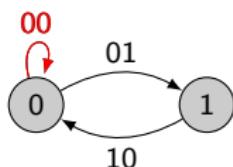
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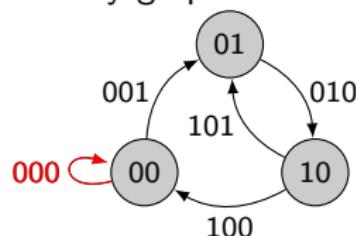
For $\mathbf{T}(\{0,1\}, 1, \{11\})$, one has:

the Rauzy graph of order 2:



$$\dots \begin{matrix} 0 \\ 0 \end{matrix} \quad \begin{matrix} 0 \\ 1 \end{matrix} \quad \begin{matrix} 1 \\ 0 \end{matrix} \quad \begin{matrix} 0 \\ 0 \end{matrix} \quad \begin{matrix} 0 \\ 1 \end{matrix} \quad \begin{matrix} 0 \\ 1 \end{matrix} \quad \begin{matrix} 1 \\ 0 \end{matrix} \quad \begin{matrix} 0 \\ 1 \end{matrix} \quad \dots \in \mathbf{T}_{\mathcal{G}_2}$$

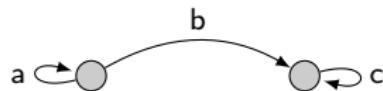
the Rauzy graph of order 3:



$$\dots \begin{matrix} 0 \\ 0 \end{matrix} \quad \begin{matrix} 0 \\ 0 \end{matrix} \quad \begin{matrix} 0 \\ 1 \end{matrix} \quad \begin{matrix} 1 \\ 0 \end{matrix} \quad \begin{matrix} 0 \\ 0 \end{matrix} \quad \begin{matrix} 0 \\ 0 \end{matrix} \quad \begin{matrix} 1 \\ 0 \end{matrix} \quad \begin{matrix} 0 \\ 1 \end{matrix} \quad \dots \in \mathbf{T}_{\mathcal{G}_3}$$

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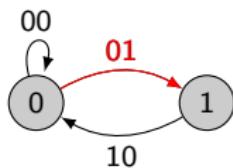
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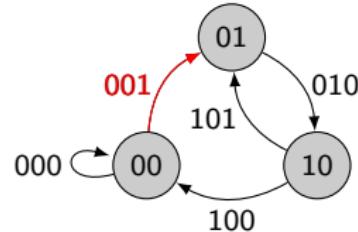
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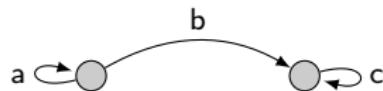
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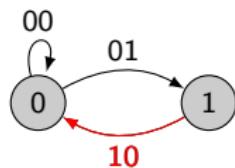
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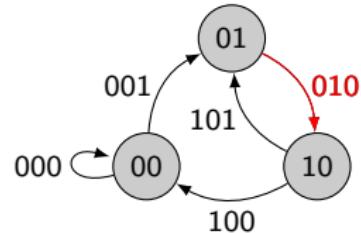
For $\mathbf{T}(\{0,1\}, 1, \{11\})$, one has:

the Rauzy graph of order 2:



$$\dots \begin{matrix} 0 & 0 & \textcolor{red}{1} & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{matrix} \dots \in \mathbf{T}_{\mathcal{G}_2}$$

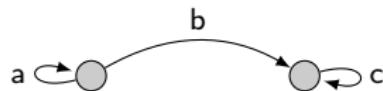
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Subshift of finite type (case of dimension 1)

- SFT associated to the oriented graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$



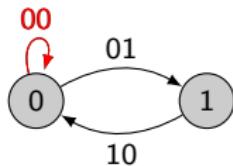
$$\begin{aligned}\mathbf{T}_{\mathcal{G}} &= \{e \in \mathcal{E}^{\mathbb{Z}} : \forall i \in \mathbb{Z}, t(e_i) = i(e_{i+1})\} \\ &= \mathbf{T}(\{a, b, c\}, 1, \{ba, ca, cb\})\end{aligned}$$

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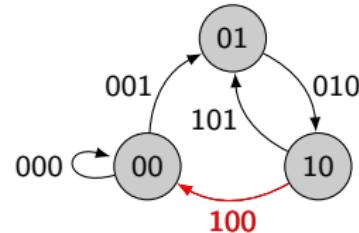
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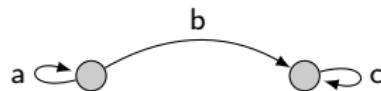
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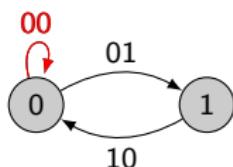
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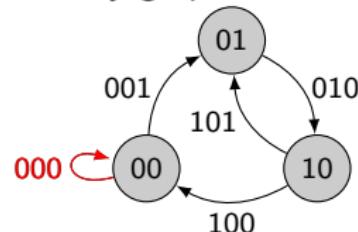
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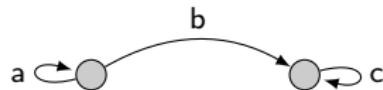
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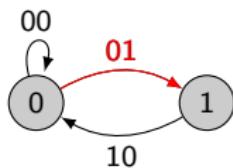
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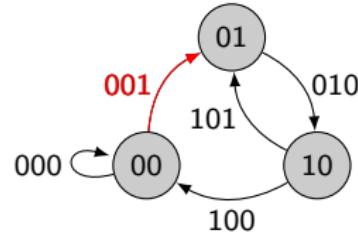
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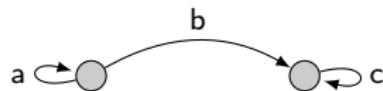
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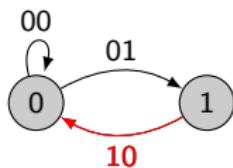
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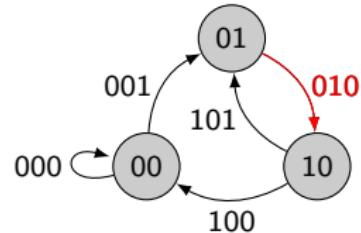
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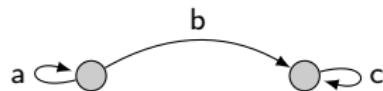
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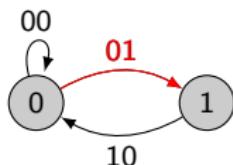
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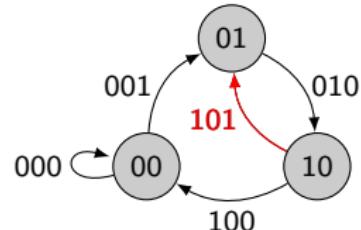
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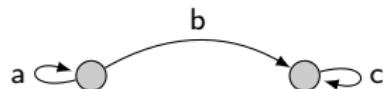
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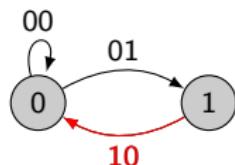
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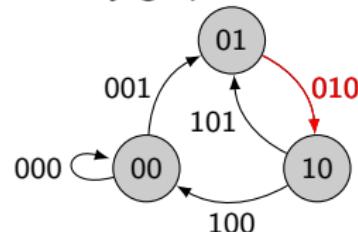
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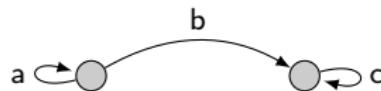
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Theorem

A subshift $\mathbf{T}(\mathcal{A}, 1, \mathcal{F})$ where $\mathcal{F} \subset \mathcal{A}^n$ is conjugate to a subshift of finite type associated to the Rauzy graph $\mathcal{G}_{\mathbf{T}}^n$.

Corollary

- In dimension 1, every nonempty SFT contains a periodic orbit.
- In dimension 1, it is decidable to know if a SFT is empty.

Sofic subshift

Définition

A subshift $\mathbf{T} \subset \mathcal{B}^{\mathbb{Z}^d}$ is *sofic* if there exist a SFT $\mathbf{T}(\mathcal{A}, d, \mathcal{F})$ and a factor $\pi : \mathcal{A}^{\mathbb{Z}^d} \longrightarrow \mathcal{B}^{\mathbb{Z}^d}$ such that $\mathbf{T} = \pi(\mathbf{T}(\mathcal{A}, d, \mathcal{F}))$.

Consider $\mathbf{T} = \left\{ x \in \{0, 1\}^{\mathbb{Z}^d} : \text{there is at most one } i \in \mathbb{Z}^d \text{ such that } x_i = 1 \right\}$.

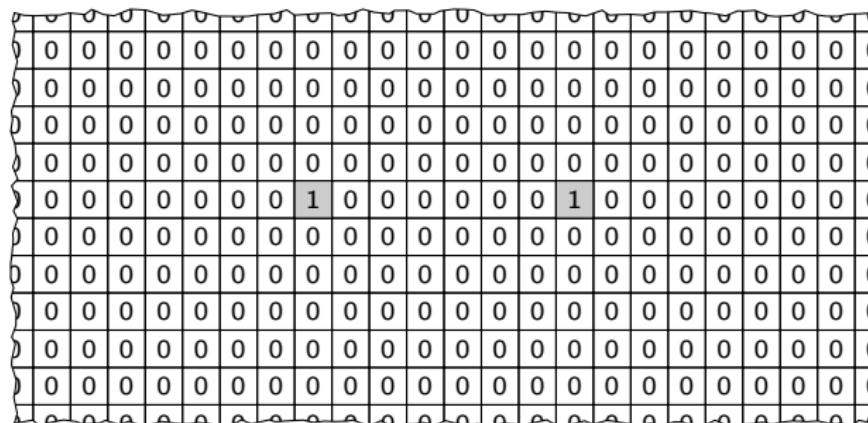
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0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

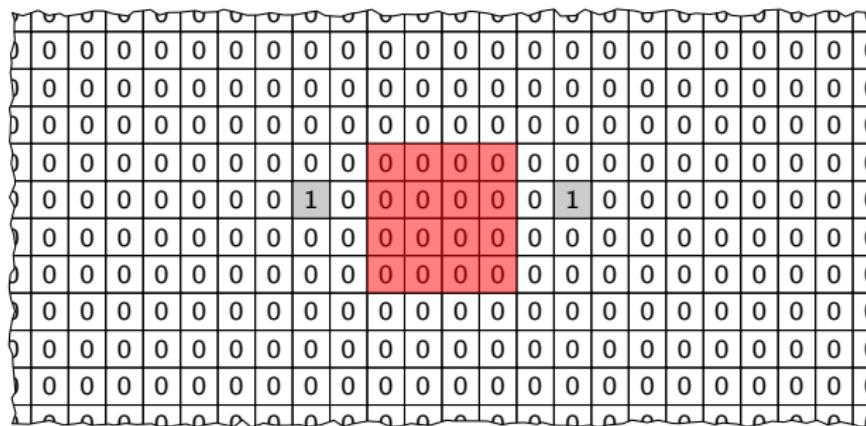
Sofic subshift

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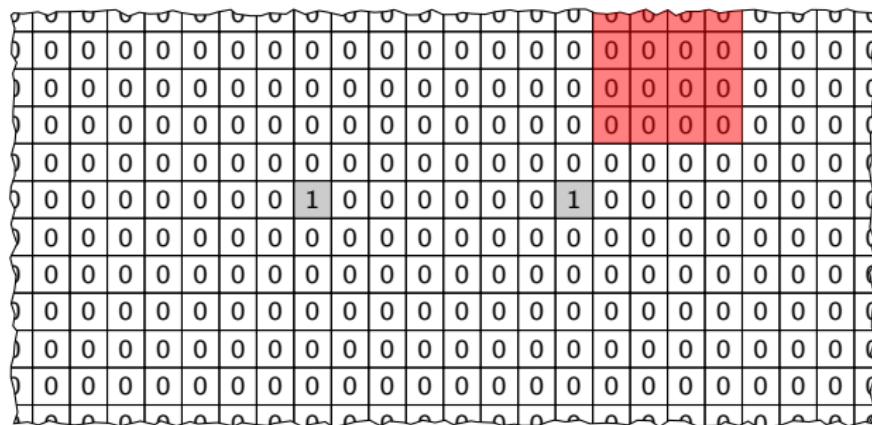
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$\in \mathbf{T}$ contradiction!

Sofic subshift

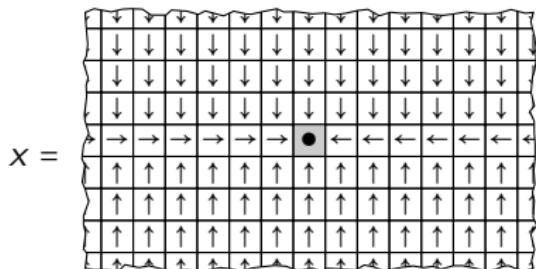
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Let $\mathcal{A}_\bullet = \{\bullet, \leftarrow, \rightarrow, \uparrow, \downarrow\}$ and $\mathcal{F}_\bullet \subset \mathcal{A}_\bullet^{\{0\} \times [0, 1]} \cup \mathcal{A}_\bullet^{[0, 1] \times \{0\}}$ which define the SFT:

$$\mathbf{T}(\mathcal{A}_\bullet, 2, \mathcal{F}_\bullet)$$



$x =$

$$\xrightarrow{\pi} \quad \pi(x) =$$



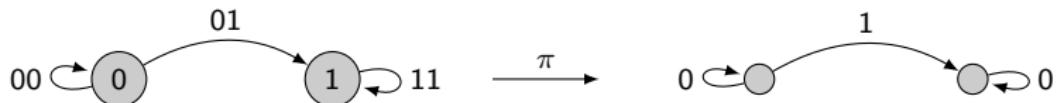
$$\begin{aligned} \pi : \quad \rightarrow, \leftarrow, \uparrow, \downarrow &\longmapsto 0 \\ \bullet &\longmapsto 1 \end{aligned}$$

Sofic subshift (case of dimension 1)

Let $\mathbf{T} = \{x \in \{0, 1\}^{\mathbb{Z}^d} : \text{there is at most } i \in \mathbb{Z}^d \text{ such that } x_i = 1\}$.

The SFT $\mathbf{T}(\{0, 1\}, 1, \{10\})$ is transformed in \mathbf{T} via π :

00	↔	0
01	↔	1
11	↔	0



Theorem (Weiss 1973)

In dimension 1, the language of a sofic is rational (i.e. recognized by a finite automaton).

Examples:

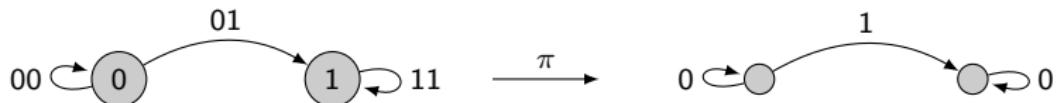
- The subshift $\mathbf{T}(\{a, b\}, 1, \{ba^{2n+1}b : n \in \mathbb{N}\})$ is ?
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Sofic subshift (case of dimension 1)

Let $\mathbf{T} = \{x \in \{0, 1\}^{\mathbb{Z}^d} : \text{there is at most } i \in \mathbb{Z}^d \text{ such that } x_i = 1\}$.

The SFT $\mathbf{T}(\{0, 1\}, 1, \{10\})$ is transformed in \mathbf{T} via π :

00	\mapsto	0
01	\mapsto	1
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Theorem (Weiss 1973)

In dimension 1, the language of a sofic is rational (i.e. recognized by a finite automaton).

Examples:

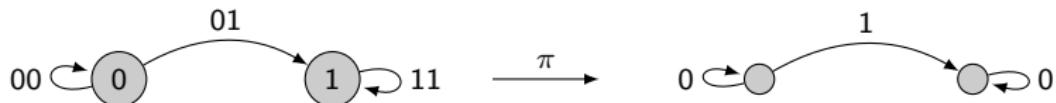
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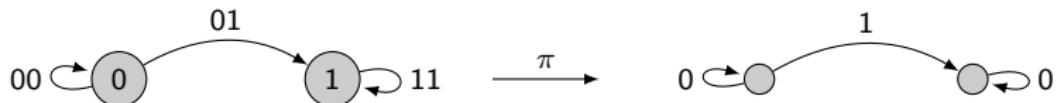
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And in two dimension?

Let $\Sigma = \mathbf{T}(\{a, b, \$\}, 1, \{ba, \beta a^n b^m \alpha : n \neq m, \alpha \neq b, \beta \neq a\})$ and consider

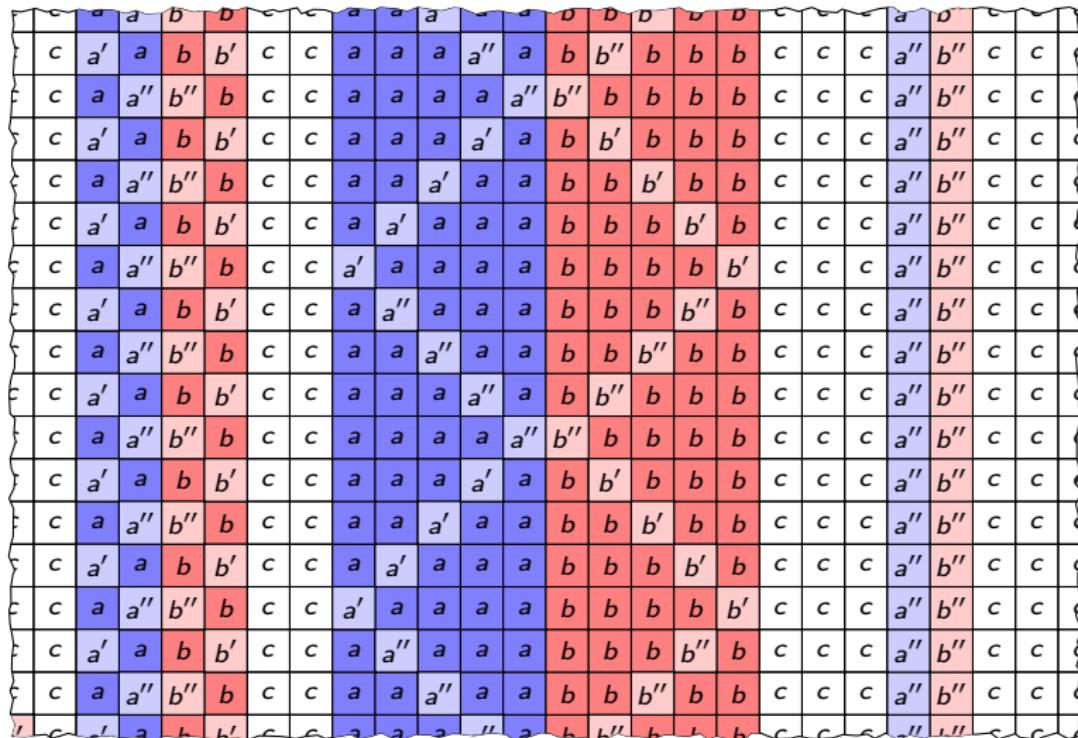
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a	a	a	b	b	c	c	a	a	a	a	b	b	b	b	b	c	c	c	a	b	c	c
c	a	a	b	b	c	c	a	a	a	a	b	b	b	b	b	c	c	c	a	b	c	c
c	a	a	b	b	c	c	a	a	a	a	b	b	b	b	b	c	c	c	a	b	c	c
c	a	a	b	b	c	c	a	a	a	a	b	b	b	b	b	c	c	c	a	b	c	c
c	a	a	b	b	c	c	a	a	a	a	b	b	b	b	b	c	c	c	a	b	c	c
c	a	a	b	b	c	c	a	a	a	a	b	b	b	b	b	c	c	c	a	b	c	c
c	a	a	b	b	c	c	a	a	a	a	b	b	b	b	b	c	c	c	a	b	c	c
c	a	a	b	b	c	c	a	a	a	a	b	b	b	b	b	c	c	c	a	b	c	c
c	a	a	b	b	c	c	a	a	a	a	b	b	b	b	b	c	c	c	a	b	c	c
c	a	a	b	b	c	c	a	a	a	a	b	b	b	b	b	c	c	c	a	b	c	c
c	a	a	b	b	c	c	a	a	a	a	b	b	b	b	b	c	c	c	a	b	c	c
c	a	a	b	b	c	c	a	a	a	a	b	b	b	b	b	c	c	c	a	b	c	c
c	a	a	b	b	c	c	a	a	a	a	b	b	b	b	b	c	c	c	a	b	c	c
c	a	a	b	b	c	c	a	a	a	a	b	b	b	b	b	c	c	c	a	b	c	c
c	a	a	b	b	c	c	a	a	a	a	b	b	b	b	b	c	c	c	a	b	c	c
c	a	a	b	b	c	c	a	a	a	a	b	b	b	b	b	c	c	c	a	b	c	c
c	a	a	b	b	c	c	a	a	a	a	b	b	b	b	b	c	c	c	a	b	c	c
c	a	a	b	b	c	c	a	a	a	a	b	b	b	b	b	c	c	c	a	b	c	c
c	a	a	b	b	c	c	a	a	a	a	b	b	b	b	b	c	c	c	a	b	c	c

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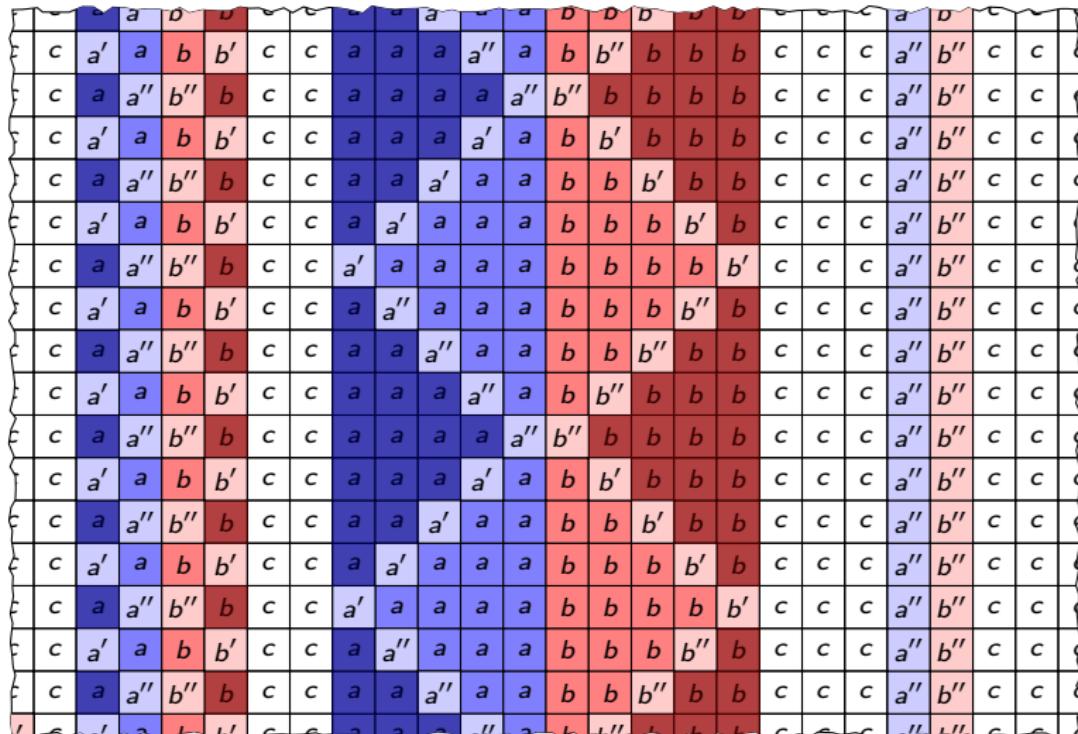
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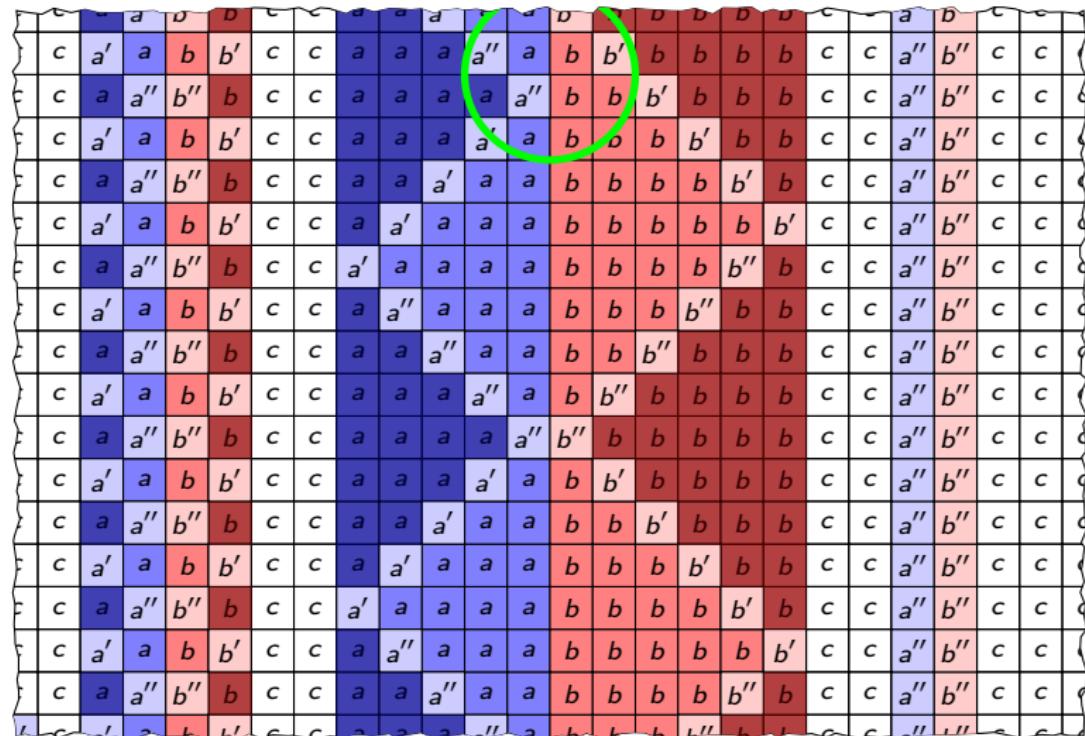
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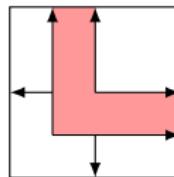
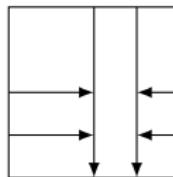
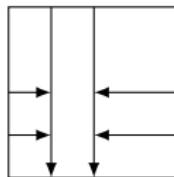
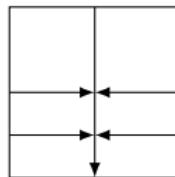
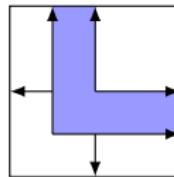
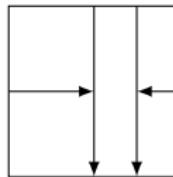
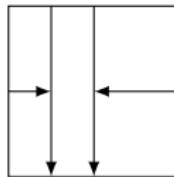
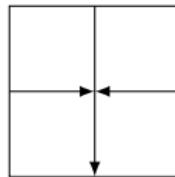
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My first aperiodic tiling: Robinson's tiling

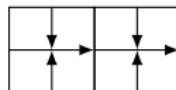
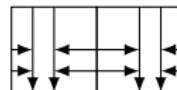
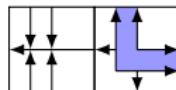
Alphabet of the tiling of Robinson

There exists different means to define the Robinson tiling. Consider $\mathcal{R}obi$ the next set of tiles modulo the rotation

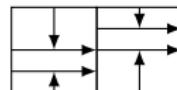


Local rules

Incoming and outgoing arrows must be respected:

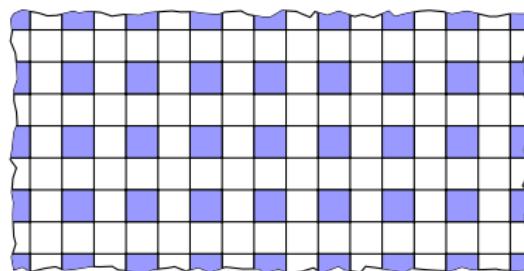


Allowed



Not allowed

We add the forbidden patterns \mathcal{F} which impose the alternating of the colors:

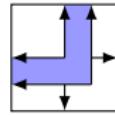
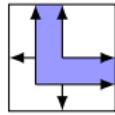
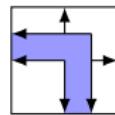
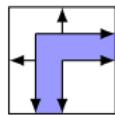


$$\text{ou } \square \in \{ \text{[forbidden pattern 1]} \text{ [forbidden pattern 2]} \text{ [forbidden pattern 3]} \text{ [forbidden pattern 4]} \}$$

Denote T_{Robi} the SFT described by these rules.

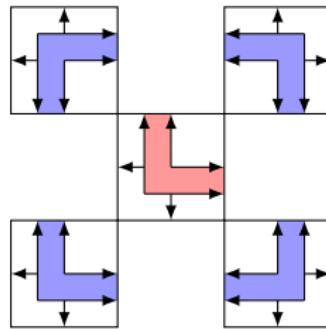
Existence of the tiling (level 1)

The Robinson tiling is based on a hierarchical structure. Nine of these tiles can be assembled to form a *super-tile of level 1*:



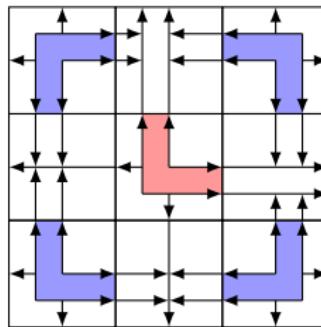
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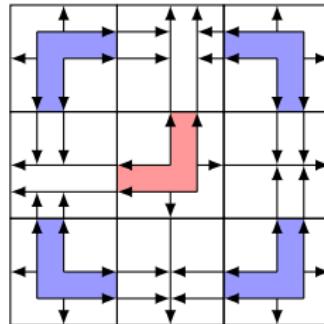
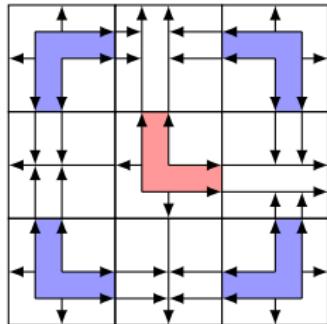
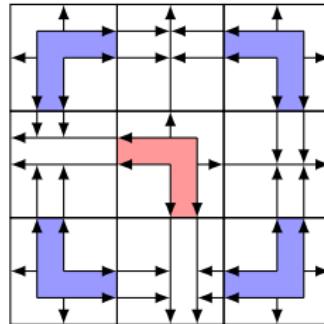
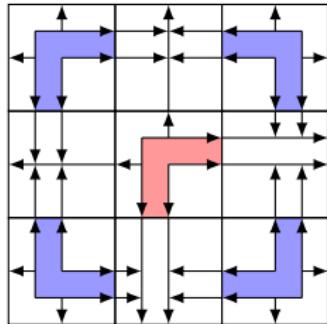
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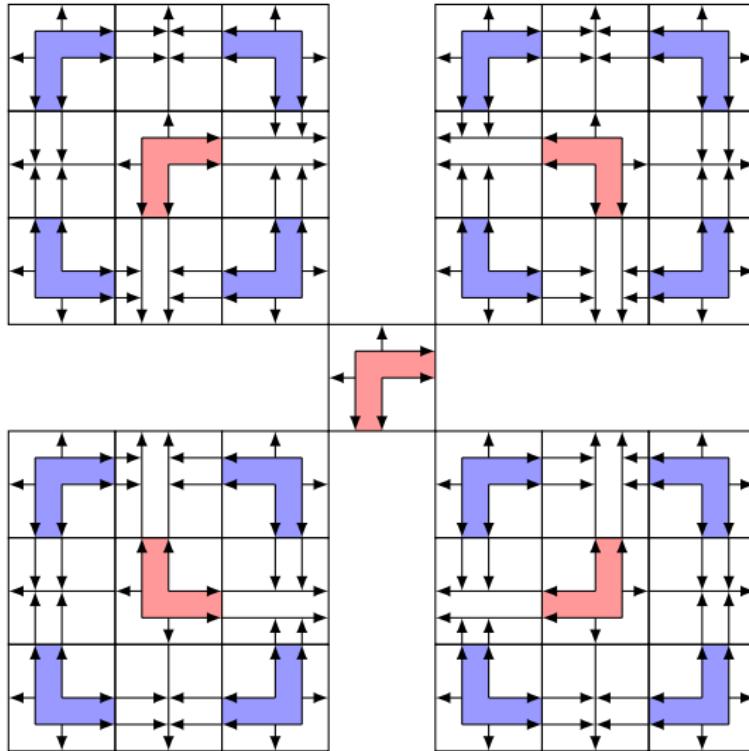
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Super-tiles of level 1 can be assembled to form *super-tiles of level 2*:



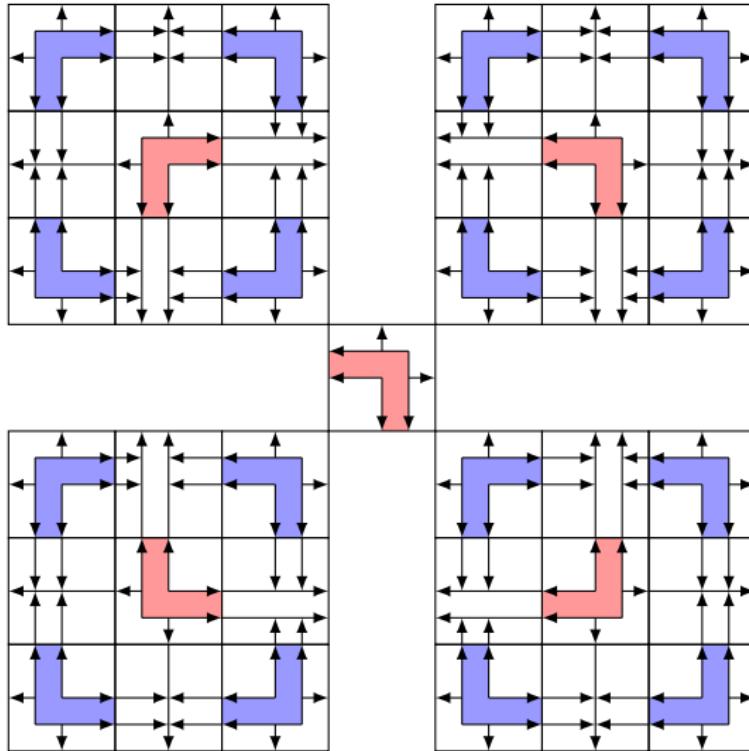
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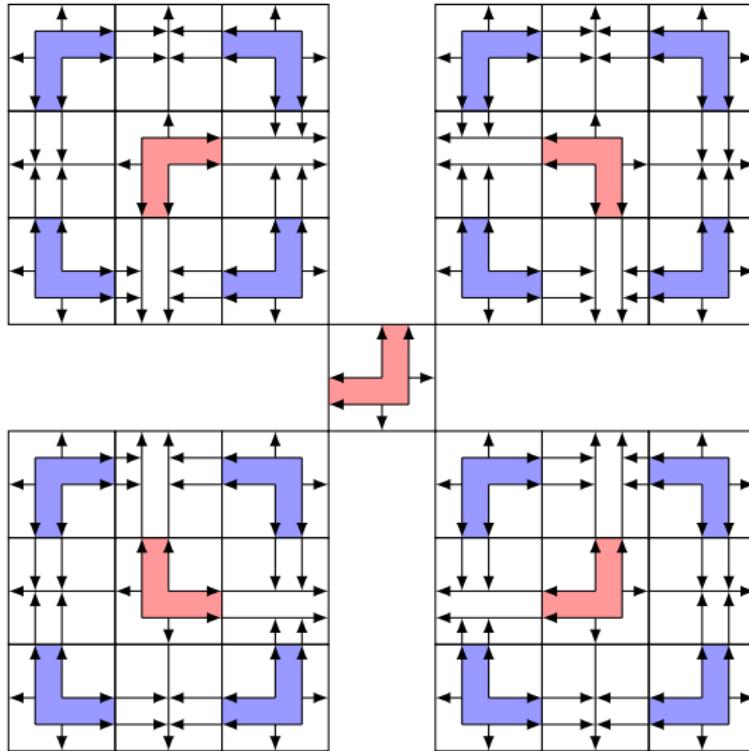
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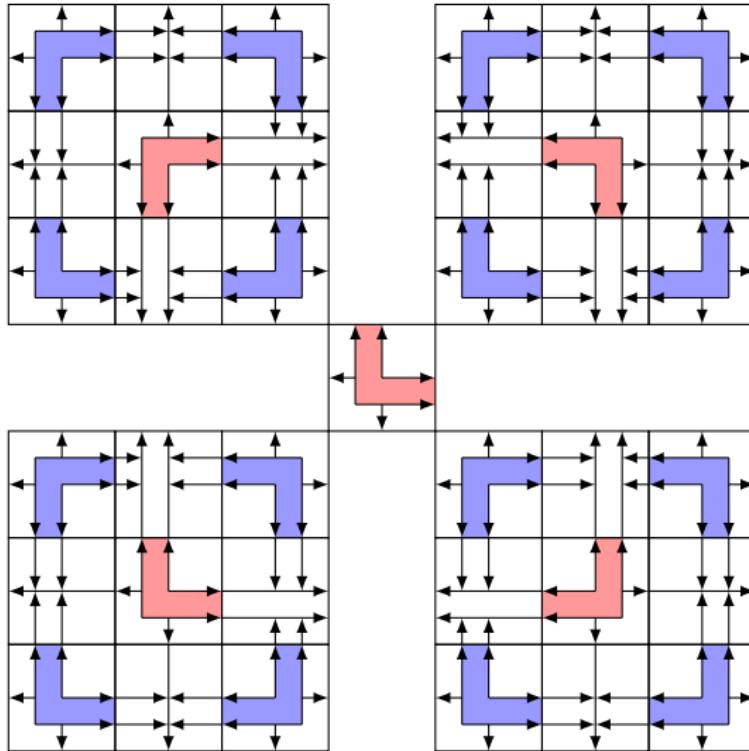
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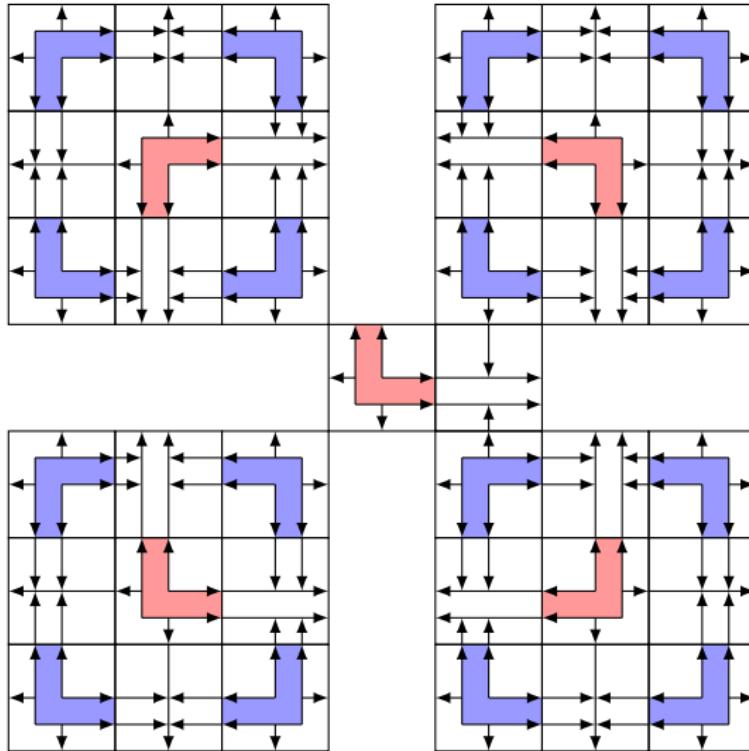
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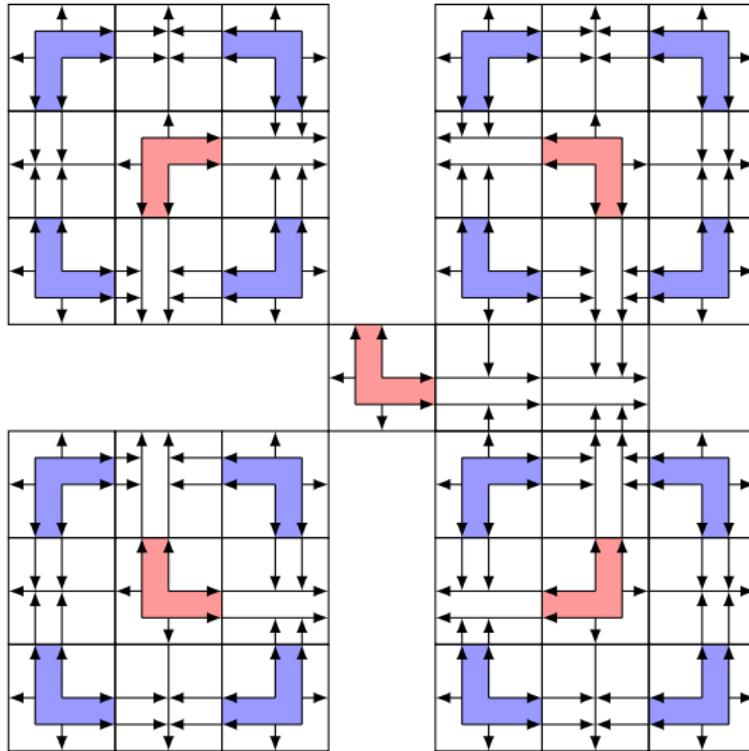
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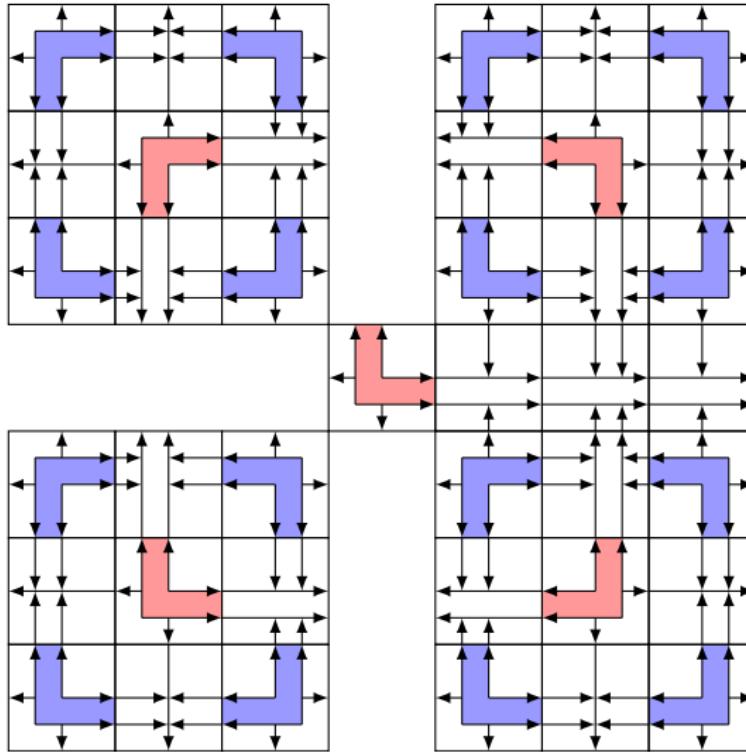
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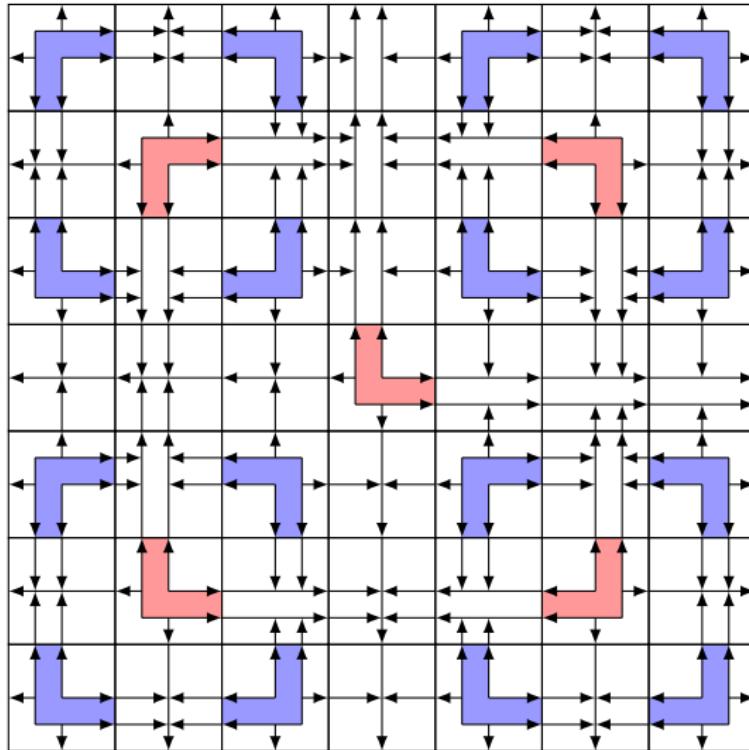
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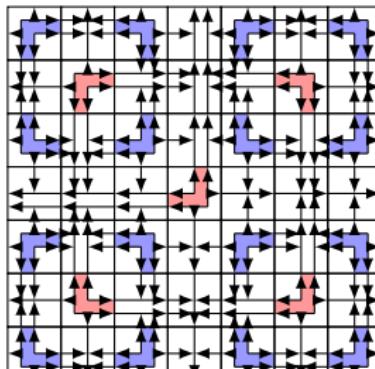
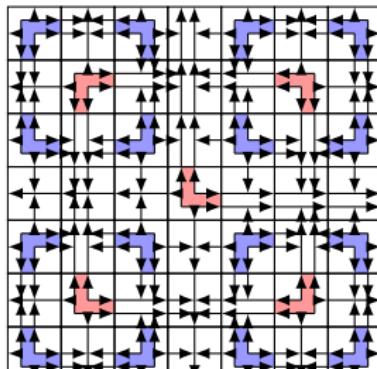
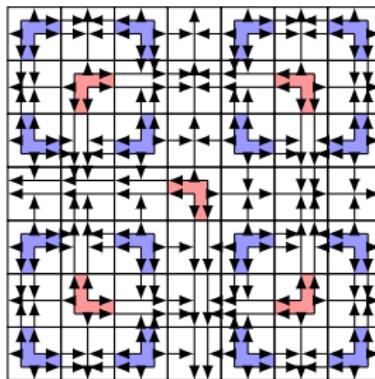
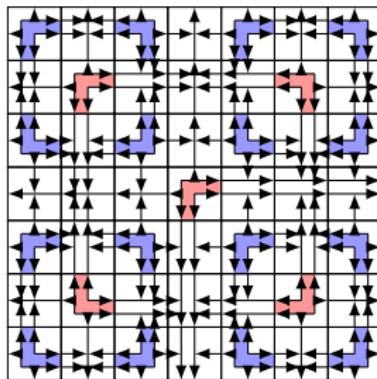
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Existence of the tiling

Iterating the operation, super-tiles of level n can be formed and by compacity we conclude that $T_{Robi} \neq \emptyset$.

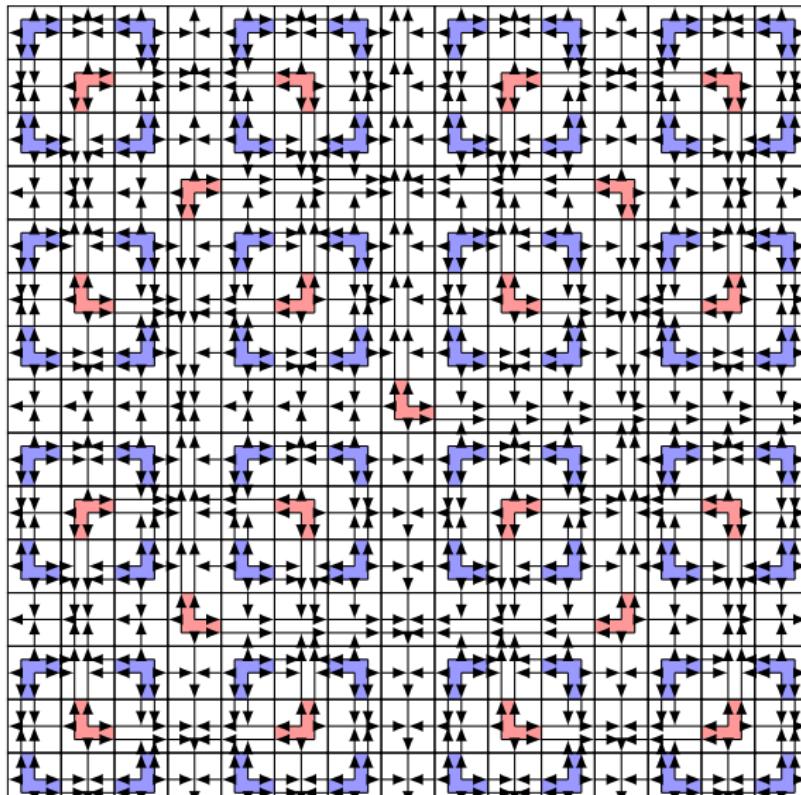


Aperiodic two-dimensional SFT

My first aperiodic tiling: Robinson's tiling

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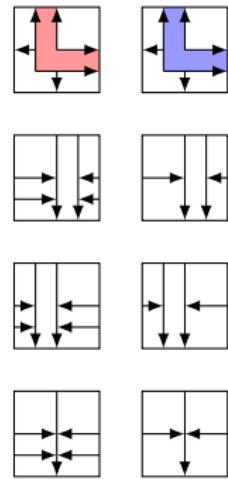
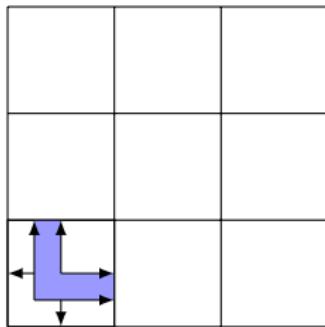
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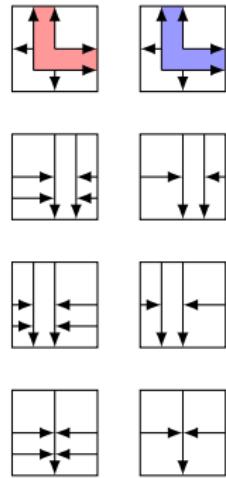
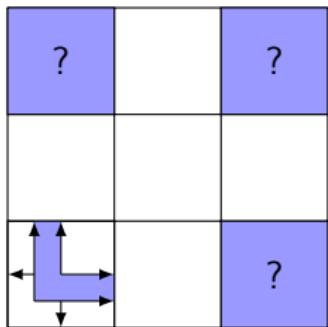
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Force the presence of super-tiles (of level 1)

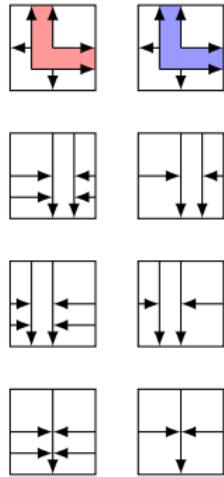
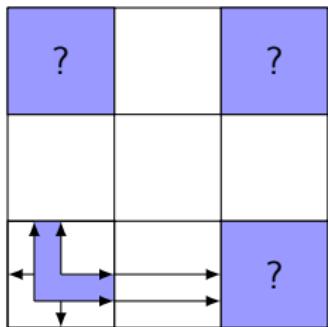


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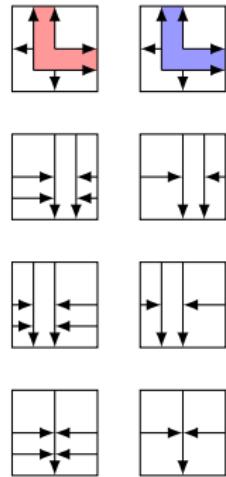
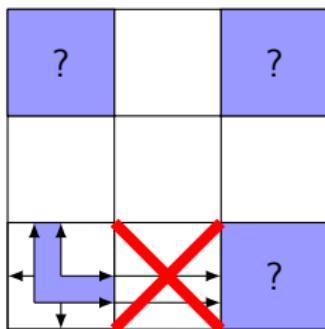
With $\boxed{?} \in \left\{ \begin{array}{c} \text{red L-shaped tile} \\ \text{blue T-shaped tile} \\ \text{blue L-shaped tile} \end{array} \right\}$

Force the presence of super-tiles (of level 1)



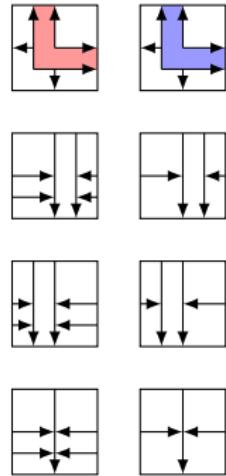
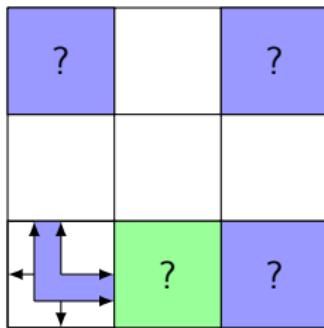
With $\boxed{?} \in \left\{ \begin{array}{c} \text{[L-shaped tile]} \\ \text{[J-shaped tile]} \\ \text{[T-shaped tile]} \\ \text{[F-shaped tile]} \end{array} \right\}$

Force the presence of super-tiles (of level 1)



With $\boxed{?} \in \left\{ \begin{array}{c} \text{[red L]} \\ \text{[blue L]} \\ \text{[red J]} \\ \text{[blue J]} \\ \text{[red T]} \\ \text{[blue T]} \\ \text{[red F]} \\ \text{[blue F]} \end{array} \right\}$

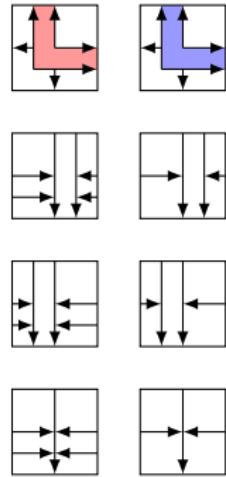
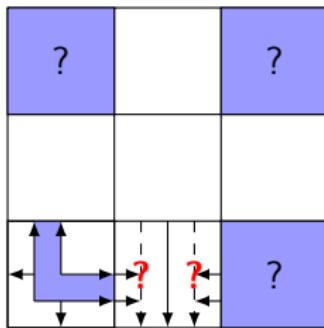
Force the presence of super-tiles (of level 1)



With $\in \left\{ \begin{array}{c} \text{red L-shaped tile} \\ \text{white 3x3 grid tile} \end{array} \right\}$

With $\in \left\{ \begin{array}{c} \text{white 3x3 grid tile} \\ \text{white 3x3 grid tile} \end{array} \right\}$

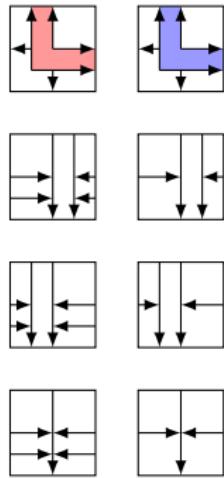
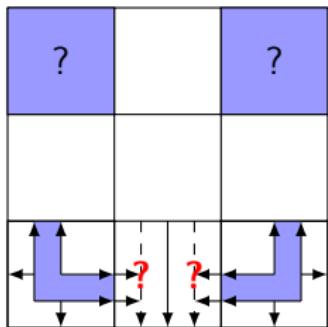
Force the presence of super-tiles (of level 1)



With $\in \left\{ \begin{array}{c} \text{red L-shaped tile} \\ \text{blue L-shaped tile} \\ \text{white tile with central arrow} \end{array} \right\}$

With $\in \left\{ \begin{array}{c} \text{white tile with central arrow} \\ \text{white tile with central arrow} \\ \text{white tile with central arrow} \end{array} \right\}$

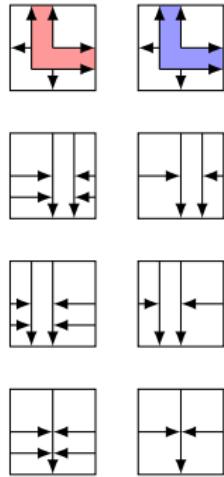
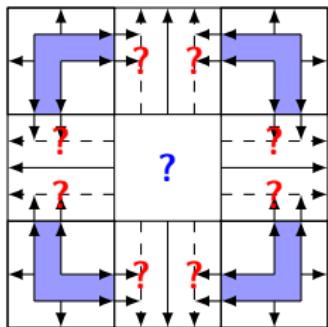
Force the presence of super-tiles (of level 1)



With $\in \left\{ \begin{array}{c} \text{red L} \\ \text{blue L} \\ \text{blue T} \\ \text{blue F} \end{array} \right\}$

With $\in \left\{ \begin{array}{c} \text{red 2x2} \\ \text{blue 2x2} \\ \text{blue 2x2} \end{array} \right\}$

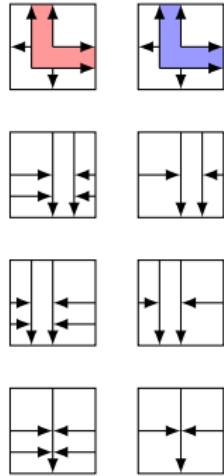
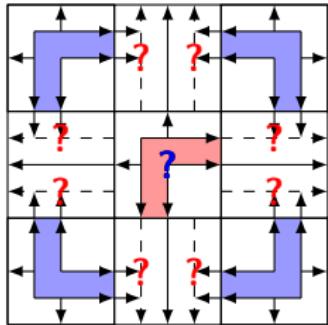
Force the presence of super-tiles (of level 1)



With $\boxed{?} \in \left\{ \begin{array}{c} \text{red L} \\ \text{red T} \\ \text{blue T} \\ \text{blue F} \end{array} \right\}$

With $\boxed{\text{? ?}} \in \left\{ \begin{array}{c} \text{red 2x2} \\ \text{blue 2x2} \\ \text{blue 2x2} \end{array} \right\}$

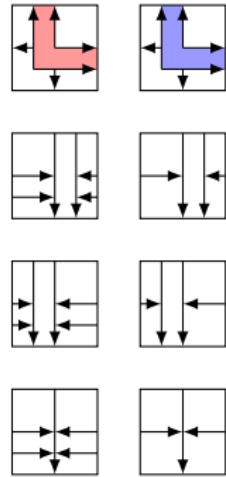
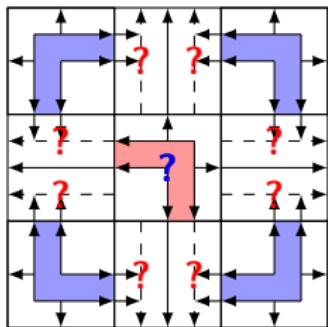
Force the presence of super-tiles (of level 1)



With $\boxed{?} \in \left\{ \begin{array}{c} \text{red L} \\ \text{blue L} \\ \text{red T} \\ \text{blue T} \end{array} \right\}$

With $\boxed{\text{??}} \in \left\{ \begin{array}{c} \text{red T} \\ \text{blue T} \\ \text{red L} \\ \text{blue L} \end{array} \right\}$

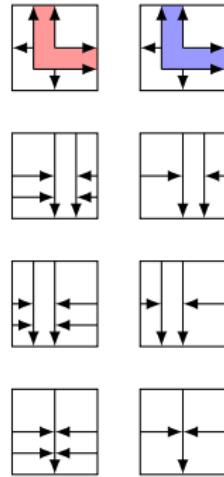
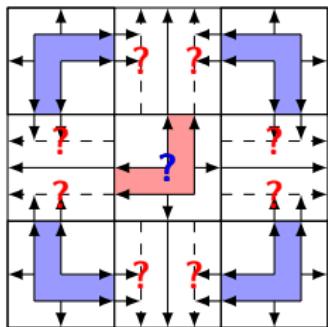
Force the presence of super-tiles (of level 1)



With $\boxed{?} \in \left\{ \begin{array}{c} \text{red L} \\ \text{red T} \\ \text{blue L} \\ \text{blue T} \end{array} \right\}$

With $\boxed{\text{??}} \in \left\{ \begin{array}{c} \text{blue V} \\ \text{red V} \end{array} \right\}$

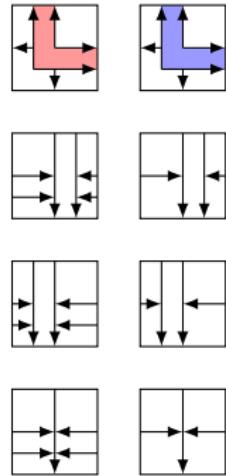
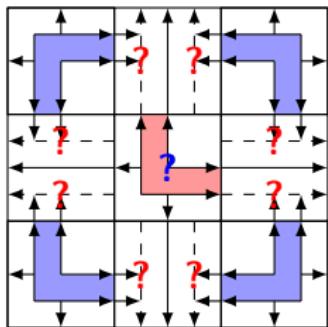
Force the presence of super-tiles (of level 1)



With $\boxed{?} \in \left\{ \begin{array}{c} \text{red L} \\ \text{blue L} \\ \text{red T} \\ \text{blue T} \end{array} \right\}$

With $\boxed{\begin{array}{cc} ? & ? \end{array}} \in \left\{ \begin{array}{c} \text{red 2x2} \\ \text{blue 2x2} \\ \text{red 2x2} \\ \text{blue 2x2} \end{array} \right\}$

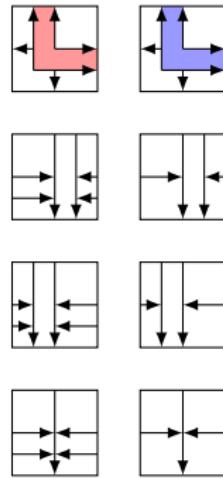
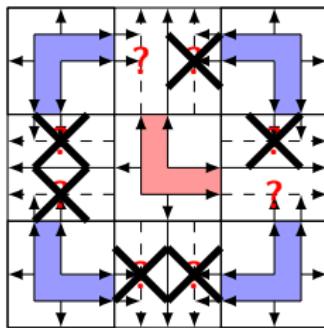
Force the presence of super-tiles (of level 1)



With $\boxed{?} \in \left\{ \begin{array}{c} \text{red L} \\ \text{blue L} \end{array} \right\}$

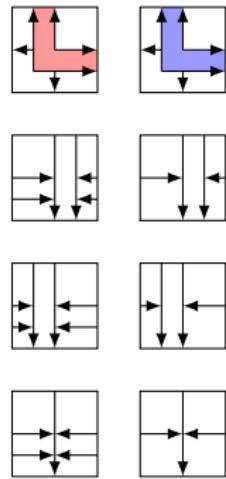
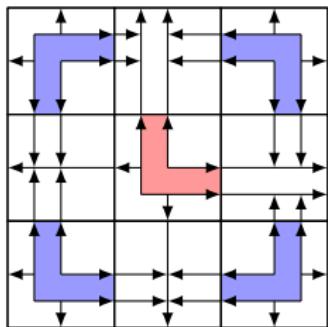
With $\boxed{\text{? ?}} \in \left\{ \begin{array}{c} \text{red L} \\ \text{blue L} \end{array} \right\}$

Force the presence of super-tiles (of level 1)

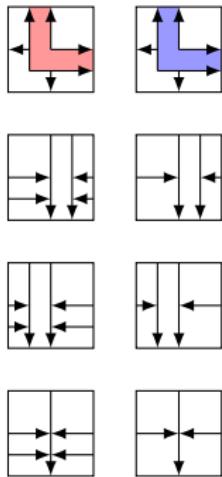
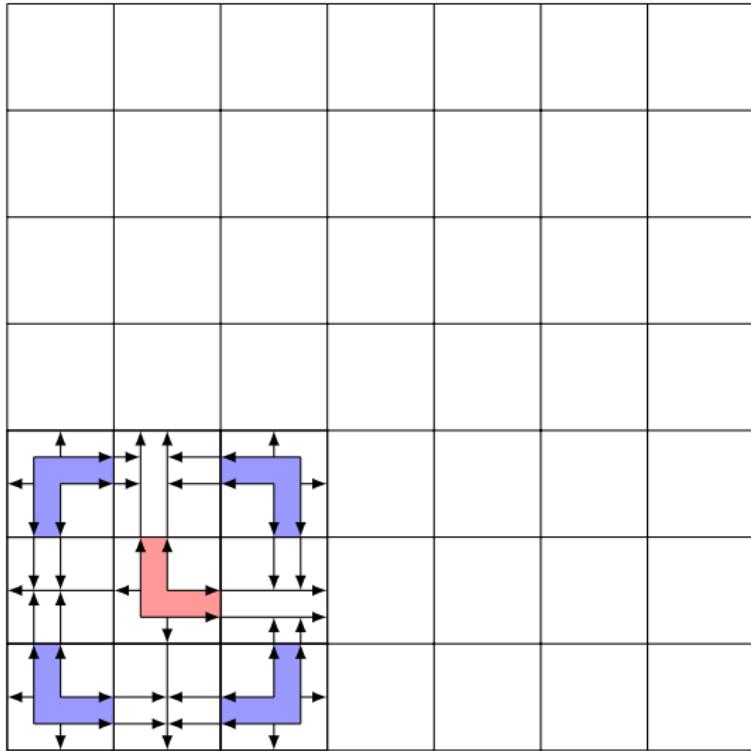


With $\begin{array}{|c|c|}\hline ? & ? \\ \hline\end{array} \in \left\{ \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \right\}$

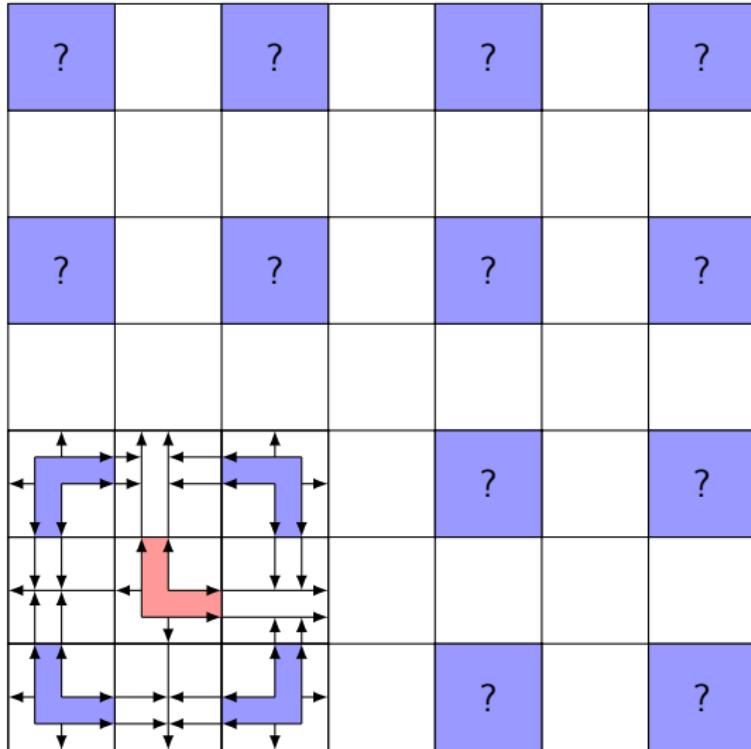
Force the presence of super-tiles (of level 1)



Force the presence of super-tiles (of level 2)

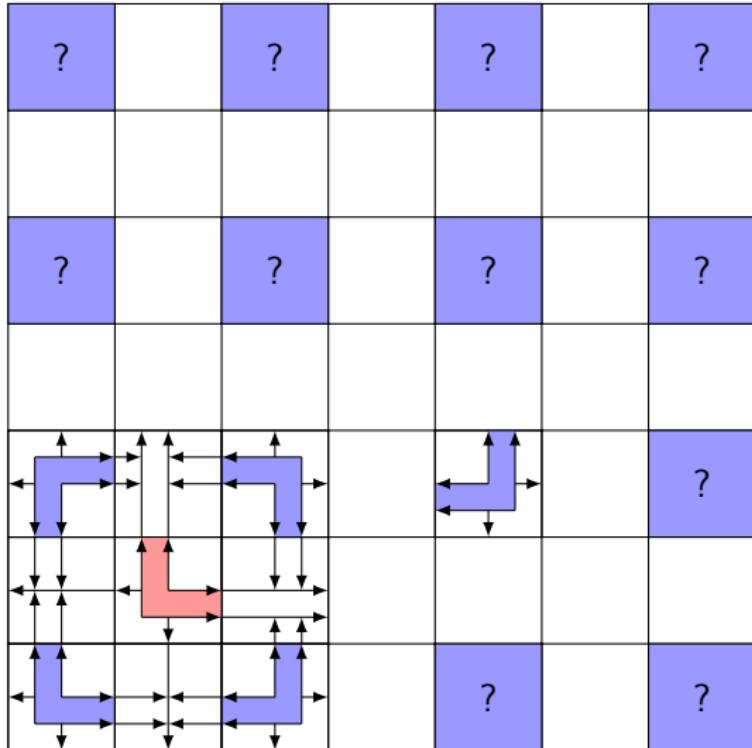


Force the presence of super-tiles (of level 2)



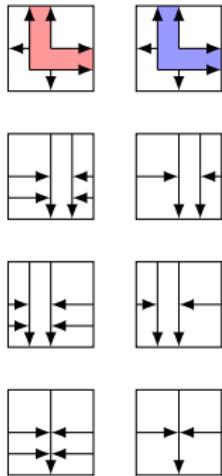
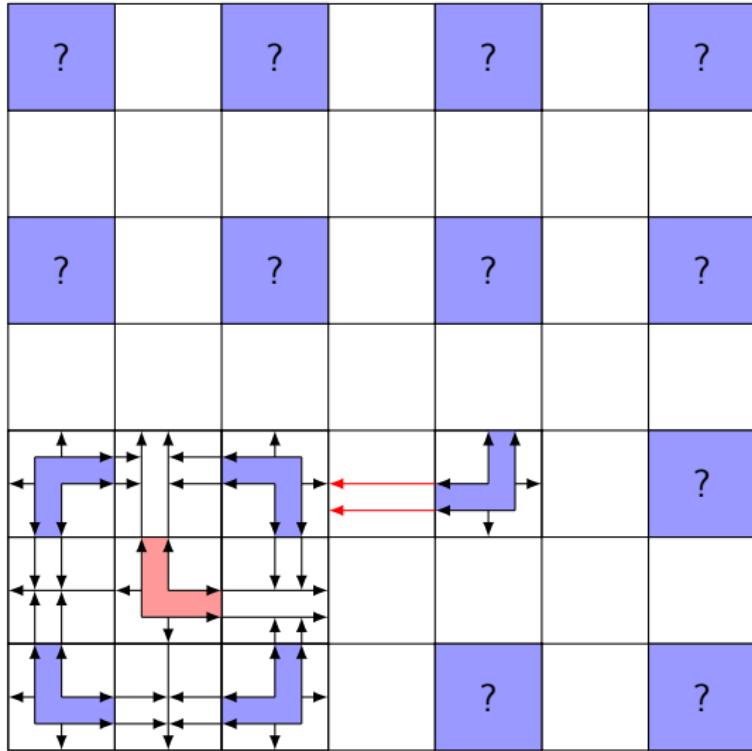
$$? \in \left\{ \begin{array}{c} \text{Red L-tile} \\ \text{Blue L-tile} \\ \text{Purple L-tile} \\ \text{Blue F-tile} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



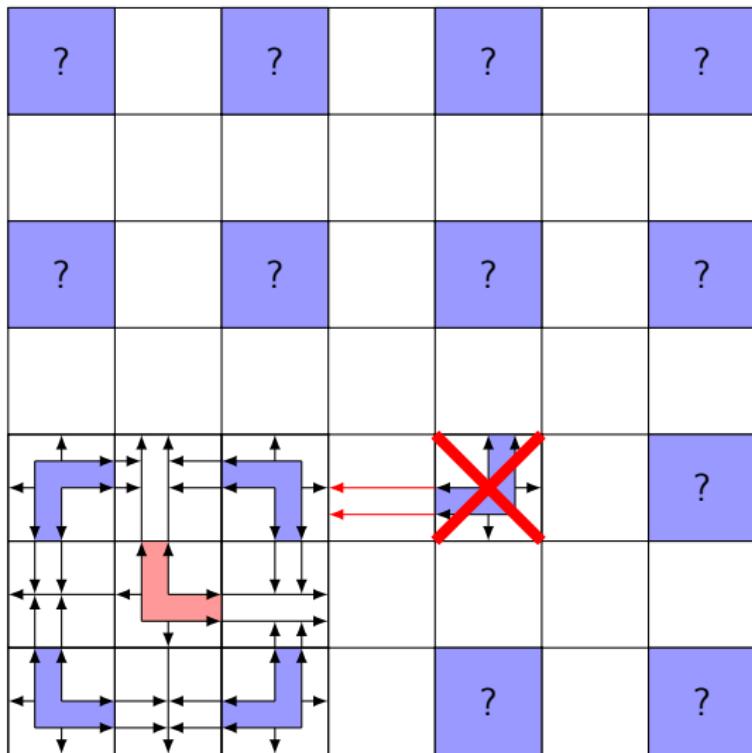
$$? \in \left\{ \begin{array}{c} \text{L-shaped tile} \\ \text{T-shaped tile} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



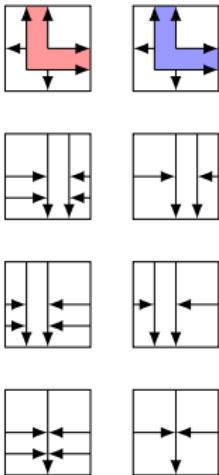
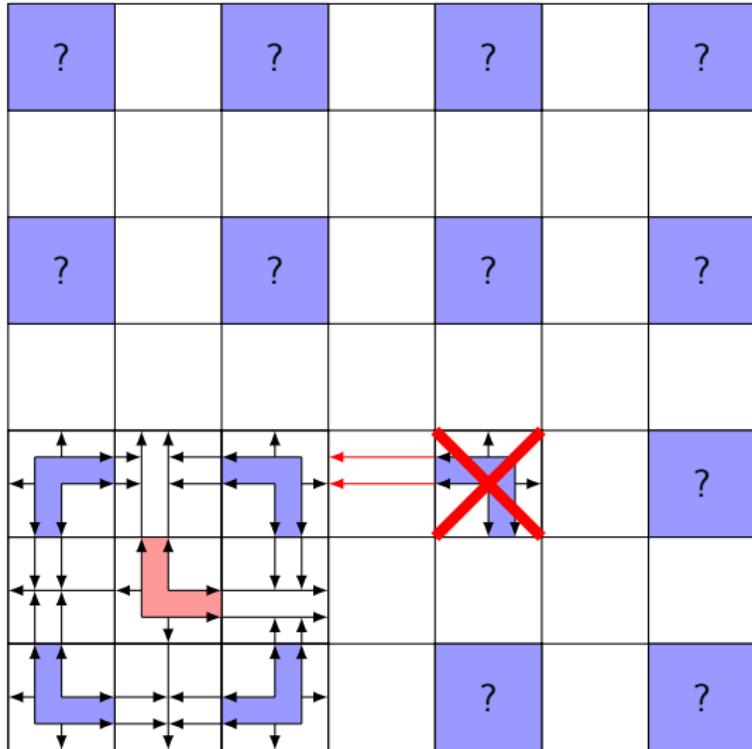
$$? \in \left\{ \begin{array}{c} \text{[blue square with red L]} \\ \text{[blue square with blue L]} \\ \text{[blue square with blue T]} \\ \text{[blue square with blue F]} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



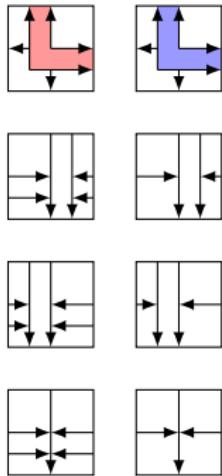
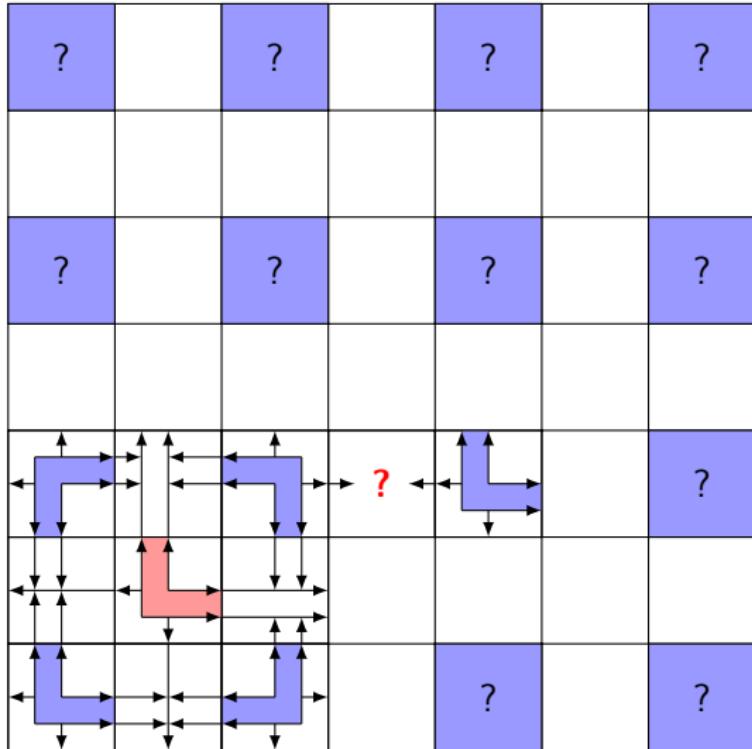
$$? \in \left\{ \begin{array}{c} \text{[blue square with L-shaped arrows]} \\ \text{[blue square with T-shaped arrows]} \\ \text{[blue square with U-shaped arrows]} \\ \text{[blue square with F-shaped arrows]} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



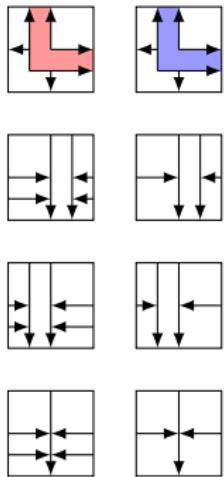
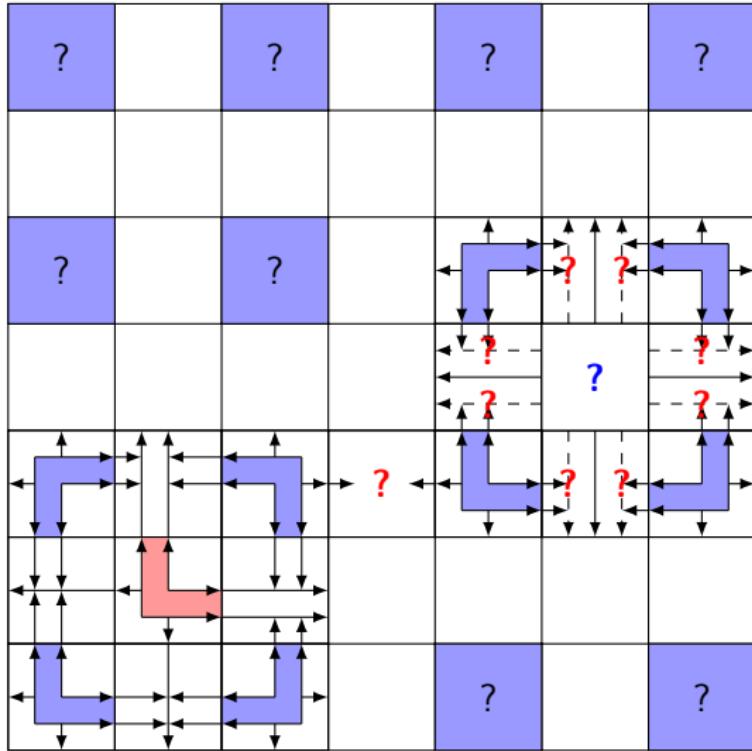
$$? \in \left\{ \begin{array}{c} \text{[blue L-shaped tile]} \\ \text{[red L-shaped tile]} \\ \text{[blue T-shaped tile]} \\ \text{[blue F-shaped tile]} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



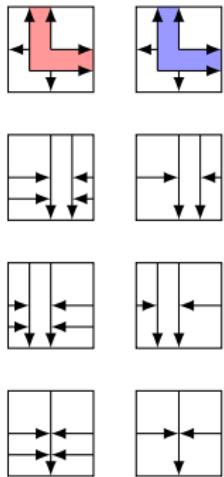
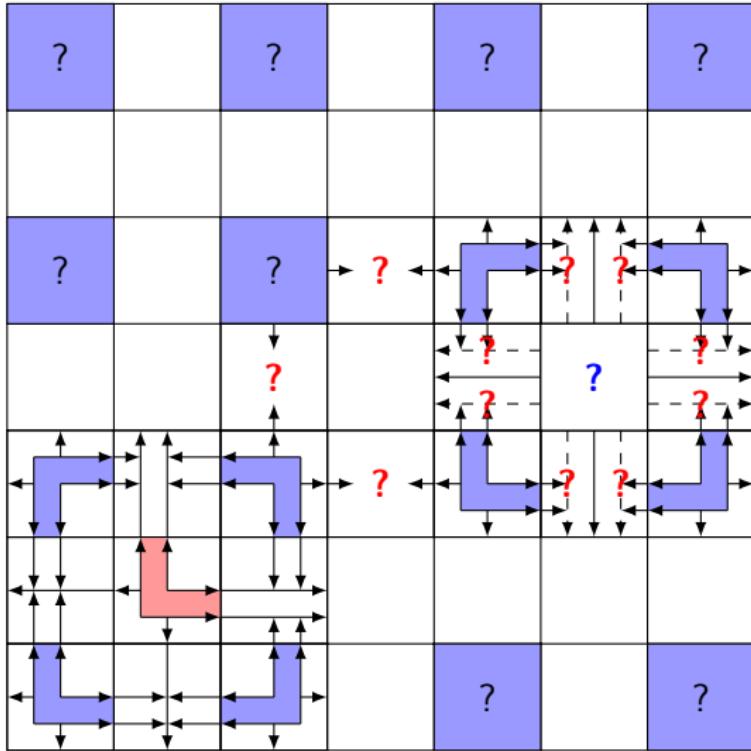
$$? \in \left\{ \begin{array}{c} \text{red L-shaped tile} \\ \text{blue L-shaped tile} \\ \text{purple L-shaped tile} \\ \text{other L-shaped tile} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



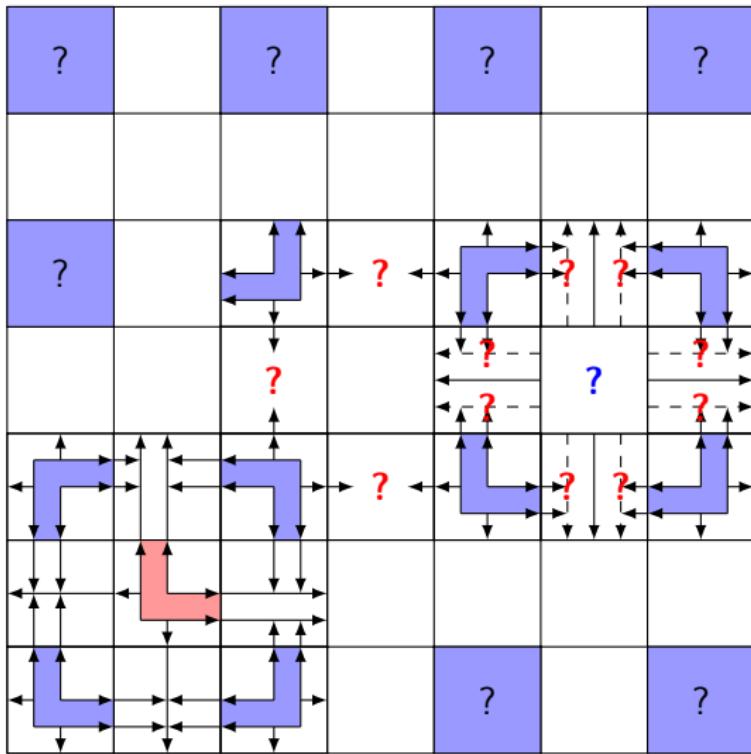
$$\boxed{?} \in \left\{ \begin{array}{c} \text{blue L-shaped tile} \\ \text{blue T-shaped tile} \\ \text{red T-shaped tile} \\ \text{blue F-shaped tile} \end{array} \right\} \boxed{?} \in \left\{ \begin{array}{c} \text{red L-shaped tile} \\ \text{red T-shaped tile} \\ \text{red F-shaped tile} \end{array} \right\} \boxed{??} \in \left\{ \begin{array}{c} \text{blue 2x2 square} \\ \text{blue 3x2 rectangle} \\ \text{blue 3x3 square} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



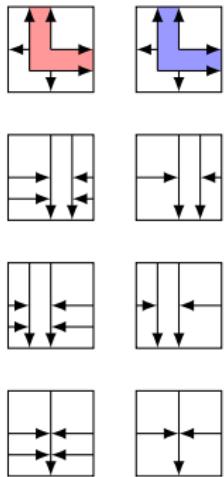
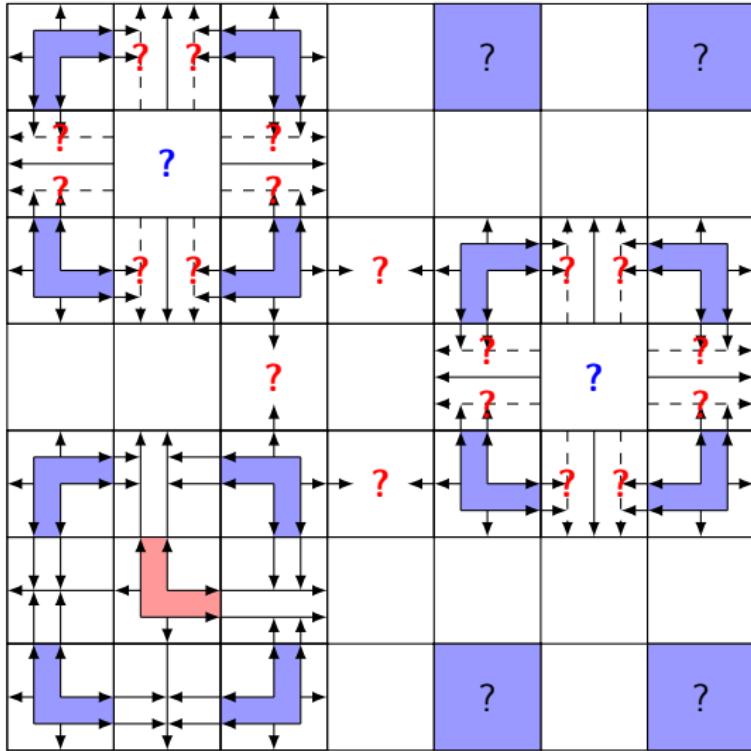
$$? \in \left\{ \begin{array}{c} \text{Red L-shaped tile} \\ \text{Blue L-shaped tile} \\ \text{Red T-shaped tile} \\ \text{Blue T-shaped tile} \end{array} \right\} ? \in \left\{ \begin{array}{c} \text{Red L-shaped tile} \\ \text{Blue L-shaped tile} \\ \text{Red T-shaped tile} \\ \text{Blue T-shaped tile} \end{array} \right\} ?? \in \left\{ \begin{array}{c} \text{Red 2x3 rectangle} \\ \text{Blue 2x3 rectangle} \\ \text{Red 3x2 rectangle} \\ \text{Blue 3x2 rectangle} \\ \text{Red 2x2 square} \\ \text{Blue 2x2 square} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



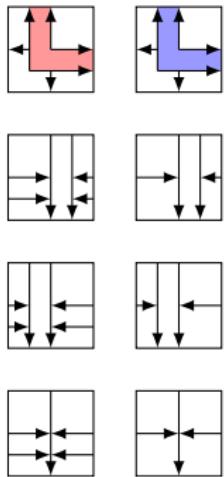
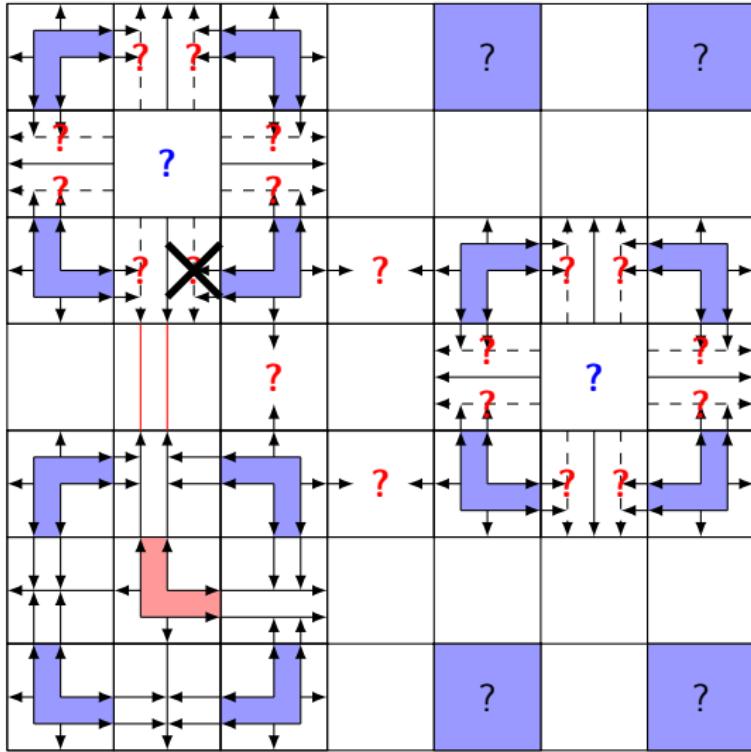
$$\boxed{?} \in \left\{ \begin{array}{c} \text{blue L-shaped tile} \\ \text{blue T-shaped tile} \\ \text{blue U-shaped tile} \\ \text{blue F-shaped tile} \end{array} \right\} \quad \boxed{?} \in \left\{ \begin{array}{c} \text{red L-shaped tile} \\ \text{red T-shaped tile} \\ \text{red U-shaped tile} \\ \text{red F-shaped tile} \end{array} \right\} \quad \boxed{??} \in \left\{ \begin{array}{c} \text{blue 2x2 square} \\ \text{blue 3x2 rectangle} \\ \text{blue 3x3 square} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



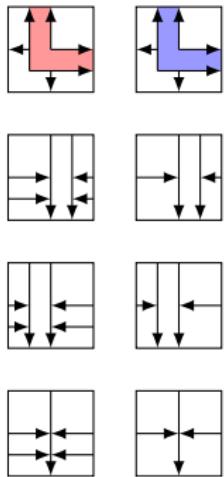
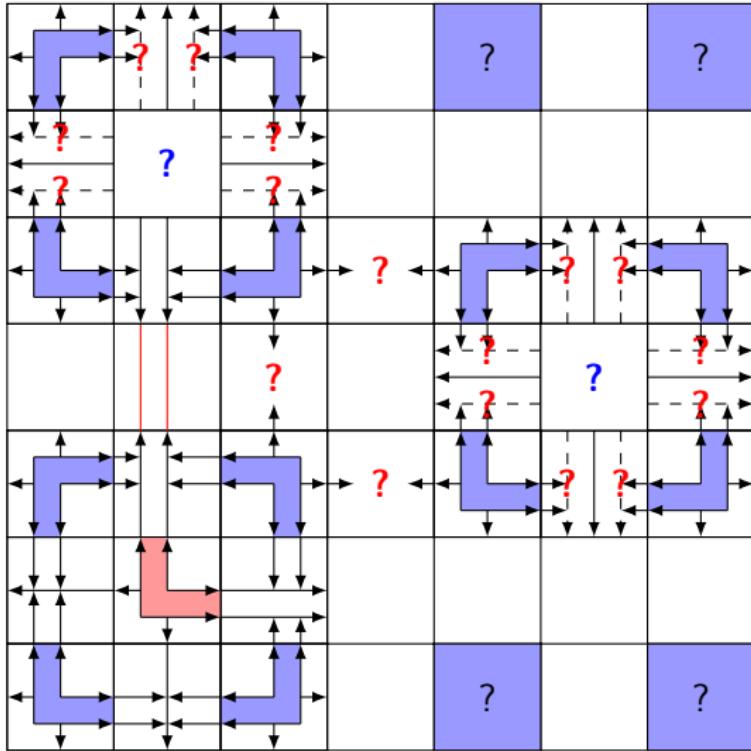
$$? \in \left\{ \begin{array}{c} \text{blue L-shaped tile} \\ \text{red L-shaped tile} \end{array} \right\} ? \in \left\{ \begin{array}{c} \text{blue L-shaped tile} \\ \text{red L-shaped tile} \end{array} \right\} ?? \in \left\{ \begin{array}{c} \text{blue L-shaped tile} \\ \text{red L-shaped tile} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



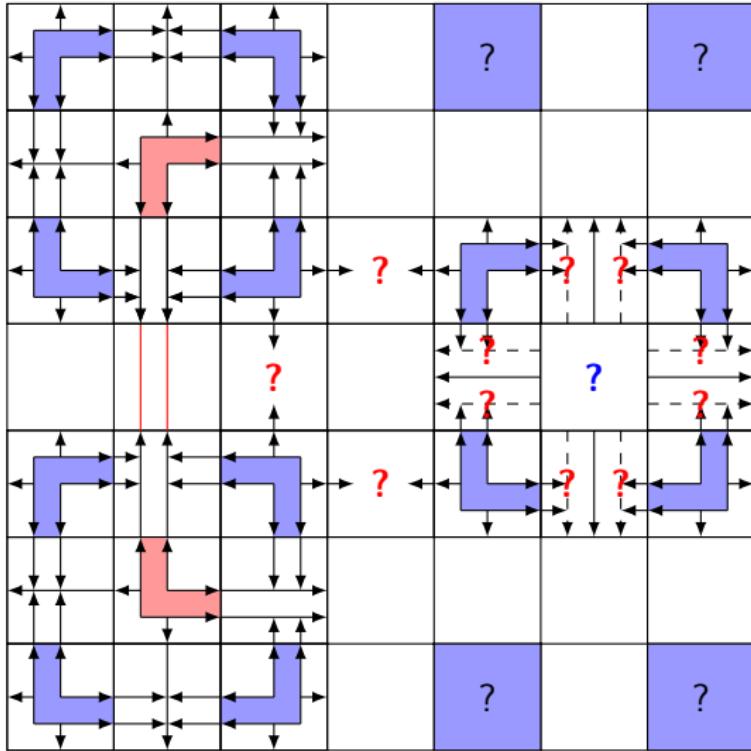
$$? \in \left\{ \begin{array}{c} \text{blue L} \\ \text{red L} \\ \text{blue T} \\ \text{red T} \\ \text{blue F} \\ \text{red F} \end{array} \right\} ? \in \left\{ \begin{array}{c} \text{blue L} \\ \text{red L} \\ \text{blue T} \\ \text{red T} \\ \text{blue F} \\ \text{red F} \end{array} \right\} ?? \in \left\{ \begin{array}{c} \text{blue 2x2} \\ \text{red 2x2} \\ \text{blue 2x2} \\ \text{red 2x2} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



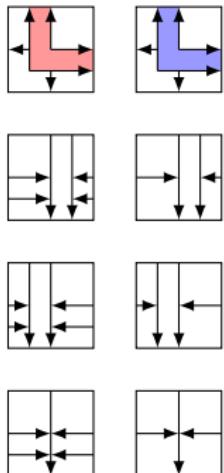
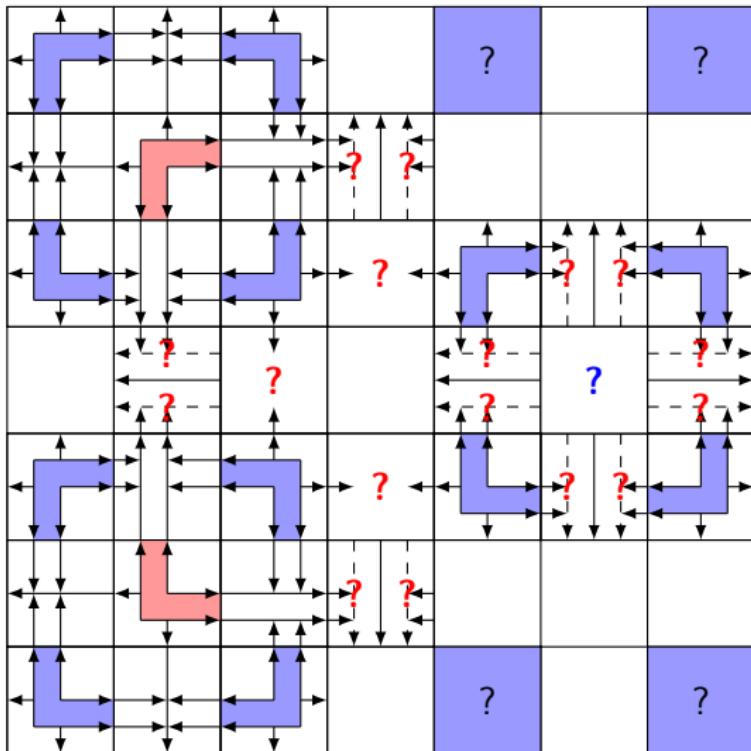
$$? \in \left\{ \begin{array}{c} \text{blue L} \\ \text{red L} \\ \text{other shapes} \end{array} \right\} ? \in \left\{ \begin{array}{c} \text{blue L} \\ \text{red L} \\ \text{other shapes} \end{array} \right\} ?? \in \left\{ \begin{array}{c} \text{blue 2x2} \\ \text{red 2x2} \\ \text{other 2x2} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



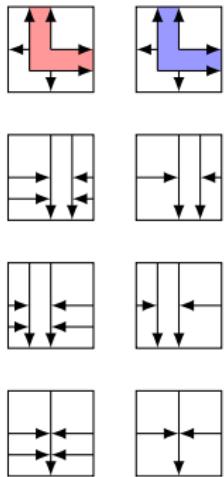
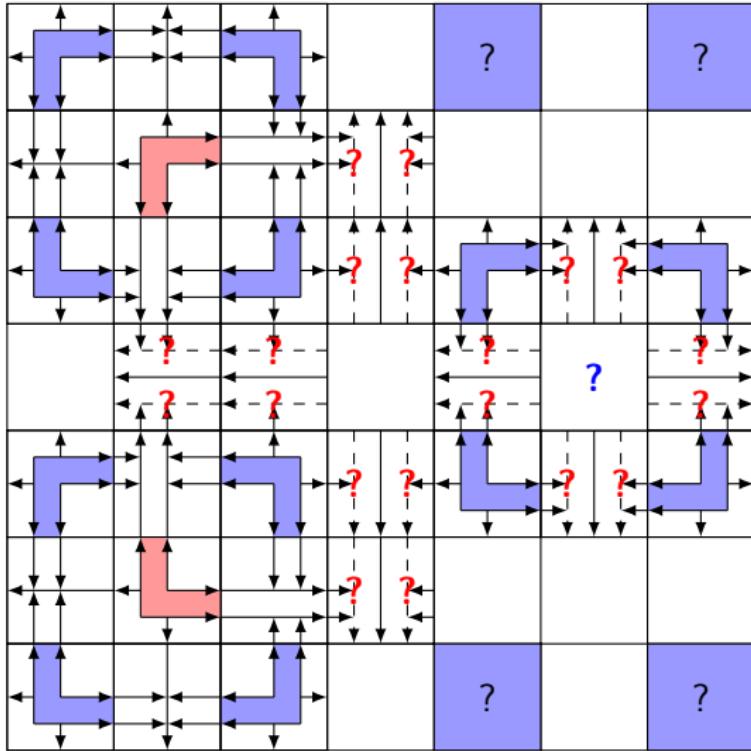
$$? \in \left\{ \begin{array}{c} \text{blue L-tile} \\ \text{red L-tile} \\ \text{blue T-tile} \\ \text{red T-tile} \end{array} \right\} ? \in \left\{ \begin{array}{c} \text{blue L-tile} \\ \text{red L-tile} \\ \text{blue T-tile} \\ \text{red T-tile} \end{array} \right\} ?? \in \left\{ \begin{array}{c} \text{blue 2x2 sub-tile} \\ \text{red 2x2 sub-tile} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



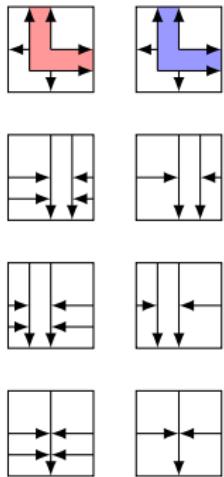
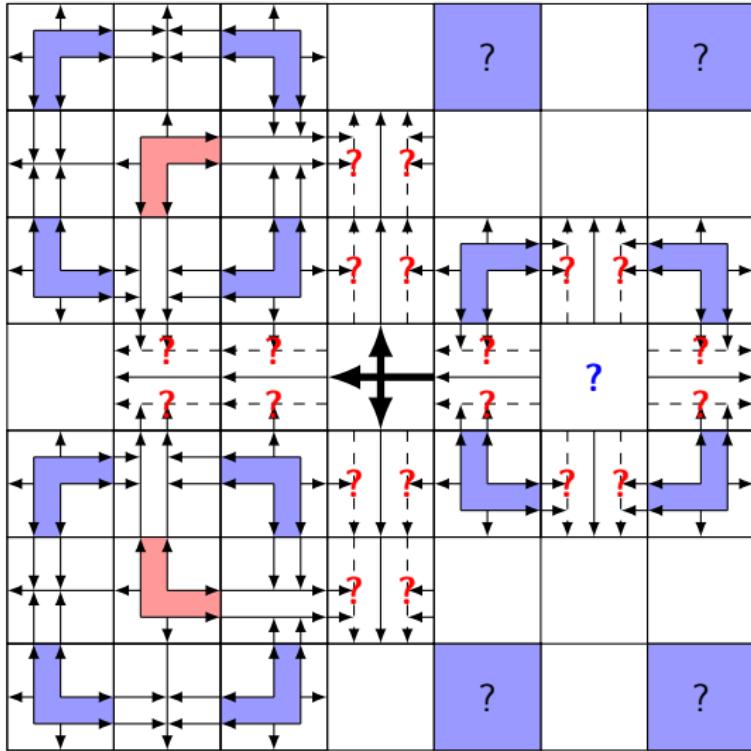
$$? \in \left\{ \begin{array}{c} \text{blue L-tile} \\ \text{red L-tile} \\ \text{blue T-tile} \\ \text{red T-tile} \end{array} \right\} ? \in \left\{ \begin{array}{c} \text{blue L-tile} \\ \text{red L-tile} \\ \text{blue T-tile} \\ \text{red T-tile} \end{array} \right\} ?? \in \left\{ \begin{array}{c} \text{blue T-tile} \\ \text{red T-tile} \\ \text{blue square} \\ \text{red square} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



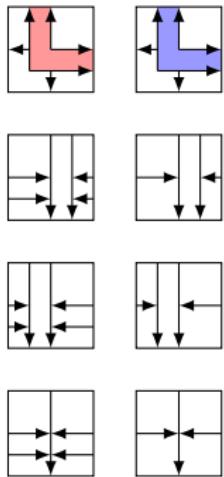
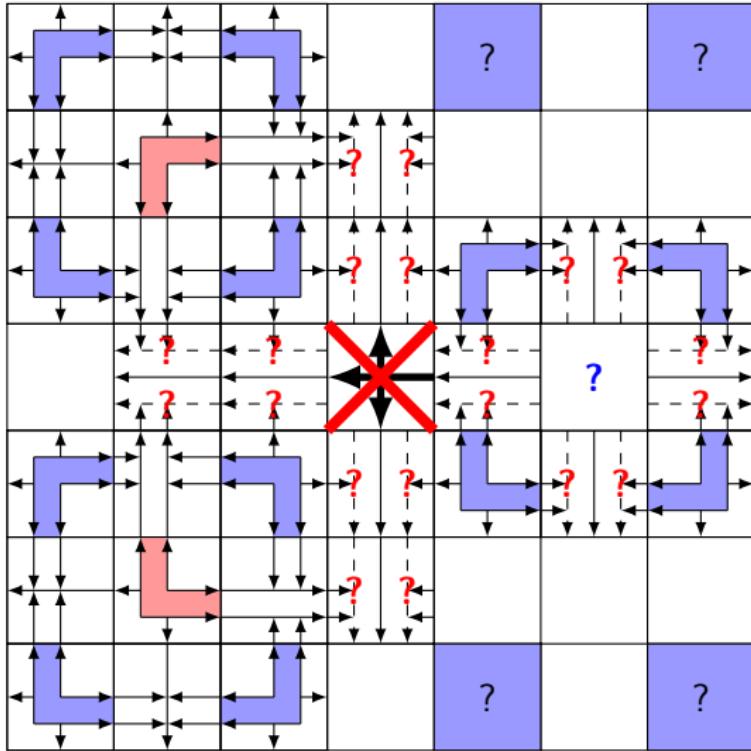
$$? \in \left\{ \begin{array}{c} \text{Blue L-shaped tile} \\ \text{Red L-shaped tile} \end{array} \right\} ? \in \left\{ \begin{array}{c} \text{Blue L-shaped tile} \\ \text{Red L-shaped tile} \end{array} \right\} ?? \in \left\{ \begin{array}{c} \text{Blue L-shaped tile} \\ \text{Red L-shaped tile} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



$$? \in \left\{ \begin{array}{c} \text{L-shaped tile configurations} \end{array} \right\} ? \in \left\{ \begin{array}{c} \text{L-shaped tile configurations} \end{array} \right\} ?? \in \left\{ \begin{array}{c} \text{L-shaped tile configurations} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)

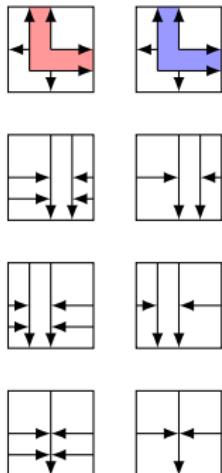
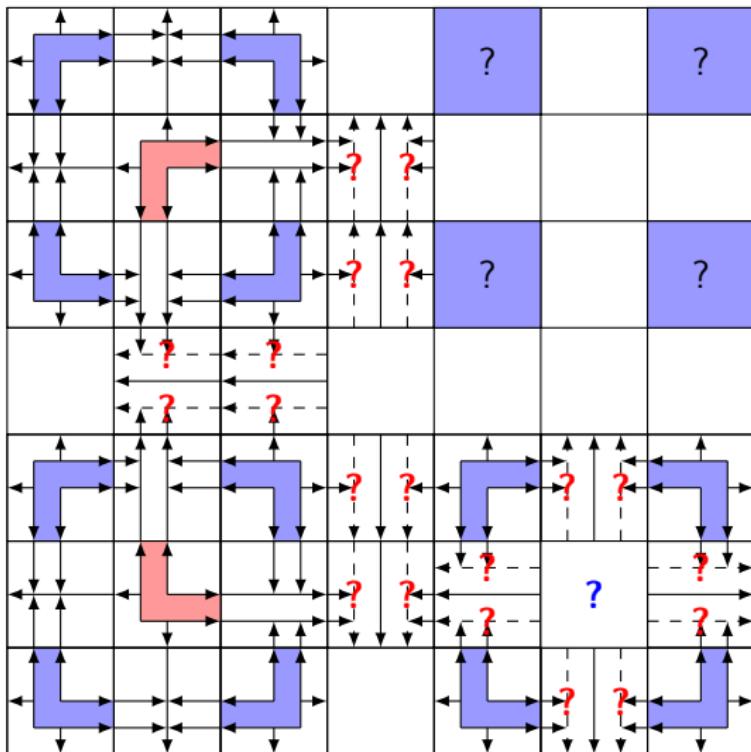


$$\text{?} \in \left\{ \begin{array}{c} \text{[blue L]} \\ \text{[blue T]} \\ \text{[blue J]} \\ \text{[blue F]} \end{array} \right\}$$

$$\text{?} \in \left\{ \begin{array}{c} \text{[red L]} \\ \text{[red T]} \\ \text{[red J]} \\ \text{[red F]} \end{array} \right\}$$

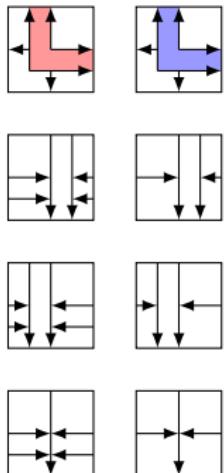
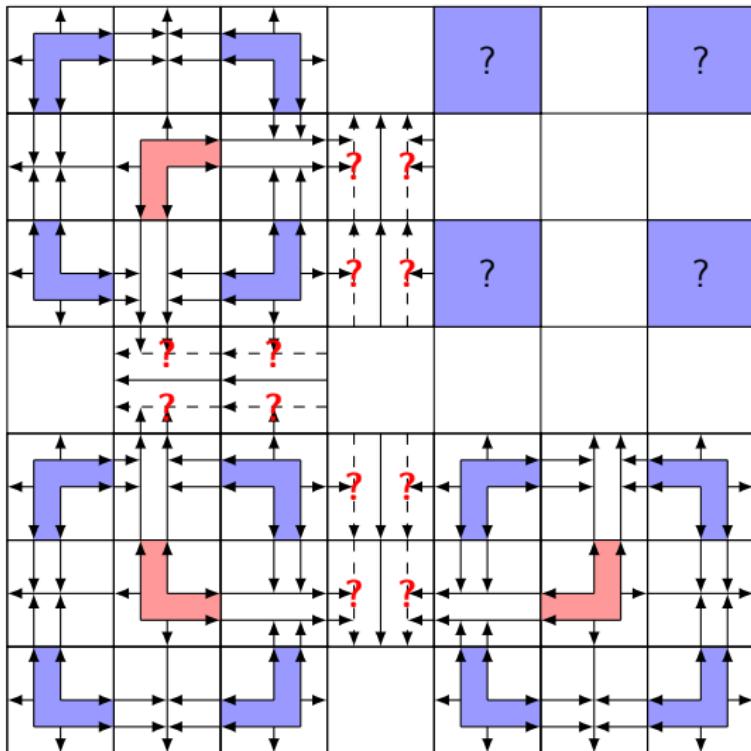
$$\text{?} \in \left\{ \begin{array}{c} \text{[blue 2x2]} \\ \text{[red 2x2]} \\ \text{[black 2x2]} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



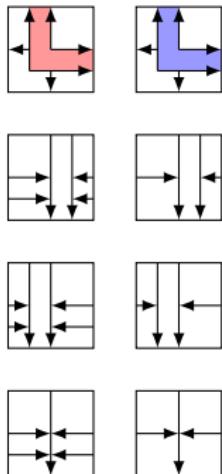
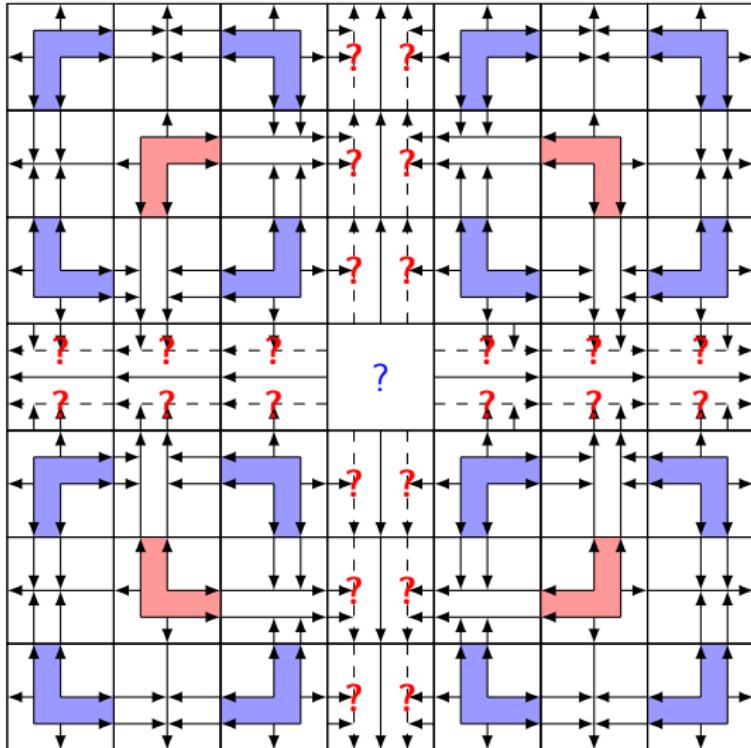
$$? \in \left\{ \begin{array}{c} \text{blue L-tile} \\ \text{blue T-tile} \\ \text{red L-tile} \\ \text{blue F-tile} \end{array} \right\} ? \in \left\{ \begin{array}{c} \text{blue L-tile} \\ \text{red L-tile} \\ \text{red T-tile} \\ \text{red F-tile} \end{array} \right\} ?? \in \left\{ \begin{array}{c} \text{blue 2x2 block} \\ \text{red 2x2 block} \\ \text{blue 2x2 block} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



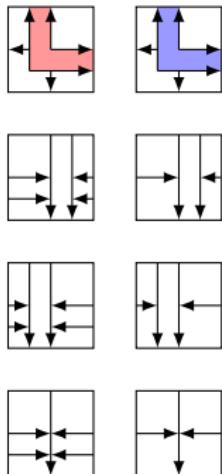
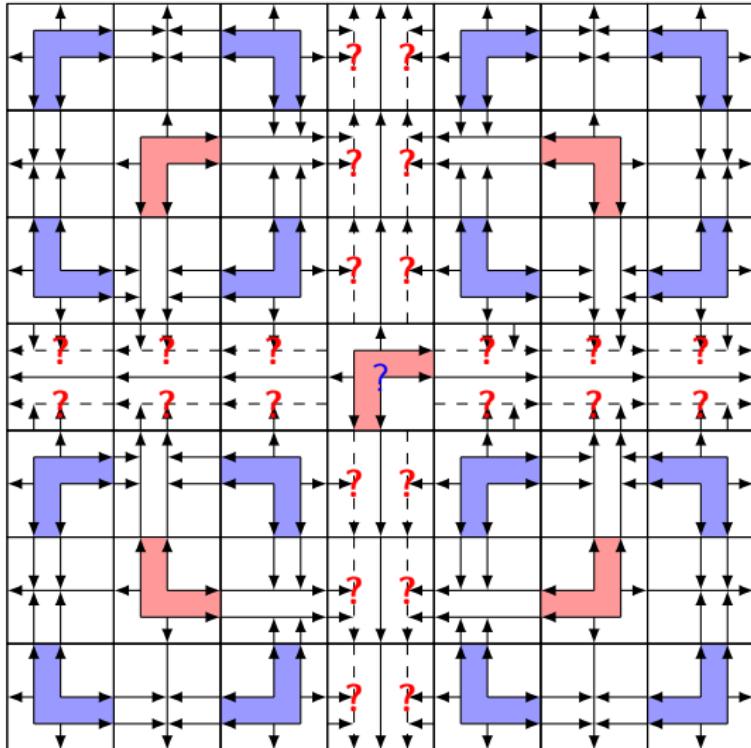
$$? \in \left\{ \begin{array}{c} \text{blue L} \\ \text{blue T} \\ \text{red L} \\ \text{red T} \end{array} \right\} ? \in \left\{ \begin{array}{c} \text{red L} \\ \text{red T} \\ \text{blue L} \\ \text{blue T} \end{array} \right\} ?? \in \left\{ \begin{array}{c} \text{blue L} \\ \text{blue T} \\ \text{red L} \\ \text{red T} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



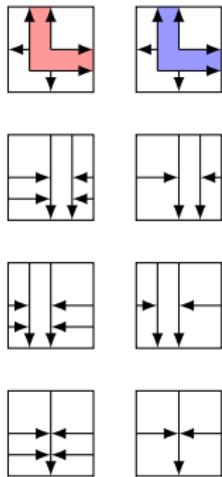
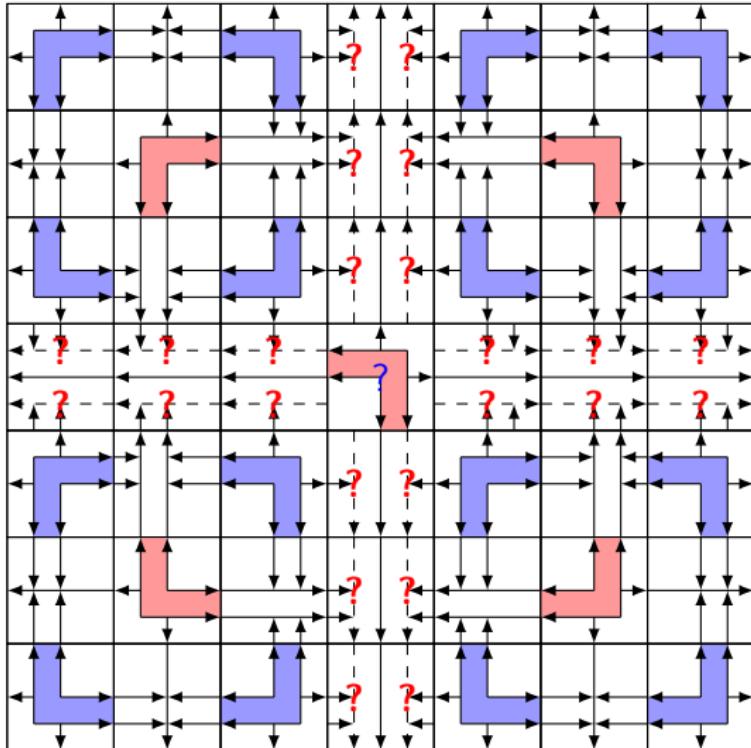
$$\boxed{?} \in \left\{ \begin{array}{c} \text{red L} \\ \text{red T} \\ \text{blue L} \\ \text{blue T} \end{array} \right\} \quad \boxed{??} \in \left\{ \begin{array}{c} \text{red 2x2} \\ \text{blue 2x2} \\ \text{black 2x2} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



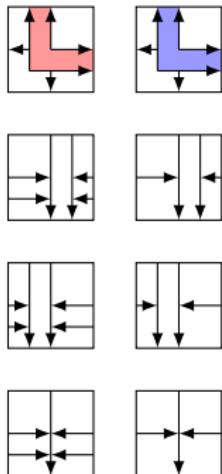
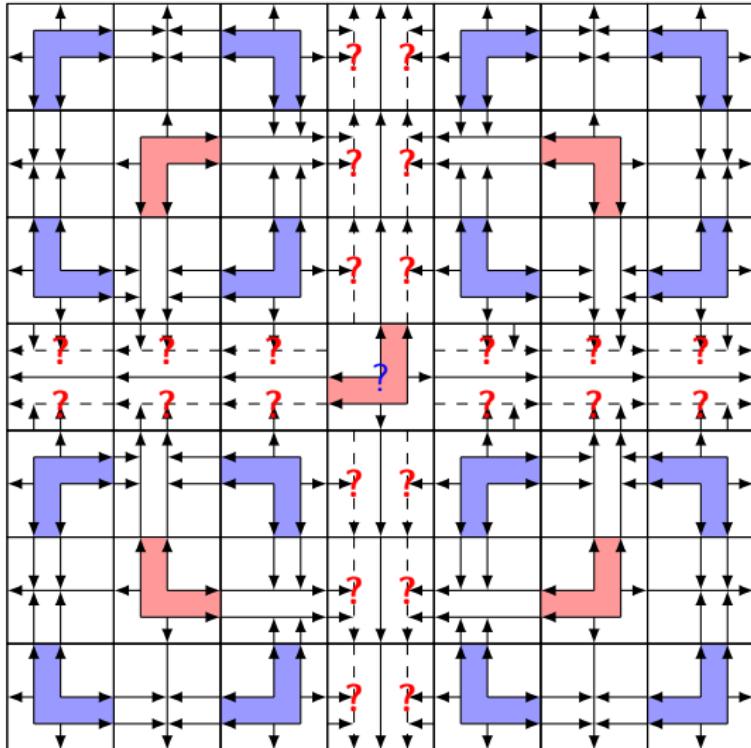
$$\boxed{?} \in \left\{ \begin{array}{c} \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \\ \text{L-shaped tile} \end{array} \right\} \quad \boxed{??} \in \left\{ \begin{array}{c} \text{2x2 grid of arrows} \\ \text{2x2 grid of arrows} \\ \text{2x2 grid of arrows} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



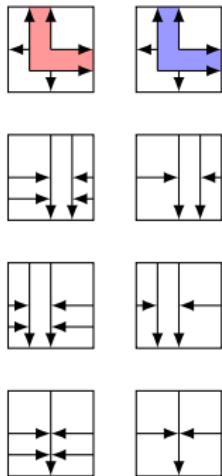
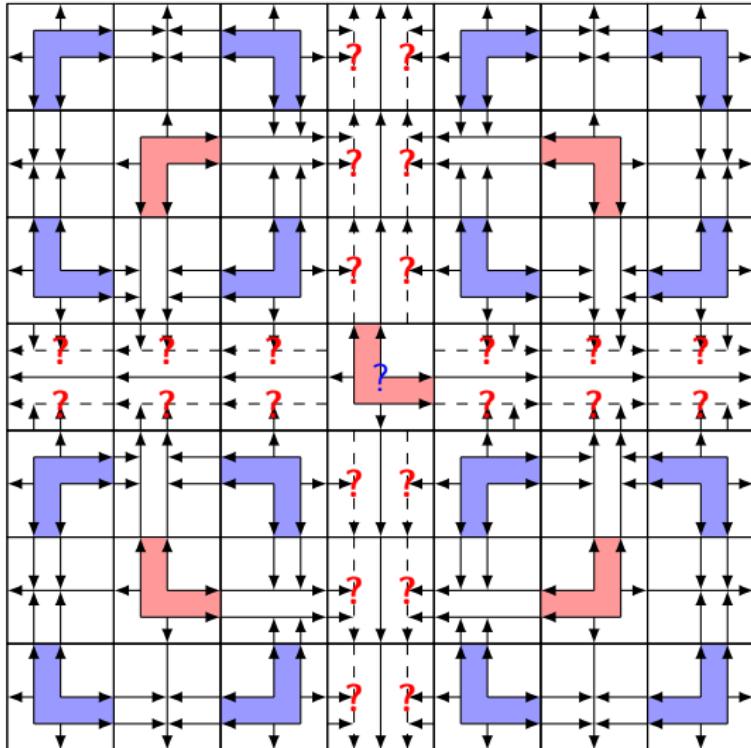
$$\boxed{?} \in \left\{ \begin{array}{c} \text{L-shaped tile} \\ \text{T-shaped tile} \end{array} \right\} \quad \boxed{??} \in \left\{ \begin{array}{c} \text{L-shaped tile} \\ \text{T-shaped tile} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



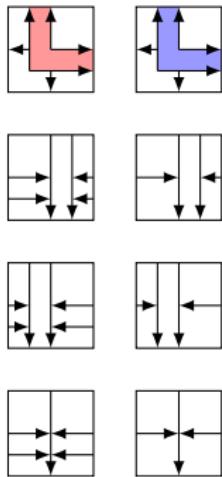
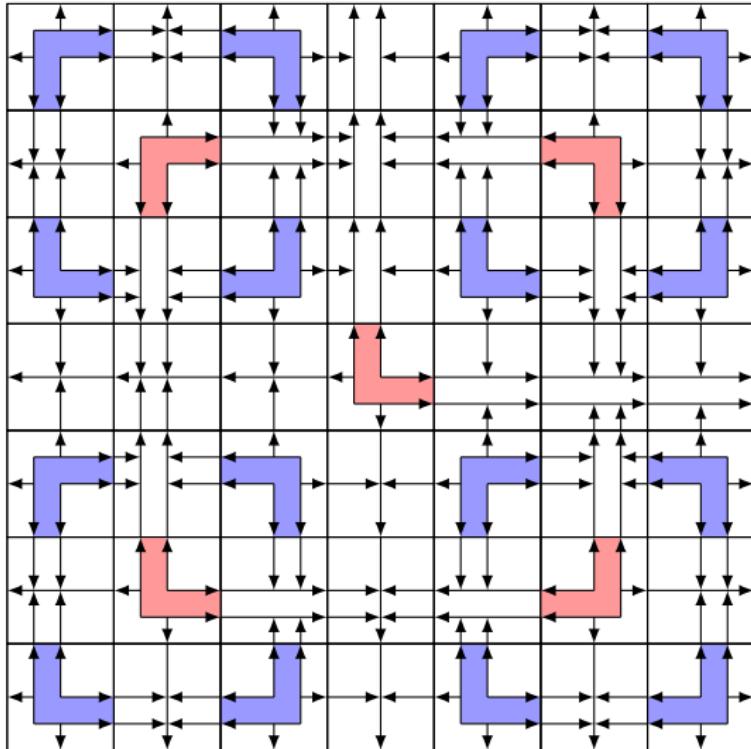
$$\boxed{?} \in \left\{ \begin{array}{c} \text{red L} \\ \text{red L} \\ \text{red L} \\ \text{red L} \end{array} \right\} \quad \boxed{??} \in \left\{ \begin{array}{c} \text{blue 2x2} \\ \text{blue 2x2} \\ \text{blue 2x2} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)



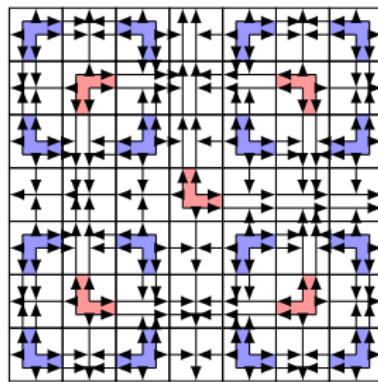
$$\boxed{?} \in \left\{ \begin{array}{c} \text{red L} \\ \text{red J} \\ \text{blue L} \\ \text{blue J} \end{array} \right\} \quad \boxed{??} \in \left\{ \begin{array}{c} \text{red 2x2} \\ \text{blue 2x2} \\ \text{blue 3x2} \end{array} \right\}$$

Force the presence of super-tiles (of level 2)

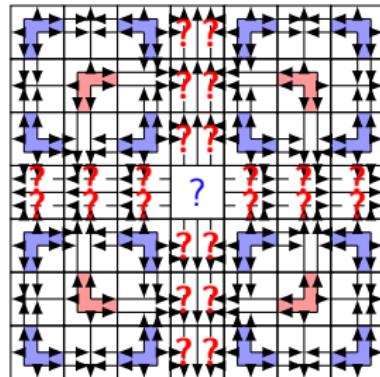
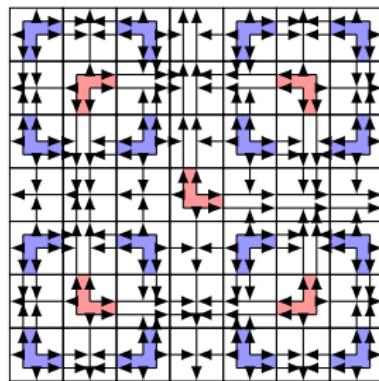


$$\boxed{?} \in \left\{ \begin{array}{c} \text{red L} \\ \text{blue L} \\ \text{red T} \\ \text{blue T} \end{array} \right\} \quad \boxed{??} \in \left\{ \begin{array}{c} \text{square 1} \\ \text{square 2} \\ \text{square 3} \end{array} \right\}$$

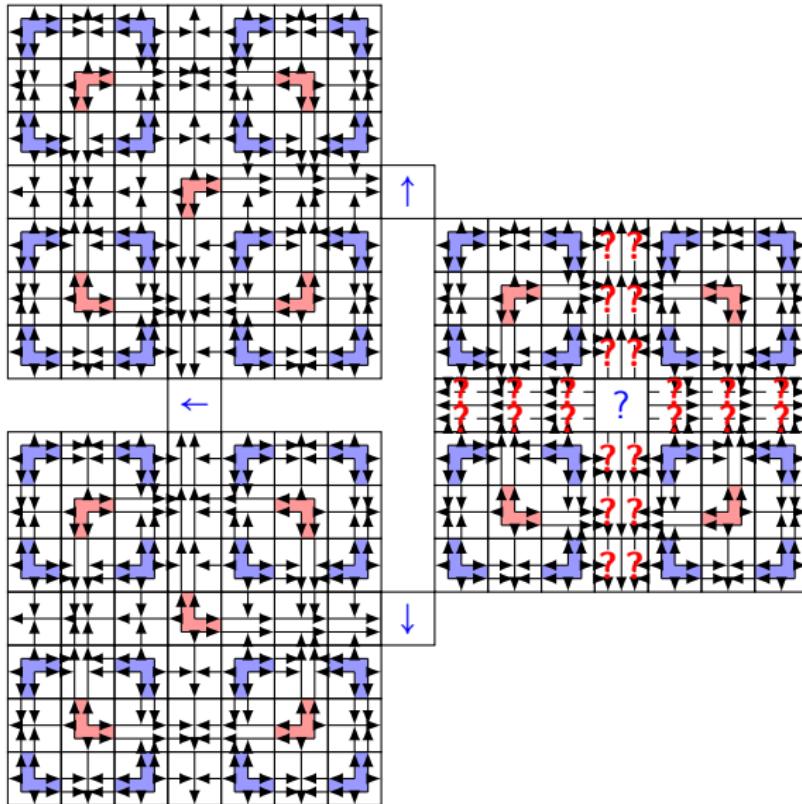
Force the presence of higher levels super-tiles



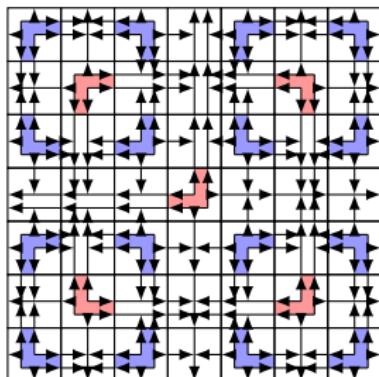
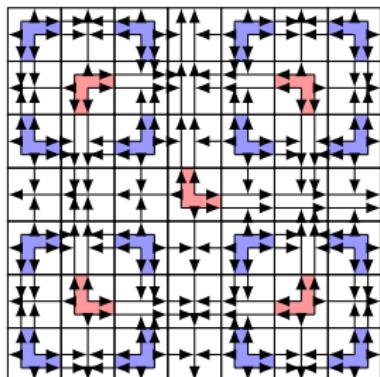
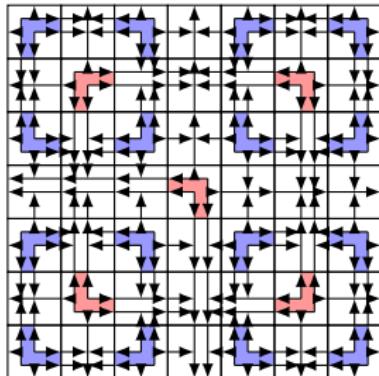
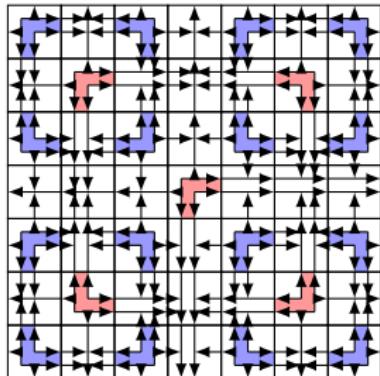
Force the presence of higher levels super-tiles



Force the presence of higher levels super-tiles



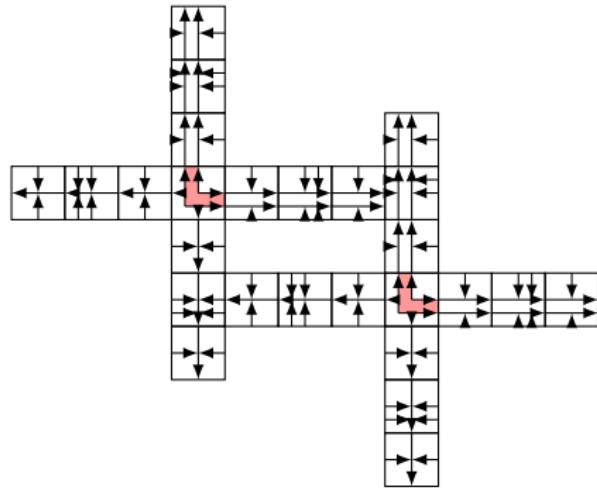
Force the presence of higher levels super-tiles



Aperiodicity

Center of super files of level n are spaced of 2^n cells.

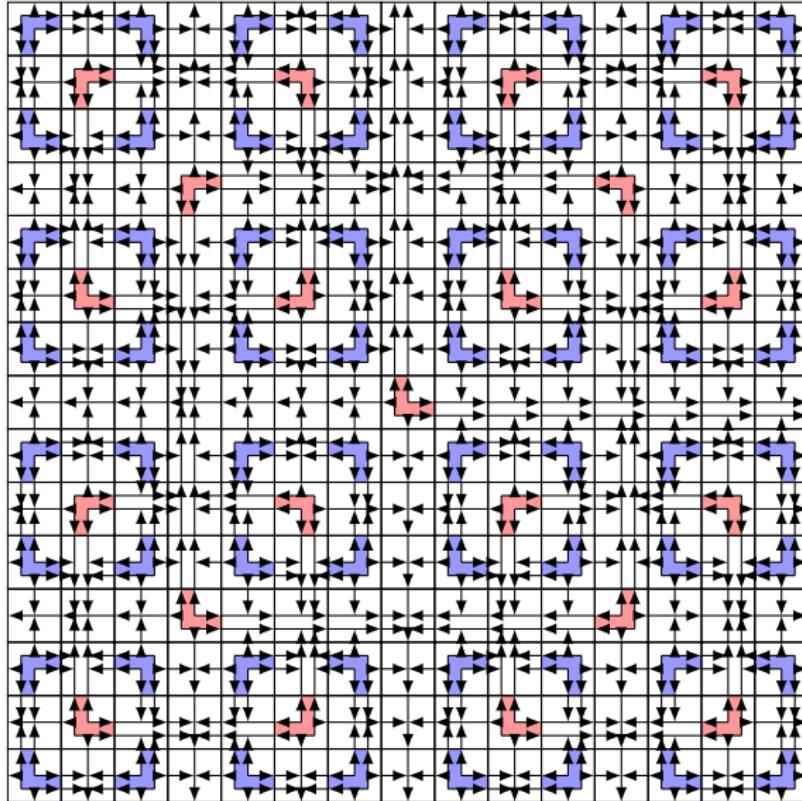
Deduce that every elements $x \in \mathbf{T}_{\text{Robi}}$ does not admit period.



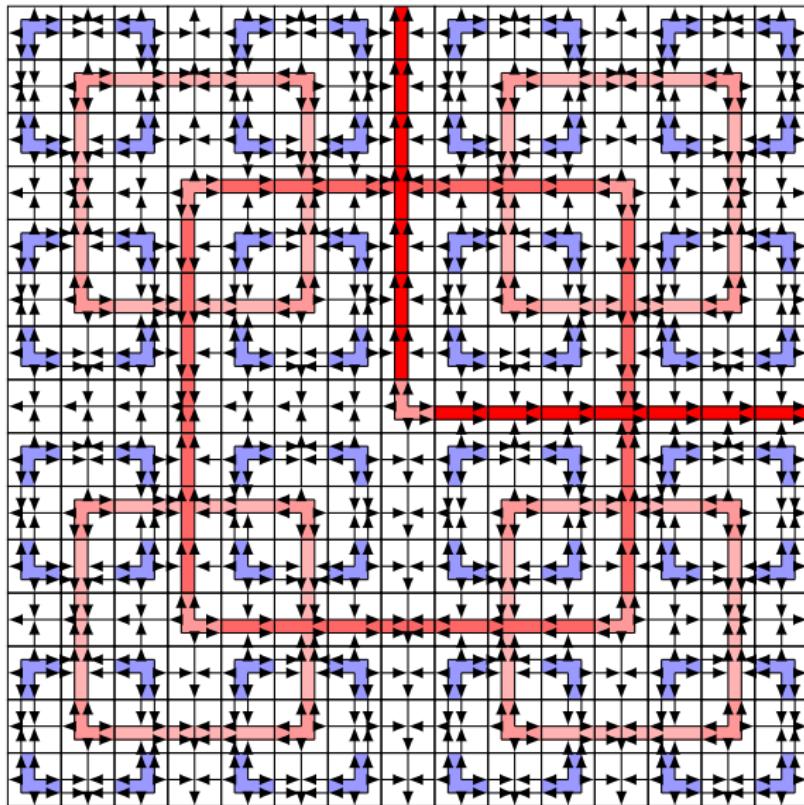
Theorem (*Robinson 1971*)

The SFT \mathbf{T}_{Robi} is not empty and all configurations are aperiodics.

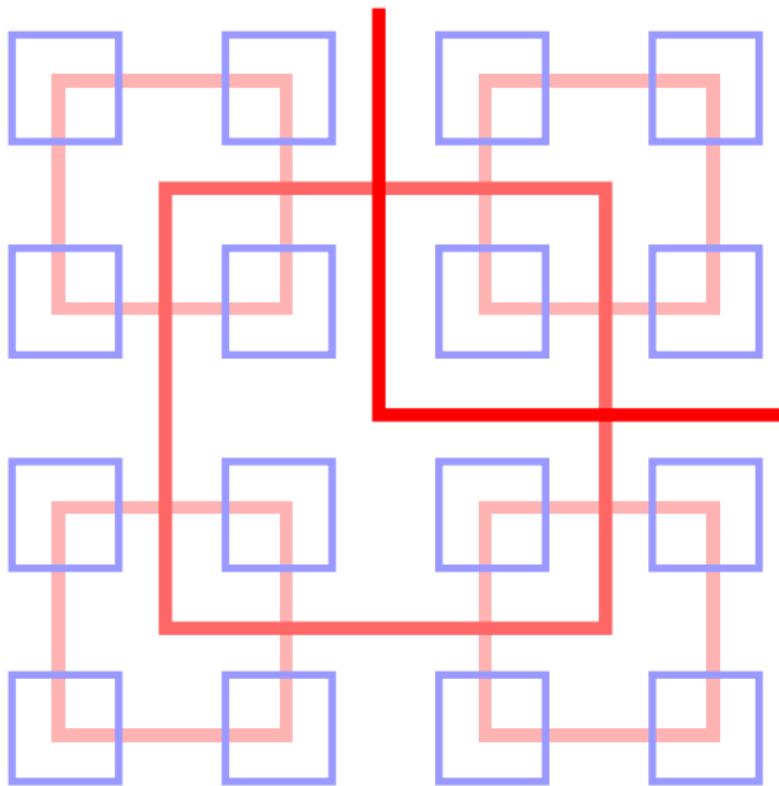
Geometric vision: Hierarchical structure



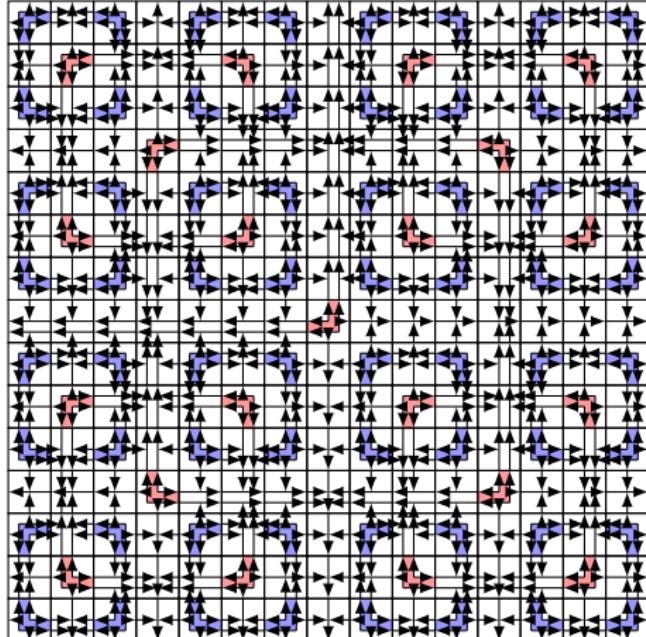
Geometric vision: Hierarchical structure



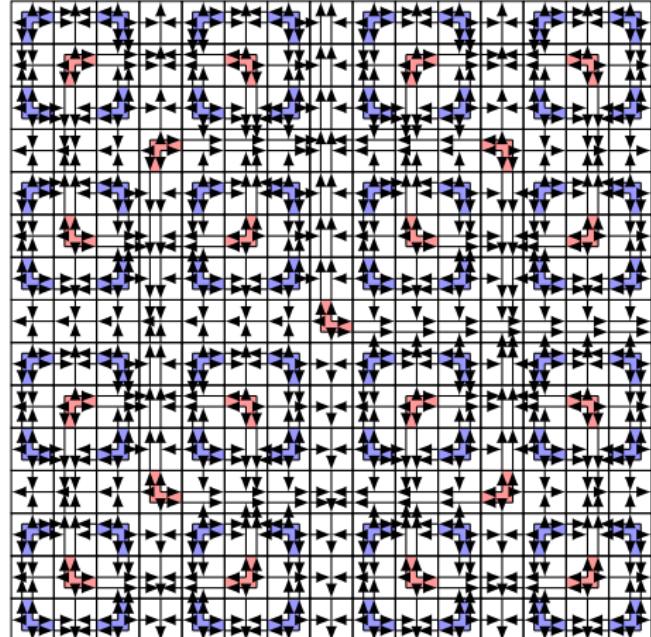
Geometric vision: Hierarchical structure



Geometric Vision: Fractured lines

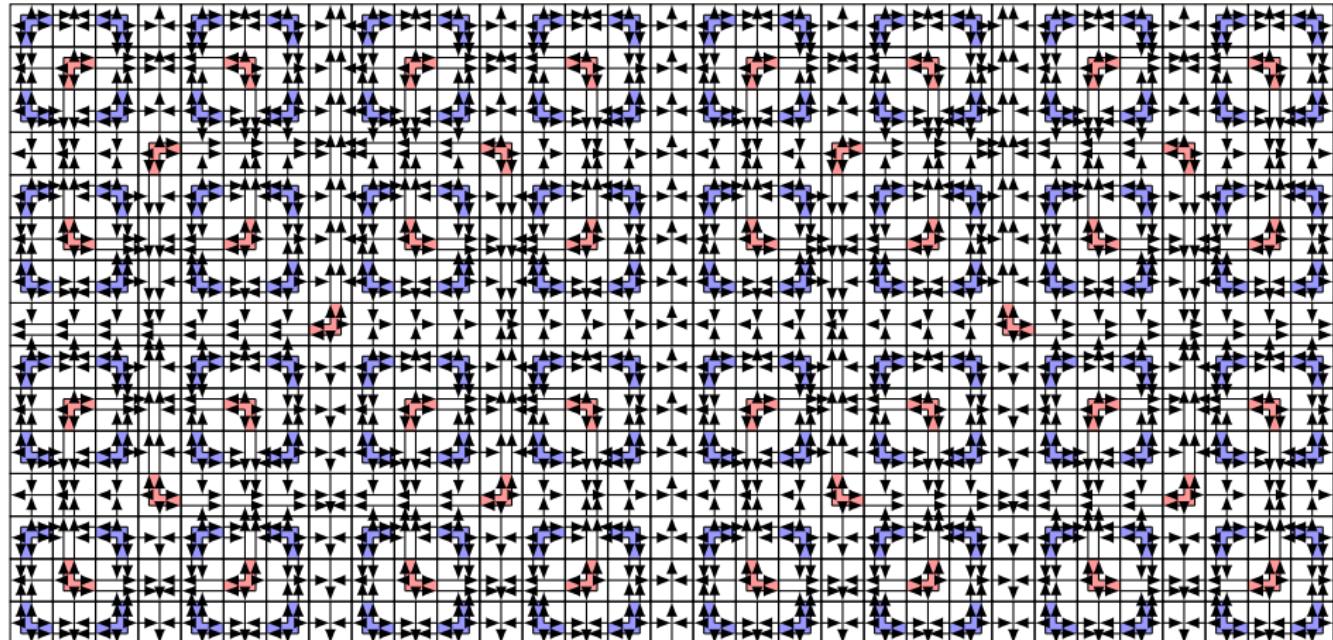


Aperiodic two-dimensional SFT

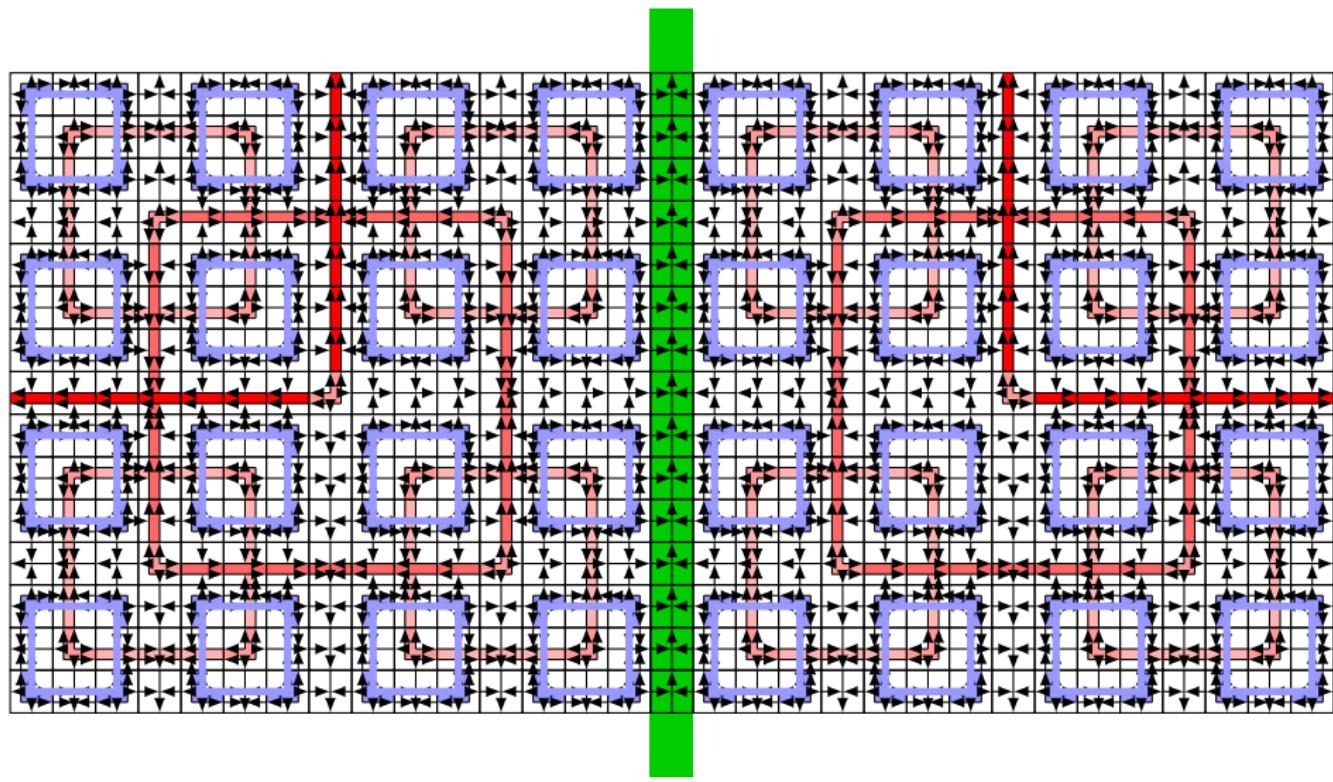


My first aperiodic tiling: Robinson's tiling

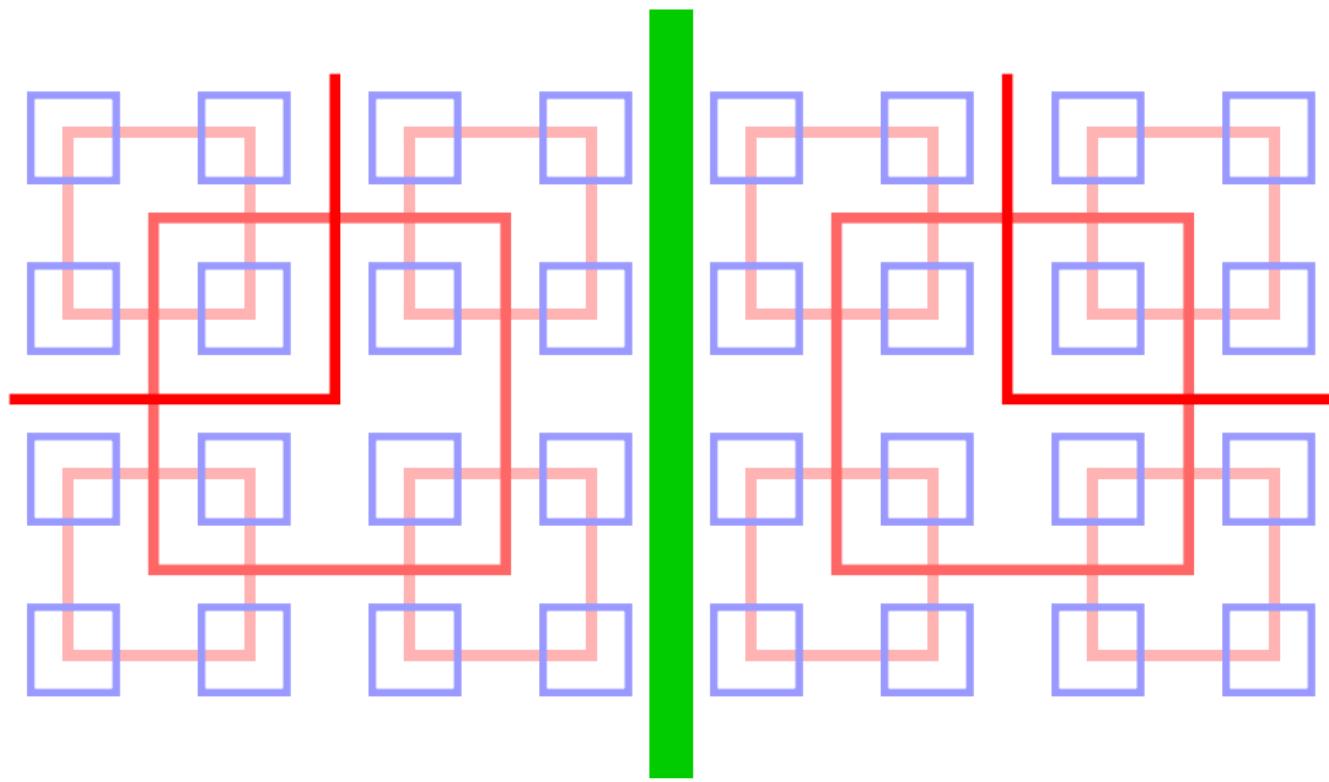
Geometric Vision: Fractured lines



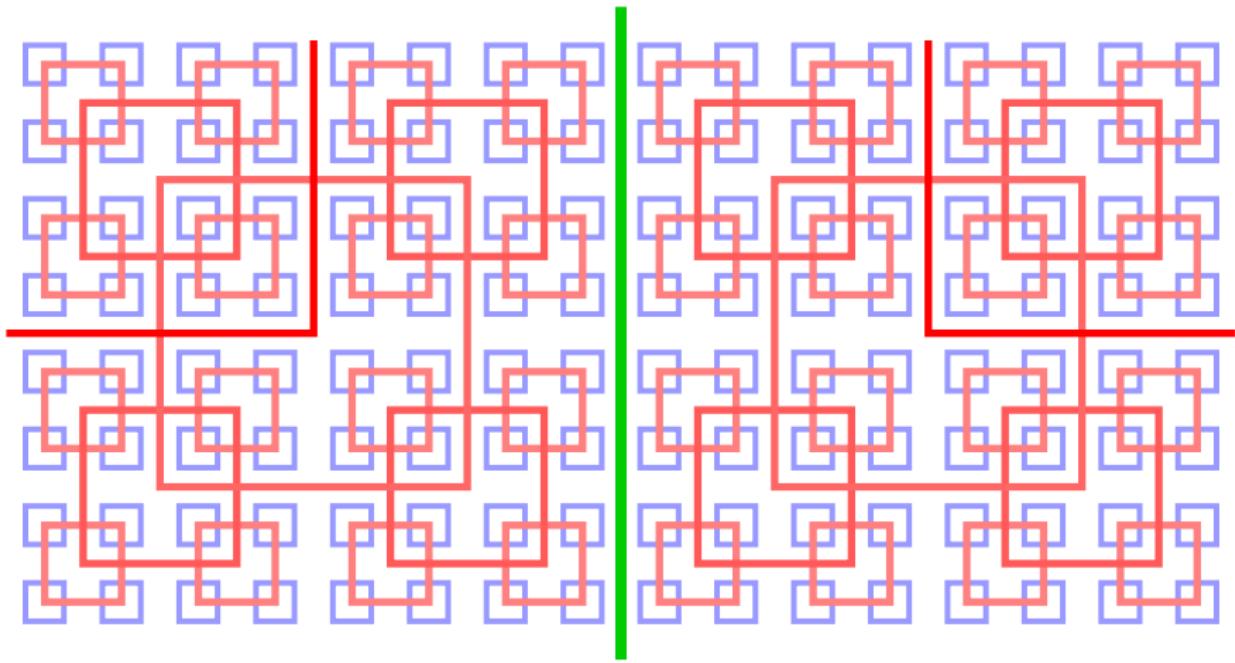
Geometric Vision: Fractured lines



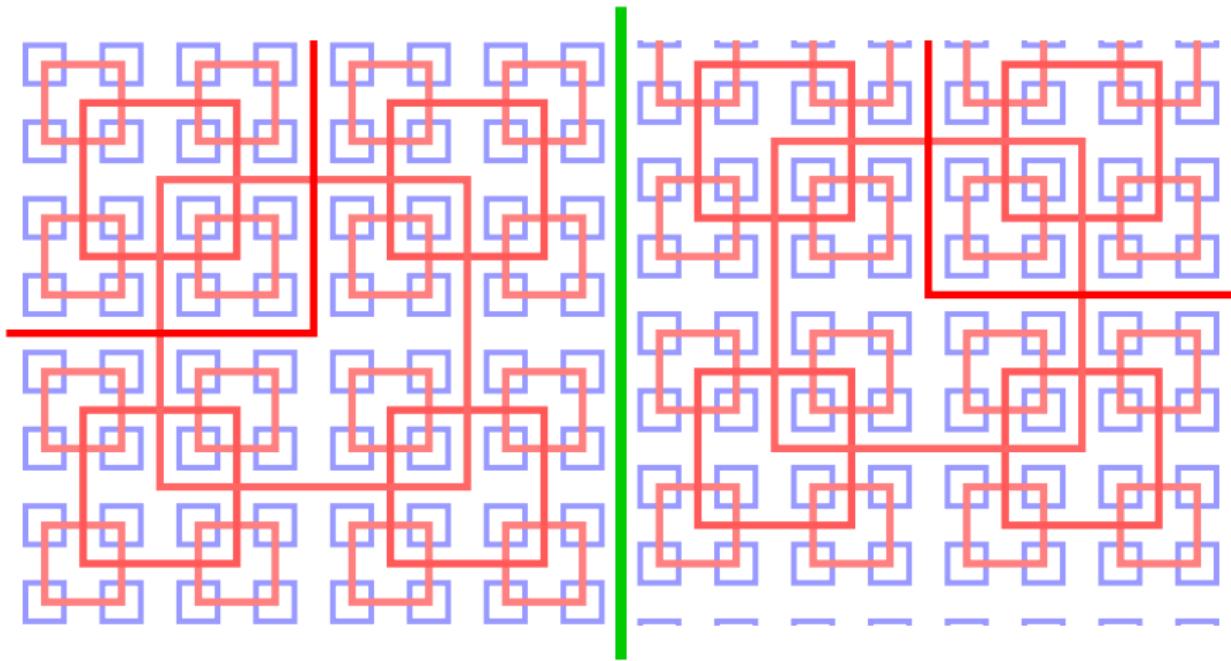
Geometric Vision: Fractured lines



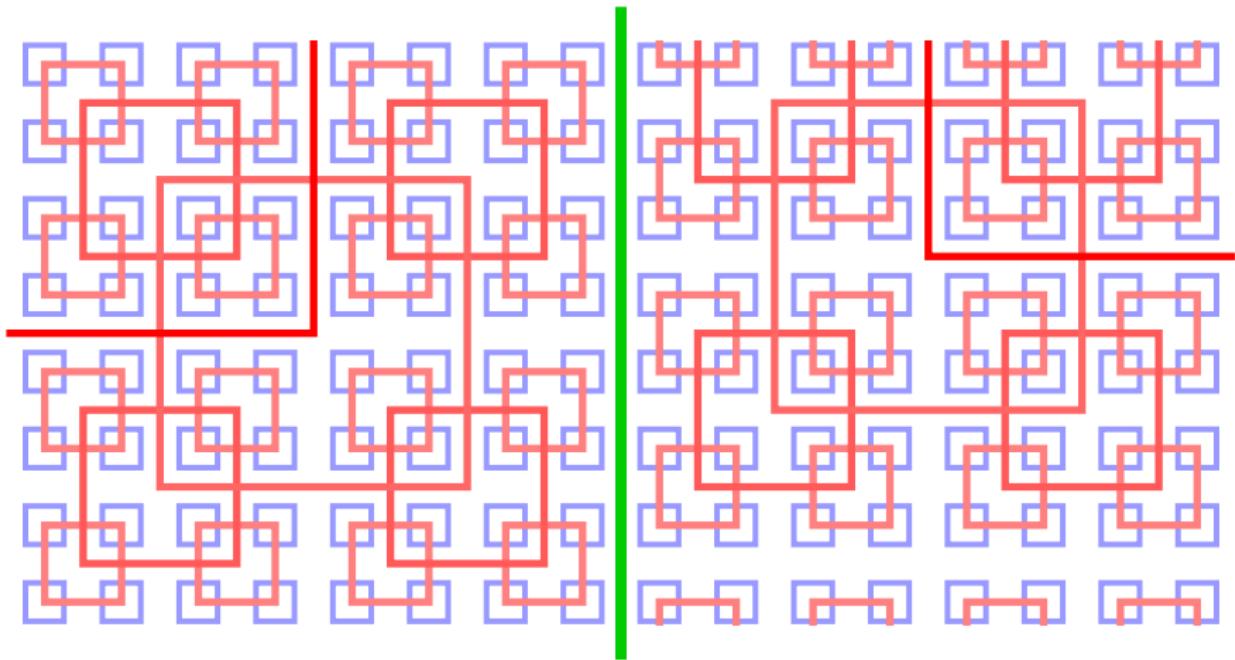
Geometric Vision: Fractured lines



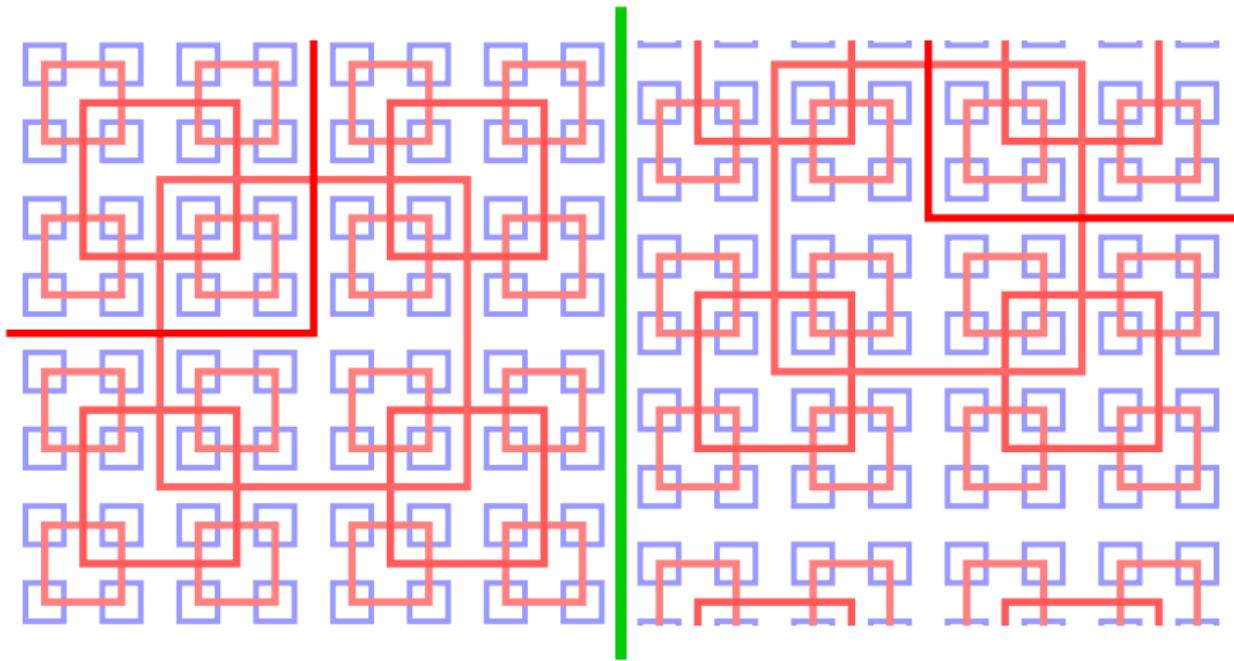
Geometric Vision: Fractured lines



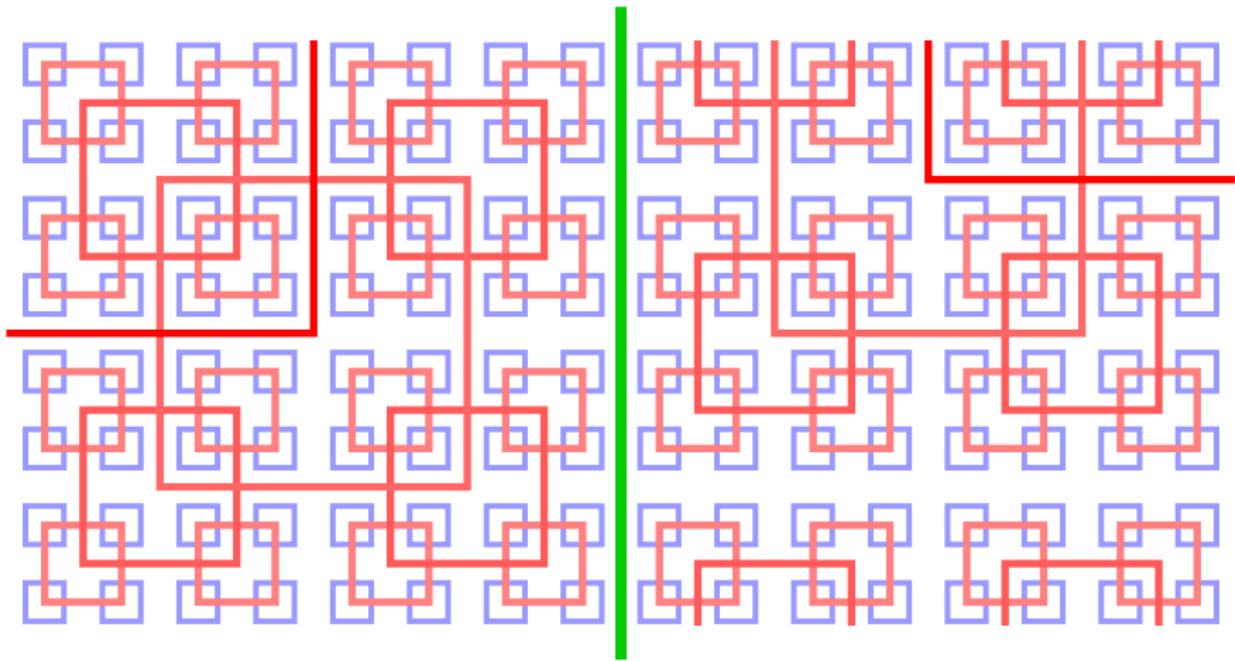
Geometric Vision: Fractured lines



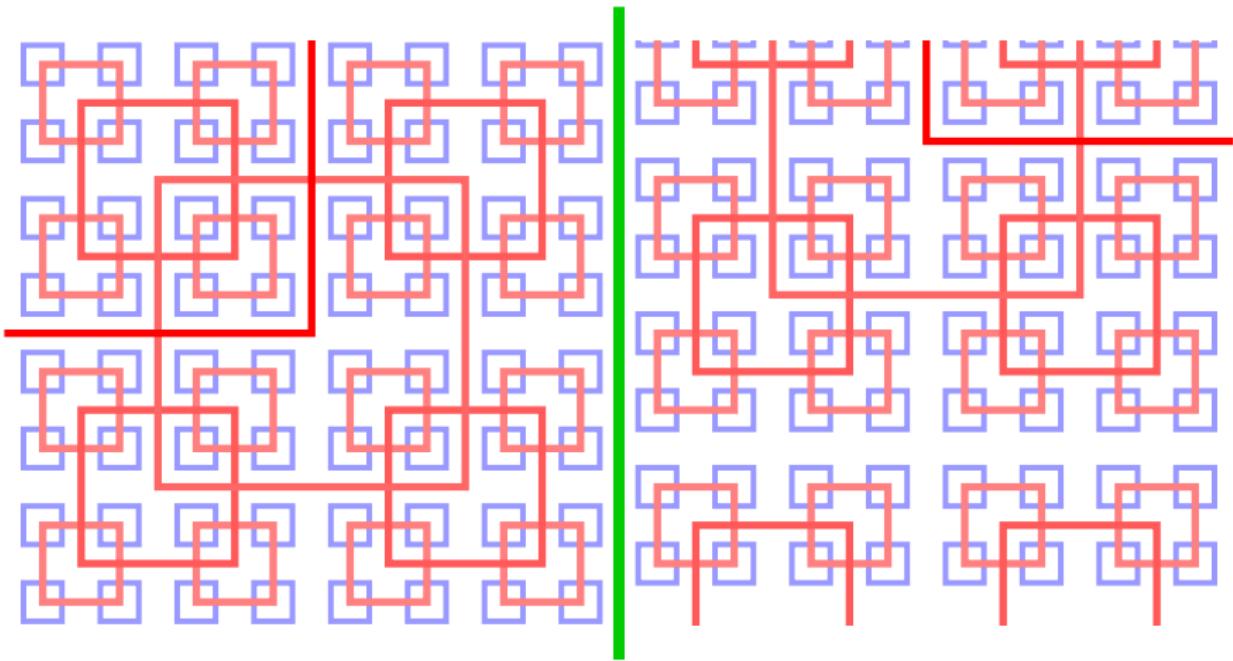
Geometric Vision: Fractured lines



Geometric Vision: Fractured lines



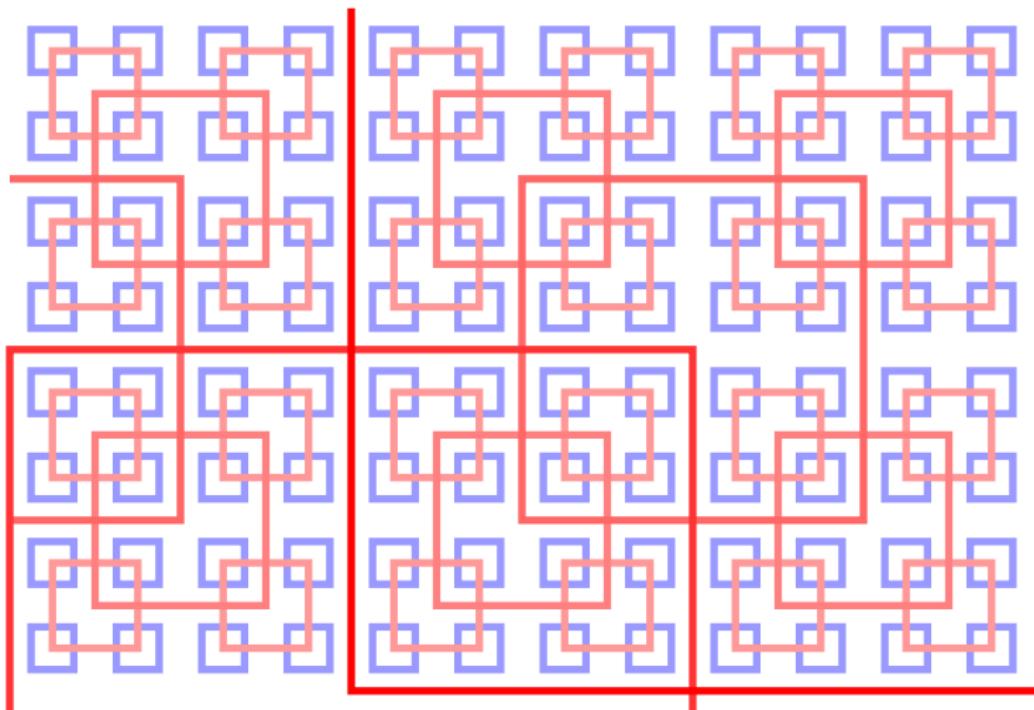
Geometric Vision: Fractured lines



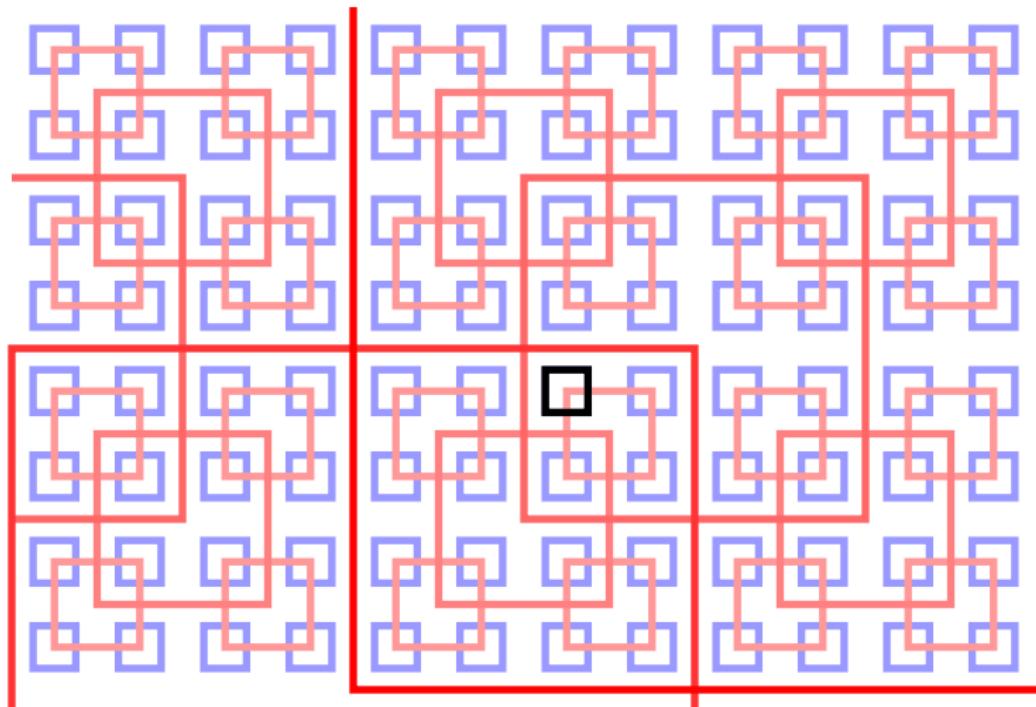
Proposition

$\mu(\{x \in \mathbf{T}_{\text{Robin}} \text{ with fracture lines}\}) = 0$ for all probability measure μ σ -invariante.

Cardinal of T_{Robi}

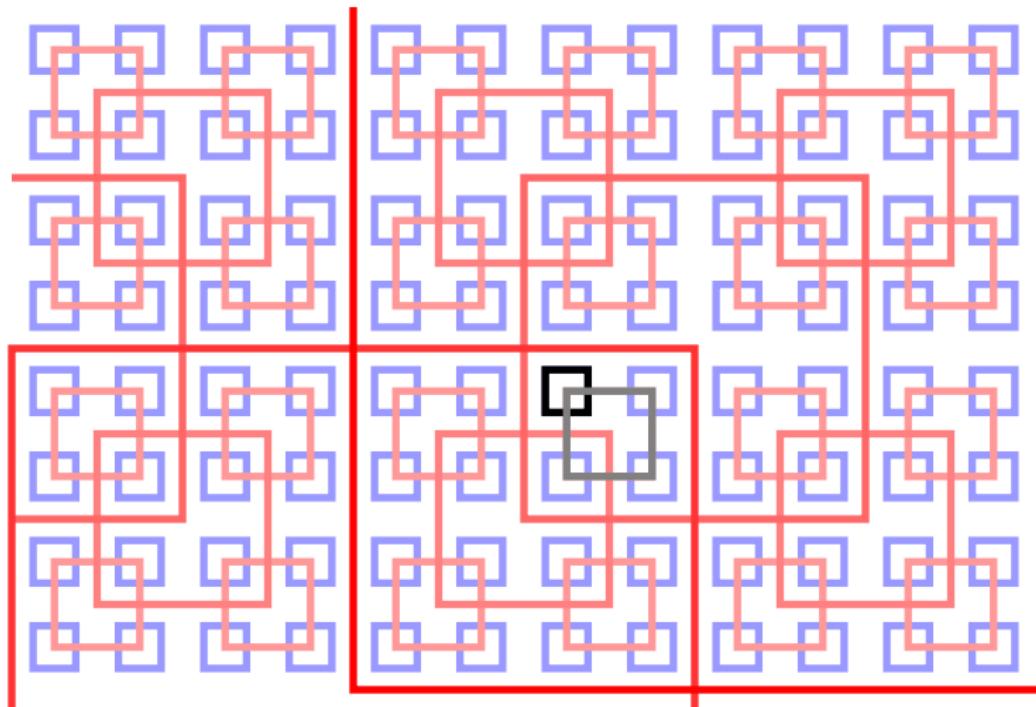


Cardinal of T_{Robi}



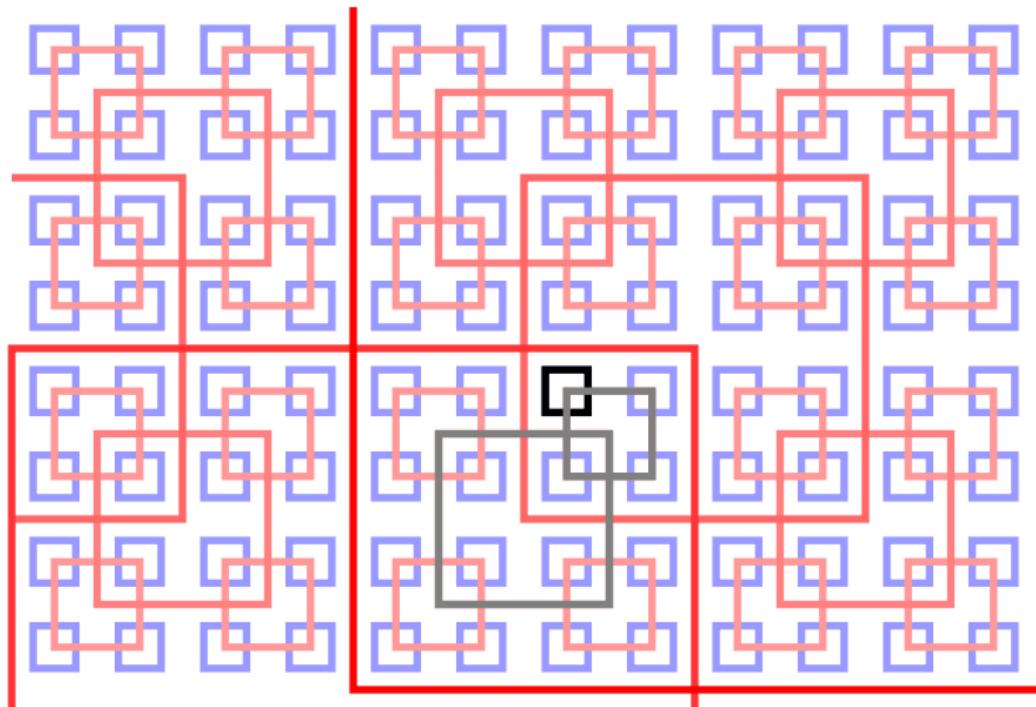
$$\rho(\square) = \begin{smallmatrix} & & \\ & \text{F} & \\ & & \end{smallmatrix}$$

Cardinal of \mathbf{T}_{Robi}



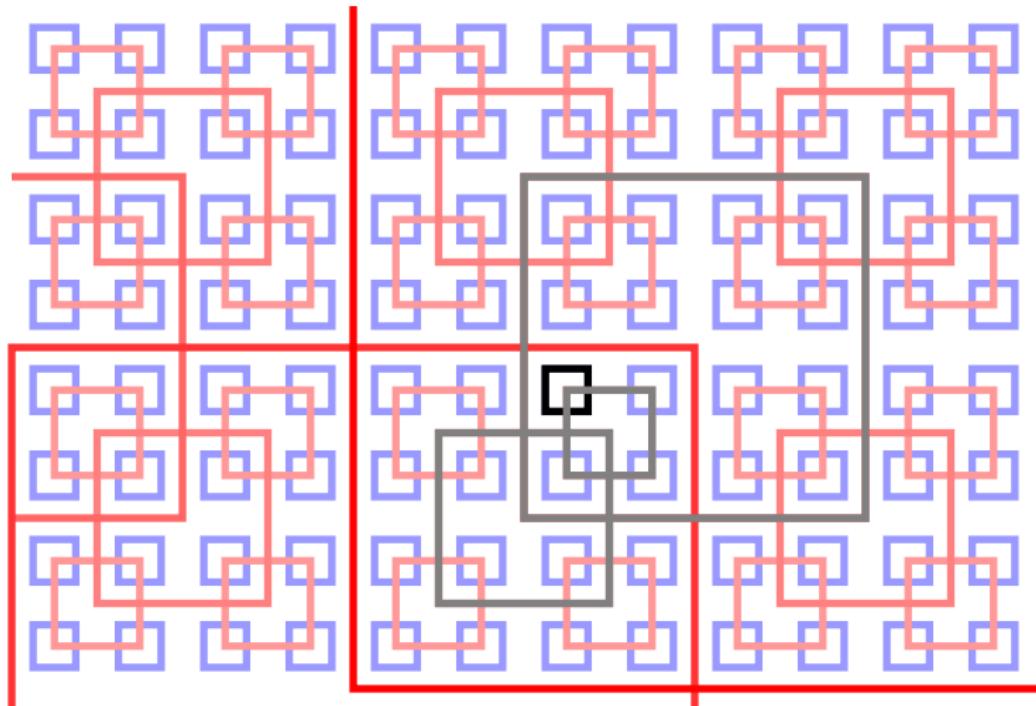
$$\rho(\square) = \begin{array}{c} \text{F} \\ \text{L} \end{array} \quad \begin{array}{c} \text{L} \\ \text{F} \end{array}$$

Cardinal of T_{Robi}



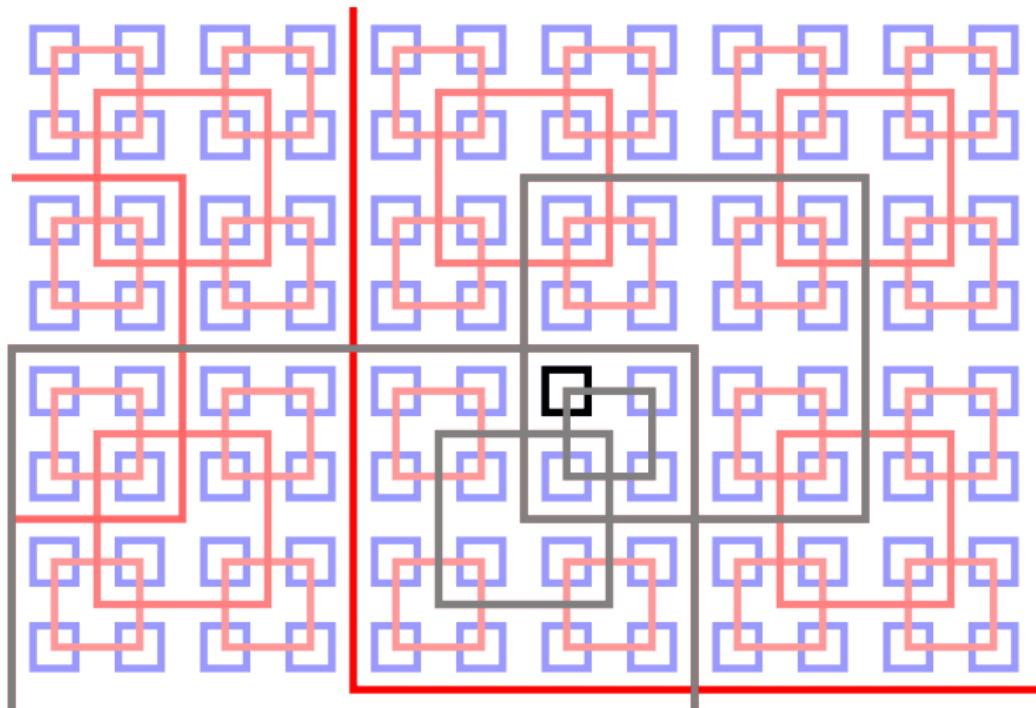
$$\rho(\square) = \begin{array}{c} \text{F} \\ \text{L} \\ \text{T} \end{array}$$

Cardinal of T_{Robi}



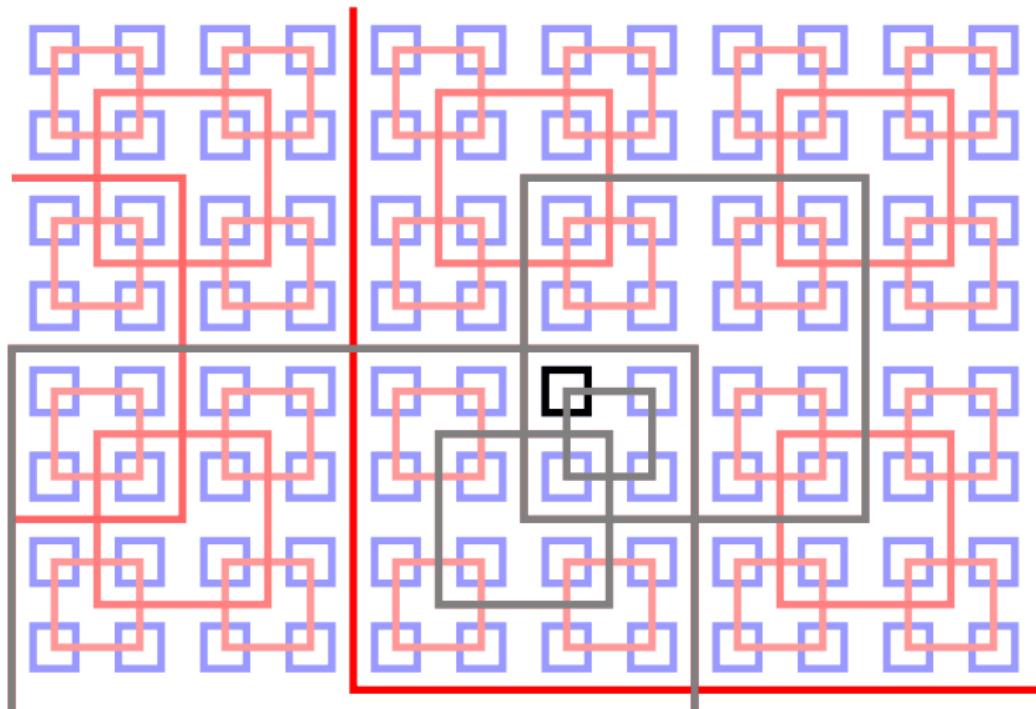
$$\rho(\square) = \begin{array}{c} \text{F} \\ \text{L} \\ \text{T} \\ \text{R} \end{array}$$

Cardinal of T_{Robi}



$$\rho(\square) = \begin{array}{c} \text{F} \\ \text{L} \\ \text{T} \\ \text{J} \\ \text{R} \end{array}$$

Cardinal of T_{Robi}

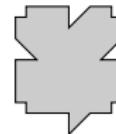
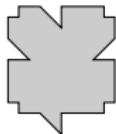
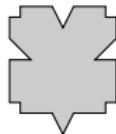
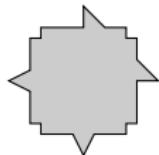
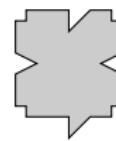
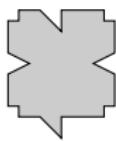
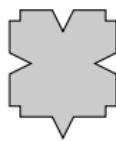
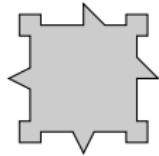
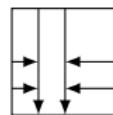
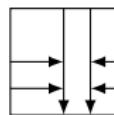
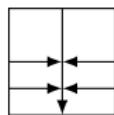
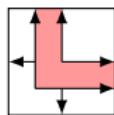
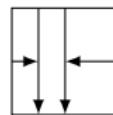
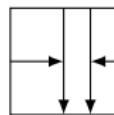
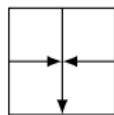
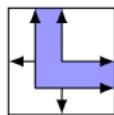


$$\rho(\square) = \begin{array}{c} \text{square with arrows pointing inwards} \\ \text{square with arrows pointing inwards} \end{array} \dots$$

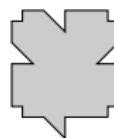
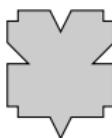
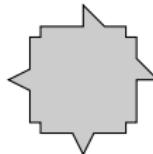
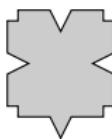
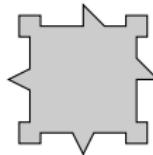
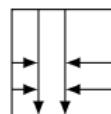
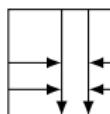
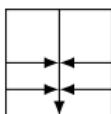
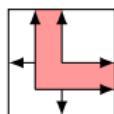
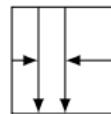
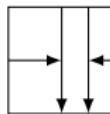
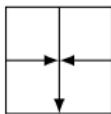
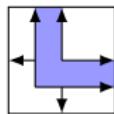
Proposition

T_{Robi} is uncountable

Robinson's tiles for children



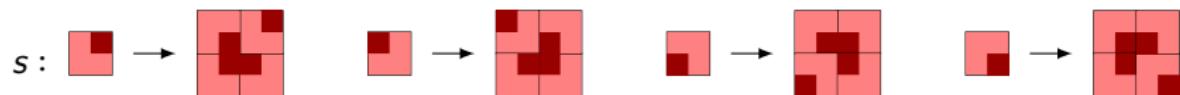
Robinson's tiles for children



Multidimensional substitutions

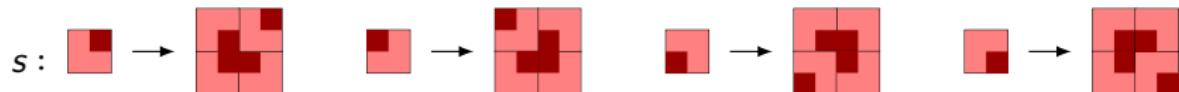
Rectangular substitution

Let $\mathcal{A} = \left\{ \begin{matrix} & \\ & \end{matrix}, \begin{matrix} & \\ & \end{matrix}, \begin{matrix} & \\ & \end{matrix}, \begin{matrix} & \\ & \end{matrix} \right\}$. Consider the next substitution:

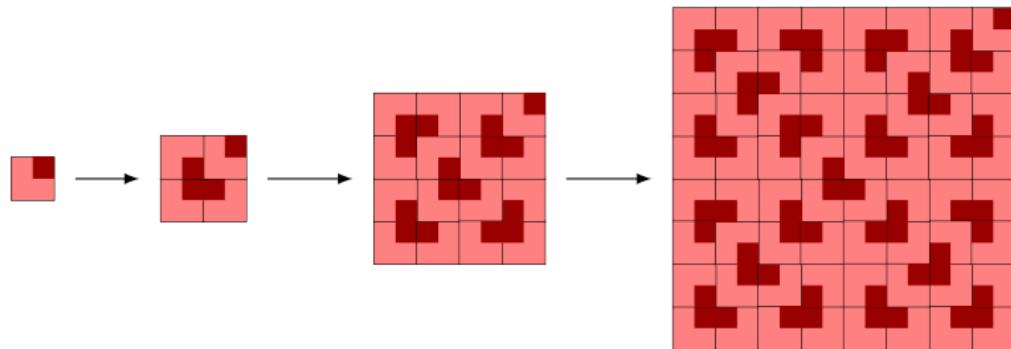


Rectangular substitution

Let $\mathcal{A} = \{\text{red square}, \text{red square with black center}, \text{red square with black top-right corner}, \text{black square}\}$. Consider the next substitution:



After iteration, we obtain:



Define *the substitutive subshift*:

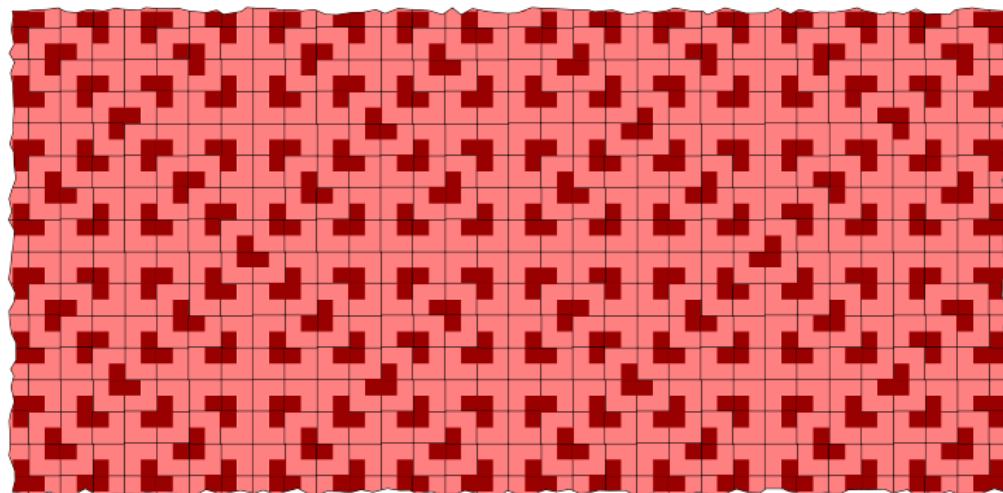
$$\mathbf{T}_s = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \forall p \sqsubset x \quad \exists a \in \mathcal{A}, \ n \in \mathbb{N}, \text{ tel que } p \sqsubset s^n(a) \right\}$$

Rectangular substitution

Let $\mathcal{A} = \left\{ \begin{matrix} & \\ & \end{matrix}, \begin{matrix} & \\ & \end{matrix}, \begin{matrix} & \\ & \end{matrix}, \begin{matrix} & \\ & \end{matrix} \right\}$. Consider the next substitution:

$$s : \begin{matrix} & \\ & \end{matrix} \rightarrow \begin{matrix} & & \\ & & \\ & & \end{matrix} \quad \begin{matrix} & \\ & \end{matrix} \rightarrow \begin{matrix} & & \\ & & \\ & & \end{matrix} \quad \begin{matrix} & \\ & \end{matrix} \rightarrow \begin{matrix} & & \\ & & \\ & & \end{matrix} \quad \begin{matrix} & \\ & \end{matrix} \rightarrow \begin{matrix} & & \\ & & \\ & & \end{matrix}$$

After iteration, we obtain:



Define *the substitutive subshift*:

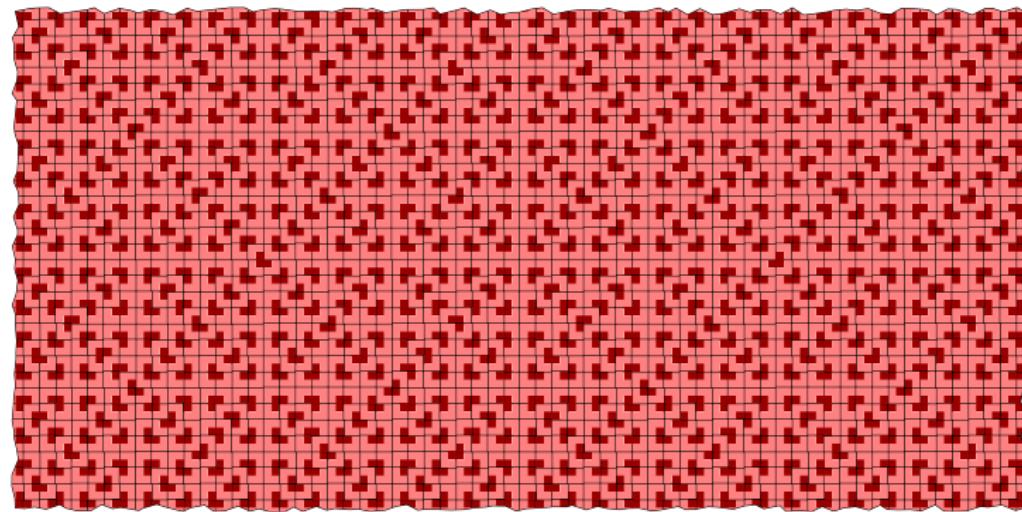
$$\mathbf{T}_s = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \forall p \sqsubset x \quad \exists a \in \mathcal{A}, \ n \in \mathbb{N}, \text{ tel que } p \sqsubset s^n(a) \right\}$$

Rectangular substitution

Let $\mathcal{A} = \left\{ \begin{matrix} & \\ & \end{matrix}, \begin{matrix} & \\ & \end{matrix}, \begin{matrix} & \\ & \end{matrix}, \begin{matrix} & \\ & \end{matrix} \right\}$. Consider the next substitution:

$$s : \begin{matrix} & \\ & \end{matrix} \rightarrow \begin{matrix} & & \\ & & \\ & & \end{matrix} \quad \begin{matrix} & \\ & \end{matrix} \rightarrow \begin{matrix} & & \\ & & \\ & & \end{matrix} \quad \begin{matrix} & \\ & \end{matrix} \rightarrow \begin{matrix} & & \\ & & \\ & & \end{matrix} \quad \begin{matrix} & \\ & \end{matrix} \rightarrow \begin{matrix} & & \\ & & \\ & & \end{matrix}$$

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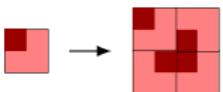
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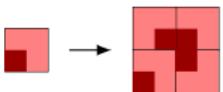
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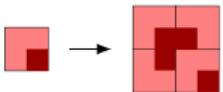
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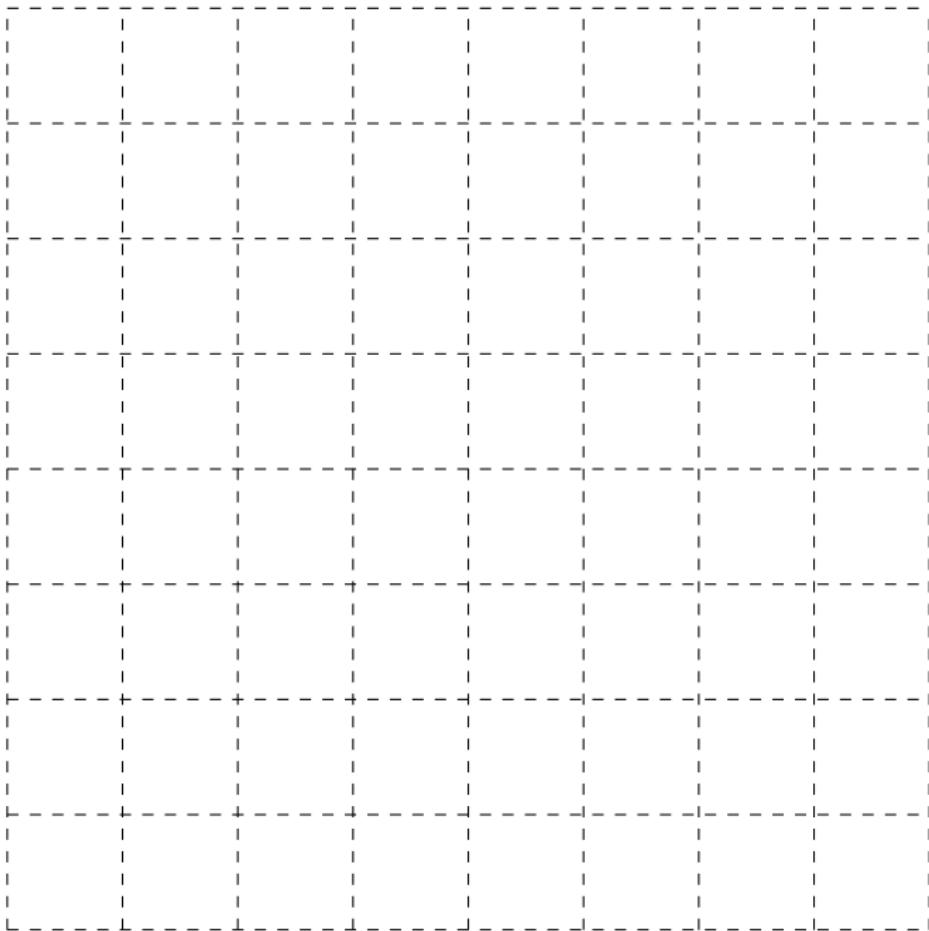
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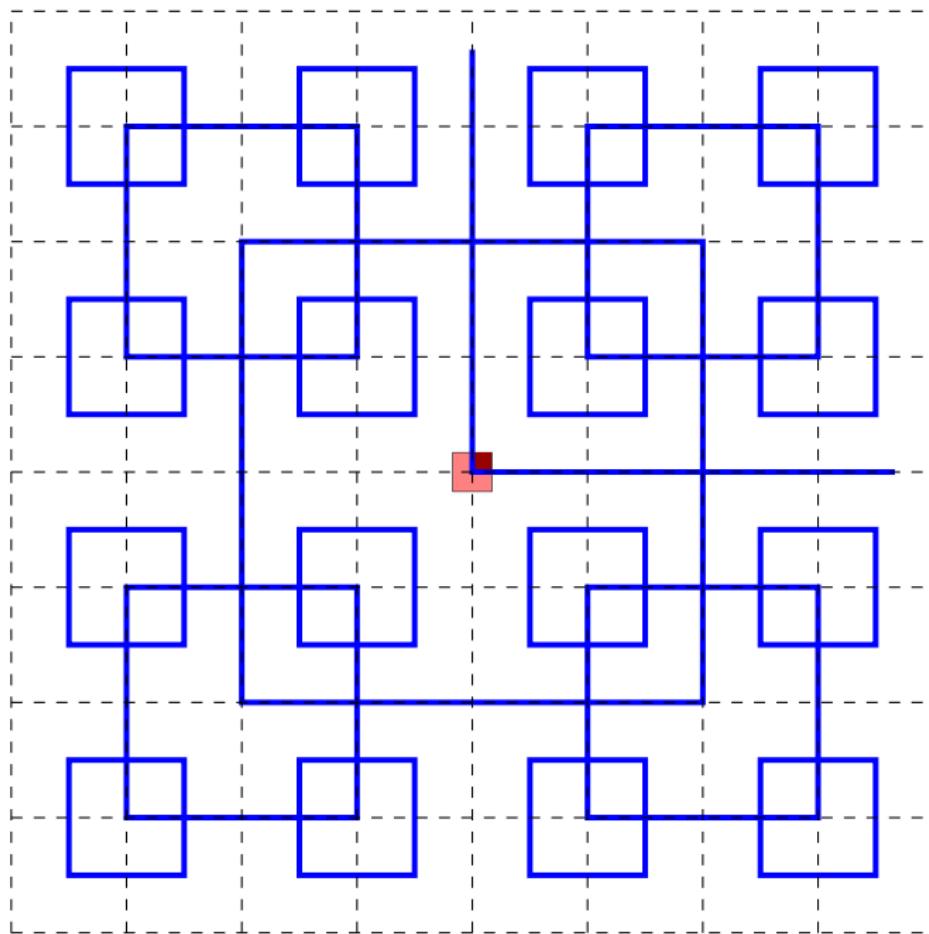
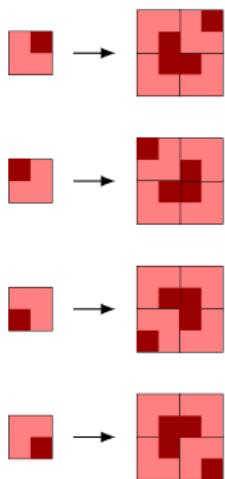
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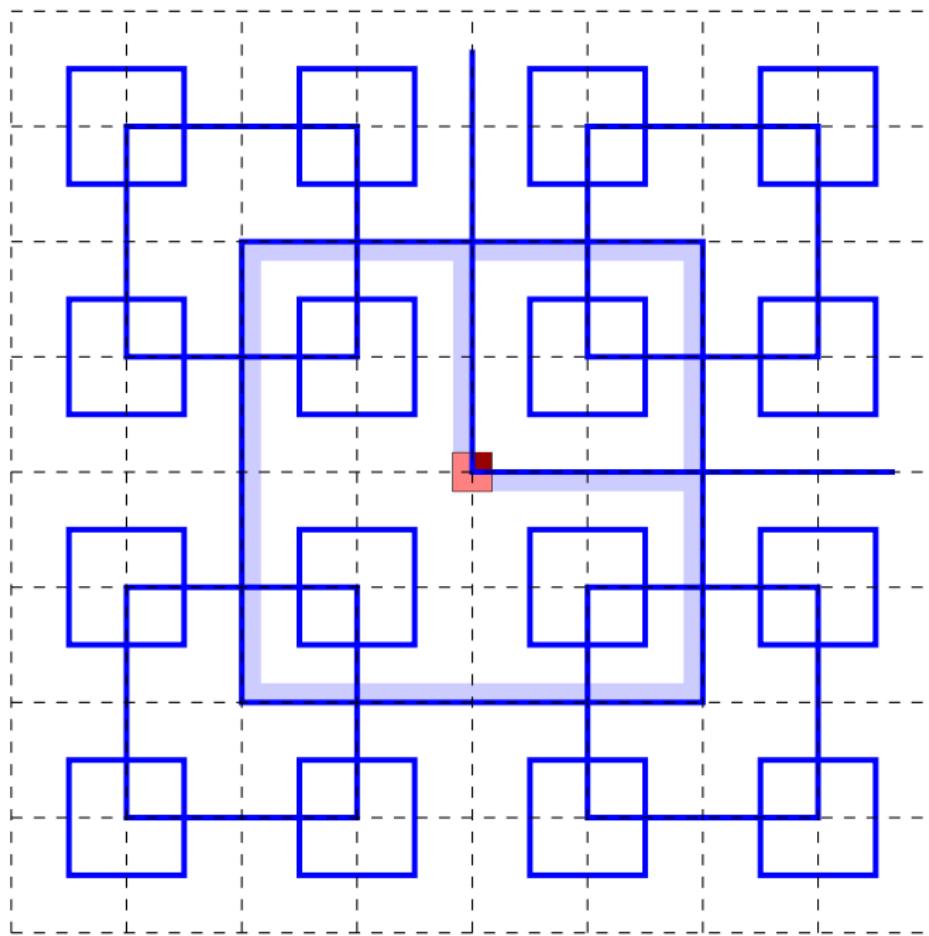
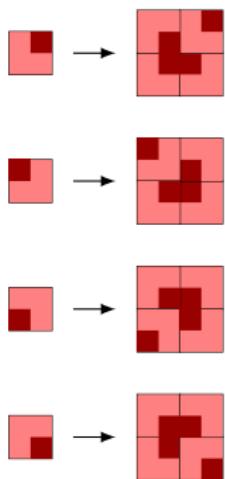
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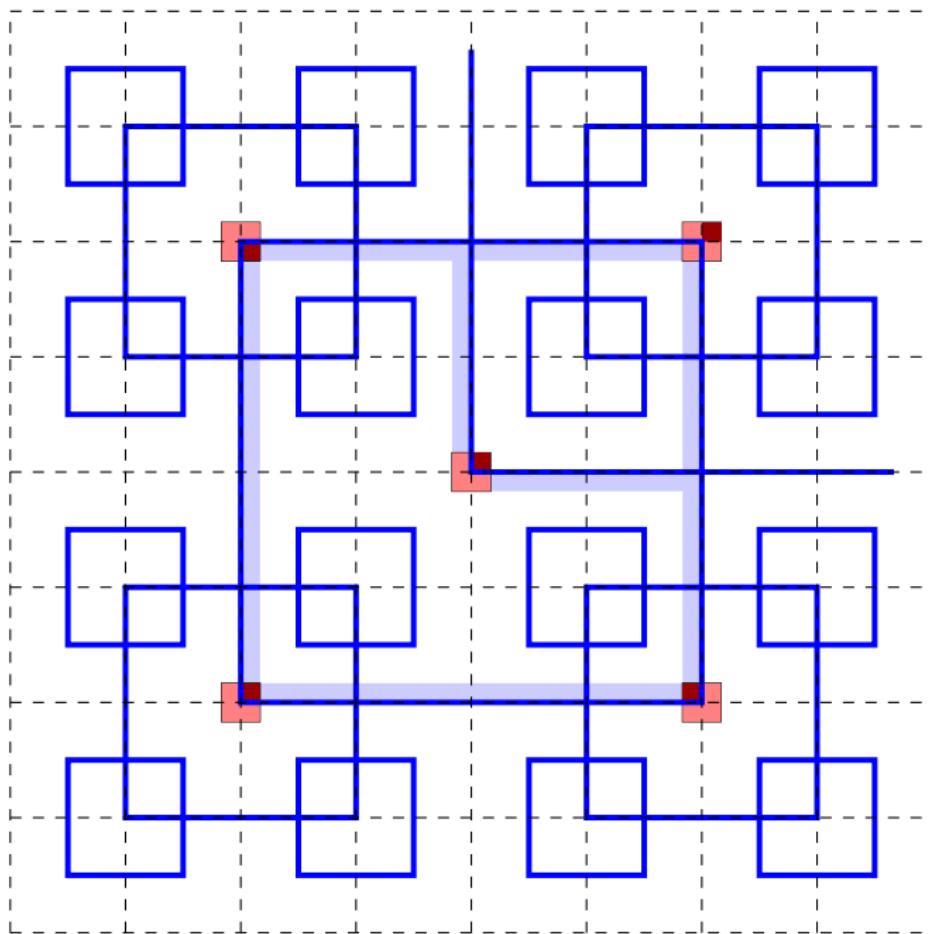
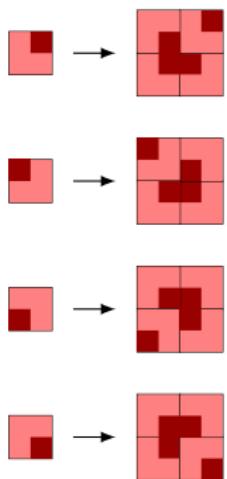
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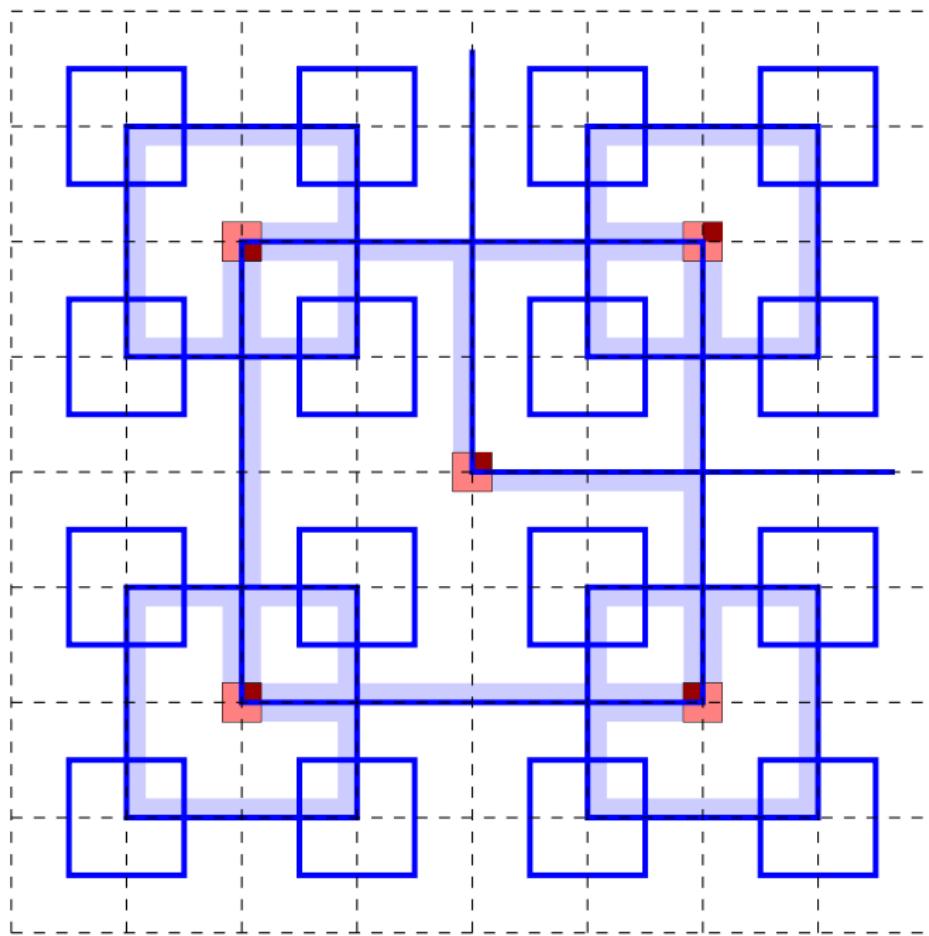
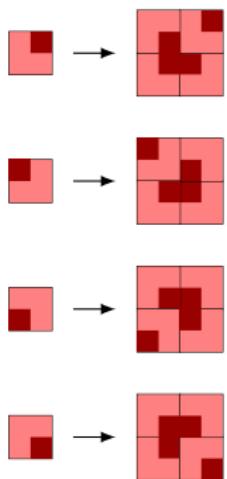
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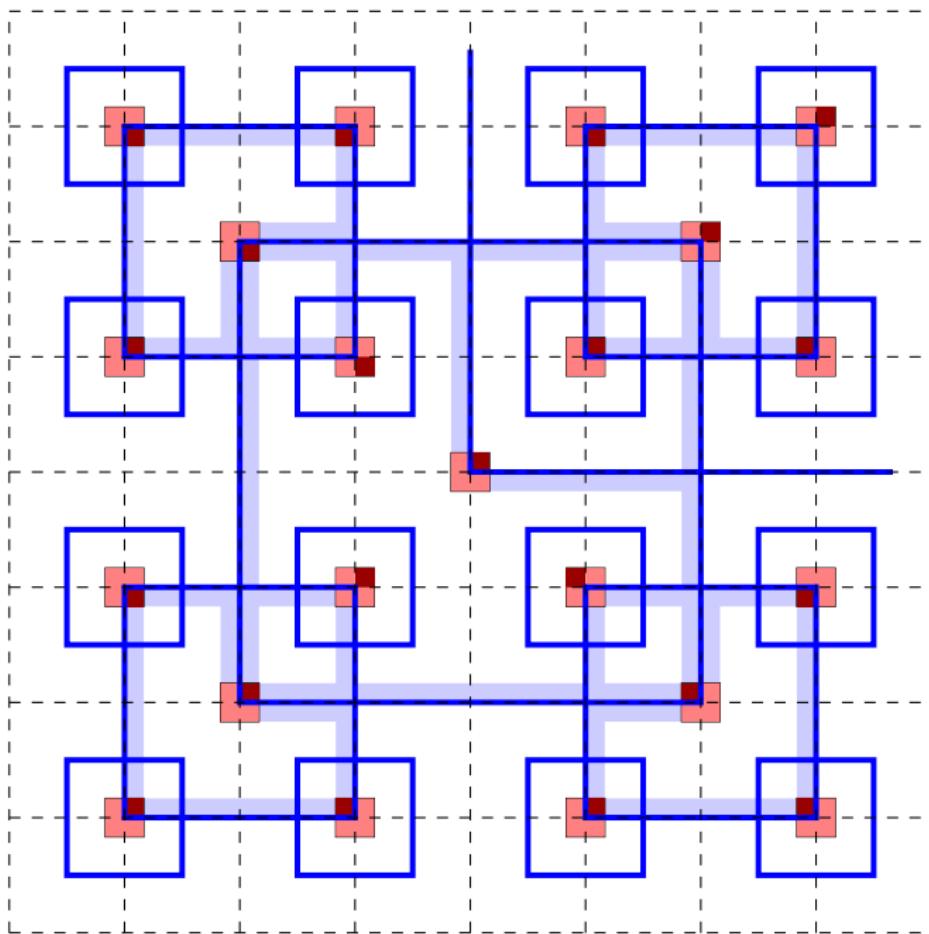
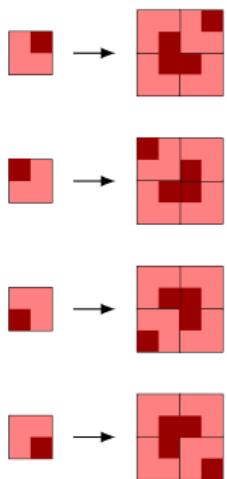
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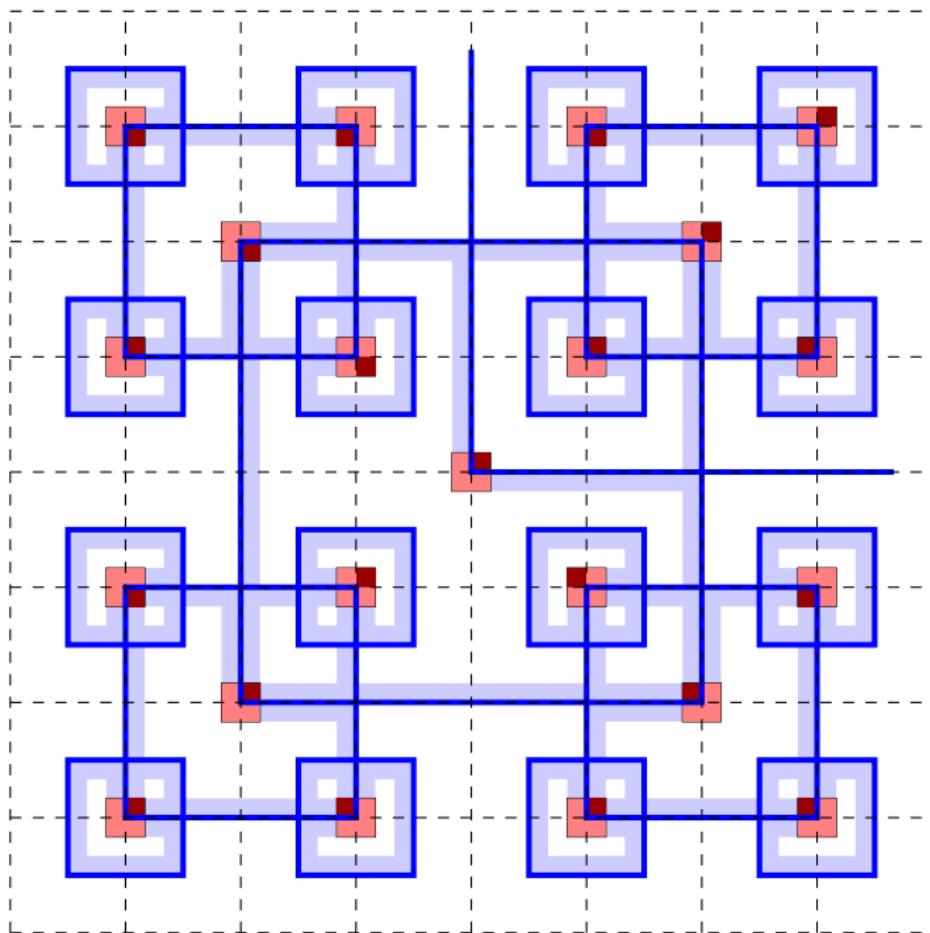
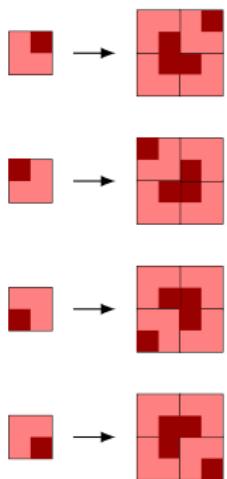
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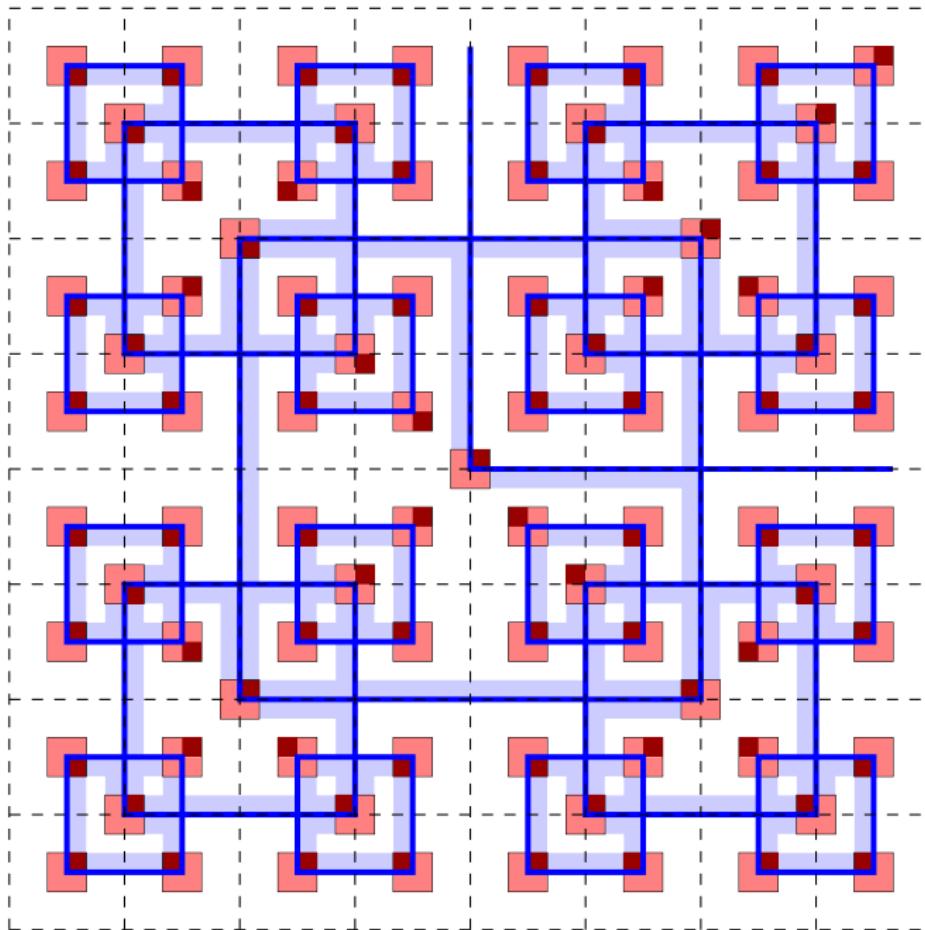
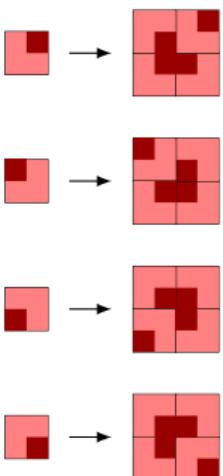
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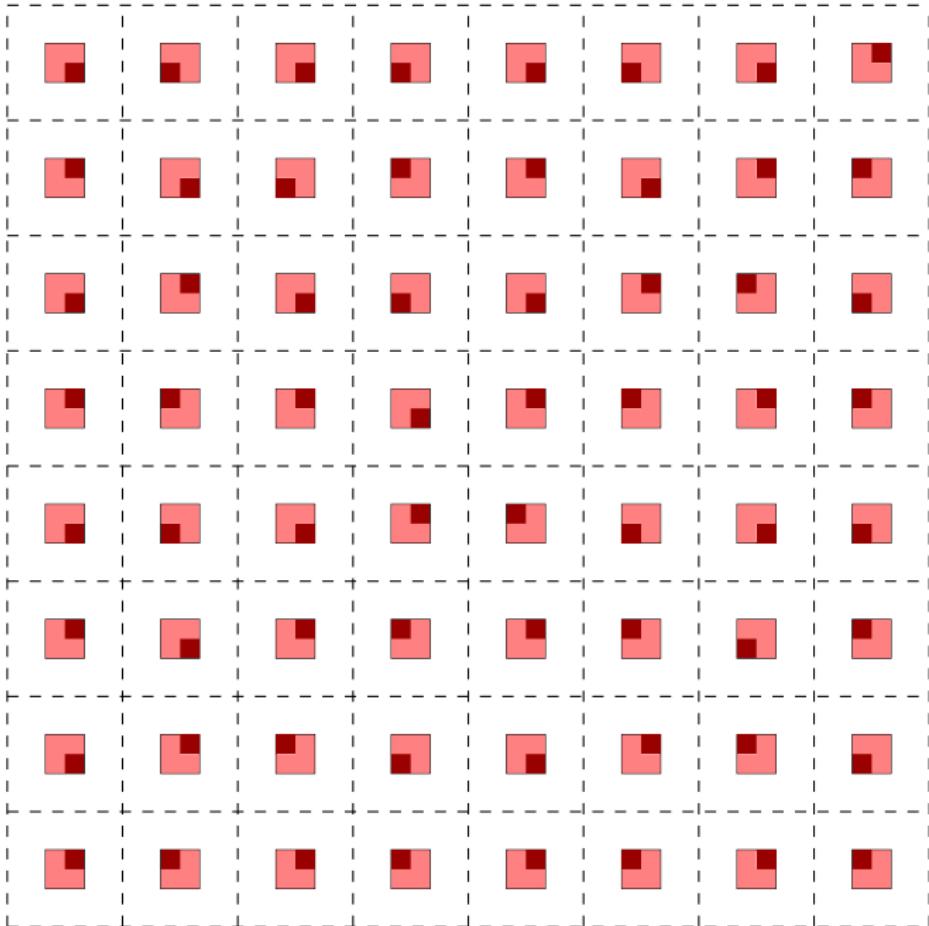
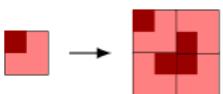
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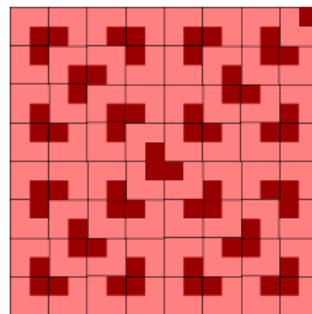
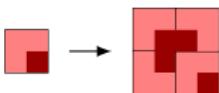
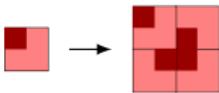
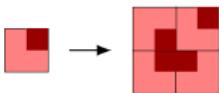
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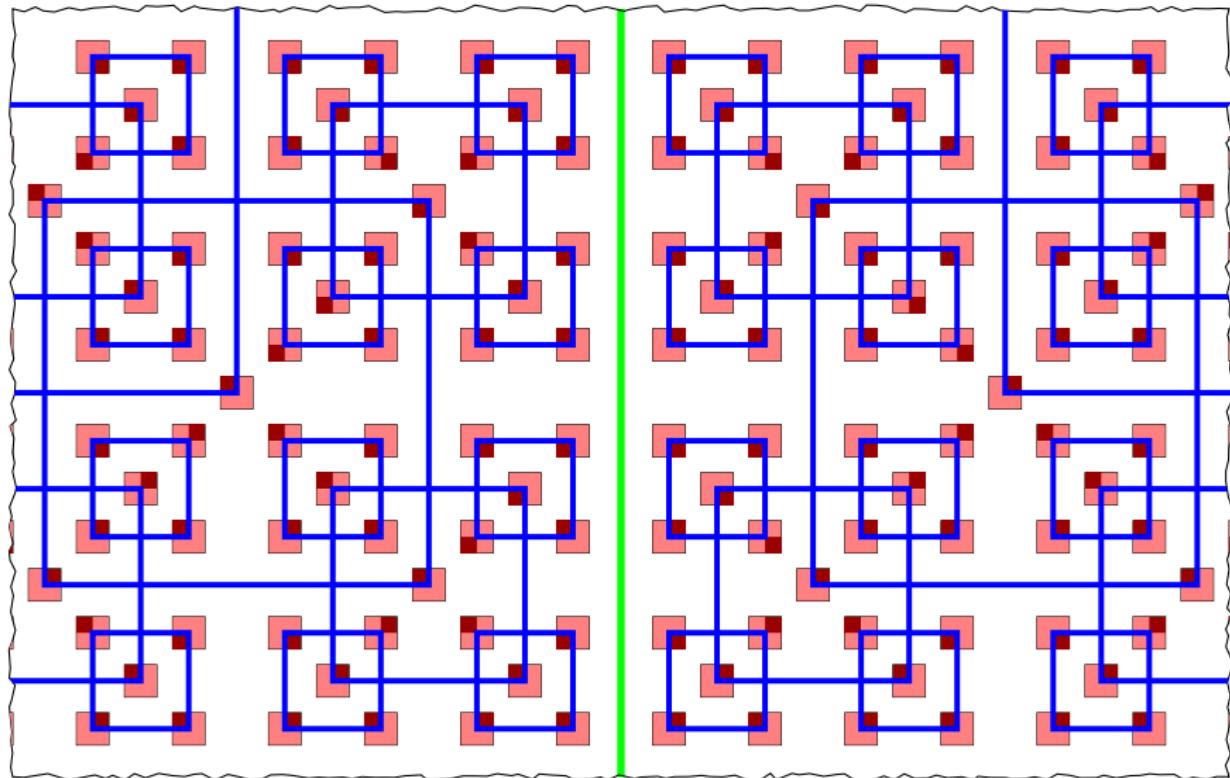
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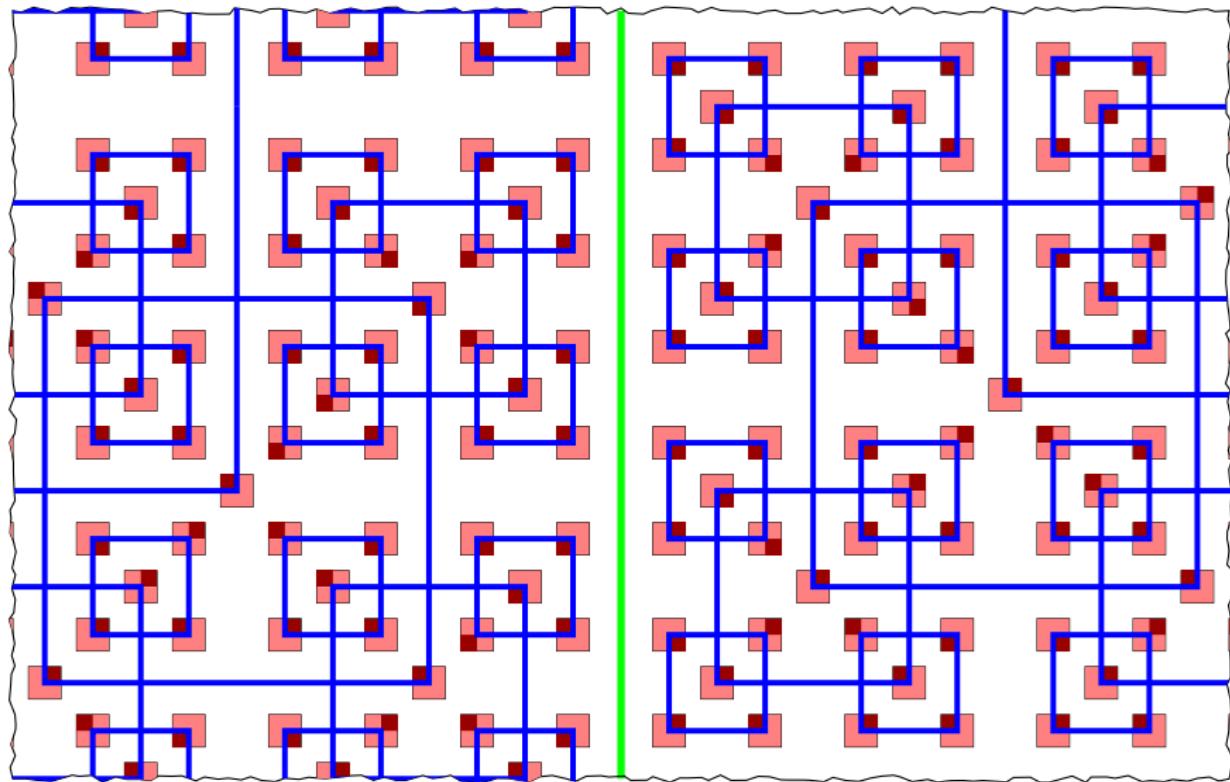
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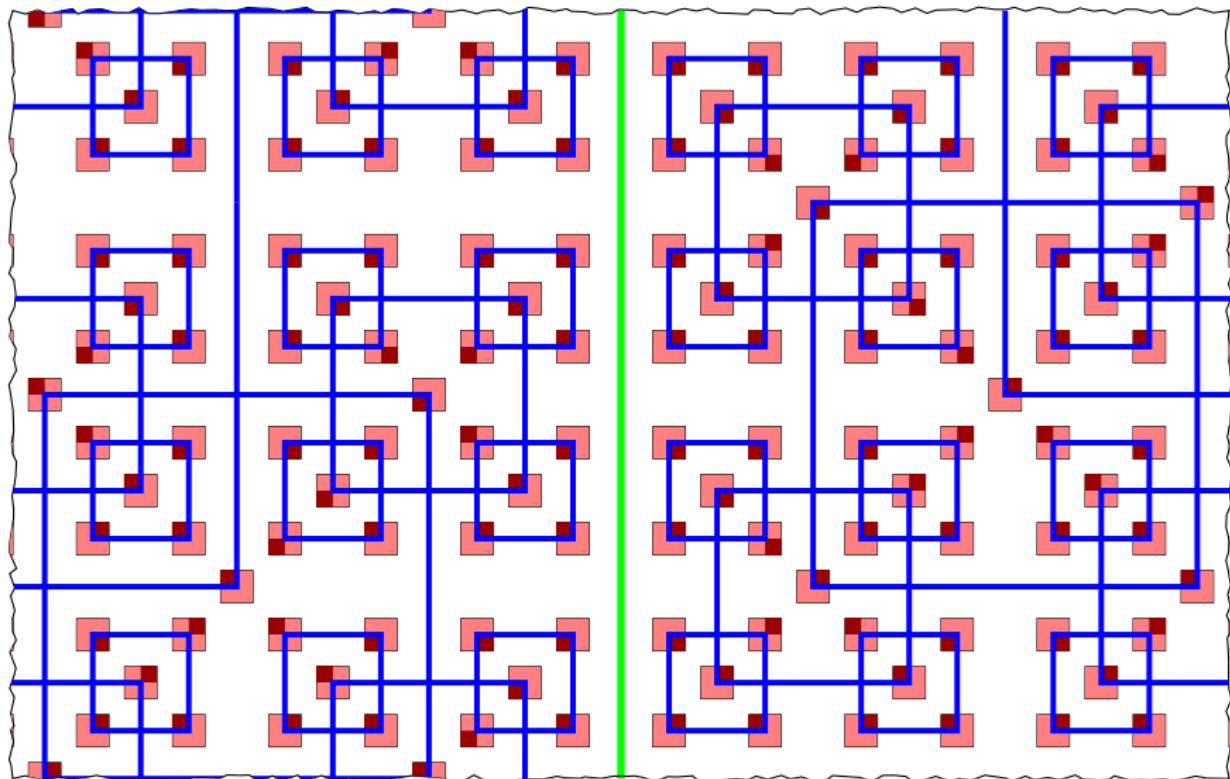
Coding of fractured lines



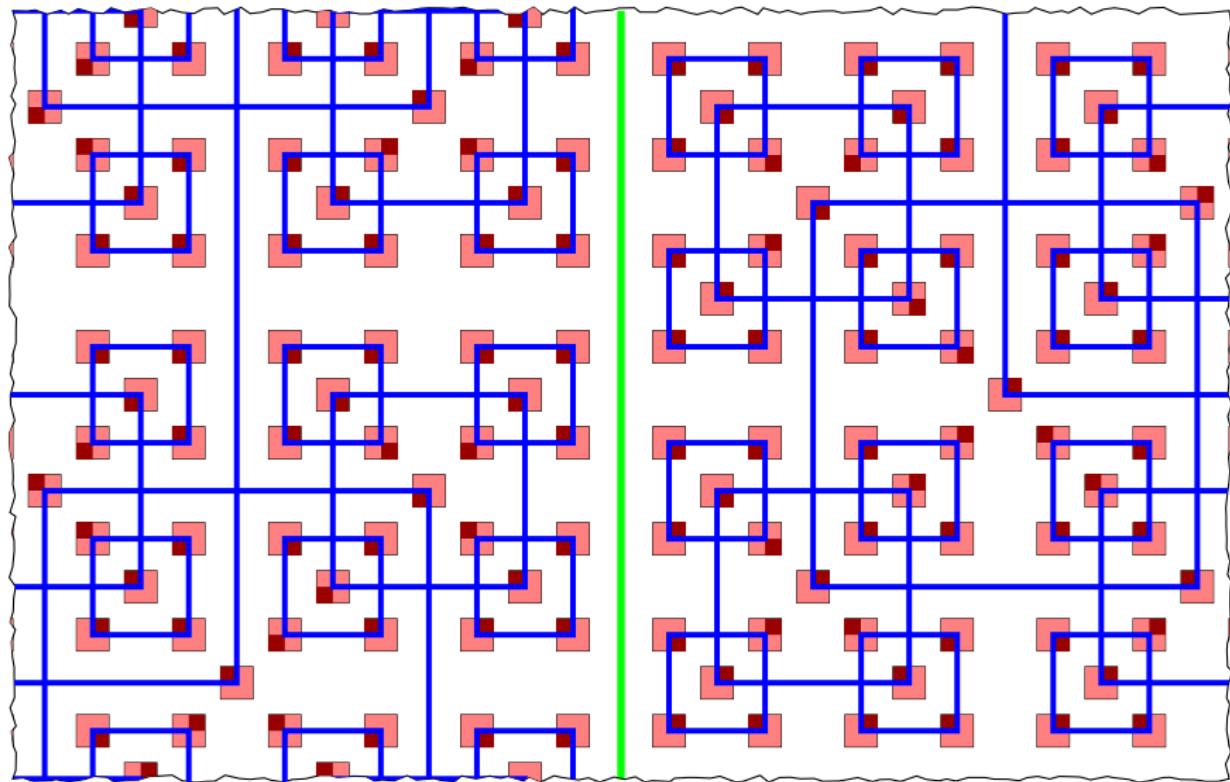
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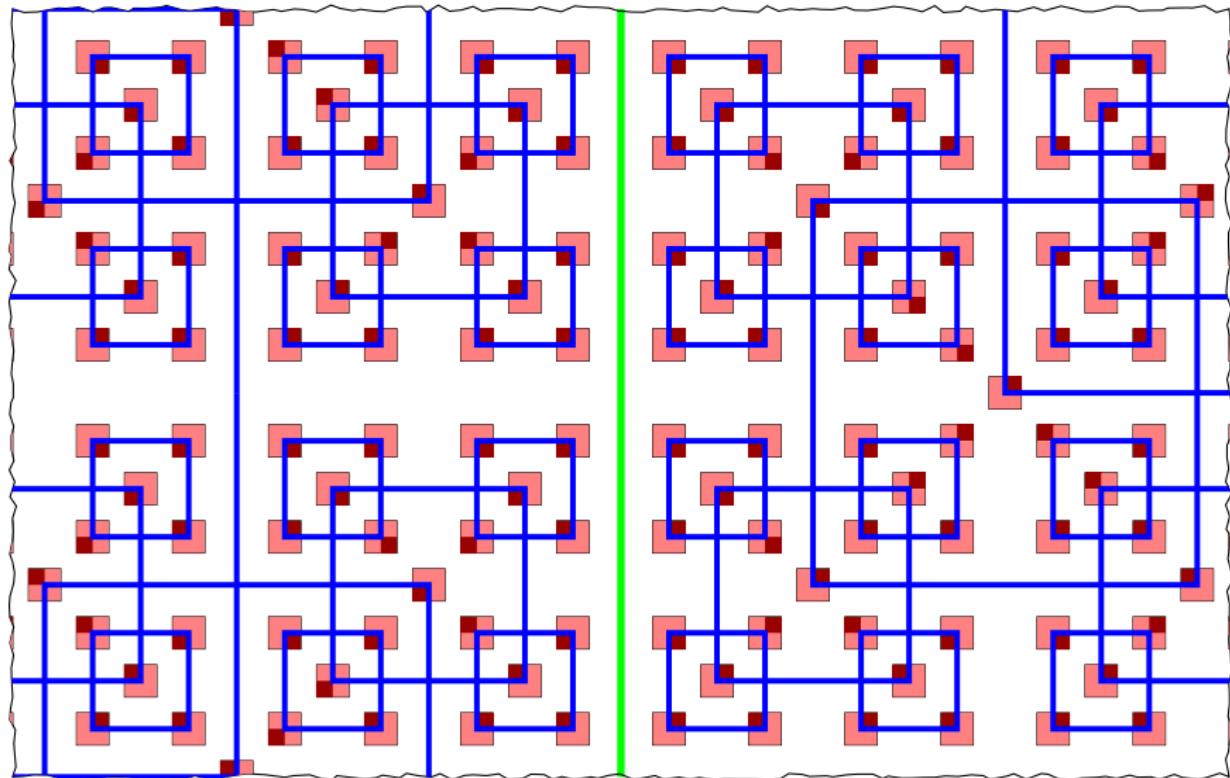
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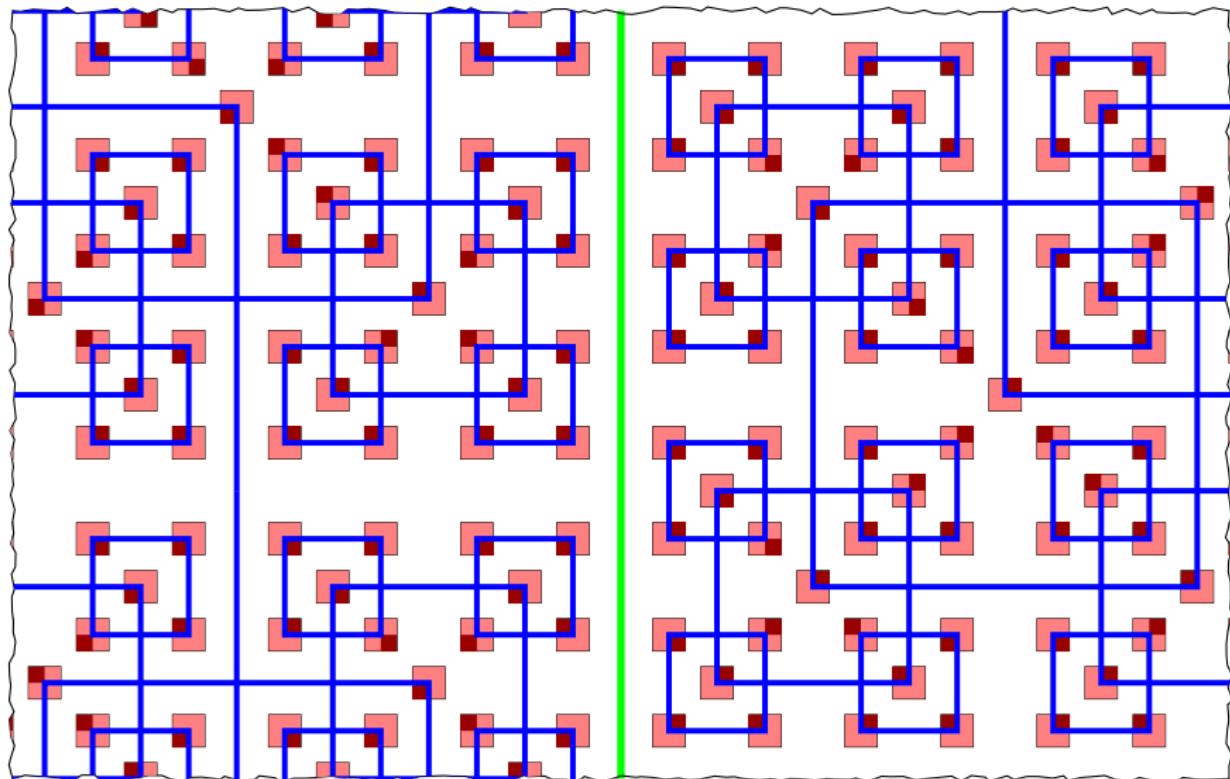
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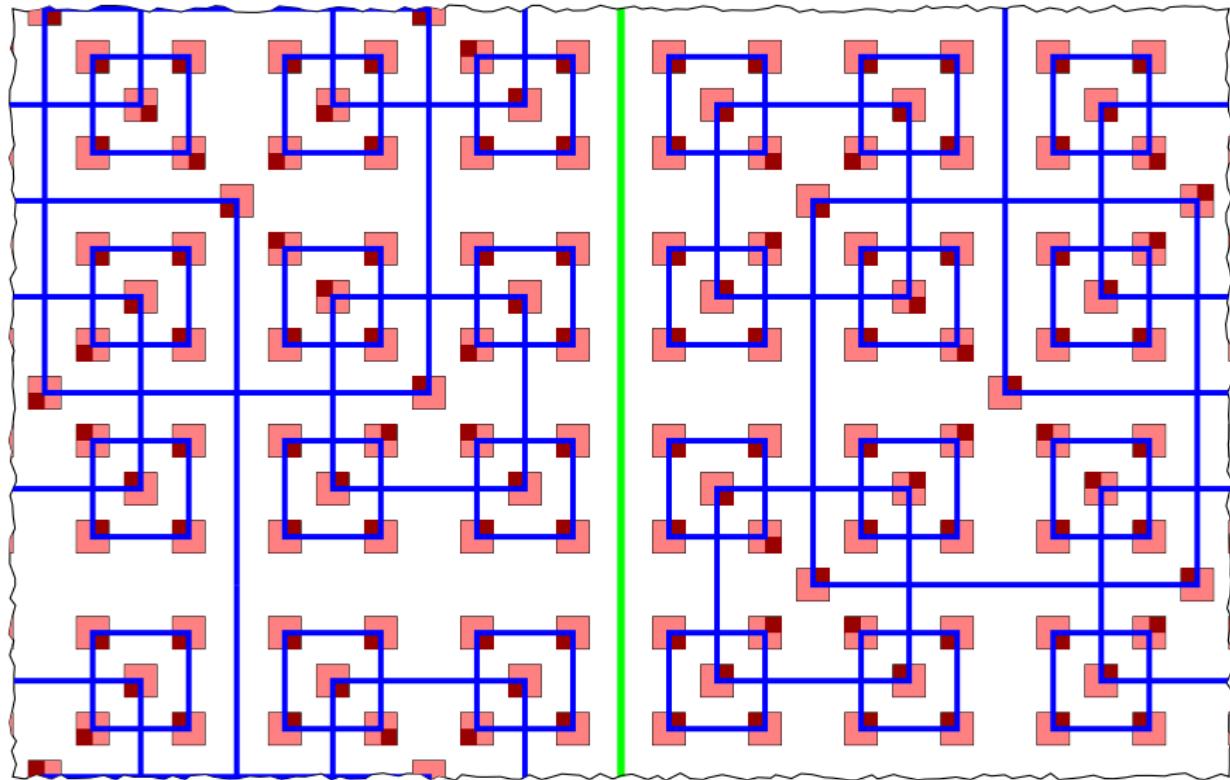
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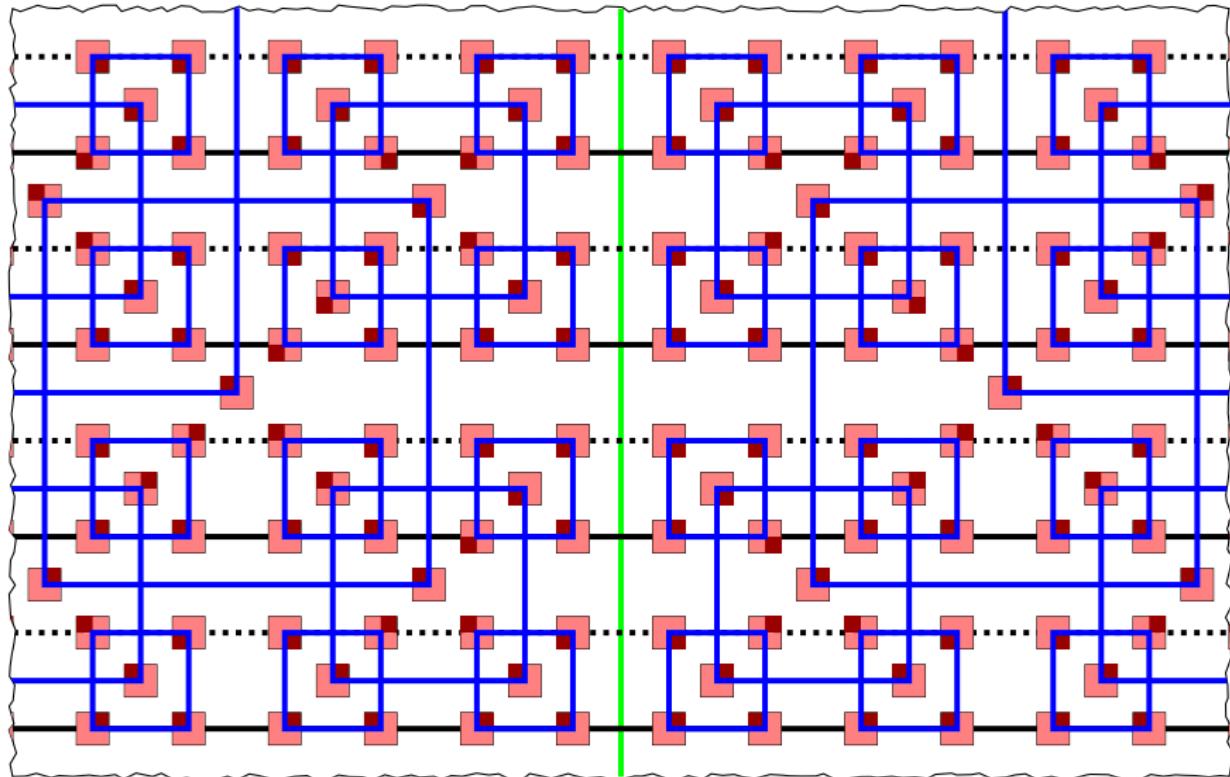
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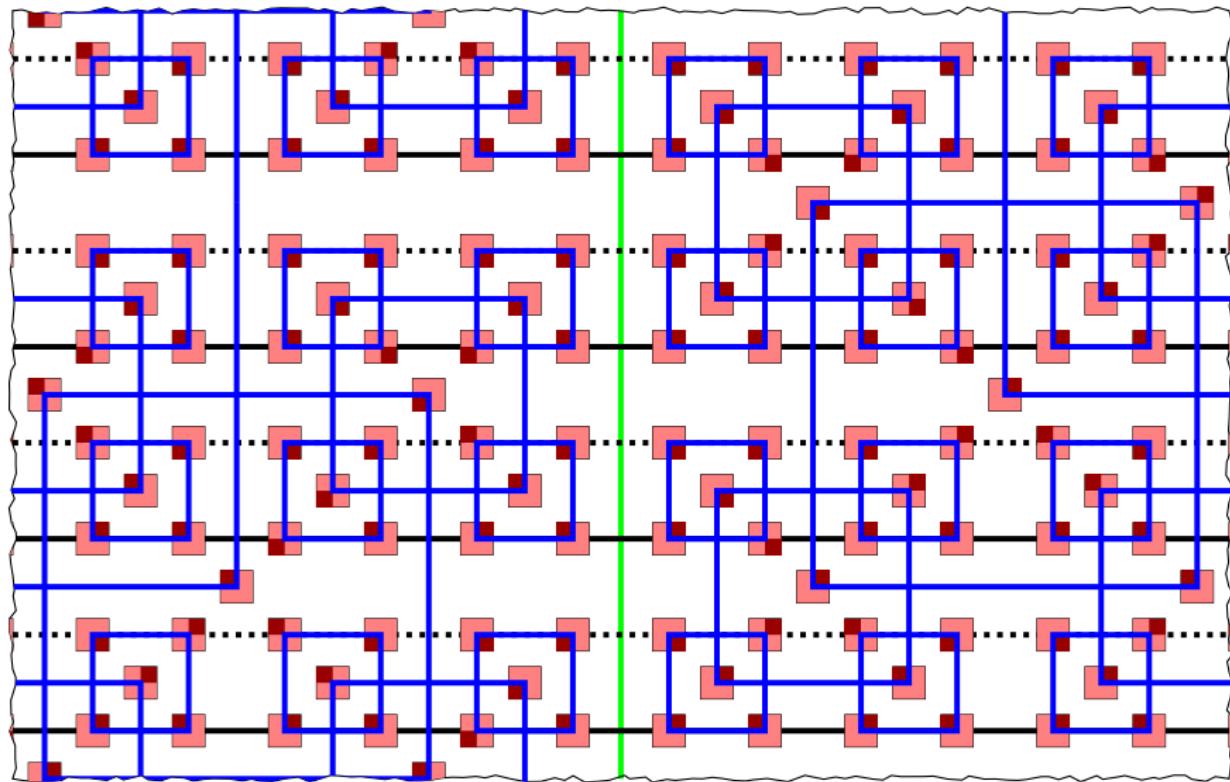
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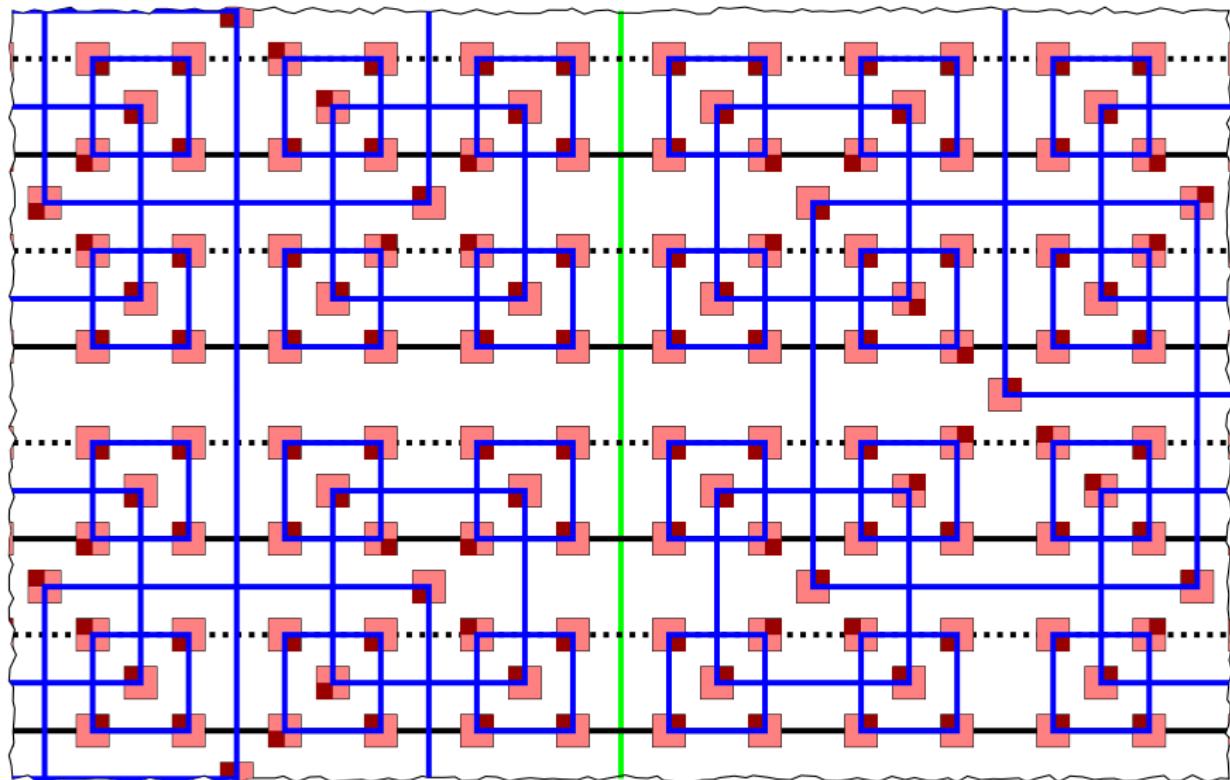
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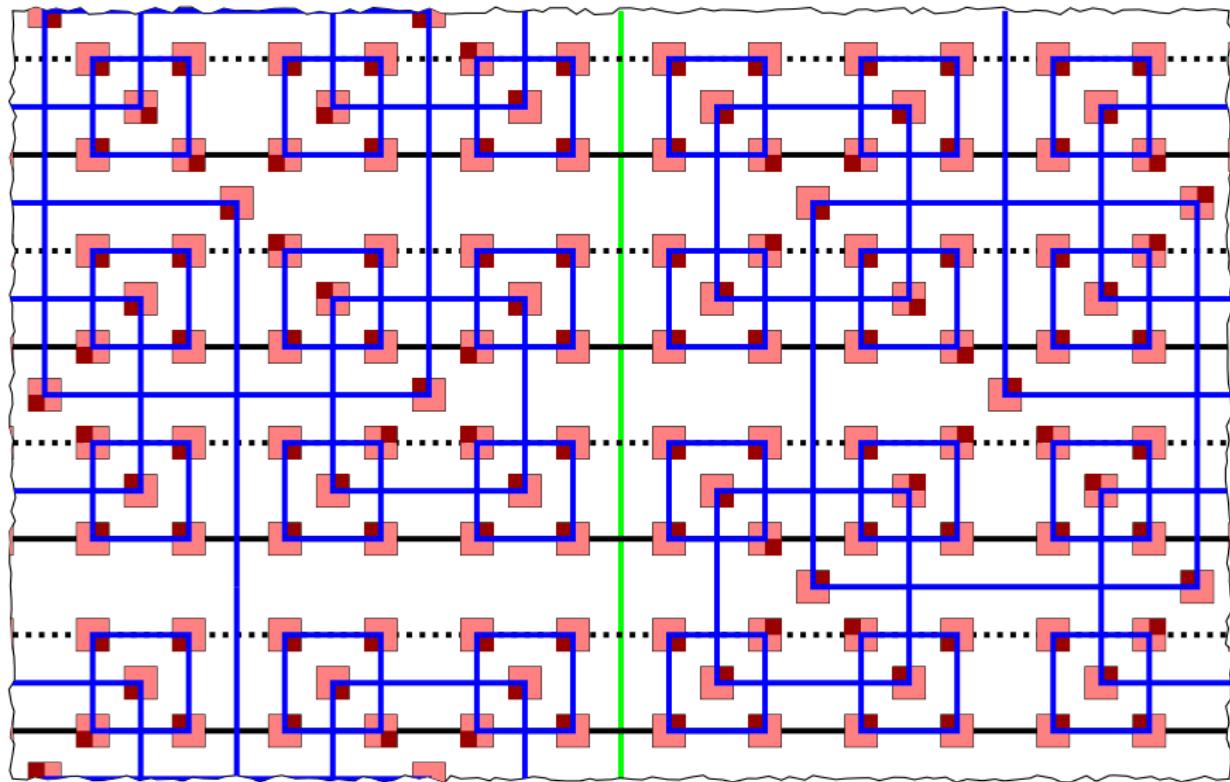
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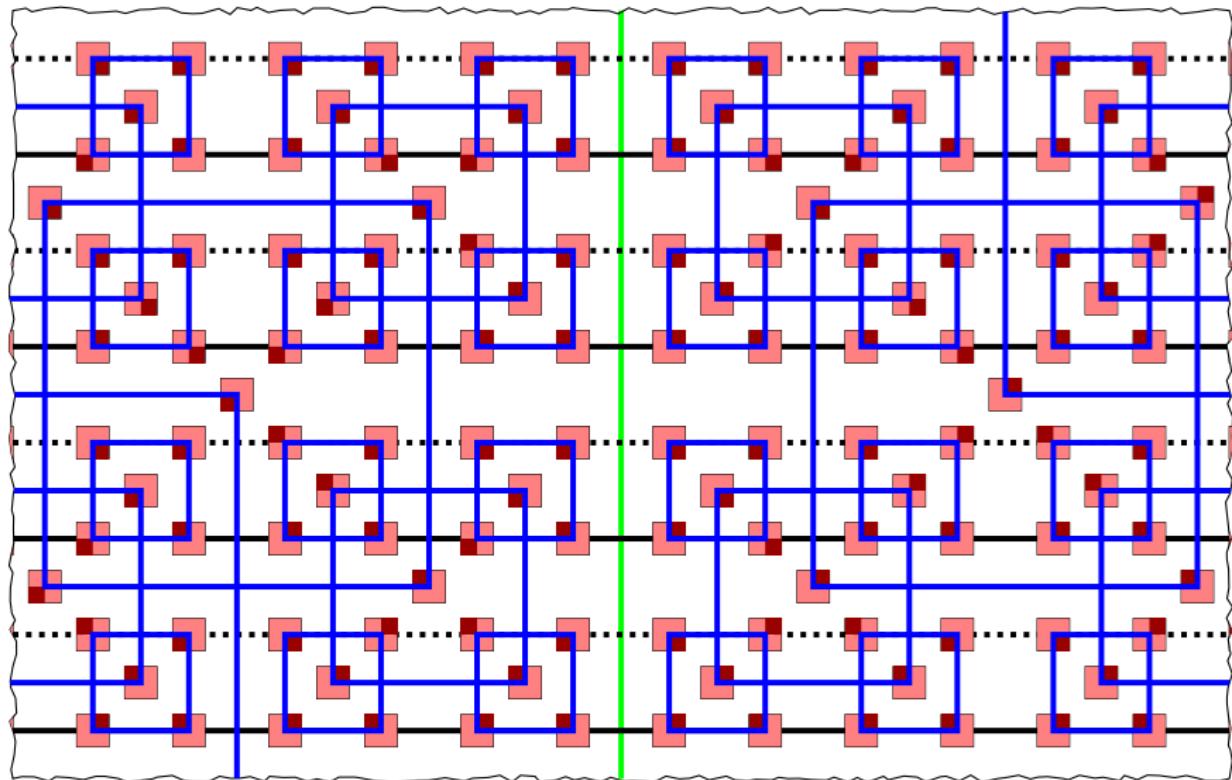
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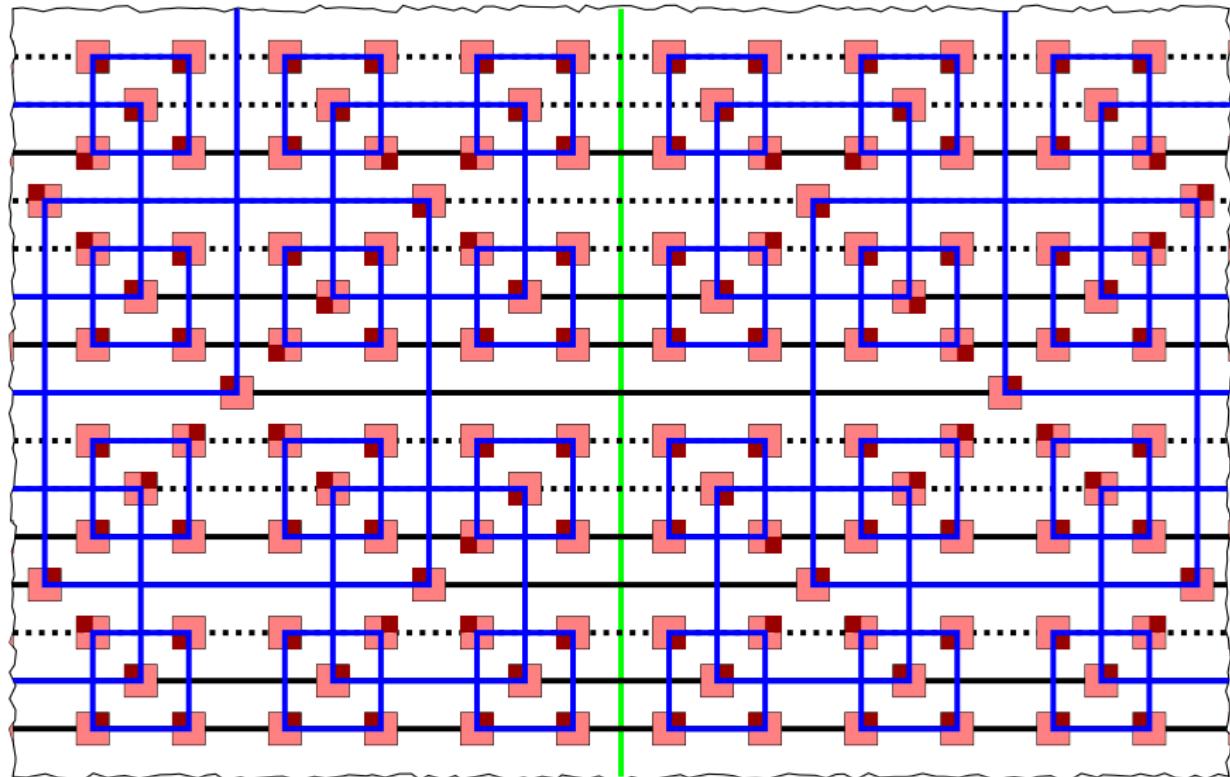
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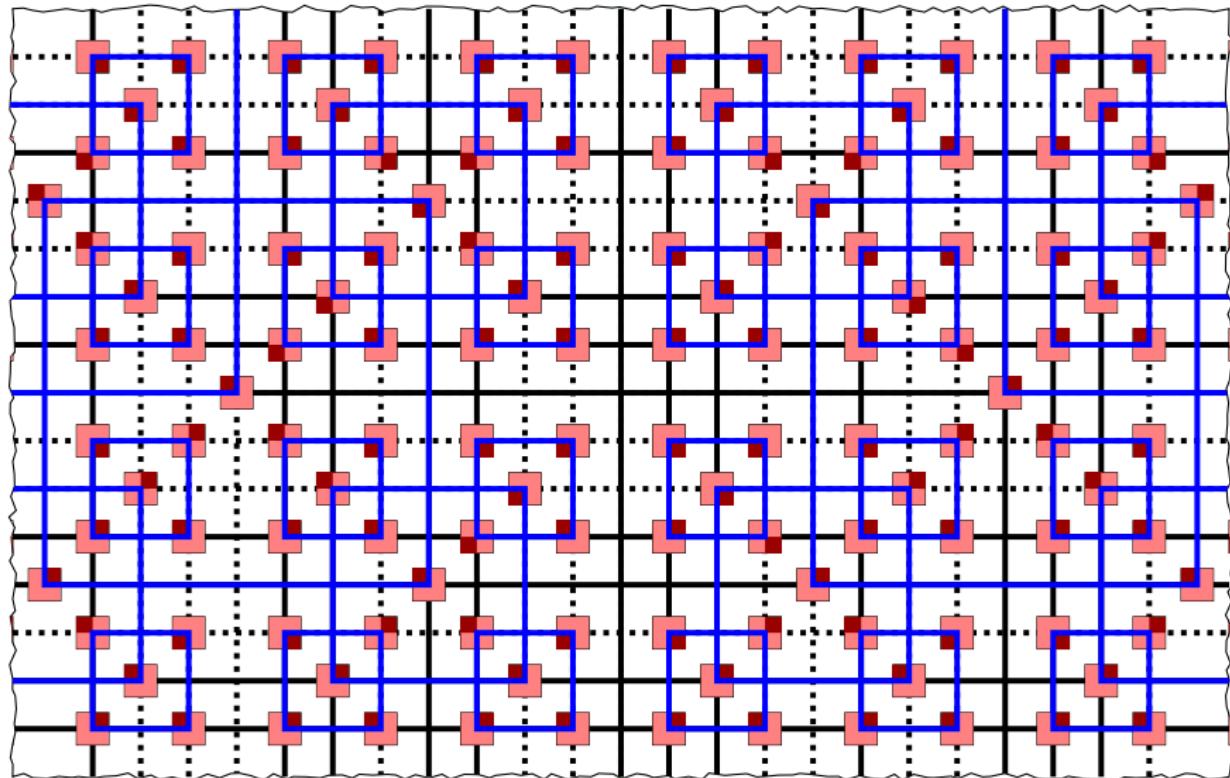
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Multidimensional substitutive tilings are sofic

Different layers to code substitute subshift with local rules:

- Alphabet of the substitution for the level 0,
- Robinson tiles to code hierarchical structure,
- alphabet of the substitution corresponding to higher levels,
- align the hierarchical structure in view to remove fractured lines.

Théorème (*Mozes 1989*)

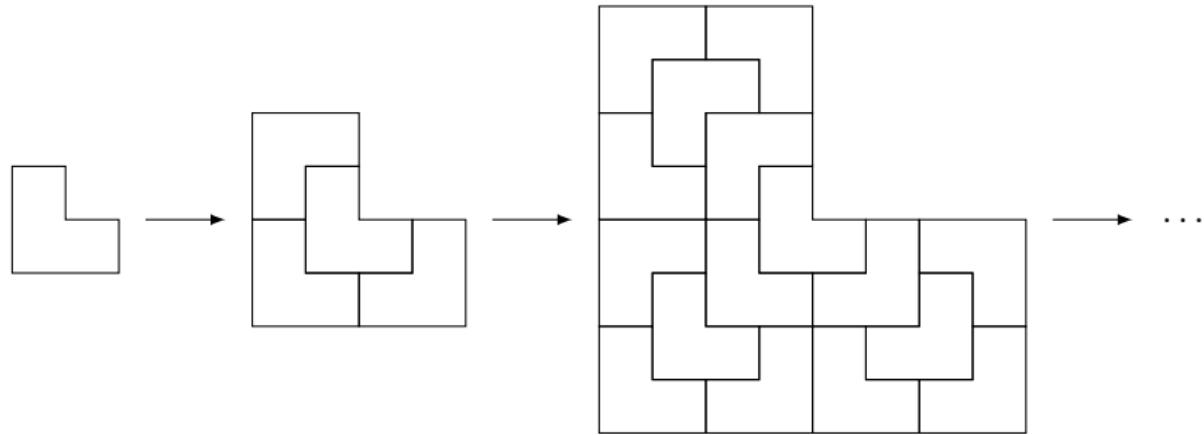
Every rectangular substitutions s on alphabet \mathcal{A} , there exists a SFT $\mathbf{T}(\mathcal{B}, d, \mathcal{F})$ and a factor map $\pi : \mathcal{B} \rightarrow \mathcal{A}$ such that

$$\pi(\mathbf{T}(\mathcal{B}, d, \mathcal{F})) = \mathbf{T}_s$$

Moreover π is a conjugacy almost everywhere.

In particular $h(\mathbf{T}(\mathcal{B}, d, \mathcal{F})) = h(\mathbf{T}_s)$.

Substitution of polygons



Theorem (*Goodman-Strauss 1998*)

The tiling space defined by substitution on polygon can be defined by local rules.

Decision problem

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Domino problem:

Given \mathcal{A} a finite alphabet and \mathcal{F} a finite set of 2-dimensional patterns.

$$\mathbf{T}(\mathcal{A}, 2, \mathcal{F}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \forall p \in \mathcal{F}, \ p \notin x \right\} \neq \emptyset?$$

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Theorem (*Berger 1966, Robinson 1971*)

The domino problem is undecidable in dimension $d \geq 2$.

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Theorem *Turing 1937*

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Consider the program Bug such that:

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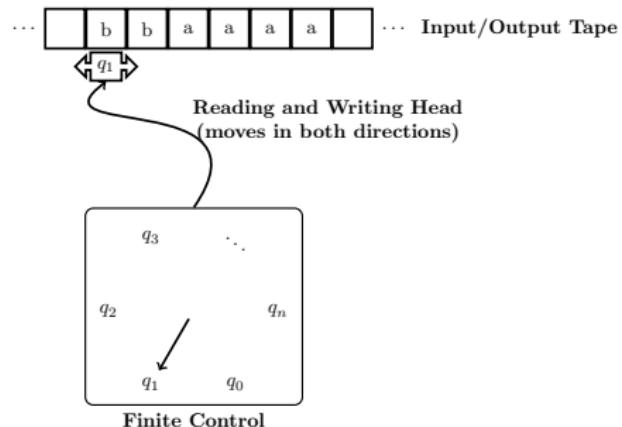
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How runs Bug on the input Bug?

Turing machine

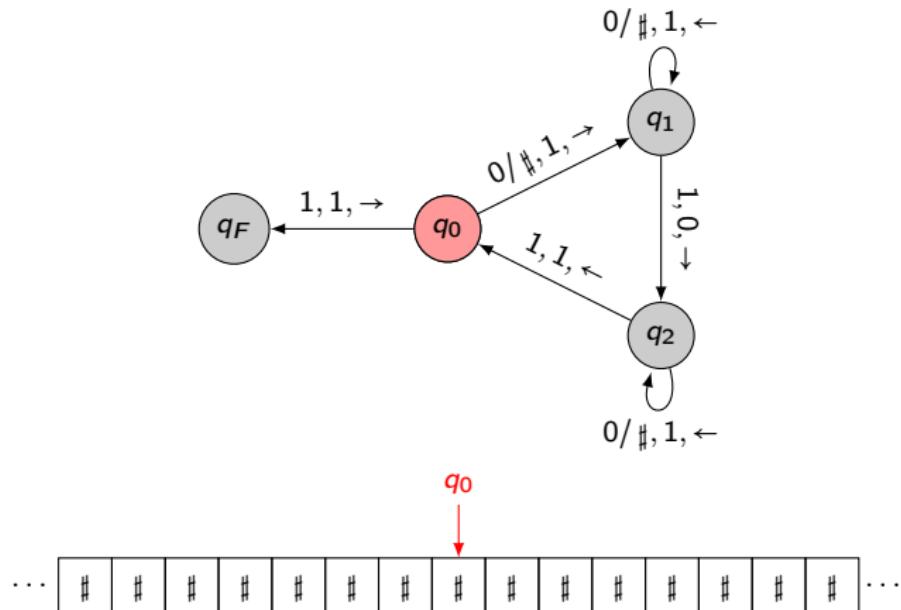
A *Turing machine* is given by
 $\mathcal{M} = (Q, \Gamma, \#, q_0, \delta, Q_F)$:

- Q : fini number of states;
- q_0 : initial state;
- q_F : final state;
- Γ fini alphabet;
- $\# \in \Gamma$ white symbol
- $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{\leftarrow, \rightarrow\}$
transition function.

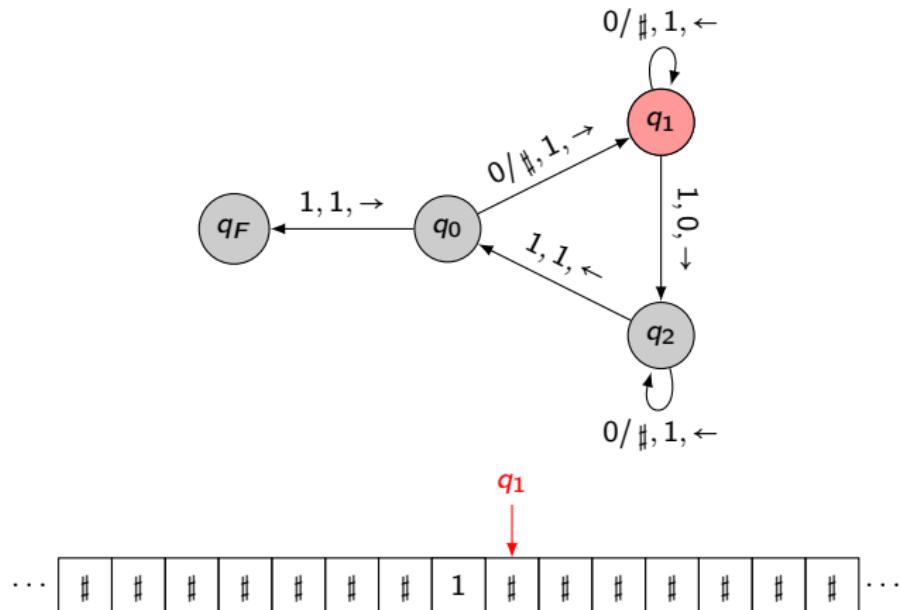


Given a tape $R \in \Gamma^{\mathbb{Z}}$, the head is at the position $i \in \mathbb{Z}$ and state $q \in Q$, the Turing machine compute $\delta(q, R_i) = (q', a, \epsilon)$. It write a at the position i , move the head at the position $i + \epsilon$ and go to the state q' .

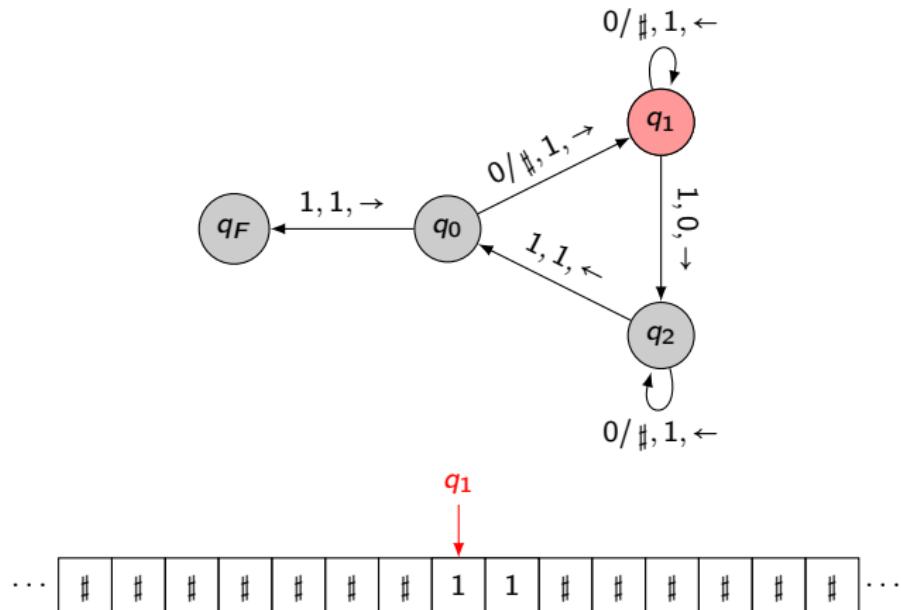
How does a turing machine work?



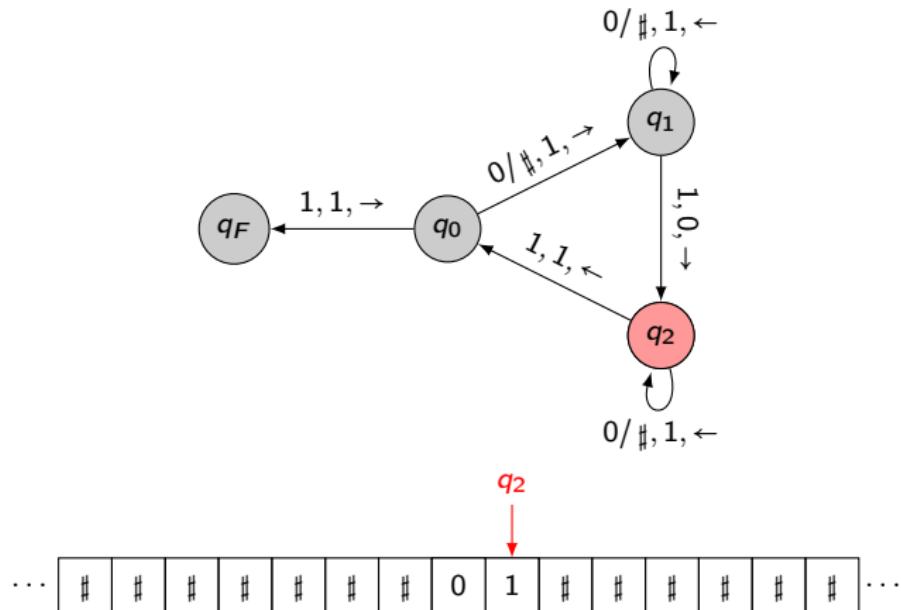
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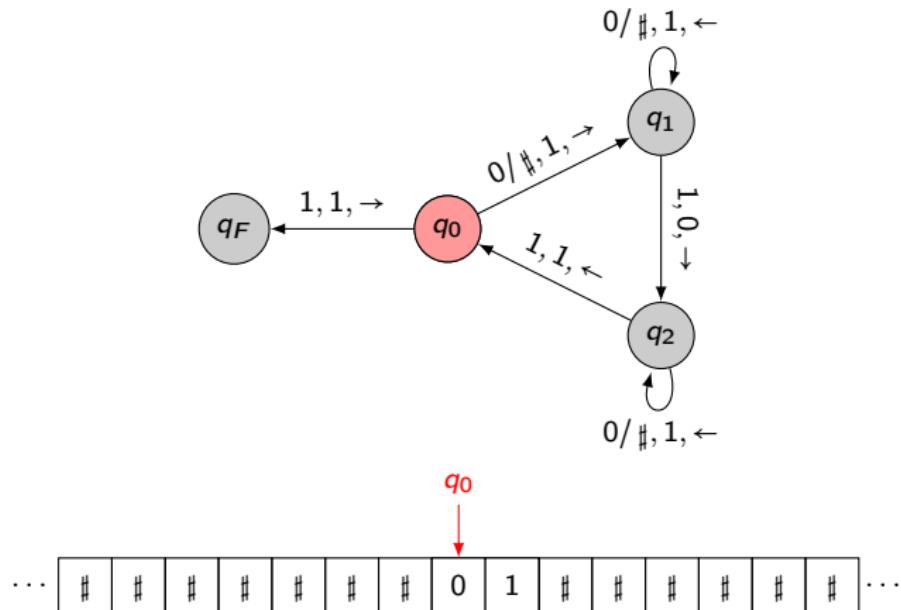
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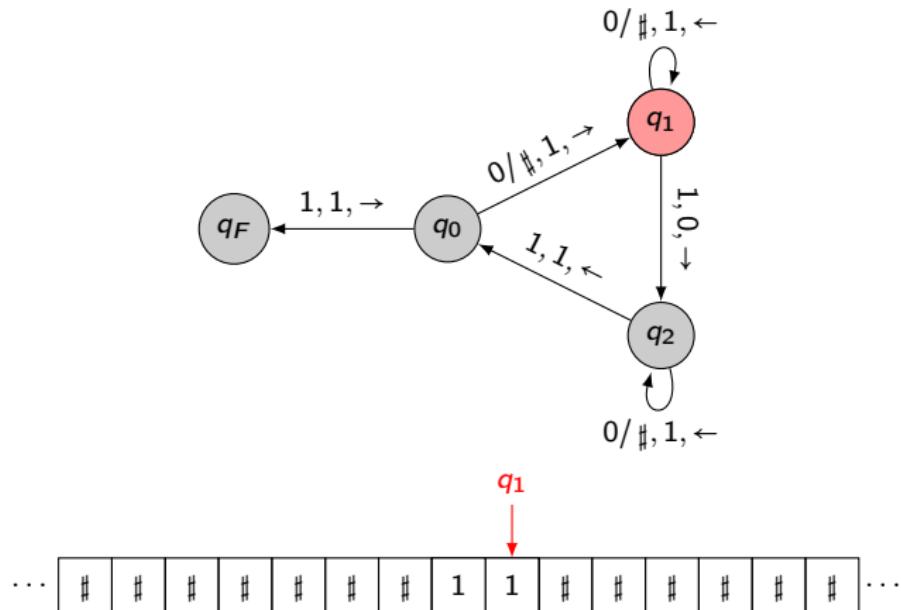
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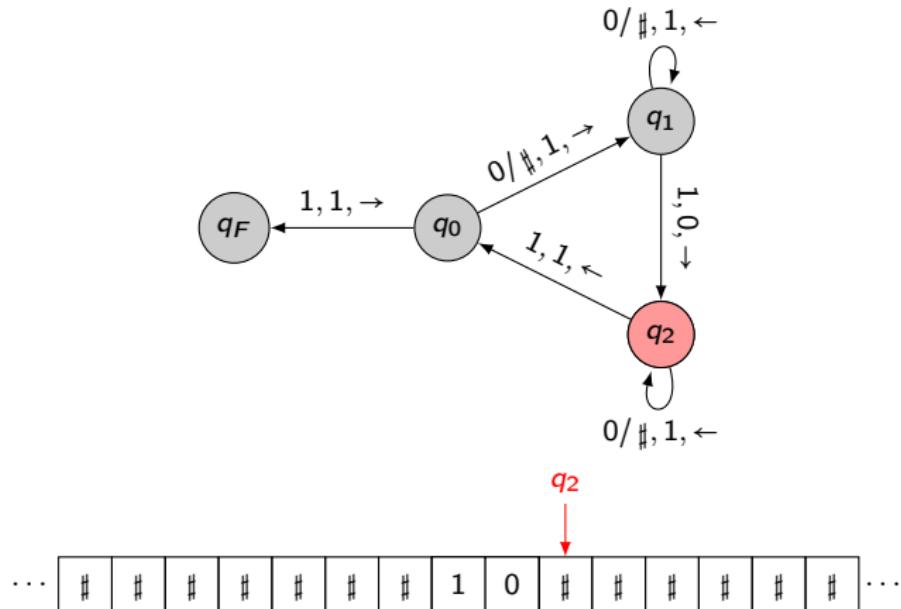
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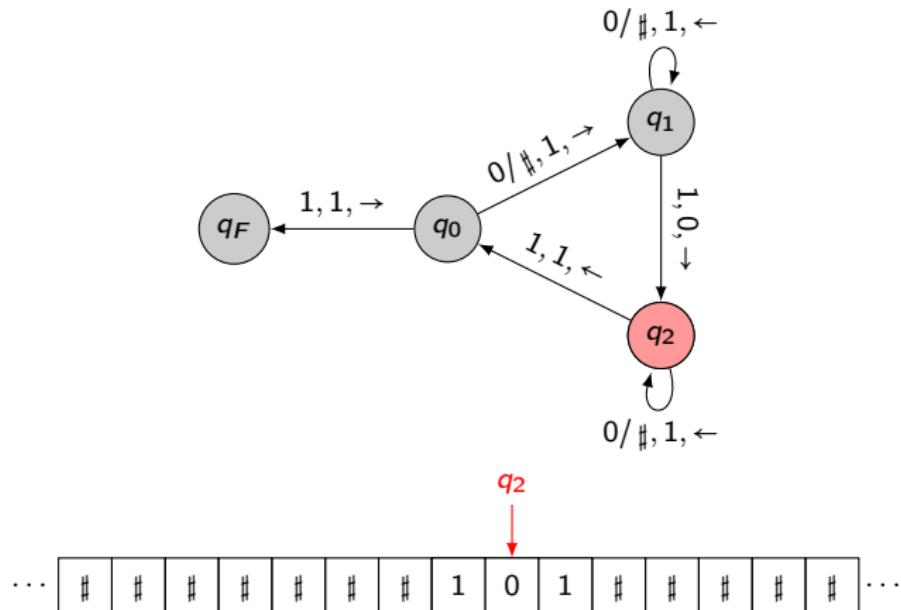
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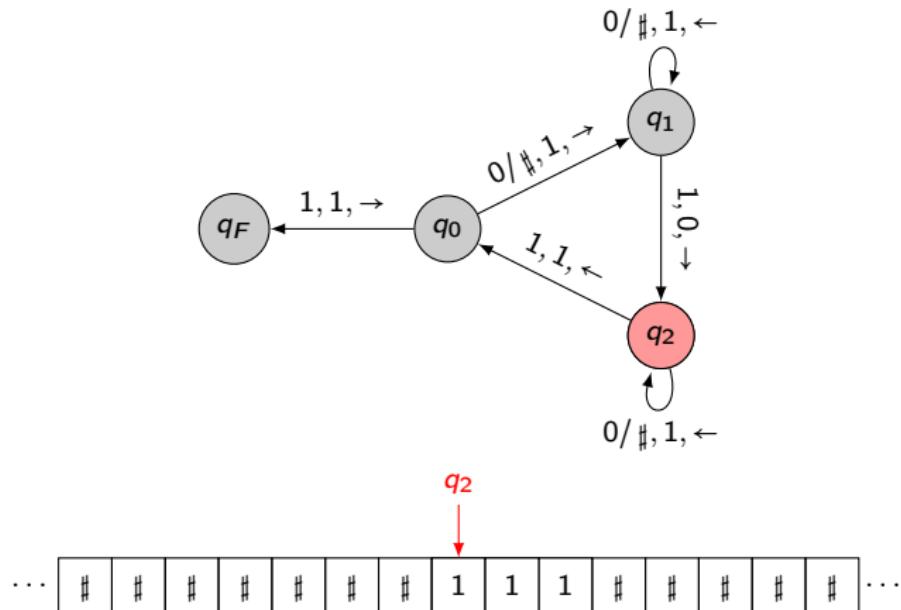
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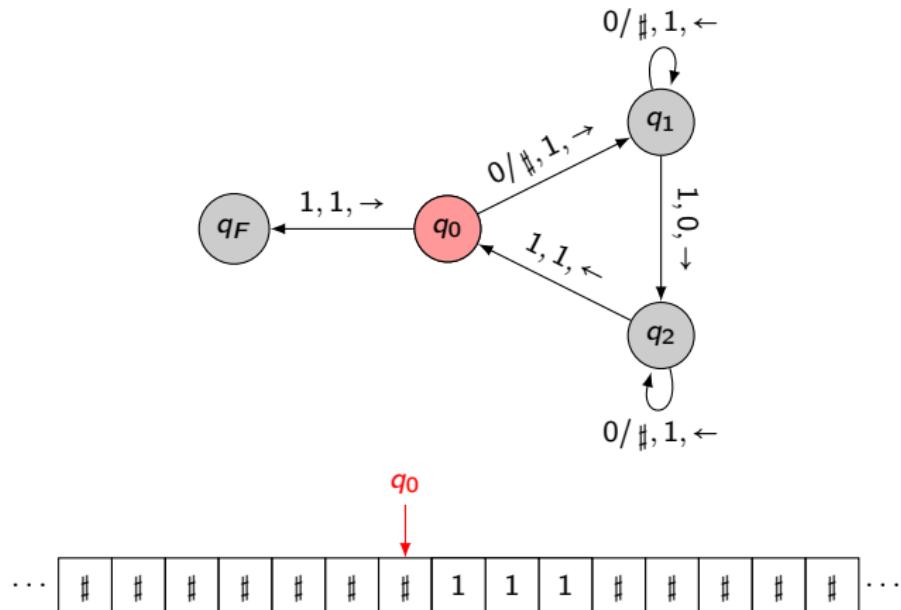
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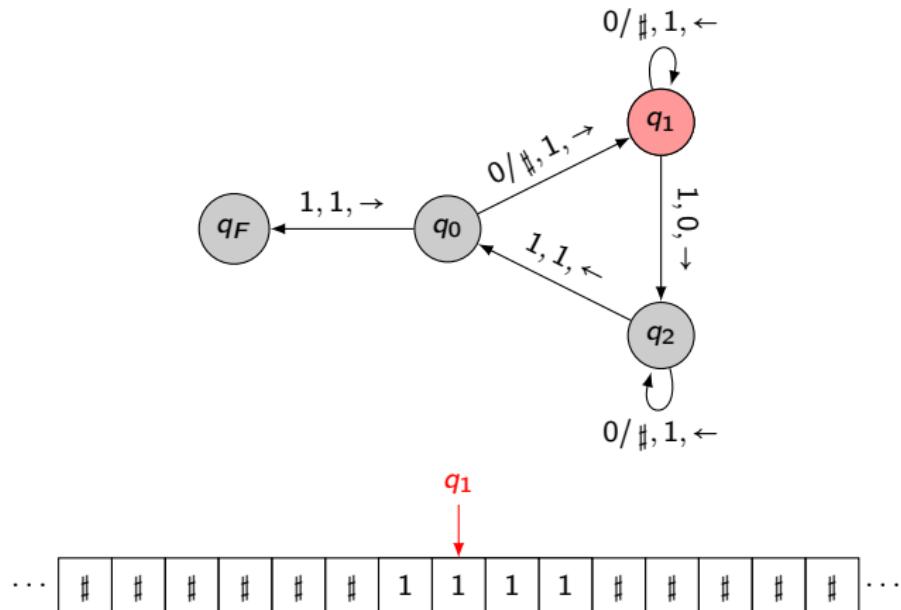
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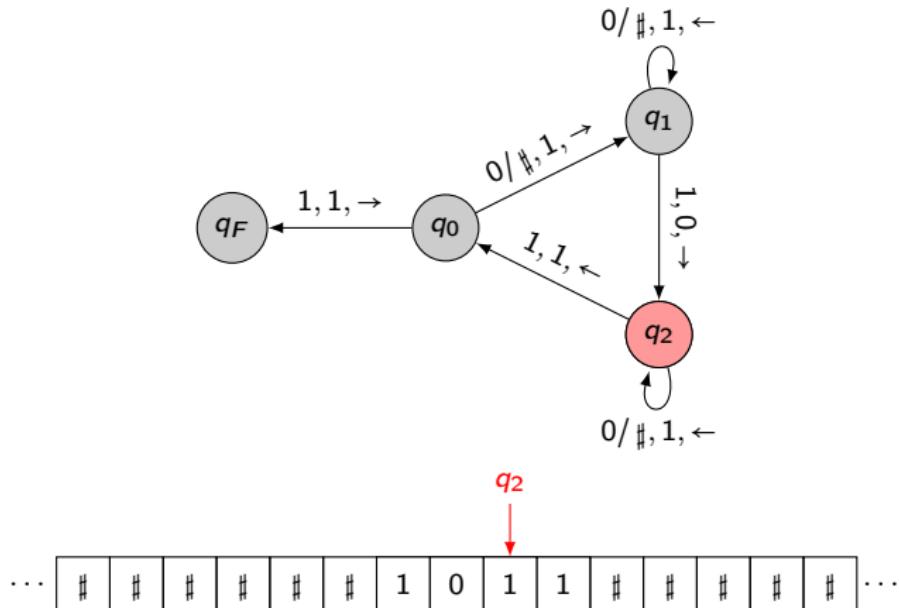
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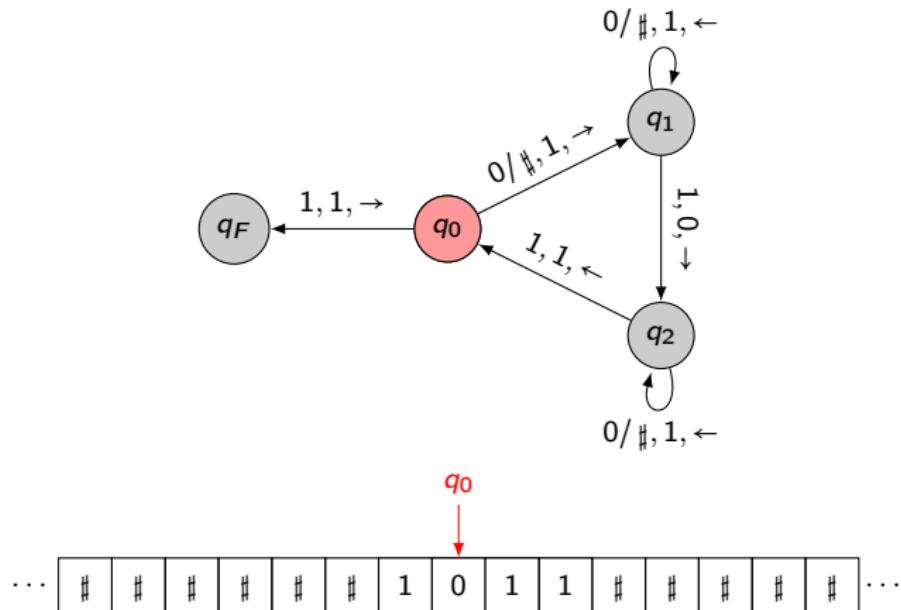
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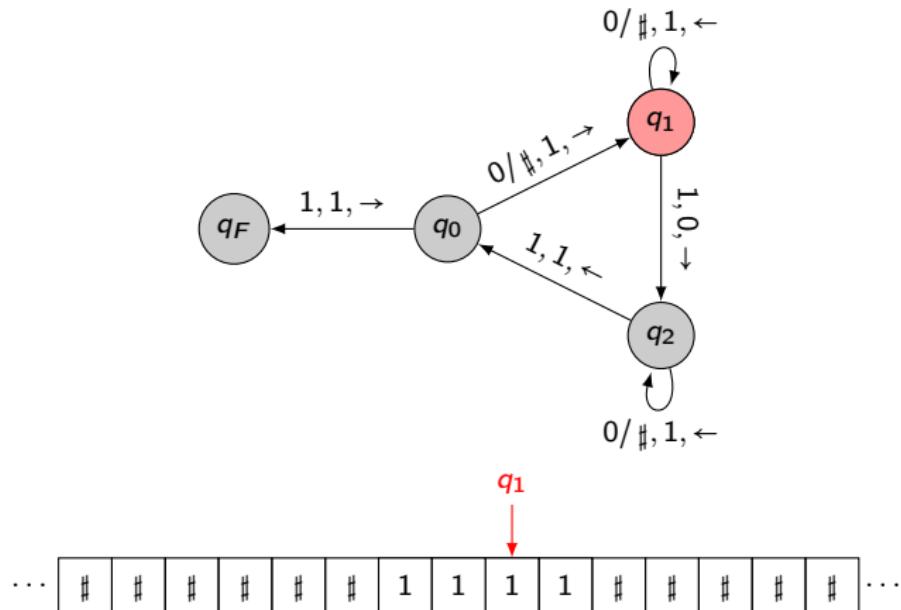
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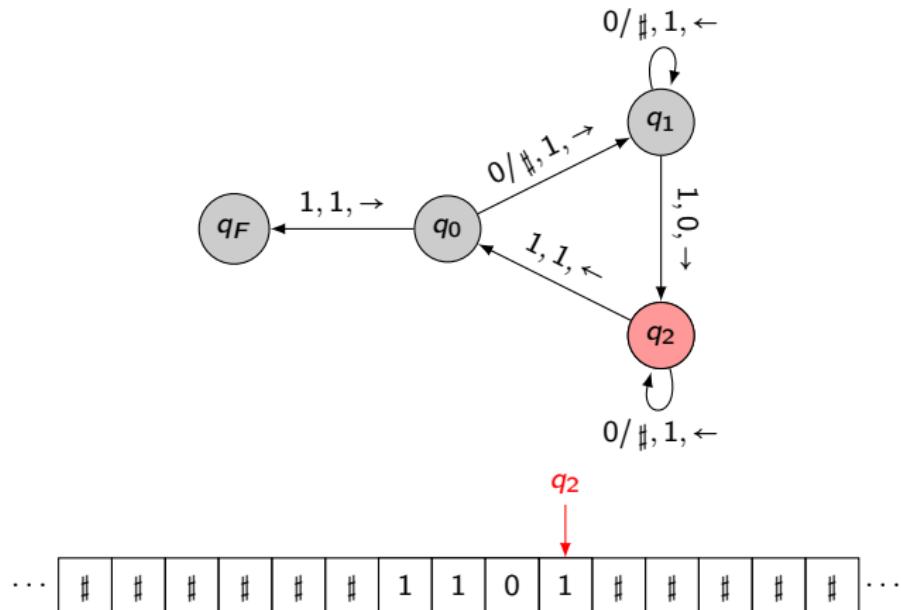
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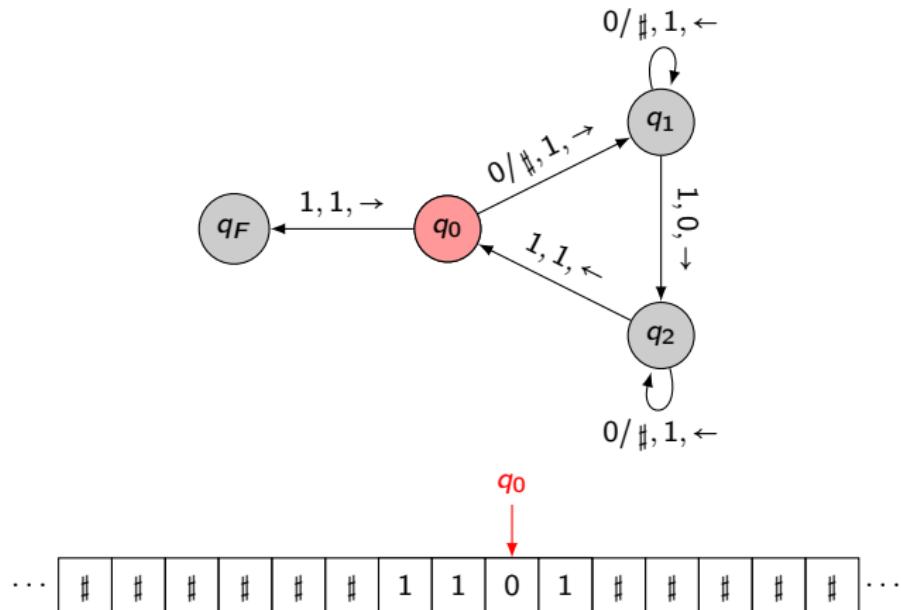
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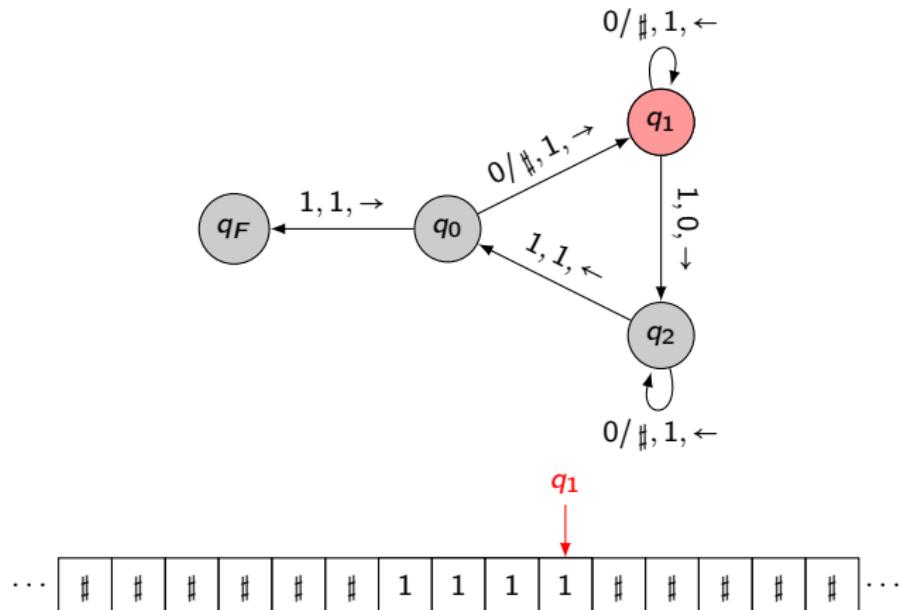
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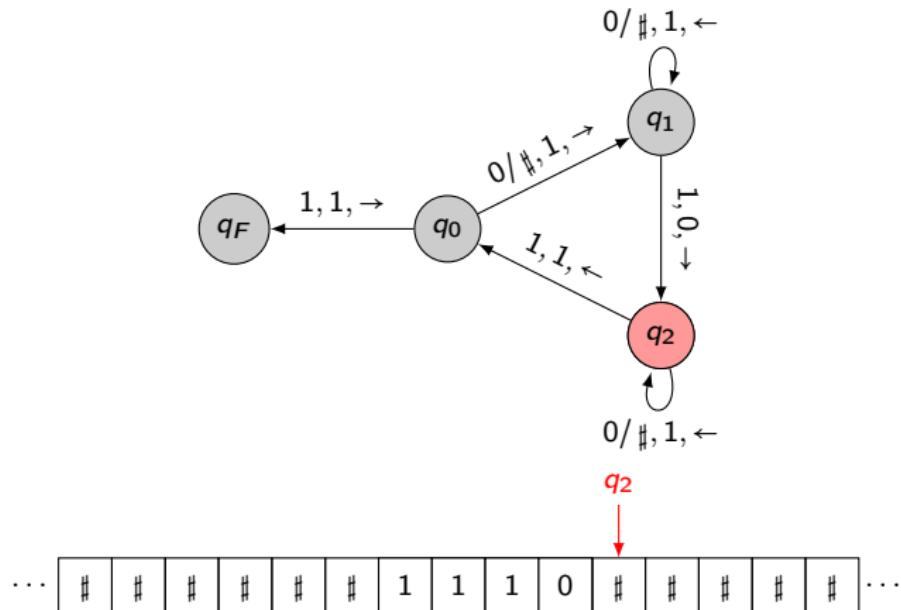
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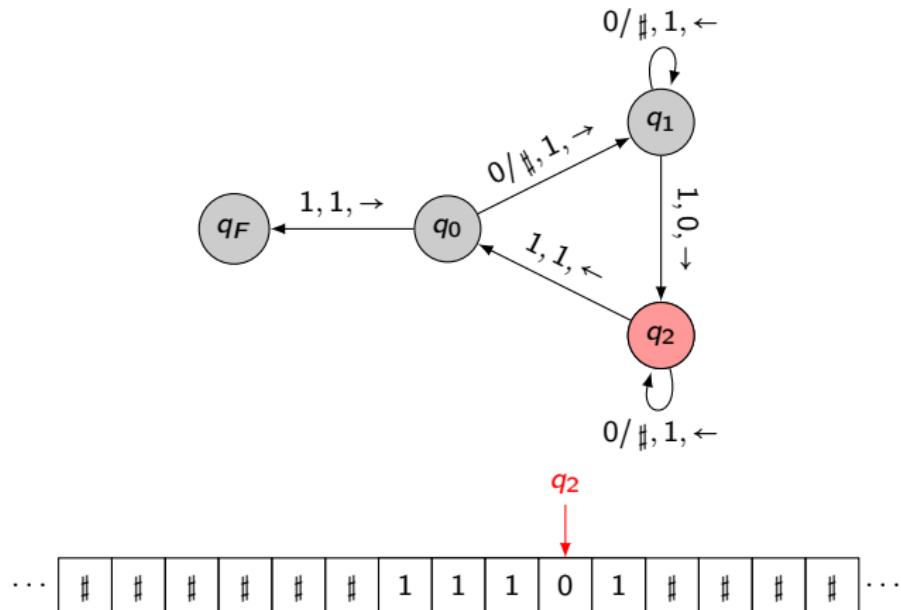
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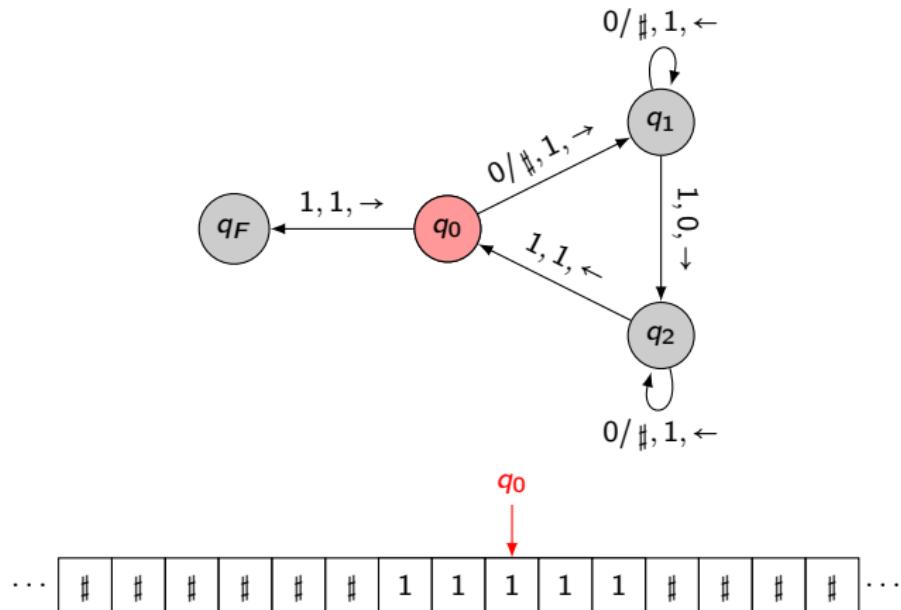
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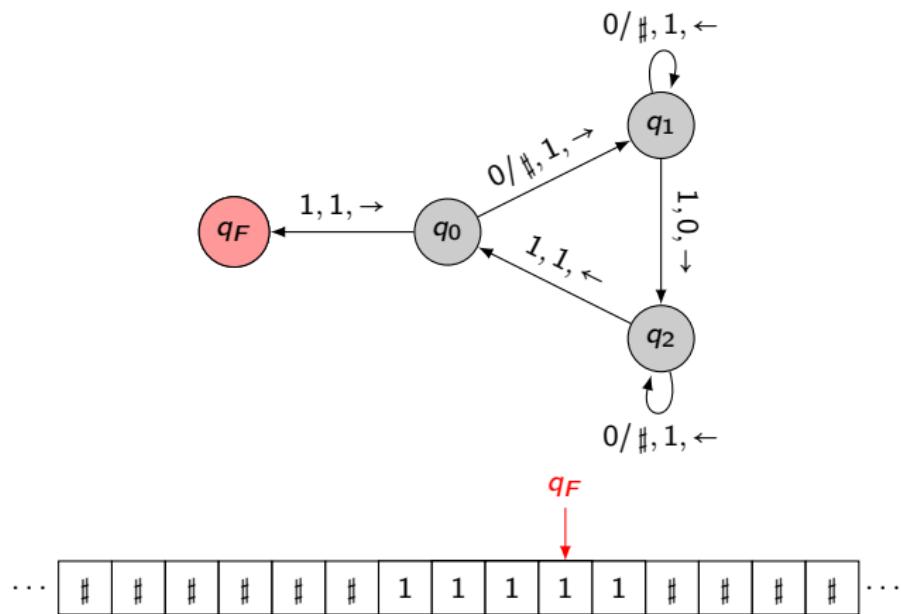
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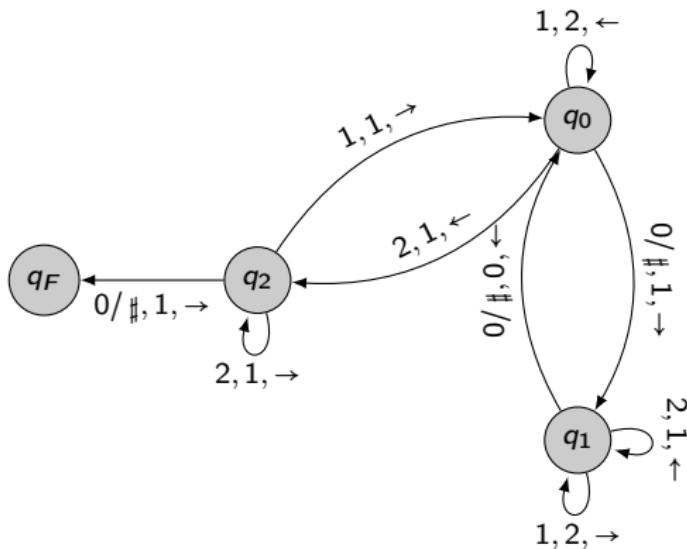


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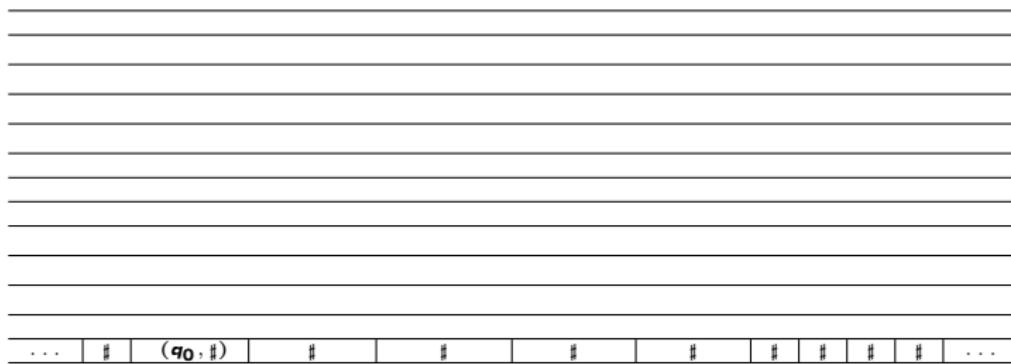
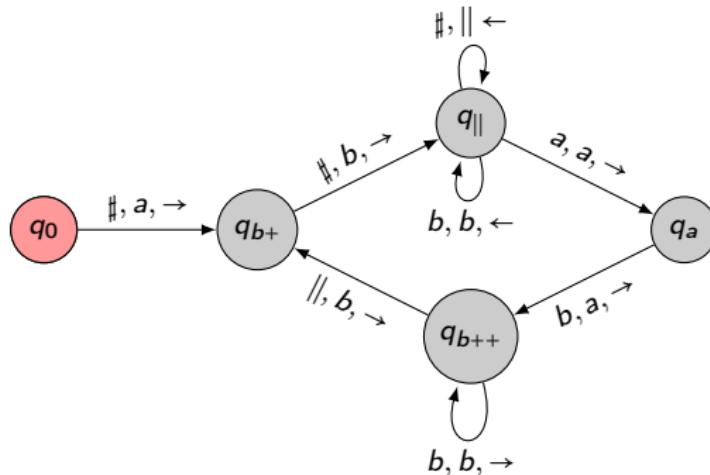
This Turing machine stops after 21 steps of computation.

Combinatory monster

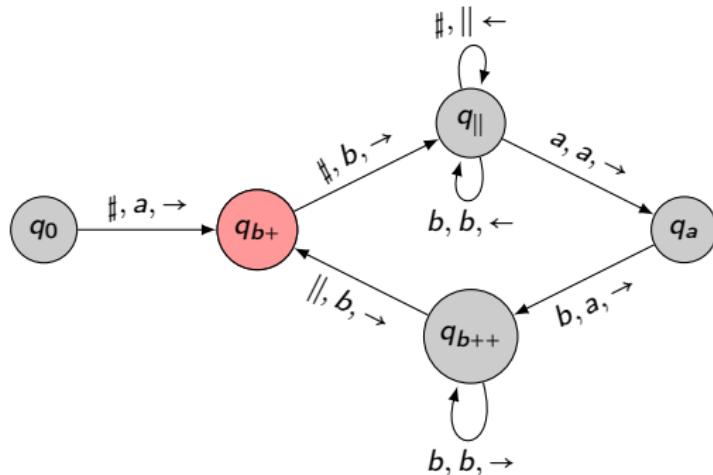


Starting from an empty tape, this Turing machine writes 374×10^6 letters in 119×10^{15} steps of computation.

From the behavior of a Turing machine to SFT

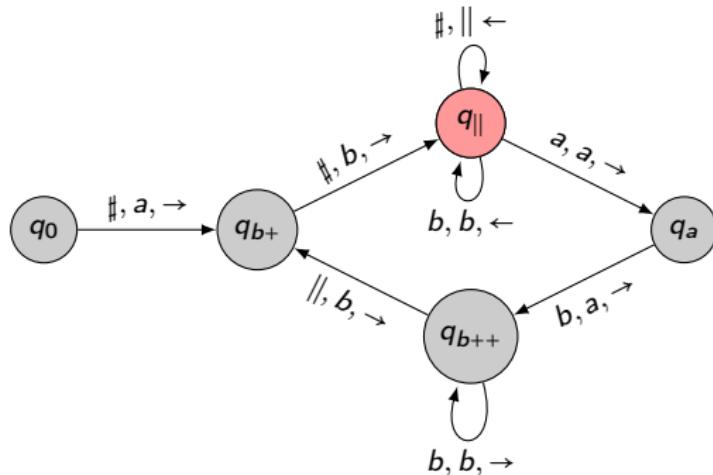


From the behavior of a Turing machine to SFT



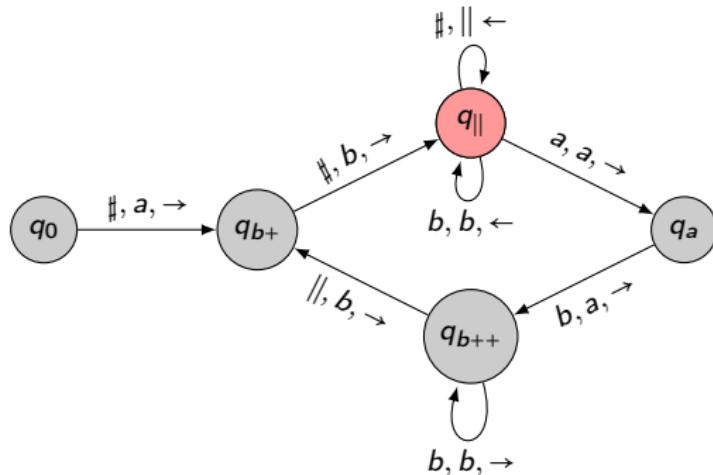
...	$\$$	a	$(q_{b^+}, \$)$	$\$$	$\$$	$\$$	$\$$	$\$$	$\$$	$\$$	$\$$	\dots
...	$\$$	$(q_0, \$)$	$\$$	$\$$	$\$$	$\$$	$\$$	$\$$	$\$$	$\$$	$\$$	\dots

From the behavior of a Turing machine to SFT



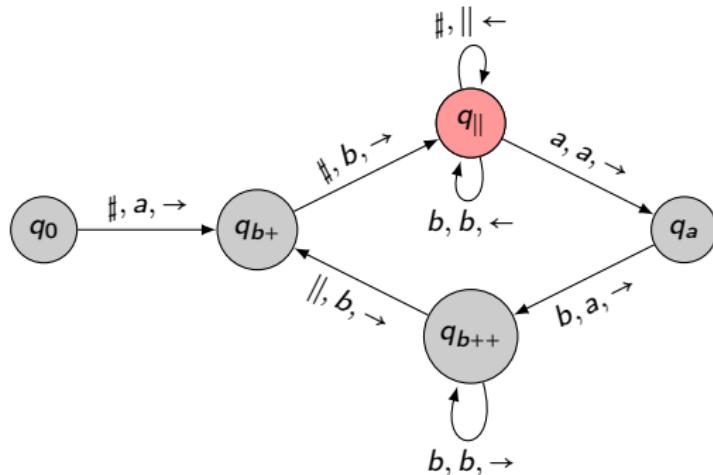
...	#	a	b	$(q_{ }, \#)$	#	#	#	#	#	#	...
...	#	a	$(q_{b+}, \#)$	#	#	#	#	#	#	#	...
...	#	$(q_0, \#)$	#	#	#	#	#	#	#	#	...

From the behavior of a Turing machine to SFT



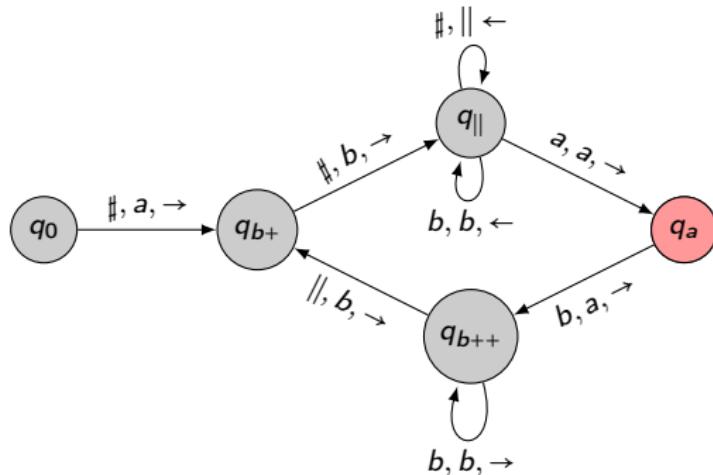
...	#	a	$(q_{ }, b)$	#	#	#	#	#	#	...
...	#	a	b	$(q_{ }, #)$	#	#	#	#	#	...
...	#	a	$(q_{b+}, #)$	#	#	#	#	#	#	...
...	#	$(q_0, #)$	#	#	#	#	#	#	#	...

From the behavior of a Turing machine to SFT



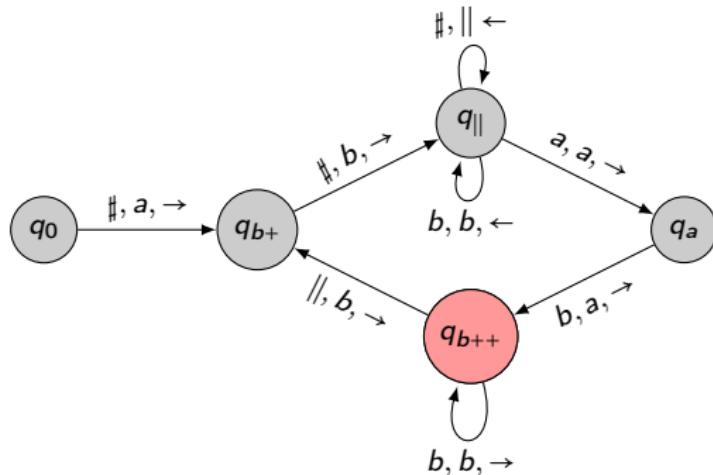
...	\$	$(q_{ }, a)$	b		\$	\$	\$	\$	\$	\$...
...	\$	a	$(q_{ }, b)$		\$	\$	\$	\$	\$	\$...
...	\$	a	b	$(q_{ }, \$)$	\$	\$	\$	\$	\$	\$...
...	\$	a	$(q_{b+}, \$)$	\$	\$	\$	\$	\$	\$	\$...
...	\$	$(q_0, \$)$	\$	\$	\$	\$	\$	\$	\$	\$...

From the behavior of a Turing machine to SFT



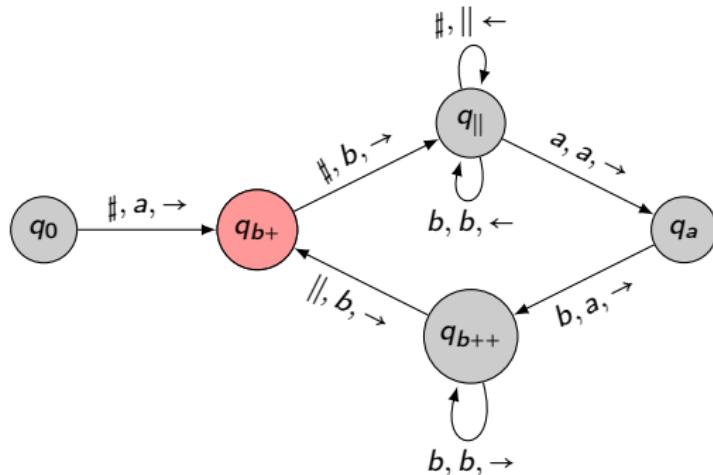
...	#	a	(q_{a+}, b)		#	#	#	#	#	...
...	#	(q_{a+}, a)	b		#	#	#	#	#	...
...	#	a	$(q_{ }, b)$		#	#	#	#	#	...
...	#	a	b	$(q_{ }, \#)$	#	#	#	#	#	...
...	#	a	$(q_{b+}, \#)$	#	#	#	#	#	#	...
...	#	$(q_0, \#)$	#	#	#	#	#	#	#	...

From the behavior of a Turing machine to SFT



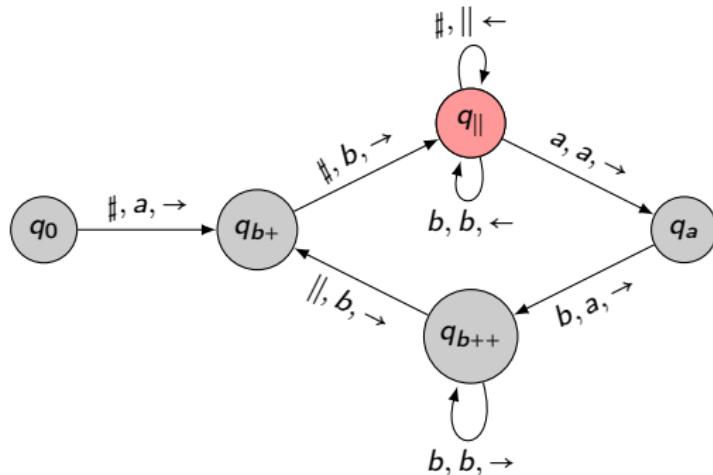
...	#	a	a	$(q_{b++}, \)$	#	#	#	#	#	#	...
...	#	a	(q_{b+}, b)	#	#	#	#	#	#	#	...
...	#	(q_{b+}, a)	b	#	#	#	#	#	#	#	...
...	#	a	$(q_{ }, b)$	#	#	#	#	#	#	#	...
...	#	a	b	$(q_{ }, \)$	#	#	#	#	#	#	...
...	#	a	$(q_{b+}, \)$	#	#	#	#	#	#	#	...
...	#	$(q_0, \)$	#	#	#	#	#	#	#	#	...

From the behavior of a Turing machine to SFT



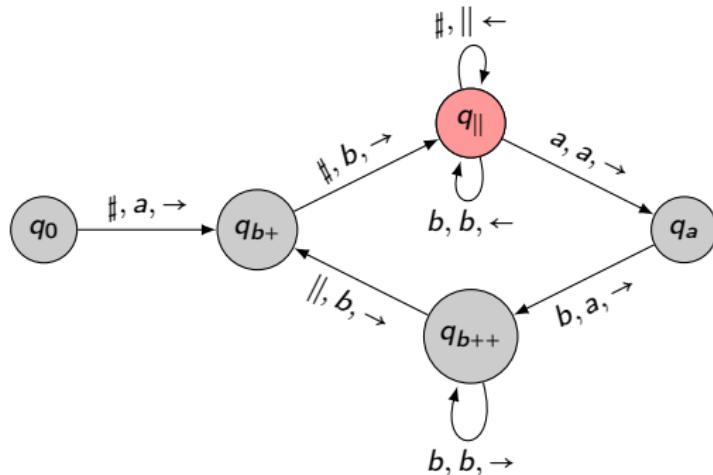
...	¶	a	a	b	$(q_{b++}, \text{¶})$	¶	¶	¶	¶	¶	...
...	¶	a	a	$(q_{b+}, \text{¶})$	¶	¶	¶	¶	¶	¶	...
...	¶	a	(q_{b+}, b)	¶	¶	¶	¶	¶	¶	¶	...
...	¶	(q_0, a)	b	¶	¶	¶	¶	¶	¶	¶	...
...	¶	a	$(q_{ }, b)$	¶	¶	¶	¶	¶	¶	¶	...
...	¶	a	b	$(q_{ }, \text{¶})$	¶	¶	¶	¶	¶	¶	...
...	¶	a	$(q_{b+}, \text{¶})$	¶	¶	¶	¶	¶	¶	¶	...
...	¶	$(q_0, \text{¶})$	¶	¶	¶	¶	¶	¶	¶	¶	...

From the behavior of a Turing machine to SFT



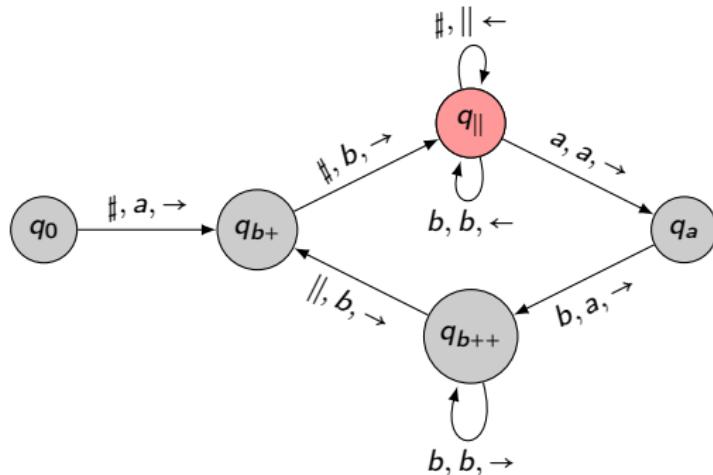
...	¶	a	a	b	b	($q_{ }$, ¶)	¶	¶	¶	¶	...
...	¶	a	a	(q_{b++} , ¶)	¶	¶	¶	¶	¶	¶	...
...	¶	a	(q_{b+} , b)	¶	¶	¶	¶	¶	¶	¶	...
...	¶	(q_0 , a)	b	¶	¶	¶	¶	¶	¶	¶	...
...	¶	a	($q_{ }$, b)	¶	¶	¶	¶	¶	¶	¶	...
...	¶	a	b	($q_{ }$, ¶)	¶	¶	¶	¶	¶	¶	...
...	¶	a	(q_{b+} , ¶)	¶	¶	¶	¶	¶	¶	¶	...
...	¶	(q_0 , ¶)	¶	¶	¶	¶	¶	¶	¶	¶	...

From the behavior of a Turing machine to SFT



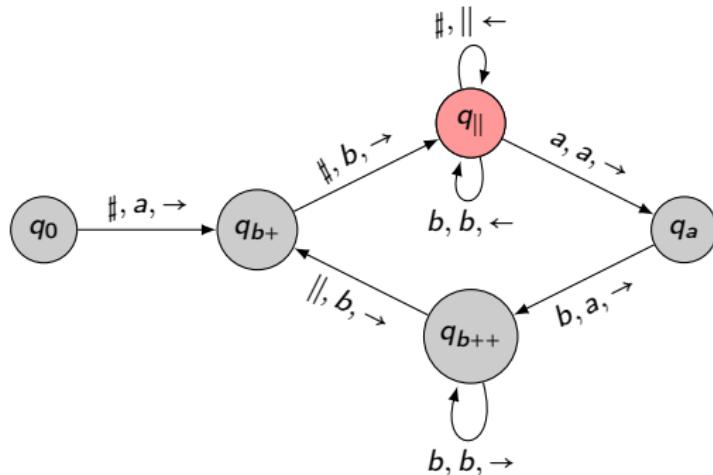
...	¶	a	a	b	(q , b)	¶	¶	¶	¶	...
...	¶	a	a	b	(qb+, ¶)	¶	¶	¶	¶	...
...	¶	a	a	b	(qb++, ¶)	¶	¶	¶	¶	...
...	¶	a	a	(qb++, ¶)	¶	¶	¶	¶	¶	...
...	¶	a	(qb+, b)	¶	¶	¶	¶	¶	¶	...
...	¶	(q , a)	b	¶	¶	¶	¶	¶	¶	...
...	¶	a	(q , b)	¶	¶	¶	¶	¶	¶	...
...	¶	a	b	(q , ¶)	¶	¶	¶	¶	¶	...
...	¶	a	(qb+, ¶)	¶	¶	¶	¶	¶	¶	...
...	¶	(q0, ¶)	¶	¶	¶	¶	¶	¶	¶	...

From the behavior of a Turing machine to SFT



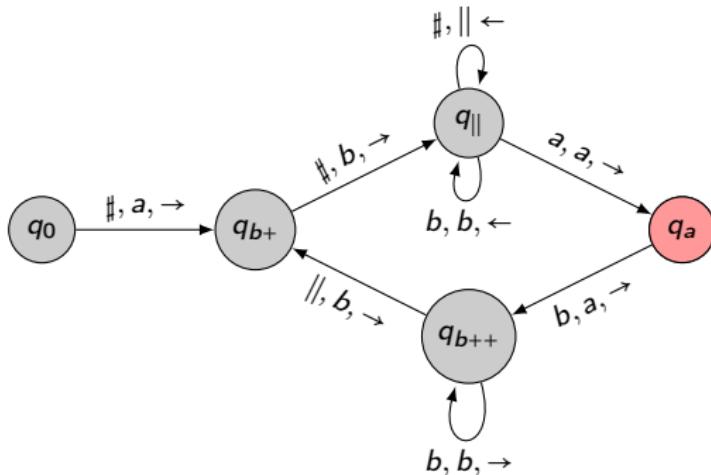
...	#	a	a	$(q_{ }, b)$	b	#	#	#	#	#	...
...	#	a	a	b	$(q_{ }, b)$	#	#	#	#	#	...
...	#	a	a	b	b	$(q_{ }, \#)$	#	#	#	#	...
...	#	a	a	$(q_{b++}, \#)$	#	#	#	#	#	#	...
...	#	a	(q_{b+}, b)	#	#	#	#	#	#	#	...
...	#	$(q_{ }, a)$	b	#	#	#	#	#	#	#	...
...	#	a	$(q_{ }, b)$	#	#	#	#	#	#	#	...
...	#	a	b	$(q_{ }, \#)$	#	#	#	#	#	#	...
...	#	a	$(q_{b+}, \#)$	#	#	#	#	#	#	#	...
...	#	$(q_0, \#)$	#	#	#	#	#	#	#	#	...

From the behavior of a Turing machine to SFT



...	#	a	$(q_{ }, a)$	b	b		#	#	#	#	...
...	#	a	a	$(q_{ }, b)$	b		#	#	#	#	...
...	#	a	a	b	$(q_{ }, b)$		#	#	#	#	...
...	#	a	a	b	b	$(q_{ }, \#)$	#	#	#	#	...
...	#	a	a	$(q_{b++}, \#)$	#	#	#	#	#	#	...
...	#	a	(q_{b+}, b)		#	#	#	#	#	#	...
...	($q_{ }$, a)	b		#	#	#	#	#	#	#	...
...	#	a	$(q_{ }, b)$		#	#	#	#	#	#	...
...	#	a	b	$(q_{ }, \#)$		#	#	#	#	#	...
...	#	a	$(q_{b+}, \#)$	#	#	#	#	#	#	#	...
...	(q_0 , #)	#	#	#	#	#	#	#	#	#	...

From the behavior of a Turing machine to SFT



...	¶	a	a	(q _{a+} , b)	b	¶	¶	¶	¶	...
...	¶	a		(q , a)	b	b	¶	¶	¶	...
...	¶	a	a	(q , b)	b	¶	¶	¶	¶	...
...	¶	a	a		b	(q , b)	¶	¶	¶	...
...	¶	a	a	b	b		(q , ¶)	¶	¶	...
...	¶	a	a		b	(q , ¶)	¶	¶	¶	...
...	¶	a	a	(q _{b++} , ¶)		¶	¶	¶	¶	...
...	¶	a	(q _{a+} , b)		¶	¶	¶	¶	¶	...
...	¶	(q , a)	b		¶	¶	¶	¶	¶	...
...	¶	a	(q , b)		¶	¶	¶	¶	¶	...
...	¶	a	b	(q , ¶)	¶	¶	¶	¶	¶	...
...	¶	a	(q _{b++} , ¶)	¶	¶	¶	¶	¶	¶	...
...	¶	(q ₀ , ¶)	¶	¶	¶	¶	¶	¶	¶	...

From the behavior of a Turing machine to SFT

Consider the Turing machine \mathcal{M}_{ex} which enumerates the language $\{a^n b^n : n \in \mathbb{N}\}$.

...
#	a	a	a	a	b	(q _{bb} , b)		#	#		
a	a	a	a	(q _a , b)	b	b		#	#		
a	a	a	(q , a)	b	b	b		#	#		
a	a	a	(q , b)	b	b	b		#	#		
a	a	a	b	(q , b)	b	b		#	#		
a	a	a	b	b	b	(q , b)		#	#		
a	a	a	b	b	b	(q , #)	#	#	#		
a	a	a	b	(q _{bb} ,)	#	#	#	#	#		
a	a	a	(q _{bb} , b)	#	#	#	#	#	#		
a	(q _a , b)	b	b		#	#	#	#	#		
a	(q , a)	b	b		#	#	#	#	#		
a	a	(q , b)	b		#	#	#	#	#		
a	a	b	(q , b)		#	#	#	#	#		
a	a	b	(q , #)	#	#	#	#	#	#		
a	a	b	(q _b , #)	#	#	#	#	#	#		
a	a	(q _{bb} ,)	#	#	#	#	#	#	#		
a	(q _a , b)		#	#	#	#	#	#	#		
(q , a)	b		#	#	#	#	#	#	#		
a	(q , b)		#	#	#	#	#	#	#		
a	b	(q , #)	#	#	#	#	#	#	#		
a	(q _b , #)	#	#	#	#	#	#	#	#		
(q ₀ , #)	#	#	#	#	#	#	#	#	#		

How code this space-time diagram with local rules?

From the behavior of a Turing machine to SFT

Consider the Turing machine \mathcal{M}_{ex} which enumerates the language $\{a^n b^n : n \in \mathbb{N}\}$.

...
#	a	a	a	a	b	(q _{bb} , b)		#	#		
a	a	a	a	(q _a , b)	b	b		#	#		
a	a	a	(q , a)	b	b	b		#	#		
a	a	a	(q , b)	b	b	b		#	#		
a	a	a	b	(q , b)	b	b		#	#		
a	a	a	b	b	(q , b)	b		#	#		
a	a	a	b	b	b	b	(q ,)	#	#		
a	a	a	b	(q _{bb} , b)		#	#	#	#		
a	a	(q _a , b)	b		#	#	#	#	#		
a	(q , a)	b	b		#	#	#	#	#		
a	a	(q , b)	b		#	#	#	#	#		
a	a	b	(q , b)		#	#	#	#	#		
a	a	b	(q , b)	b	(q , #)	#	#	#	#		
a	a	b	(q _{bb} ,)	#	#	#	#	#	#		
a	(q _a , b)		#	#	#	#	#	#	#		
(q , a)	b		#	#	#	#	#	#	#		
a	(q , b)		#	#	#	#	#	#	#		
a	b	(q , #)	#	#	#	#	#	#	#		
a	(q _b , #)	#	#	#	#	#	#	#	#		
(q ₀ , #)	#	#	#	#	#	#	#	#	#		

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Consider the Turing machine \mathcal{M}_{ex} which enumerates the language $\{a^n b^n : n \in \mathbb{N}\}$.

...
#	a	a	a	a	b	(q _{bb} , b)		#	#		
a	a	a	a	a	(q _a , b)	b		#	#		
a	a	a	a	(q , a)	b	b		#	#		
a	a	a	a	(q , b)	b	b		#	#		
a	a	a	b	(q , b)	b	b		#	#		
a	a	a	b	(q , b)	b	b		#	#		
a	a	a	b	(q , b)	b	b		#	#		
a	a	a	b	(q , b)	b	b		#	#		
a	a	a	b	(q _{bb} , b)	b	b		#	#		
a	a	(q _a , b)	b								
a	(q , a)	b	b								
a	a	(q , b)	b								
a	a	b	(q , b)								
a	a	b	(q , b)								
a	a	b	(q _{bb} , b)								
a	(q _a , b)										
(q , a)	b										
a	(q , b)										
a	b	(q , b)									
a	(q _{bb} , b)										
(q ₀ , #)											

How code this space-time diagram with local rules?

From the behavior of a Turing machine to SFT

Let $\mathcal{A}_M = \Gamma \cup (Q \times \Gamma)$.

A Turing machine M give a set P_M of allowed 3×2 -patterns:

- if $x, y, z \in \Gamma$ (no head in the neighborhood), we allow :

x	y	z
x	y	z

- if there is a head in the neighborhood, for example $\delta(q_1, x) = (q_2, y, \leftarrow)$ is coded by:

v	w	(q_2, z)
v	w	z

w	(q_2, z)	y
w	z	(q_1, x)

(q_2, z)	y	z'
z	(q_1, x)	z'

y	z'	z''
(q_1, x)	z'	z''

- if q_F is a final state, the machine stops computation:

z	(q_F, x)	z'
z	(q_F, x)	z'

Consider the SFT $T_M = T(\mathcal{A}_M, 2, \mathcal{A}_M^{[0,2] \times [0,1]} \setminus P_M)$.

The SFT T_M contains every space time diagram produced by M but also no well initialized configurations.

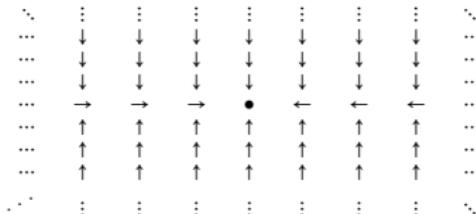
The completion problem

Completion problem

Given a SFT \mathbf{T} and a pattern p , it is possible to find $x \in \mathbf{T}$ such that $p \sqsubset x$?

Let $\mathcal{A}_\bullet = \{\bullet, \leftarrow, \rightarrow, \uparrow, \downarrow\}$ and consider the SFT:

$$\mathbf{T}(\mathcal{A}_\bullet, 2, \mathcal{F}_\bullet) = \overline{\mathcal{O}(x)}$$
 where $x =$



Let $\mathbf{T} = \mathbf{T}(\mathcal{A}_\bullet \times \mathcal{A}_M, 2, \mathcal{F}_\bullet \times \mathcal{F}_M \cup \mathcal{F}_{Syncro} \cup \{q_F\})$ where \mathcal{F}_{Syncro} synchronise:

- • with $(q_0, \#)$,
- \uparrow, \rightarrow et \leftarrow avec $\#$

$(\bullet, (q_0, \#))$ can be completed in a configuration of $\mathbf{T} \iff M$ does not halt

Theorem (Wang 1961)

The completion problem is undecidable.

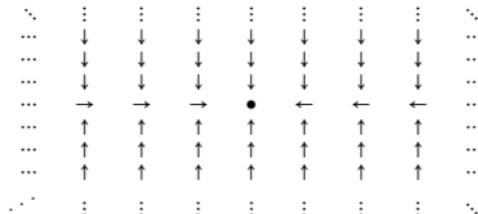
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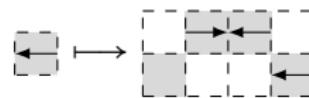
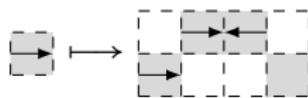
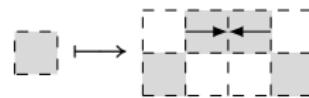
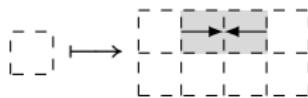
Theorem (Wang 1961)

The completion problem is undecidable.

There is no links with the domino problem: by compacity there is no subshift such that where a tile appear exactly one times.

How to generate zones of computation?

We start from a result of *Mozes-89* (every 2D substitution is sofic) and the following substitution :



How the substitutive subshift obtained can describe computation zones?

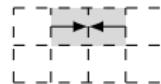
Description of the zones of computation



white square communication tile

grey square computation tile

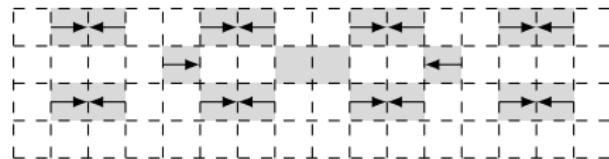
Description of the zones of computation



white square = communication tile

gray square = computation tile

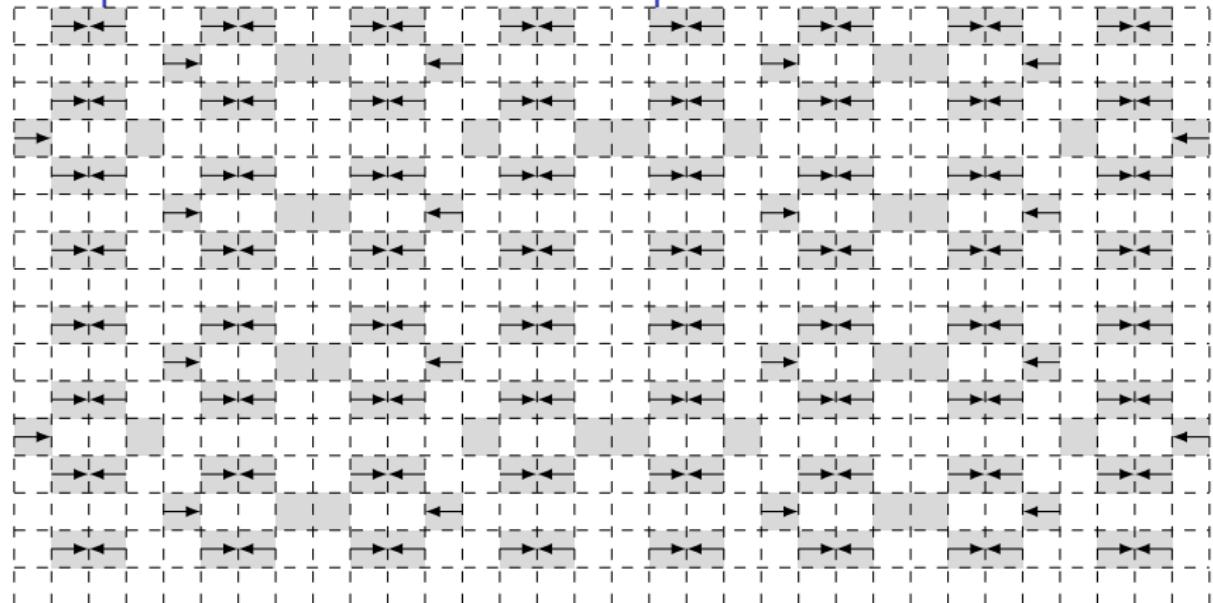
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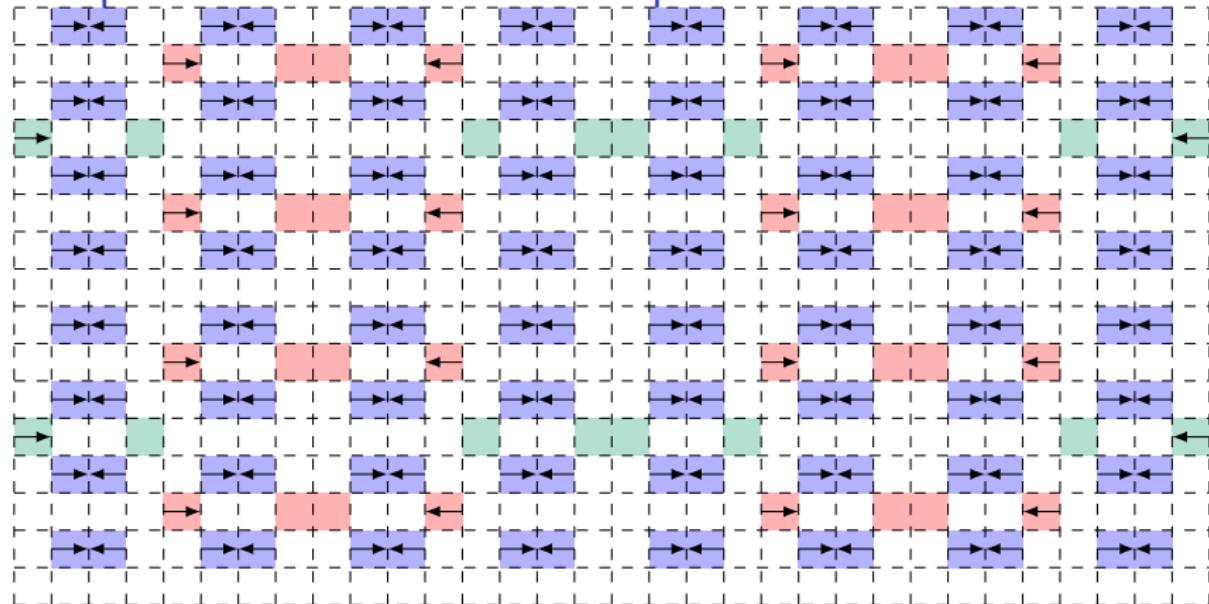
Description of the zones of computation



□ communication tile

■ computation tile

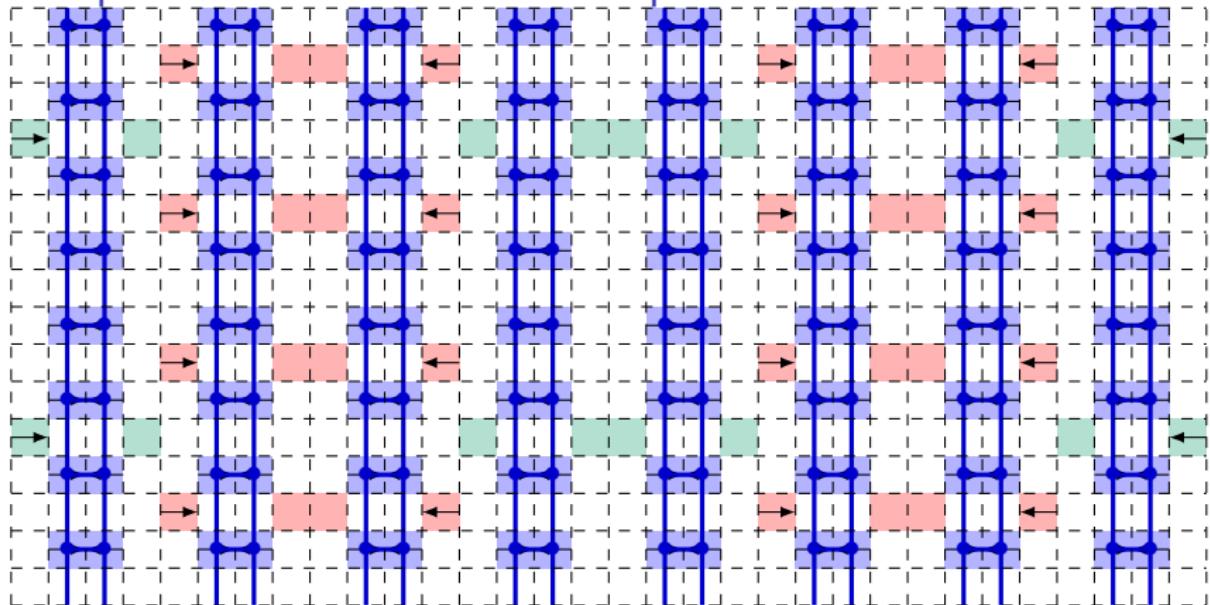
Description of the zones of computation



□ communication tile

■ computation tile

Description of the zones of computation

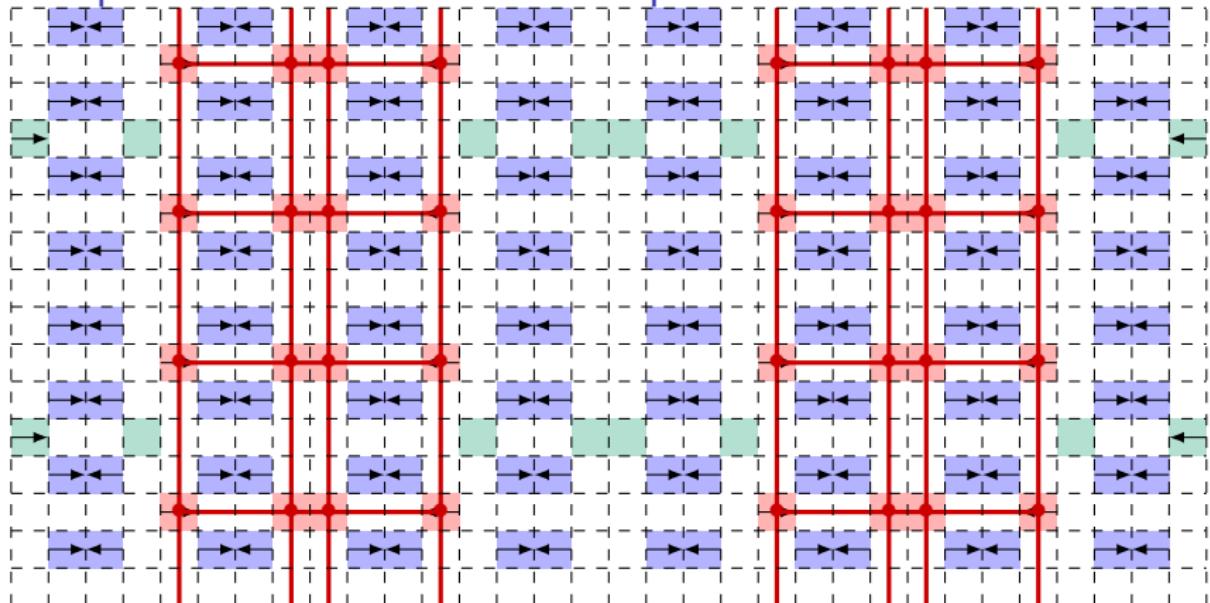


Properties of a fractionated strip of level n :

- communication tile
- computation tile

- the width is 2^n ,
- one line of computation every 2^n

Description of the zones of computation

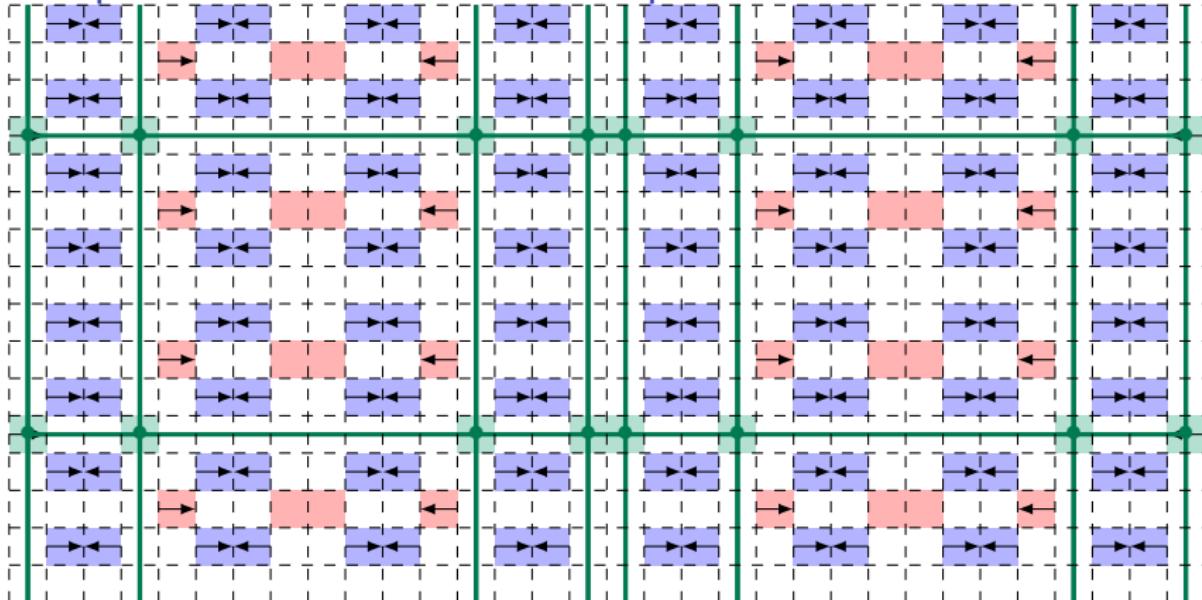


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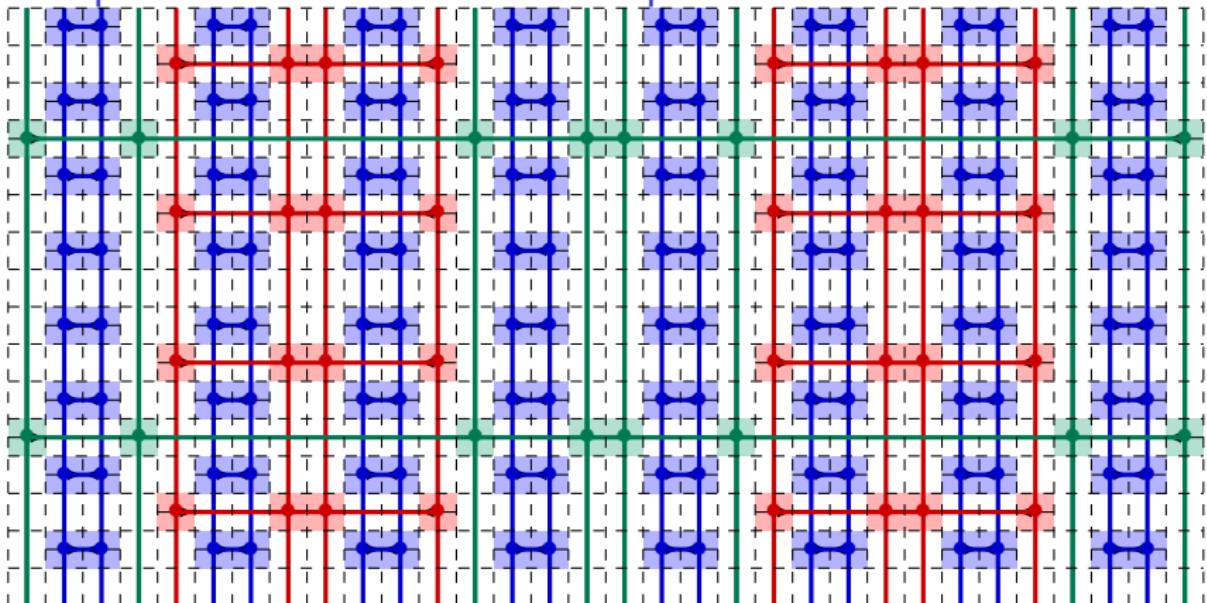


Properties of a fractionated strip of level n :

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Description of the zones of computation



Properties of a fractionated strip of level n :

communication tile

computation tile

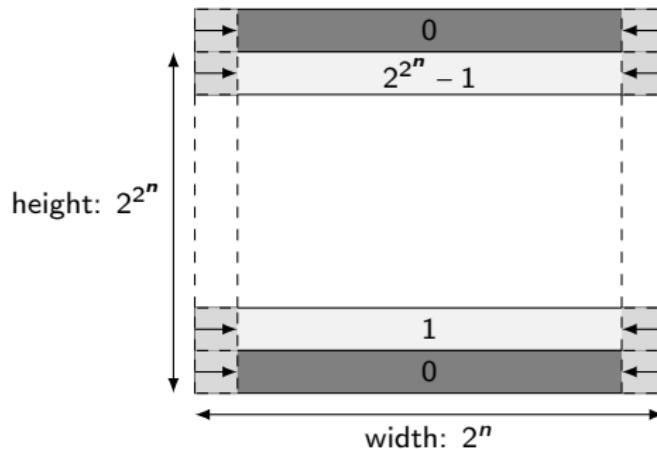
- the width is 2^n ,
- one line of computation every 2^n
- communication lines are not superposed.

How initialize computation in infinite strips?

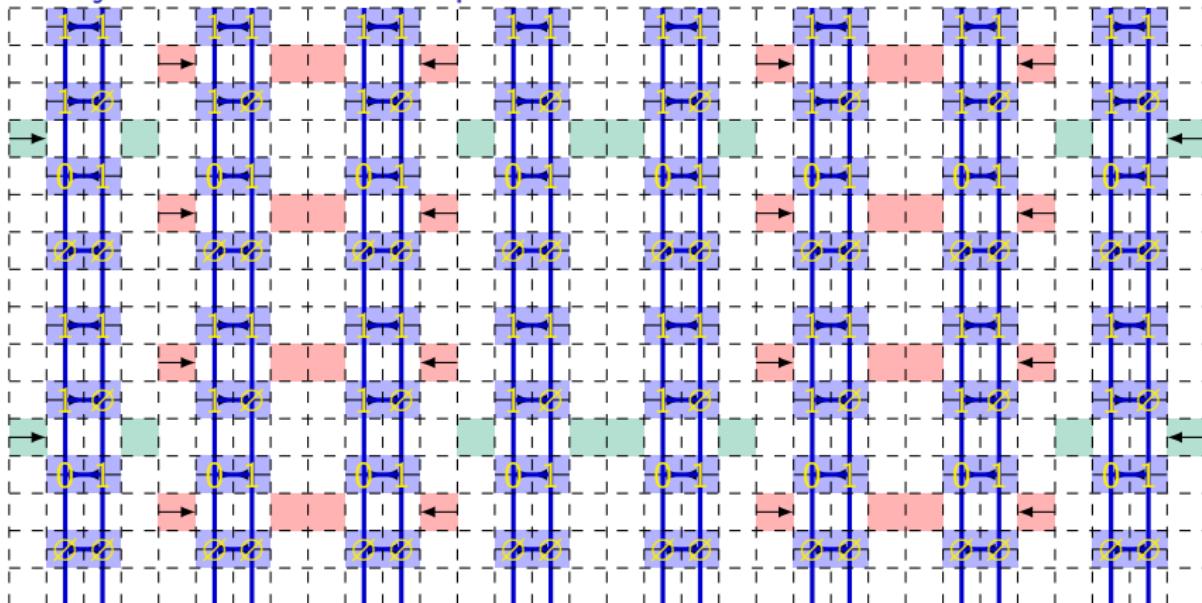
Use a binary counter as a **clock**:

- every computation tile contains 0 or 1;
- finite automaton code addition by 1.

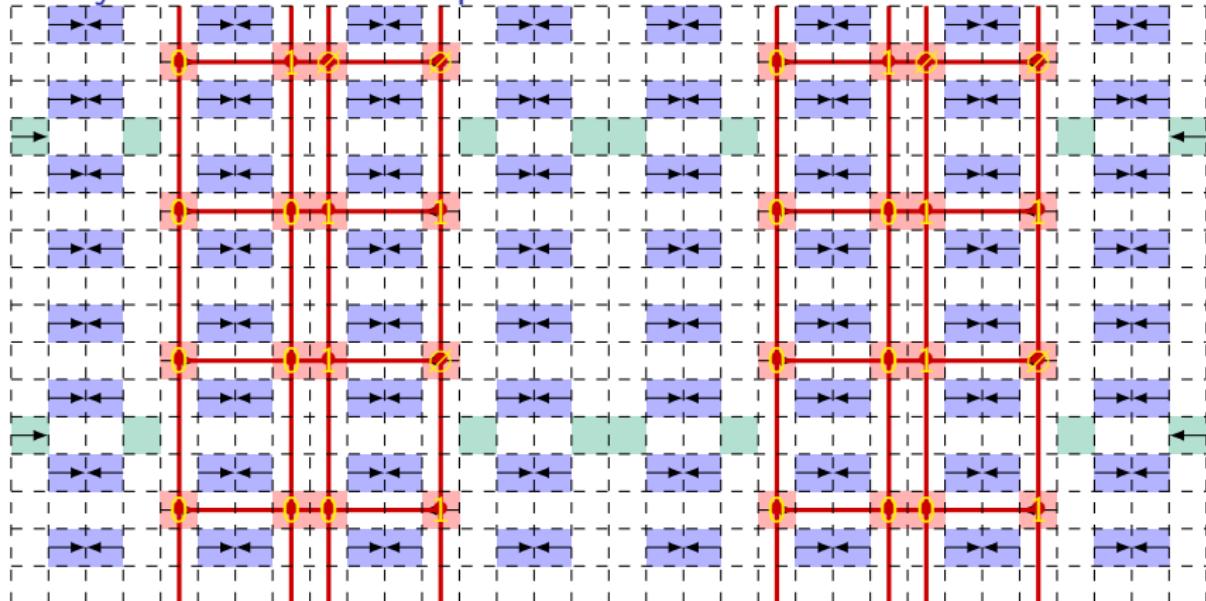
Strip of order n :



Binary counter for strips of level 1



Binary counter for strips of level 2

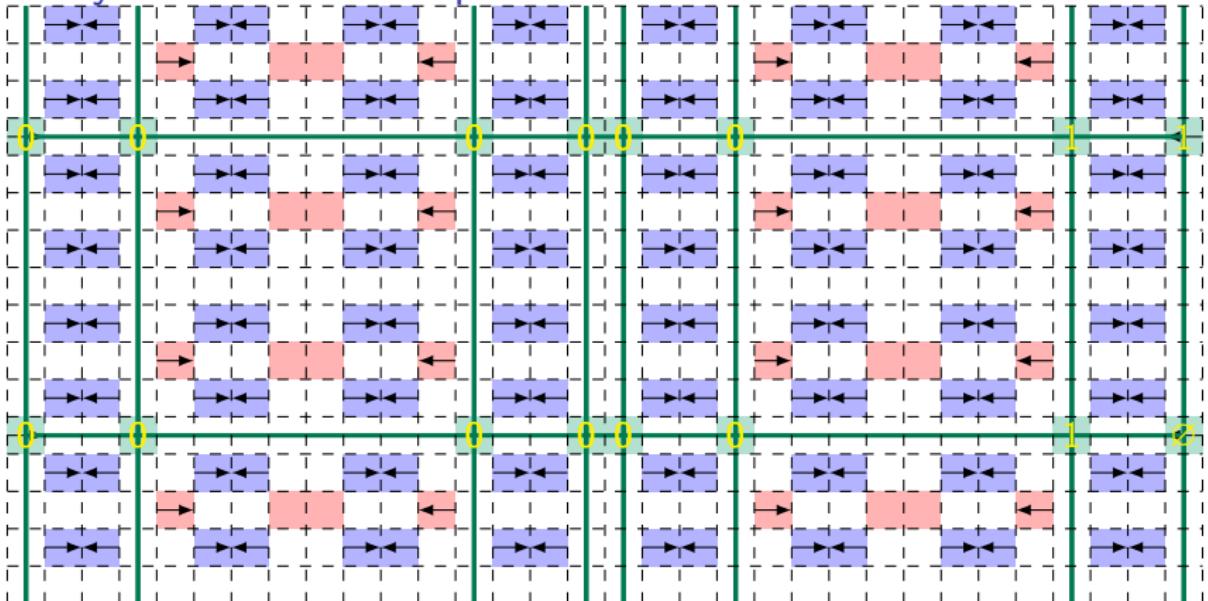


□ communication tile

■ computation tile

⋮	⋮	⋮	⋮
1	0	0	1
1	∅	∅	∅
0	1	1	1
0	1	1	∅
0	1	0	1
0	1	∅	∅
0	0	1	1
0	0	1	∅
0	0	0	1
∅	∅	∅	∅
⋮	⋮	⋮	⋮

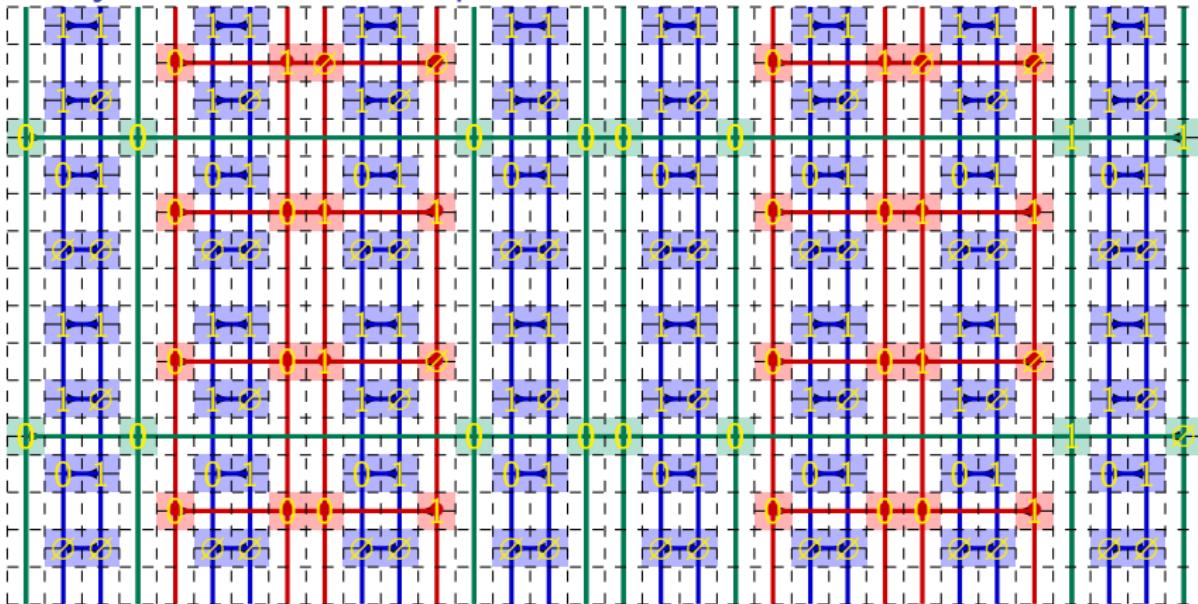
Binary counter for strips of level 3



communication tile

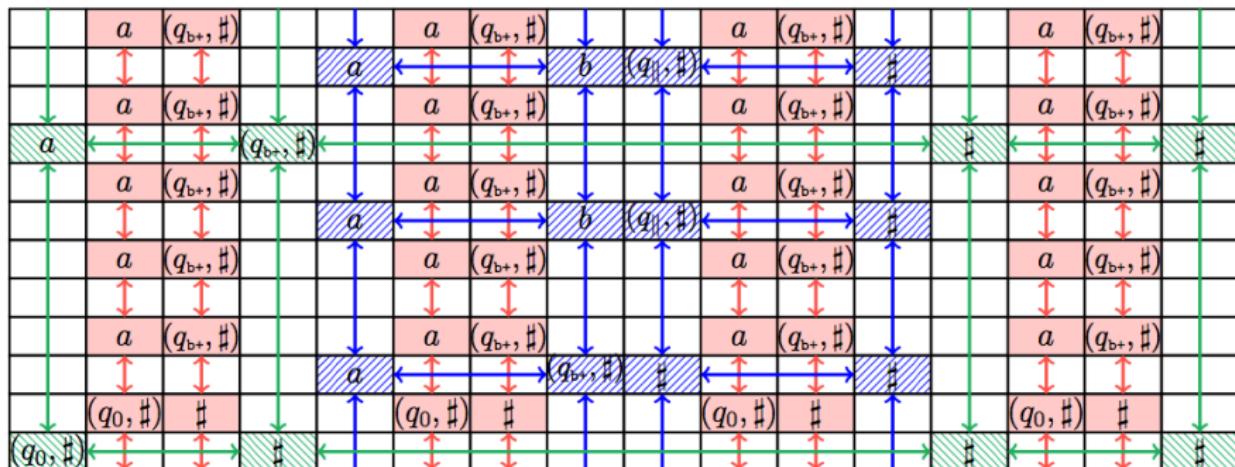
computation tile

Binary counter for strips



T_{Grid} : SFT where each $x \in T_{\text{Grid}}$ contains strips of level n initialized periodically by a clock for all $n \in \mathbb{N}$.

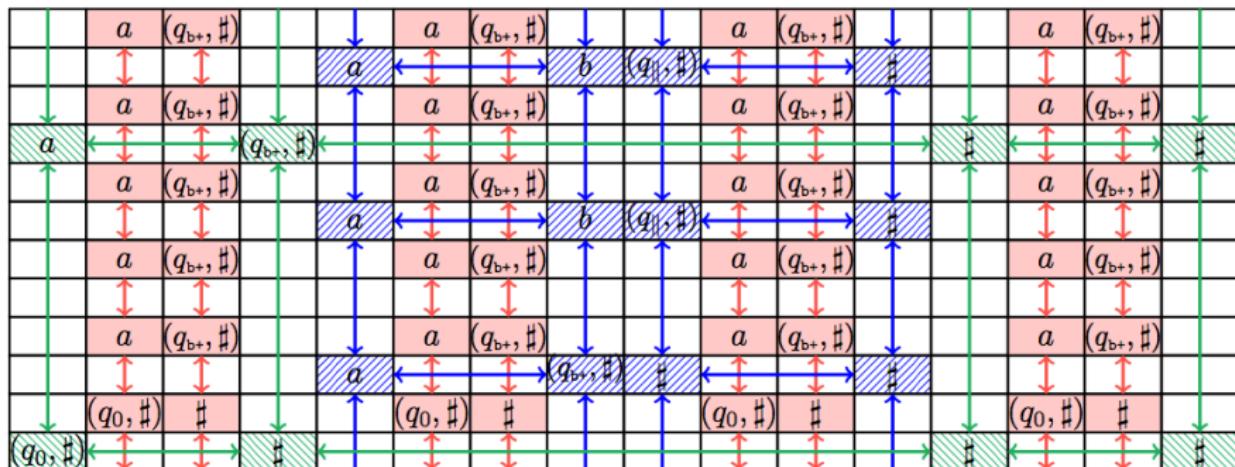
A configuration of $\mathbf{T}_{\text{Calcul}(\mathcal{M})}$



Construction of $\mathbf{T}_{\text{Calcul}(\mathcal{M})} \subset \mathbf{T}_{\text{Grid}} \times \mathcal{A}_{\mathcal{M}}^{\mathbb{Z}^2}$:

- Layer 1: \mathbf{T}_{Grid}
- Layer 2: $\mathcal{F}_{\mathcal{M}}$ is verified on each fractionated strips with boundaries conditions
- We forbid the tile which contains q_F .

A configuration of $\mathbf{T}_{\text{Calcul}(\mathcal{M})}$



A strip of level n allows to code the space-time diagram of \mathcal{M} of size $2^n \times 2^{2^n}$, thus:

$$\mathcal{M} \text{ halts} \iff \mathbf{T}_{\text{Calcul}(\mathcal{M})} = \emptyset$$

Theorem (Berger 1966, Robinson 1971)

The domino problem is undecidable in dimension $d \geq 2$.

Little historic of aperiodic tilings

Little historic of aperiodic tilings

Auteurs	Nombre de tuiles de Wang	Nombre de tuiles à translation près	Nombres de tuiles à isométrie près
R. Breger 1966	20 426		
R. Breger 1966	104		
D. E. Knuth 1966	92		
R. Penrose 1978		20	2
R. M. Robinson 1971	56	32	6
R. Ammann 1978		16	2

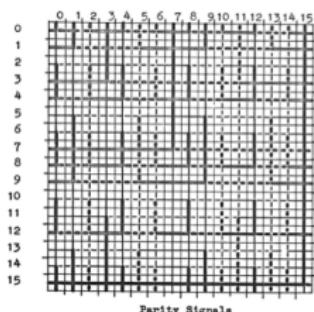
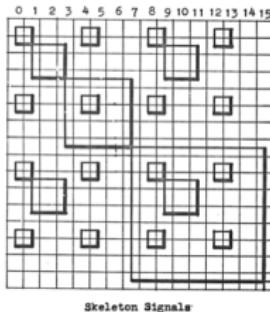
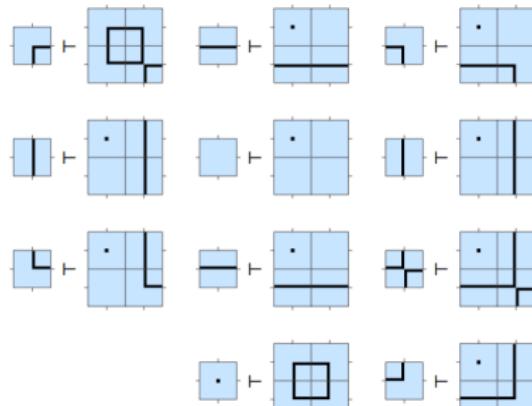
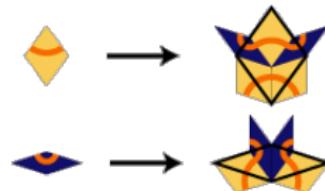
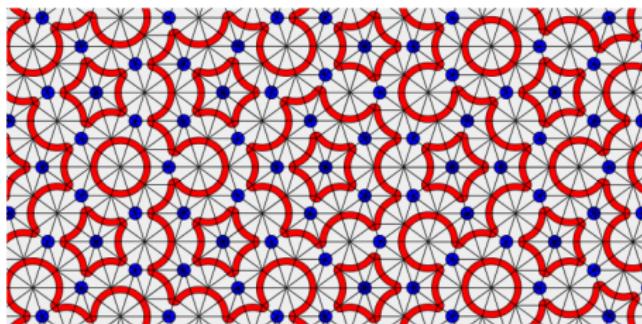


Figure 24 Part of the Solution of Q



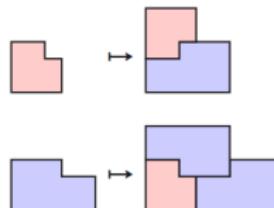
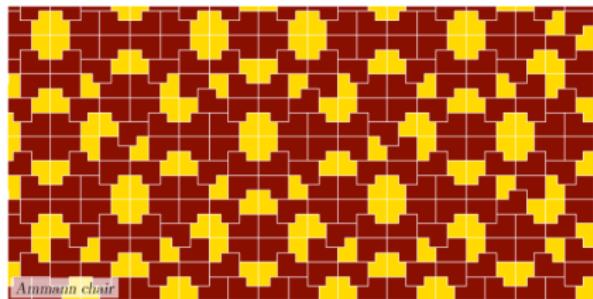
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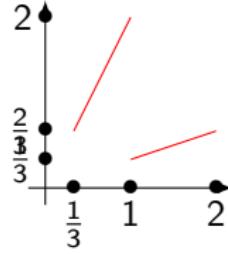
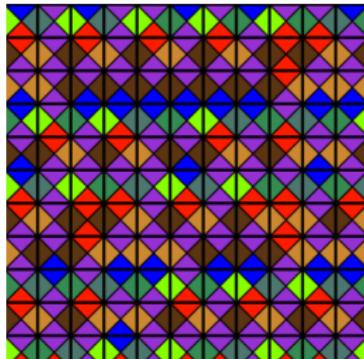
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<i>J. Kari 1996</i>	14		
<i>K. Culick 1996</i>	13		

Problematic

There exists other type of aperiodic tilings?