Quantified block gluing for multidimensional subshift of finite type: aperiodicity and entropy

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Abstract

It is possible to extend the notion of block gluing for subshifts studied in [PS15] adding a gap function which gives the distance which allows to concatenate two rectangular blocks of the langage. In this article, we study the interplay between this intensity and computational properties. In particular, we prove that there block gluing SFTs with linear gap which are aperiodic and that all the non-negative right-recursively enumerable (Π_1-computable) numbers can be realized as entropy of such subshifts of finite type. As block gluing with linear gap implies transitivity, this last point provides a solution to Problem 9.1 in [HM10] about the characterization of the entropies of transitive subshift of finite type.

1 Introduction

It appeared recently that it is possible to understand many dynamical properties of multidimensional subshifts of finite type through characterizations results, beginning with the entropy [HM10]. This interplay between dynamical properties and computability or decidability properties was shown by a lot of recent works: a characterization of the possible projective sub-actions of a SFT [Hoc09, AS10, DRS], measure of the computationnally simplest configurations with Medvedev degrees [Sim14] and sets of Turing degrees [JV13], characterization of the possible sets of periods in terms of complexity theory [JV15], etc. The importance of computability considerations in these models has been clearly established, and recent research direction comes from the observation that dynamical restrictions can prevent embedding universal computing.

One can observe this phenomenon for some mixing-like properties. To be more precise, the result proved by M. Hochman and T. Meyerovitch [HM10] states that the set of numbers that are entropy of a multidimensional SFT is exactly the set of non-negative right-recursively enumerable (or Π_1-computable) real numbers. When the SFT is strongly irreducible, the entropy becomes computable [HM10]. Only some step results towards a characterization are known [PS15]. We interpret this obstruction as a reduction of the computational power of the model under this restriction, which is manifested by the reduction of possible entropies.

In [PS15] the authors studied SFT which are block gluing. This means that there exists a constant c such that for any couple of square blocks in the language of the subshift, the pattern obtained by gluing these two blocks in any way respecting a distance c between them is also in the language.

We propose in this text to study similar properties that consist in imposing the gluing property when the two blocks are spaced by $f(n)$, where n is the length of two square blocks having the same size, and $f : \mathbb{N} \to \mathbb{N}$. We got interested in the influence of this property on the possibility of non-existence of periodic orbits (which is the first ingredient for embedding computations in SFT), and on the set of possible entropies, according to the gap function f. We observed two regimes:

- **Sub-logarithmic regime:** if $f \in o(\log(n))$ and $f \leq id$, then the set of periodic orbits of any f-block gluing SFT is dense (Proposition 19) and its language is decidable. Moreover, the entropy is computable (Proposition **??**).

- **Linear regime:**
  - There exists some $f \in O(n)$ such that there exists an aperiodic f-block gluing SFT (Theorem 22).
  - There exists $f \in O(n)$ such that there exists a f-block gluing SFT with non decidable language (Proposition 21).
  - The possible entropies of f-block gluing SFT with $f \in O(n)$ are the right recursively enumerable non-negative real numbers (Theorem 38). Since $O(n)$-block gluing SFT are transitive, this result characterizes the set of possible entropy of transitive SFT which is an open question of [HM10].

The results in the sub-logarithmic regime regime extend to a larger class of intensities some known result about the density of the set of periodic orbits under block gluing condition (meaning finite block gluing considering intensity) obtained in [PS15].

The linear regime regime is more interesting because it leads to more complex and structured constructions. The proof of Theorem 22 rely on the 'net gluing' property (notion introduced in this article) of the Robinson subshift and on transformations on subshift of finite type over some fixed alphabet that misshape the configuration of a subshift and permit to transform a net gluing subshift into a block gluing one. An important point is that the entropy of the image of a subshift by the transformation is a function of the entropy of this subshift which has a closed form.

The proof of Theorem 38 rely on the construction of [HM10], using the Robinson subshift to implement machines in computation zones defined by this subshift that control the frequency of some 'frequency bits' $0, 1$ that are identified in columns, and adding random bits $1, 1'$ over the 1 symbols that generate the entropy. The two obstacles to the transitivity property in their construction are the identification of the frequency bits in columns, and that the behaviors occurring in infinite computation zones are very specific to these zones. We solve these problems identifying the frequency bits inside every computation zone to solve the first problem, and simulating machines having the 'bad behaviors' occurring in infinite computation zones in every finite one, aside machines that have the 'good behavior' to solve the second problem.

From considerations on the computability properties of subshifts, arise some 'natural' tools and principles of the organization, stocking, and exchanges of information of various 'objects' observable in the system. In the construction presented in this article, structures extracted from the Robinson subshift permit at the same time to attribute areas for computation and control agents (Turing machines), as this is done in [AS10], and allow various signals to propagate, so that the computation agents synchronize or communicate, without interfering between each other. These agents are organized as a hierarchy in [AS10] in the strong sense that the results of Turing machines at some level of the hierarchy will be transmitted to the Turing machine immediately above in the hierarchy for its proper computation (in our construction this hierarchy is only geographic).

The article is organized as follows :

- In Section 2 we recall symbolic dynamics general definitions, define some "block gluing" notions, and recall the Robinson subshift definition and properties.
- In Section 3, we explore the property of periodic orbits.
- In Section 4, we explore the property of entropy.
- The other Sections detail the construction of the proof of Theorem 38.

2 First properties of block gluing with gap function

In this section we recall some definitions on symbolic dynamics and we introduce the notion of block gluing with intensity function. Then we give some examples of subshifts of finite type which are block gluing for various intensity functions.

2.1 Subshifts as dynamical systems

2.1.1 Subshifts and patterns

Let $\mathcal{A}$ be a finite set called the **alphabet**. A **configuration** x is an element of $\mathcal{A}^{\mathbb{Z}^2}$. In this article we focus on two dimensional configurations but all the following definitions can be generalized to $\mathbb{Z}^d$, $d \geq 2$. The space $\mathcal{A}^{\mathbb{Z}^2}$ is endowed by the product topology derived from the discrete topology on $\mathcal{A}$. For this topology, $\mathcal{A}^{\mathbb{Z}^2}$ is a compact metric space on which $\mathbb{Z}^2$ acts continually by translation via the **shift** map, denoted σ, which is defined for all $\mathbf{i} \in \mathbb{Z}^2$ by:

$$\begin{array}{cccll} \sigma^{\mathbf{i}}: & \mathcal{A}^{\mathbb{Z}^2} & \longrightarrow & \mathcal{A}^{\mathbb{Z}^2} & \\ & x & \longmapsto & \sigma^{\mathbf{i}}(x) & \text{such that } \forall \mathbf{u} \in \mathbb{Z}^2,\ \sigma^{\mathbf{i}}(x)_{\mathbf{u}} = x_{\mathbf{i}+\mathbf{u}} \end{array}$$

Let $\mathbb{U}$ be a finite subset of $\mathbb{Z}^2$. Denote $x_{\mathbb{U}}$ the **restriction** of $x \in \mathcal{A}^{\mathbb{Z}^2}$ to $\mathbb{U}$. A **pattern** p on **support** $\mathbb{U}$, denoted $\text{supp}\,(p)$, is an element of $\mathcal{A}^{\mathbb{U}}$. Define $\mathbb{U}_n = [\![0; n-1]\!]^2$ the **elementary support** of size $n \in \mathbb{N}$. A pattern on support $\mathbb{U}_n$ is a **n-block**. As well, a pattern with support $[\![0; n-1]\!] \times [\![0, m-1]\!]$ is a $n \times m$**-rectangle**. A pattern p having support $\mathbb{U}$ **appears** at position $\mathbf{i} \in \mathbb{Z}^2$ in a configuration $x \in \mathcal{A}^{\mathbb{Z}^2}$ if for all $\mathbf{j} \in \mathbb{U}$, $p_{\mathbf{j}} = x_{\mathbf{i}+\mathbf{j}}$, denote $p \sqsubset x$. A pattern p on support $\mathbb{U}$ is a **sub-pattern** of a pattern q on support $\mathbb{V}$ when $\mathbb{U} \subset \mathbb{V}$ and $q_{\mathbb{U}} = p$.

A **subshift** X is a closed subset of $\mathcal{A}^{\mathbb{Z}^2}$ which is invariant under the action of the shift, meaning $\sigma(X) \subset X$. The couple (X, σ) is a dynamical system. Any subshift X can be defined by a set of **forbidden patterns**, as the set of configurations where no element of this set appears. Formally there exists $\mathcal{F}$ a set of patterns such that :

$$X = X_{\mathcal{F}} := \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} : \text{ for all } p \in \mathcal{F},\ p \not\sqsubset x \right\}.$$

If the subshift can be defined by a finite set of forbidden patterns, it is called a **subshift of finite type** (SFT for short). The order of a SFT is the smallest r such that it can be defined by forbidden r-blocks.

A configuration $x \in \mathcal{A}^{\mathbb{Z}^2}$ is **periodic** if there exists $m, n > 0$ such that $\sigma^{(m,0)}(x) = \sigma^{(0,n)}(x) = x$. A subshift is **aperiodic** when none of its configurations is periodic.

A pattern **appears** in a subshift X if there is a configuration of X in which it appears. The set of patterns which appear in X is called the **language** of X, denoted $\mathcal{L}(X)$. Denote $\mathcal{L}_n(X)$ the set of n-blocks that appears in X.

In this article the construction of subshifts is obtained on an alphabet $\mathcal{A}$ which is a product of alphabets $\mathcal{A} = \mathcal{A}_1 \times ... \times \mathcal{A}_k$. We call informally the i^{th} layer of this subshift the space of the projections of a configuration written on the i^{th} alphabet $\mathcal{A}_i$.

2.1.2 Morphisms

A **morphism** between two subshifts X and Y on alphabets $\mathcal{A}_X$ and $\mathcal{A}_Y$ is a continuous map $\varphi : X \to Y$ such that $\varphi \circ \sigma = \sigma \circ \varphi$. Equivalently by Hedlund's Theorem [Hed69], φ can be defined with a **local function** $\overline{\varphi} : \mathcal{A}_X^{[\![-r,r]\!]^2} \to \mathcal{A}_Y$ of **radius** $r \in \mathbb{N}$ by

$$\varphi(x)_{\mathbf{i}} = \overline{\varphi}(x_{\mathbf{i}+[\![-r,r]\!]^2}) \qquad \text{for all } x \in X, \text{ and } \mathbf{i} \in \mathbb{Z}^2.$$

A **factor** is a morphism which is onto, and it is a **conjugacy** if it is invertible, the inverse map being also a morphism in this case. Two subshifts are **conjugated** if there exists a conjugacy between them. In this case we considerate that they have the same dynamical behavior.

2.2 Block gluing notions

2.2.1 Definitions

In this section, X is a subshift on the alphabet $\mathcal{A}$ and $f : \mathbb{N} \to \mathbb{N}$ is a non decreasing function. We denote $||.||_\infty$ the norm defined by

$$||\mathbf{i}||_\infty = \max\{\mathbf{i}_1, \mathbf{i}_2\}$$

for all $\mathbf{i} \in \mathbb{Z}^2$. We denote d_∞ the associated distance function.

Definition 1. *Let $n \in \mathbb{N}$ be an integer. The **gluing set** in the subshift X of some n-block p relative to some other n-block q is the set of $\boldsymbol{u} \in \mathbb{Z}^2$ such that there exists a configuration in X where q appears in position $(0,0)$ and p appears in position $\boldsymbol{u}$ (see Figure 1). This set is denoted $\Delta_X(p,q)$. Formally*

$$\Delta_X(p,q) = \left\{\boldsymbol{u} \in \mathbb{Z}^2 : \ \exists x \in X \text{ such that } x_{[\![0,n-1]\!]^2} = q \text{ and } x_{\boldsymbol{u}+[\![0,n-1]\!]^2} = p\right\}$$

*When the intersection of the sets $\Delta_X(p,q)$ for (p,q) couples of n-blocks is non empty, we denote this intersection $\Delta_X(n)$. This set is called the **gluing set of n-blocks** in X.*

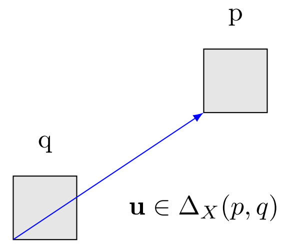


Figure 1: Illustration of Definition 1.

Definition 2. *A subshift X is said to be f**-block transitive** if for all $n \in \mathbb{N}$ one has*

$$\Delta_X(n) \cap \left\{\boldsymbol{u} \in \mathbb{Z}^2 : ||\boldsymbol{u}||_\infty \leq n + f(n)\right\} \neq \emptyset.$$

*The function f is called the **gap function**.*

Remark 1. *The condition on the vectors $\boldsymbol{u} \in \mathbb{Z}^2$ is $||\boldsymbol{u}||_\infty \leq n + f(n)$ since we consider the distance between the positions where the patterns appear instead of the space between them.*

Definition 3. *A subshift X is said to be f **net gluing** if there exists a function $\boldsymbol{u} : \mathcal{L}(X)^2 \to \mathbb{Z}^2$ and a function $\widetilde{f} : \mathcal{L}(X)^2 \to \mathbb{N}$ such that for all $n \in \mathbb{N}$ and for all n-blocks p and q,*

$$\boldsymbol{u}(p,q) + (n + \widetilde{f}(p,q))(\mathbb{Z}^2 \backslash \{0\}) \subset \Delta_X(p,q)$$

and

$$\max_{p,q \in \mathcal{L}_n(X)} \widetilde{f}(p,q) \leq f(n).$$

Remark 2. *This property is different from quasiperiodicity properties in the sense that the configuration where two patterns appears can depend on the relative position of the two patterns.*

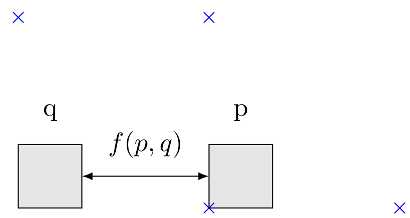


Figure 2: Illustration of Definition 3. The crosses designate elements of the gluing set of p relatively to q in X.

Definition 4. *A subshift X is f-**block gluing** when*

$$\left\{\boldsymbol{u} \in \mathbb{Z}^2, ||\boldsymbol{u}||_\infty \geq f(n) + n\right\} \subset \Delta_X(n).$$

For any function f, one has

$$f\text{-block gluing} \Longrightarrow f\text{-net gluing} \Longrightarrow f\text{-block transitive}$$

A subshift is said $O(f)$**-block gluing** (resp. $O(f)$**-net gluing**, $O(f)$**-block transitive**) if it is g (block gluing) (resp. g-net gluing, g-block transitive) for a function $g : \mathbb{N} \to \mathbb{N}$ such that there exists $C > 0$ such that $g(n) \leq C\, f(n)$ for all $n \in \mathbb{N}$. A property verified on the class of $O(f)$ block gluing (resp. g net gluing, g block transitive) subshifts is said to be **sharp** if the property is false for all $h \in o(f)$ (this means that for all $\epsilon > 0$ there exists n_0 such that $h(n) \leq \epsilon f(n)$ for all $n \geq n_0$).

A subshift is **linearly block gluing** (resp. **linearly net gluing**, **linearly transitive**) if it is $O(n)$ block gluing (resp. $O(n)$ net gluing, $O(n)$ block transitive).

2.2.2 Equivalent definition

The following proposition gives an equivalent definition for linear block gluing and net gluing subshifts using some exceptional values:

Proposition 5. *A subshift $X \subset \mathcal{A}^{\mathbb{Z}^2}$ is linearly block gluing if and only if there exists a function $f \in O(n)$, $c \geq 2$ an integer and $m \in \mathbb{N}$ such that*

$$\{\boldsymbol{u} \in \mathbb{Z}^2, ||\boldsymbol{u}||_\infty \geq f(c^l + m) + c^l + m\} \subset \Delta_X(c^l + m) \qquad \forall l \geq 0.$$

A similar assertion is true for net gluing.

Proof. Clearly a linear-block gluing subshift verifies this property. Reciprocally, let p and q be two n-blocks, and consider $l(n) = \lceil \log_c(n-m) \rceil$, where for all real number x, $\lceil x \rceil$ designates the smallest integer greater than x. Consider p' and q' some $c^{l(n)} + m$-blocks whose restrictions on $[\![0, n-1]\!]^k$ are respectively p and q. The set $\Delta_X(p', q')$ contains $\{\mathbf{u} \in \mathbb{Z}^2, ||\mathbf{u}||_\infty \geq f(c^{l(n)} + m) + c^{l(n)} + m\}$. As a consequence, $\Delta_X(p, q)$ contains $\{\mathbf{u} \in \mathbb{Z}^2, ||\mathbf{u}||_\infty \geq g(n) + n\}$, where $g(n) = f(c^{l(n)} + m) + c^{l(n)} - n + m$. Since $c^{l(n)} \leq c * (n + |m|)$, the function g is in $O(n)$, hence X is $O(n)$-block gluing. ☐

The proofs of block gluing and net gluing properties often rely on this proposition. In our construction, we first complete patterns into patterns over cells, whose sizes are given by an exponential sequence as in the statement of the proposition. Then we prove the gluing property for two cells.

2.2.3 Gluing and morphisms

The following proposition shows that a factor of a block gluing (resp. net gluing) subshift is also block gluing (resp. net gluing) and gives a precise gap function.

Proposition 6. *Let $\varphi : X \to Y$ be some onto r-block map between two $\mathbb{Z}^2$-subshifts, and $f : \mathbb{N} \to \mathbb{N}$ a non decreasing function. If the subshift X is f-block gluing (resp. f- net gluing), then Y is g-block gluing (resp. g-net gluing) where $g : n \longmapsto f(n + 2r) + 2r$.*

Proof. Denote $\overline{\varphi} : \mathcal{A}_X^{[\![-r,r]\!]^2} \to \mathcal{A}_Y$ the local rule of φ.

Let p', q' be two n-blocks in the language of Y. There exist p and q two $(n+2r)$-blocks in the language of X such that p' and q' are respectively the image of p and q by $\overline{\varphi}$. Let $\mathbf{u} \in \Delta_X(p, q)$. There exists $x \in X$ such that $x_{[\![0,n+2r-1]\!]^2} = p$, and $x_{\mathbf{u}+[\![0,n+2r-1]\!]^2} = q$. Applying φ to $\sigma^{r(1,1)}(x)$, we obtain some $y \in Y$ such that $y_{[\![0,n-1]\!]^k} = p'$, and $y_{\mathbf{u}+[\![0,n-1]\!]^k} = q'$. We deduce that

$$\Delta_X(p, q) \subset \Delta_Y(p', q') \quad \text{ so } \quad \Delta_X(n + 2r) \subset \Delta_Y(n).$$

Thus if X is f-block gluing then Y is g-block gluing where $g : n \longmapsto f(n+2r)+2r$.

If X is f-net gluing, then the gluing set of two $(2n+r)$-blocks p, q contains

$$\mathbf{u}(p,q) + (n + 2r + \widetilde{f}(p,q))(\mathbb{Z}^2 \backslash \{(0,0)\}),$$

such that $\widetilde{f}(p,q) \leq f(n+2r)$. Hence the gluing set of p', image of p by $\overline{\varphi}$, relative to q', image of q by $\overline{\varphi}$, in Z contains this set. One deduces that Y is g-net gluing where $g : n \longmapsto f(n+2r)+2r$. □

We deduce that the classes of subshifts defined by these properties are invariant of conjugacy under some assumption on f. The set functions that verify this assumption includes all the possible gap functions we already know:

Corollary 7. *Let f be some non decreasing function. If for all $r \in \mathbb{N}$, there is a constant C such that for all $n \geq 0$, $Cf(n) \geq f(n+2r)$ then the following classes of subshifts are invariant under conjugacy: $O(f)$-block transitive, $O(f)$-net gluing, $O(f)$-block gluing, sharp $O(f)$-net gluing and sharp $O(f)$-block gluing subshifts.*

In particular it is verified when f is constant or $n \mapsto n^k$ with $k > 0$ or $n \mapsto e^n$ or $n \mapsto \log(n)$.

2.3 Some examples

We say that two blocks p, q having respective supports $\mathbb{U}, \mathbb{V}$ are **spaced by distance** k when

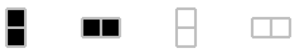
$$\max_{\mathbf{u}\in\mathbb{U}} \min_{\mathbf{v}\in\mathbb{V}} ||\mathbf{u}-\mathbf{v}||_\infty \geq k.$$

Since the subshifts that we consider in this text are bi-dimensional, this means that there are at least k column or at least k lines between the two blocks.

2.3.1 First examples

We present here some examples of block gluing SFT.

Example 8. *Consider the SFT $X_{\texttt{Chess}}$ defined by the following set of forbidden patterns:*



This subshift has two configurations (see Figure 3 for an example) and both of them are periodic. It is 1-net gluing, but not block gluing. Indeed, the gluing set of the pattern ■ relatively to itself is

$$\Delta_{X_{\texttt{Chess}}}(■, ■) = 2\mathbb{Z}^2 \setminus \{(0,0)\} \cup \left(2\mathbb{Z}^2 + (1,1)\right)$$

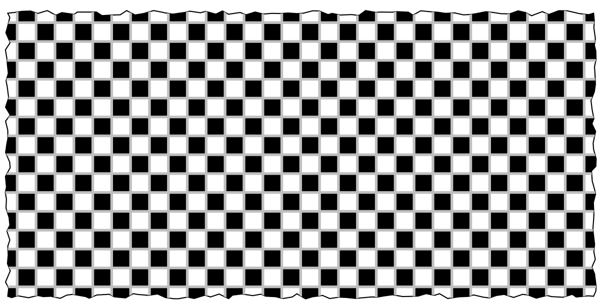


Figure 3: An example of configuration of $X_{\texttt{Chess}}$.

Example 9. *Consider the SFT $X_{\texttt{Even}}$ defined by the following set of forbidden patterns:*

An example of configuration in this subshift is given in Figure 4. This subshift is 1*-block gluing since two blocks in its language can be glued with distance 1, filling the configuration with* $\square$ *symbols.*

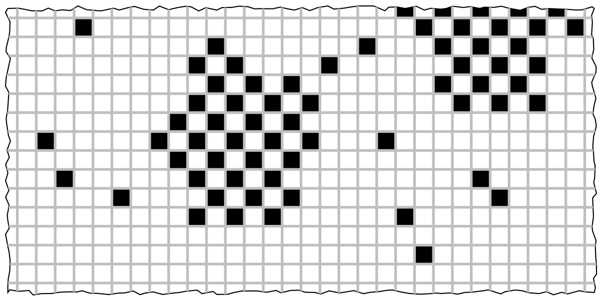


Figure 4: An example of configuration of $X_{\texttt{Even}}$.

Example 10. *Consider the SFT* X_{Linear} *defined by the following set of forbidden patterns:*

The local rules imply that if a configuration contains the pattern $\square\blacksquare^n\square$ *then it contains* $*\square\blacksquare^{n-2}\square*$ *just above, where* $* \in \{\square, \blacksquare\}$*. Thus a configuration of* X_{Linear} *can be seen as a layout of triangles of made of symbols* $\blacksquare$ *on a background of* $\square$ *symbols (an example of configuration is given on Figure 5).*

This subshift is sharp linearly block gluing. Indeed, consider two n*-blocks in its language separated horizontally or vertically by* $2n$ *cells. They contain pieces of triangles that we complete with the smallest triangle possible, the other symbols of the configuration being all* $\square$ *symbols. The worst case for gluing two* n*-blocks is when the blocks are filled with the symbol* $\blacksquare$*. In this case we can complete each of the two blocks by a triangle which base is constituted by* $\blacksquare^{3n}$*. Hence every couple of blocks can be glued horizontally and vertically with linear distance. To prove that* X_{Linear} *is not* f*-block gluing with* $f(n) \in o(n)$*, we consider the rectangle*

$$\square\blacksquare^n\square$$

that we would like to glue above itself. To do that we need to separate the two copies of this pattern by about $\lceil \frac{n}{2} \rceil$ *cells.*

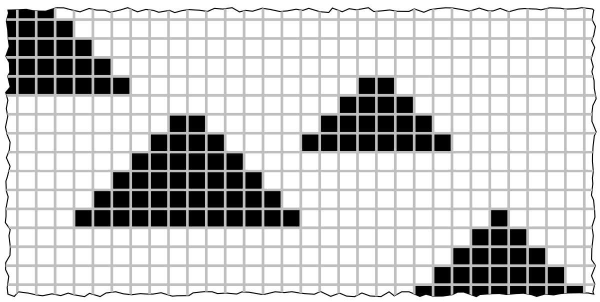


Figure 5: An example of configuration of $X_{\texttt{Linear}}$.

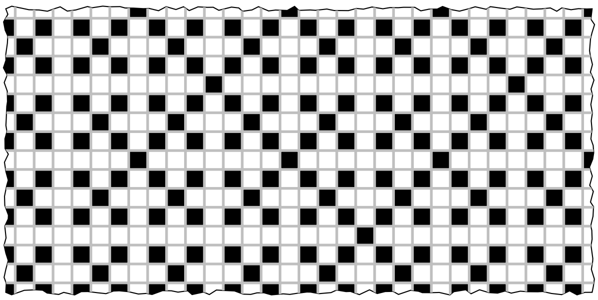


Figure 6: A part of a configuration of X_s.

2.3.2 Linearly net gluing subshifts given by substitutions

Let $\mathcal{A}$ be a finite alphabet. A **substitution** rule is a map $s : \mathcal{A} \to \mathcal{A}^{\mathbb{U}_m}$, for some $m \geq 1$. This function can be extended naturally on blocks in view to iterate it. The subshift X_s associated to this substitution is the set of configurations such that any pattern appearing in it appears as a sub-pattern of some $s^n(a)$ with $n \geq 0$ and $a \in \mathcal{A}$.

Consider the following substitution s defined by

where an exemple of configuration is given in Figure 6. Since $\blacksquare$ appears on position $(0,0)$ in $s(\square)$ and $s(\blacksquare)$, we deduce that for any configuration x, there exists $\mathbf{i}_1 \in [\![0,1]\!]^2$ such that $x_{\mathbf{i}_1+2\mathbb{Z}^2} = \blacksquare$. By induction, for all $n \geq 1$, there exists $\mathbf{i}_n \in [\![0,2^n-1]\!]^2$ such that $x_{\mathbf{i}_n+2^n\mathbb{Z}^2} = s^n(\blacksquare)$. Since every pattern of X_s appears in $s^n(\blacksquare)$ for some $n \in \mathbb{N}$, we deduce that X_s has the linear net-gluing property, using Proposition 5.

This argument can be easily generalized for substitution s for which there exists $i \in \mathbb{N}$, a subset $\mathcal{Z} \subset [\![0,m^i-1]\!]^2$ and an invertible map $\nu : \mathcal{A} \to \mathcal{Z}$ such that $a \in \mathcal{A}$ appears on the same position $\nu(a)$ in any pattern of the patterns $s^i(d)$ with $d \in \mathcal{A}$.

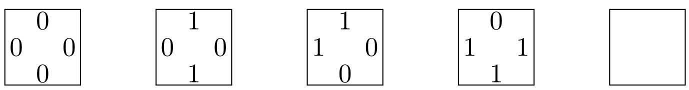
2.3.3 Intermediate intensities

Here we present an example of block gluing SFT whose gap function is strictly between linear and constant classes.

Consider the SFT $X_{\texttt{Log}}$ having two layers, with the following characteristics:

Symbols:

The first layer has symbols $\blacksquare$ and $\square$, and the second one the symbols:



The first four of these symbols are thought as coding for the adding machine. Each one contains four symbols: the west one is the initial state of the machine, the east one the forward state, the south one the input letter, and the last symbol is the output.

Local rules:

- **First layer:**

  The following patterns are forbidden in the first layer:

These local rules imply that if a configuration contains the pattern $\square\blacksquare^n\square$ in the first layer, then it contains $\square\blacksquare^n\square$, $\square\blacksquare^{n+1}$, or $\square^{n+2}$ just above. Thus a configuration of the first layer of $X_{\texttt{Log}}$ can be seen as triangular shapes of symbols $\blacksquare$ on a background of $\square$ symbols (an example of configuration is given in Figure 7).

- **Second layer:**
  - the adding machine symbols are superimposed on black squares, the other ones on blank squares.
  - for two adjacent machine symbols, the symbols on the sides have to match.
  - on a pattern $\blacksquare\square$, on the machine symbol over the black square, the east symbol have to be 0.
  - on a pattern $\begin{smallmatrix}\blacksquare & \blacksquare \\ \blacksquare & \square\end{smallmatrix}$, the machine have a south symbol being 0 on the north west black square.

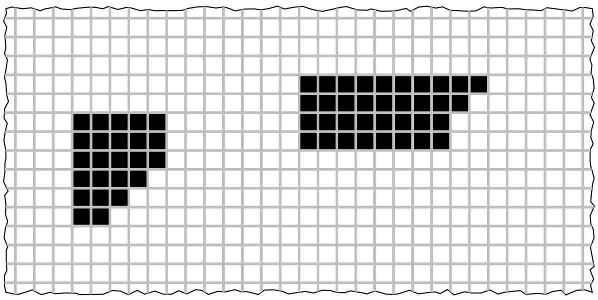


Figure 7: An example of configuration that respects the rules of the first layer of $X_{\texttt{Log}}$.

This subshift is sharp $O(log)$-block gluing. Indeed, any two n-blocks in its language can be glued vertically with distance 1. For the horizontal gluing, the worst case for gluing two n-blocks is when the two blocks are filled with black squares and the adding machine symbols on the leftmost column of the blocks are only 1 (thus maximizing the number of lines where the rectangular shape into which we complete the block have to be greater in length than the one just below). In this case, we can complete the block such that each line (from the bottom to the top) is extended from the one below with one $\blacksquare$ symbol on the right when the machine symbol have a 1 on its west side, and adding blank squares to obtain a rectangle. The number of columns added is smaller than the maximal number of bits added by the adding machine to a length n string of $0, 1$ symbols in n steps, which is $O(log(n))$. This means that two n-blocks can be glued horizontally with distance $O(log(n))$. To see that this property is sharp, consider the horizontal gluing of two $1 \times n$ rectangles of black squares, similarly as in the linear case.

Remark 3. *The set of possible tight gap functions of block gluing SFT seems restricted. We don't know for instance if, when f is the square root function, there exist subshifts for which the f block gluing property is tight.*

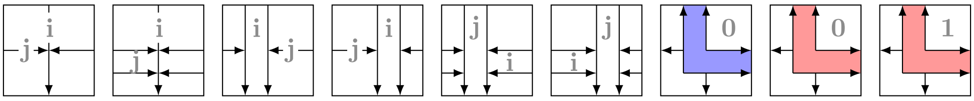
2.4 A linearly net gluing version of the Robinson subshift

The Robinson subshift was constructed by R. Robinson [Rob71] in order to prove undecidability results. It has been used by in other constructions of subshifts of finite type as a structure layer to implement computation. We recall some properties and refer to [Rob71] for the proofs.

We present here a version of this subshift which is adapted to constructions under the dynamical constraints that we consider. Let us denote X_{adR} this subshift, which is constructed as the product of two layers. We present the first layer in Subsection 2.4.1, then we describe some hierarchical structures appearing in this layer in Subsection 2.4.2. In Subsection 2.4.3, we describe the second layer. The subshift X_{adR} obtained is linearly net gluing.

2.4.1 Robinson layer

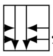
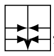
The first layer has the following *symbols*, and their transformation by rotations by $\frac{\pi}{2}$, π or $\frac{3\pi}{2}$:



The symbols i and j can have value $0,1$ and are attached respectively to vertical and horizontal arrows. In the text, we refer to this value as the value of the $0,1$-**counter**. In order to simplify the representations, these values will often be omitted on the figures.

In the text we will often designate as **corners** the two last symbols. The other ones are called **arrows symbols** and are specified by the number of arrows in the symbol. For instance a six arrows symbols are the images by rotation of the fifth and sixth symbols.

The *rules* are the following ones:

1. the outgoing arrows and incoming ones correspond for two adjacent symbols.
2. in every 2×2 square there is a blue symbol and the presence of a blue symbol in position $\mathbf{u} \in \mathbb{Z}^2$ forces the presence of a blue symbol in the positions $\mathbf{u} + (0,2), \mathbf{u} - (0,2), \mathbf{u} + (2,0)$ and $\mathbf{u} - (2,0)$.
3. on a position having mark (i,j), the first coordinate is transmitted to the horizontally adjacent positions and the second one is transmitted to the vertically adjacent positions.
4. on a six arrows symbol, like , or a five arrow symbol, like , the marks i and j are different.

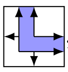
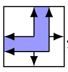
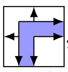
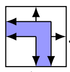
The Figure 8 shows some pattern in the language of this layer. The subshift on this alphabet and generated by these rules is denoted X_R: this is the Robinson subshift.

Theorem 11 ([Rob71])**.** *The subshift X_R is non-empty and aperiodic.*

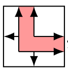
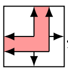
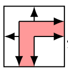
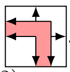
2.4.2 Hierarchical structures

In this section we describe some observable hierarchical structures in the elements of the Robinson subshift. Recall that $\mathbb{U}_k = [\![0, k-1]\!]^2$.

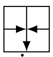
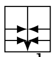
Finite supertiles Let us define by induction the south west (resp. south east, north west, north east) **supertile of order** $n \in \mathbb{N}$. For $n = 0$, one has

$St_{sw}(0) =$ , $St_{se}(0) =$ , $St_{nw}(0) =$ , $St_{ne}(0) =$ .

For $n \in \mathbb{N}$, the support of the supertile $St_{sw}(n+1)$ (resp. $St_{se}(n+1)$, $St_{nw}(n+1)$, $St_{ne}(n+1)$) is $\mathbb{U}^{(2)}_{2^{n+2}-1}$. On position $\mathbf{u} = (2^{n+1}-1, 2^{n+1}-1)$ write

$St_{sw}(n+1)_{\mathbf{u}} =$ , $St_{se}(n+1)_{\mathbf{u}} =$ , $St_{nw}(n+1)_{\mathbf{u}} =$ , $St_{ne}(n+1)_{\mathbf{u}} =$ .

Then complete the supertile such that the restriction to $\mathbb{U}^{(2)}_{2^{n+1}-1}$ (resp. $(2^{n+1},0) + \mathbb{U}^{(2)}_{2^{n+1}-1}$, $(0,2^{n+1}) + \mathbb{U}^{(2)}_{2^{n+1}-1}$, $(2^{n+1},2^{n+1}) + \mathbb{U}^{(2)}_{2^{n+1}-1}$) is $St_{sw}(n)$ (resp. $St_{se}(n)$, $St_{nw}(n)$, $St_{ne}(n)$).

Then complete the cross with the symbol  or with the symbol  in the south vertical arm with the first symbol when there is one incoming arrow, and the second when there are two. The other arms are completed in a similar way. For instance, Figure 8 shows the south west supertile of order two.

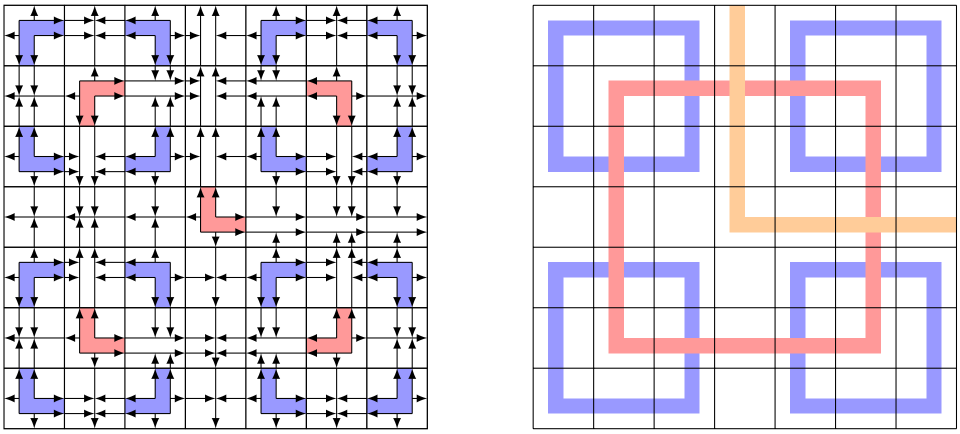


Figure 8: The south west order 2 supertile denoted $St_{sw}(2)$ and petals intersecting it.

Proposition 12 ([Rob71])**.** *For all configuration x, if an order n supertile appears in this configuration, then there is an order $n+1$ supertile, having this order n supertile as sub-pattern, which appears in the same configuration.*

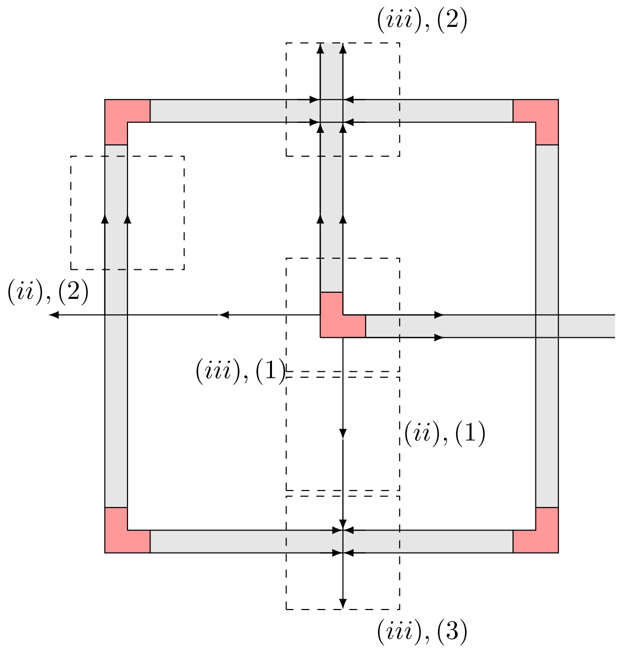


Figure 9: Correspondence between infinite supertiles and sub-patterns of order n supertiles. The whole picture represents a schema of some finite order supertile.

Infinite supertiles Let x be a configuration in the first layer and consider the equivalence relation $\sim_x$ on $\mathbb{Z}^2$ defined by $\mathbf{i} \sim_x \mathbf{j}$ if there is a finite supertile in x which contains $\mathbf{i}$ and $\mathbf{j}$. An **infinite order** supertile is an infinite pattern over an equivalence class of this relation. Each configuration is amongst the following types (with types corresponding with types numbers on Figure 9):

(i) A unique infinite order supertile which covers $\mathbb{Z}^2$.

(ii) Two infinite order supertiles separated by a line or a column with only three-arrows symbols (1) or only four arrows symbols (2). In such a configuration, the order n finite supertiles appearing in the two infinite supertiles are not necessary aligned, whereas this is the case in a type (i) or (iii) configuration.

(iii) Four infinite order supertiles, separated by a cross, whose center is superimposed with:

- a red symbol, and arms are filled with arrows symbols induced by the red one. (1)
- a six arrows symbol, and arms are filled with double arrow symbols induced by this one. (2)
- a five arrow symbol, and arms are filled with double arrow symbols and simple arrow symbols induced by this one. (3)

Informally, the types of infinite supertiles correspond to configurations that are limits (for type (*ii*) infinite supertiles this will be true after alignment [Section 2.4.3]) of a sequence of configurations centered on particular sub-patterns of finite supertiles of order n. This correspondence is illustrated on Figure 9. We notice this fact so that it helps to understand how patterns in configurations having multiple infinite supertiles are sub-patterns of finite supertiles.

We say that a pattern p on support $\mathbb{U}$ appears periodically in the horizontal (resp. vertical) direction in a configuration x of a subshift X when there exists some $T > 0$ and $\mathbf{u}_0 \in \mathbb{Z}^2$ such that for all $k \in \mathbb{Z}$,

$$x_{\mathbf{u}_0 + \mathbb{U} + kT(1,0)} = p$$

(resp. $x_{\mathbf{u}_0 + \mathbb{U} + kT(0,1)} = p$). The number T is called the period of this periodic appearance.

Lemma 13 ([Rob71])**.** *For all n and m integers such that $n \geq m$, any order m supertile appears periodically, horizontally and vertically, in any supertile of order $n \geq m$ with period 2^{m+2}. This is also true inside any infinite supertile.*

Petals For a configuration x of the Robinson subshift some finite subset of $\mathbb{Z}^2$ which has the following properties is called a **petal**.

- this set is minimal with respect to the inclusion,
- it contains some symbol with more than three arrows,
- if a position is in the petal, the next position in the direction, or the opposite one, of the double arrows, is also in it,
- and in the case of a six arrows symbol, the previous property is true only for one couple of arrows.

These sets are represented on the figures as squares joining four corners when these corners have the right orientations.

Petals containing blue symbols are called order 0 petals. Each one intersect a unique greater order petal. The other ones intersect four smaller petals and a greater one: if the intermediate petal is of order $n \geq 1$, then the four smaller are of order $n-1$ and the greatest one is of order $n+1$. Hence they form a hierarchy, and we refer to this in the text as the **petal hierarchy** (or hierarchy).

We usually call the petals valued with 1 **support petals**, and the other ones are called **transmission petals**.

Lemma 14 ([Rob71])**.** *For all n, an order n petal has size $2^{n+1}+1$.*

We call order n **two dimensional cell** the part of $\mathbb{Z}^2$ which is enclosed in an order $2n+1$ petal, for $n \geq 0$. We also sometimes refer to the order $2n+1$ petals as the cells borders. In particular, order $n \geq 0$ two-dimensional cells have size $4^{n+1}+1$ and repeat periodically with period 4^{n+2}, vertically and horizontally, in every cell or supertile having greater order. See an illustration on Figure 8.

2.4.3 Alignment positioning

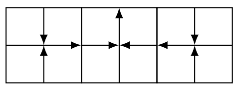
If a configuration of the first layer has two infinite order supertiles, then the two sides of the column or line which separates them are non dependent. The two infinite order supertiles of this configuration can be shifted vertically (resp. horizontally) one from each other, while the configuration obtained stays an element of the subshift. This is an obstacle to dynamical properties such as minimality or transitivity, since a pattern which crosses the separating line can not appear in the other configurations. In this section, we describe additional layers that allow aligning all the supertiles having the same order and eliminate this phenomenon.

Here is a description of the second layer:

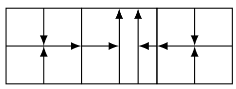
Symbols: nw, ne, sw, se, and a blank symbol.

The *rules* are the following ones:

- **Localization:** the symbols nw, ne ,sw and se are superimposed only on three arrows and five arrows symbols in the Robinson layer.
- **Induction of the orientation:** on a position with a three arrows symbol such that the long arrow originate in a corner is superimposed a symbol corresponding to the orientation of the corner.
- **Transmission rule:** on a three or five arrows symbol position, the symbol in this layer is transmitted to the position in the direction pointed by the long arrow when the Robinson symbol is a three or five arrows symbol with long arrow pointing in the same direction.
- **Synchronization rule:** On the pattern

  or

  in the Robinson layer, if the symbol on the left side is ne (resp. se), then the symbol on the right side is nw (resp. sw). On the images by rotation of these patterns, we impose similar rules.
- **Coherence rule:** the other couples of symbols are forbidden on these patterns.

Global behavior: the symbols ne, nw, sw, se designate orientations: north east, north west, south west and south east. We will re-use this symbolisation in the following. The localization rule implies that these symbols are superimposed on and only on straight paths connecting the corners of adjacent order n cells for some integer n.

The effect of transmission and synchronization rules is stated by the following lemma:

Lemma 15. *In any configuration x of the subshift X_{adR}, any order n supertile appears periodically in the whole configuration, with period 2^{n+2}, horizontally and vertically.*

Proof.
- This property is true in an infinite supertile: this is the statement of Lemma 13. Hence the statement is true in a type (i) configuration. This is also true in a type (iii) configuration, since the infinite supertiles are aligned, and that the positions where the order n supertiles appear are the same in any infinite supertile. This statement uses the property that an order n supertile forces the presence of an order $n+1$ one.
- Consider a configuration of the subshift X_{adR} which is of type (ii). Let us assume that the separating line is vertical, the other case being similar. In order to simplify the exposition we assume that this column intersects $(0, 0)$.

1. **Positions of the supertiles along the infinite line:**
From Lemma 13, there exists a sequence of numbers $0 \leq z_n < 2^{n+2} - 1$ and $0 \leq z'_n < 2^{n+2} - 1$ such that for all $k \in \mathbb{Z}$, the orientation symbol on positions $(-1, z_n + k.2^{n+2})$ (in the column on the left of the separating one) is se and the orientation symbols on positions $(1, z'_n + k.2^{n+2})$ is sw. The symbol on positions $(-1, z_n + k.2^{n+2} + 2^{n+1})$ is then ne and is nw on positions on positions $(1, z'_n + k.2^{n+2} + 2^{n+1})$: this comes from the fact that an order n petal has size $2^{n+1} + 1$.
Let us prove that for all n, $z_n = z'_n$. This means that the supertiles of order n on the two sides of the separating line are aligned.
2. **Periodicity of these positions:**
Since for all n, there is a space of 2^n columns between the rightmost or leftmost order n supertile in a greater order supertile and the border of this supertile (by a recurrence argument), this means that the space between the rightmost order n supertiles of the left infinite supertile and the leftmost order n supertile of the right infinite supertile is $2^{n+1} + 1$. Since two adjacent of these supertiles have opposite orientations, this implies that each supertile appears periodically in the horizontal direction (and hence both horizontal and vertical directions) with period 2^{n+2}.

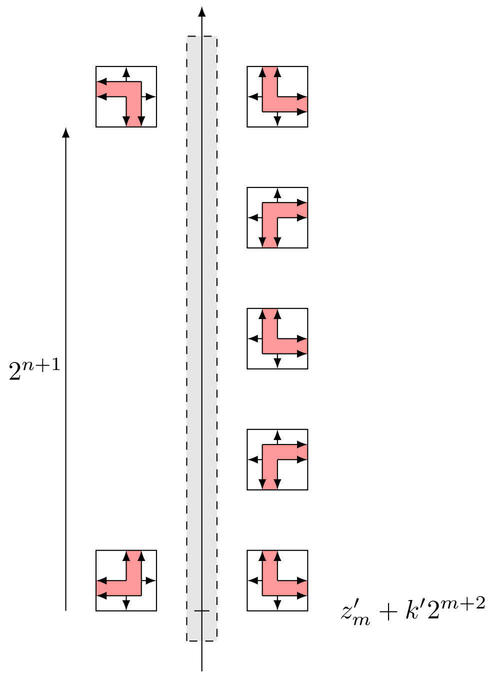


Figure 10: Schema of the proof. The separating line is colored gray.

3. **The orientation symbols force alignment:**
Assume that there exists some n such that $z_n \neq z'_n$. Since $\{z_n + k.2^{n+2}, (n,k) \in \mathbb{N} \times \mathbb{Z}\} = \mathbb{Z}$, this implies that there exist some $m \neq n$ and some k, k' such that
$$z_n + k.2^{n+2} = z'_m + k'.2^{m+2}.$$
One can assume without loss of generality that $m < n$, exchanging m and n if necessary. Then the position $(-1, z_n + k.2^{n+2} + 2^{n+1})$ has orientation symbol equal to ne. As a consequence, the position $(1, z'_m + k'.2^{m+2} + 2^{n-m-1}.2^{m+2})$ has the same symbol. However, by definition, this position has symbol se: there is a contradiction. This situation is illustrated on Figure 10.

$\square$

2.4.4 Completing blocks

Let $\chi : \mathbb{N}^* \to \mathbb{N}^*$ such that for all $n \geq 1$,

$$\chi(n) = \left\lceil \log_2(n) \right\rceil + 4.$$

Let us also denote χ' the function such that for all $n \geq 1$,

$$\chi'(n) = \left\lceil \frac{\left\lceil \log_2(n) \right\rceil}{2} \right\rceil + 2.$$

The following lemma will be extensively used in the following of this text, in order to prove dynamical properties of the constructed subshifts:

Lemma 16. *For all $n \geq 1$, any n-block in the language of X_{adR} is sub-pattern of some order $\chi(n)$ supertile, and is sub-pattern of some order $\chi'(n)$ order cell.*

Proof.

1. **Completing into an order $2^{\lceil \log_2(n) \rceil + 1} - 1$ block:**

   Consider some n-block p that appears in some configuration x of the SFT X_{adR}. We can complete it into a $2^{\lceil \log_2(n) \rceil + 1} - 1$ block, since $2^{\lceil \log_2(n) \rceil + 1} - 1 \geq 2n - 1 \geq n$ for all $n \geq 1$.

2. **Intersection with four order $\lceil \log_2(n) \rceil$ supertiles:**

   From the periodic appearance property of the order $\lceil \log_2(n) \rceil$ supertiles in each configuration, this last block intersects at most four supertiles having this order. Let us complete p into the block whose support is the union of the supports of the supertiles and the cross separating these.

3. **Possible patterns after this completion according to the center symbol:**

   Since this pattern is determined by the symbol at the center of the cross and the orientations of the supertiles, the possibilities for this pattern are listed on Figure Figure 11. Indeed, when the orientations of the supertiles are like 1. on Figure 11, each of the supertiles forcing the presence of an order $\lfloor \log_2(n) \rfloor + 1$ supertile, the center is a red corner. When the orientations of the supertiles are like 2, 3, 4,5 on Figure 11, the center of the block can not be superimposed with a red corner since the two west supertiles force an order $\lfloor \log_2(n) \rfloor + 1$ supertile, as well as the two east supertiles. This forces a non-corner symbol on the position considered.

   For type $4, 5, 9, 10$ patterns, there are two possibilities: the values of the two arms of the central cross are equal or not. Hence the notation $4, 4'$, where $4'$ designates the case where the two values are different.

   One completes the alignment layer on p according to the restriction of the configuration x.

   On these patterns, the value of symbols on the cross is opposed to the value of the symbols on the crosses of the four supertiles composing it.

4. **Localization of these patterns as part of a greater cell:**

   The way to complete the obtained pattern is described as follows:

   (a) When the pattern is 1. on Figure 11, this is an order $\lfloor \log_2(n) \rfloor + 1$ supertile and the statement is proved. Indeed, any order $\lceil \log_2(n) \rceil + 1$ supertile is a sub-pattern of any order $\lfloor \log_2(n) \rfloor + 4$ one.

   (b) One can see the other patterns on Figure 11 in an order $\lfloor \log_2(n) \rfloor + 1$, $\lfloor \log_2(n) \rfloor + 2$, $\lfloor \log_2(n) \rfloor + 3$, or $\lfloor \log_2(n) \rfloor + 4$ supertile, depending on how was completed the initial pattern thus far (this correspondance is shown on Figure 12), hence a sub-pattern of an order $\lfloor \log_2(n) \rfloor + 4$ supertile.

The orientation of the greater order supertiles implied in this completion are chosen according to the symbols of the alignment layer. This layer is then completed.

5. This implies that any n-block is the sub-pattern of an order $2(\left\lceil \frac{1}{2}\lceil \log_2(n)\rceil \right\rceil + 2) + 1$ supertile, which is included into an order $\left\lceil \frac{1}{2}\lceil \log_2(n)\rceil \right\rceil + 2$ cell.

$\square$

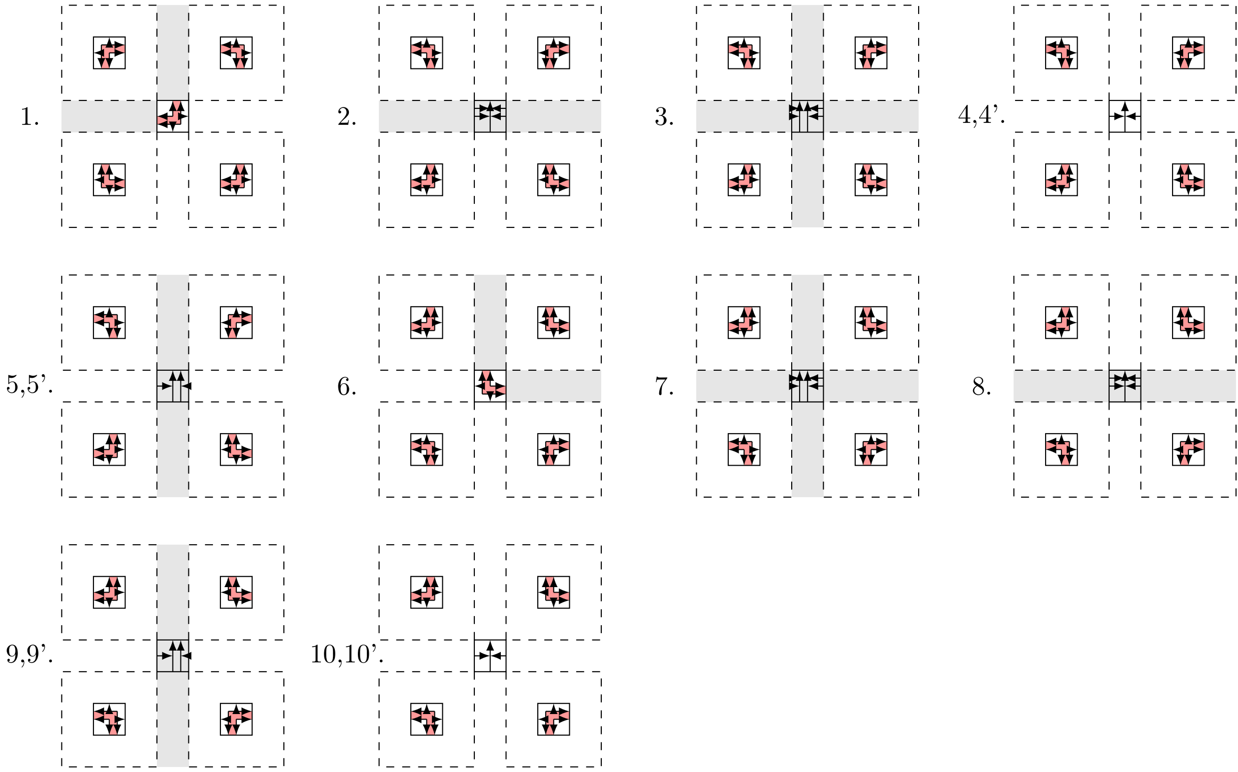


Figure 11: Possible orientations of four neighbor supertiles having the same order.

2.4.5 Linear net gluing property of X_{adR}

Let us denote $id : \mathbb{N} \to \mathbb{N}$ the function defined by $id(n) = n$ for all n.

Proposition 17. *The subshift X_{adR} is $32id$-net gluing, hence linearly net gluing.*

Proof. Let p, q be two n-blocks in the language of X_{adR} with $n \geq 1$. There exists $m \in \mathbb{N}$ such that $2^{m+1} - 1 < n \leq 2^{m+2} - 1$. Hence, there is a supertile S of order $m + 5$ (this comes from the completion result on this subshift) where p appears. Consider a configuration $x \in X_{adR}$ in which the pattern q appears in position $(0, 0)$. The supertile S appears periodically in x with period $2^{m+6} = 32.2^{m+1} \leq 32n$. Thus the gluing set of p relatively to q in X_{adR} contains a set $\mathbf{u} + 2^{m+6}(\mathbb{Z}^2 \backslash \{(0, 0)\})$ for some $\mathbf{u} \in \mathbb{Z}^2$. Thus X_{adR} is 32id net gluing. $\square$

3 Existence of periodic points for f-block gluing SFT

In this section we study the existence of a periodic point in f-block gluing SFT according to the gap function f.

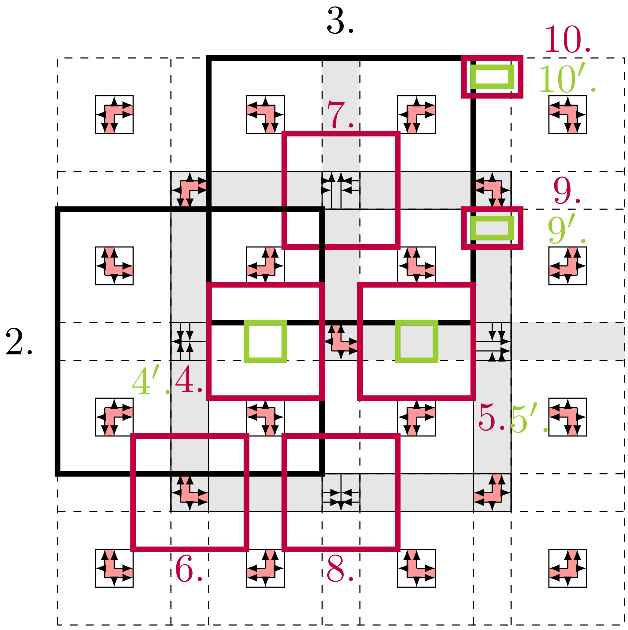


Figure 12: Illustration of the correspondance between patterns of Figure 11 and parts of a supertile.

3.1 Under some threshold for the gap function, there exist periodic points

In [PS15], the authors show that any constant block gluing SFT admits a periodic point. Using a similar argument, we obtain an upper bound on the gap functions forcing the existence of periodic points.

Proposition 18. *Let $X \subset \mathcal{A}^{\mathbb{Z}^2}$ be some SFT having rank $r \geq 2$ which is f-block gluing for some function f. If this function verifies that there exists $n \in \mathbb{N}$ such that*

$$f(n) < \frac{\log_{|\mathcal{A}|}(n-r+2)}{r-1} - r + 2,$$

then X admits a periodic point.

Proof. Let w be a $n \times (r-1)$ pattern in the language of X.

1. **Gluing the pattern w over itself:**

   By the f-block gluing property, there exists $x \in X$ such that

   $$x_{[\![0,n-1]\!] \times [\![0,r-2]\!]} = w = x_{[\![0,n-1]\!] \times [\![f(n)+r-1, f(n)+2r-3]\!]}.$$

2. **Taking w long enough, two of the columns in the obtained pattern are equal:**

   Consider the sub-patterns of $x_{[\![0,n-1]\!] \times [\![0,f(n)+r-2]\!]}$ over supports $[\![k, k+r-2]\!] \times [\![0, (f(n)+r-2)]\!]$. There are $n-(r-2)$ of them and the number of possibilities is $|\mathcal{A}|^{(r-1)(f(n)+r-2)}$. Since we have

   $$f(n) < \frac{\log_{|\mathcal{A}|}(n-r+2)}{r-1} - r + 2,$$

   by the pigeon hole principle, there exists $k \in [\![0, n-r+1]\!]$ and $l \geq 1$ such that

   $$x_{[\![k,k+r-2]\!] \times [\![0,f(n)+r-2]\!]} = x_{[\![k+l,k+l+r-2]\!] \times [\![0,f(n)+r-2]\!]}$$

   (see Figure 13).

3. **Construction of a periodic configuration:**

Consider the configuration defined by

$$z_{(il,j(f(n)+r-2))+[\![0,l-1]\!]\times[\![0,f(n)+r-2]\!]} = x_{[\![k,k+l-1]\!]\times[\![0,f(n)+r-2]\!]}$$

for all $(i,j) \in \mathbb{Z}^2$. It consists in covering $\mathbb{Z}^2$ with the pattern $x_{[\![k,k+l-1]\!]\times[\![0,f(n)+r-2]\!]}$. This configuration is periodic by definition. Moreover, it satisfies the local rules of X. We deduce that $z \in X$. Thus X admits a periodic point.

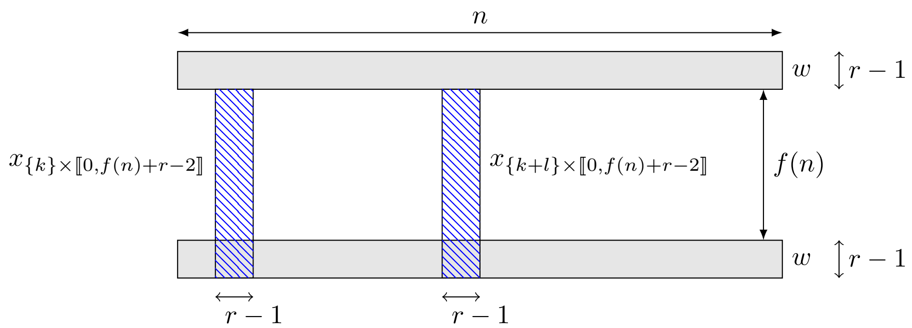


Figure 13: If $n - r + 2 > |\mathcal{A}|^{(r-1)(f(n)+r-2)}$, it is possible to find a pattern of x for which the horizontal and vertical borders are similar.

□

3.2 Under some smaller threshold, the set of periodic points is dense and the language is decidable

Using a similar argument as in [PS15], we obtain also an upper bound on the gap functions that force the density of periodic points, and as a consequence the decidability of the language.

Proposition 19. *Let X be some f-block gluing $\mathbb{Z}^2$-SFT, where f is a function such that*

$$f(n) \in o(\log(n))$$

and $f \leq id$. Then X has a dense set of periodic points.

Proof. 1. **Gluing multiple times the same block P horizontally:**

Consider P some n-block in the language of X, and take 2^k copies of it. We group them by two and glue the couples horizontally, at distance $f(n)$. Then glue the obtained patterns after grouping them by two, at distance $f(2n + f(n))$, and repeat this operation until having one rectangular block Q. The size of this block is equal to $(2id + f)^{\circ k}(n)$.

2. **Glue the same rectangular pattern over and under the obtained one:**

Then consider some $(2id + f)^{\circ k}(n) \times (r-1)$ pattern R, where r is the rank of the SFT X. Glue it on the top of Q with $f(\max((2id + f)^{\circ k}(n), n, r))$ lines between the two rectangles. Then glue the rectangle R under the obtained pattern with

$$f(\max(f(\max((2id + f)^{\circ k}(n), n, r)) + r + n, (2id + f)^{\circ k}(n)))$$

lines between them. For k great enough (depending on n), these two last distances are equal to $f((2id + f)^{\circ k}(n))$, and $f(f((2id + f)^{\circ k}(n)) + n + r)$ respectively. By the gluing property, the obtained pattern is in the language of X (see Figure 14).

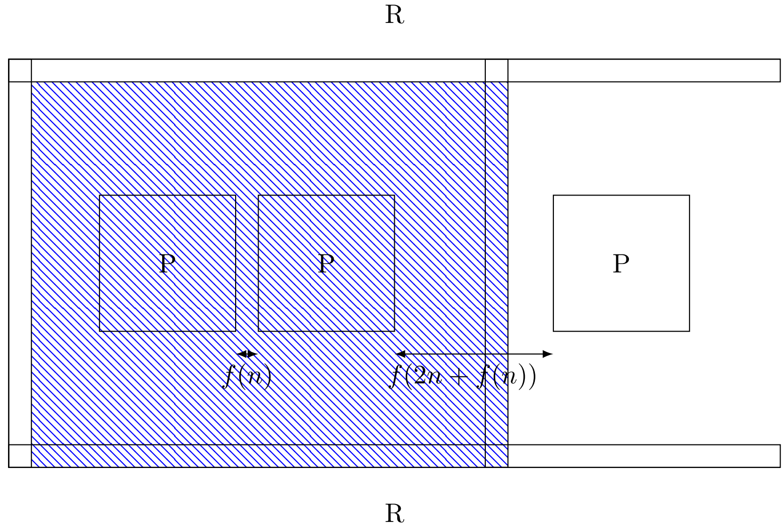


Figure 14: An illustration of the proof of Proposition 19. The result of the procedure is the colored rectangle.

3. **Pigeon hole principle on the columns of the obtained pattern:**

   Consider the $(r-1) \times (r+n+f((2id+f)^{\circ k}(n)) + f(f((2id+f)^{\circ k}(n)+n+r))$ sub-patterns that appear on the bottom of the columns just on the right of each occurrence of the pattern P. There are 2^k of them, and there are at most $(|\mathcal{A}|)^{r+n+f((2id+f)^{\circ k}(n)+f(f((2id+f)^{\circ k}(n)+n+r))}$ different possibilities. From the fact that $f \leq id$, it follows that

   $$(|\mathcal{A}|)^{r+n+f((2id+f)^{\circ k}(n)+f(f((2id+f)^{\circ k}(n)+n+r))} \leq (|\mathcal{A}|)^{2(r+n)+2f(3^k n)} \leq 2^k$$

   for k great enough.

   By the pigeon hole principle, two of these patterns are equal. Consider the rectangle between these two occurrences (including the second one).

4. **Construction of a periodic configuration containing P:**

   This rectangle can be repeated on the whole plane to get a periodic configuration which is in X.

   The set of periodic configurations obtained by this method is dense in X (for every pattern in its language appear in such a configuration).

□

Corollary 20. *A subshift X verifying the conditions of Proposition 19 has decidable language.*

Proof. The complementary of the langage of a SFT can be enumerated. Since it is possible to enumerate periodic configurations of a SFT and that under the conditions of Proposition 19 this set is dense, we conclude that it is also possible to enumerate the langage. We conclude that the subshift has a decidable language. □

3.3 An example of linearly block gluing SFT with non decidable language

The property obtained in Proposition 20 is no longer true considering linearly block gluing subshifts of finite type. In this section we provide an example of linearly block gluing subshift of finite type having non decidable language.

To construct such example we need to introduce the notion of Turing machine. A Turing machine is an automaton with a finite number of internal states which reads and writes letters on an one-sided infinite tape. The computation begins with the machine in a special initial state and the head located over the leftmost symbol. Initially, the tape contains some data which is the input of the computation. The state of the data tape along with the location and internal state of the machine are called a configuration of the Turing machines. A configuration uniquely determines all the future configurations by a discrete time computation process. At each iteration the machine is located over some symbol of the tape, reads it and based on this data and on its internal state, performs the following actions : it replaces the current data symbol by a new one, updates its internal state and moves to the the left or right. The computation may halt after a finite number of steps if the machine either moves off the tape or enters a halting state. A machine is formally some $\mathcal{M} = (Q, q_0, q_h, \mathcal{A}_\mathcal{M}, \#, \delta)$ where Q refers to the set of internal states of the machine, q_0 the initial state, q_h the halting state, $\mathcal{A}_\mathcal{M}$ the tape alphabet with a blank symbol $\#$ and $\delta : Q \times \mathcal{A}_\mathcal{M} \to Q \times \mathcal{A}_\mathcal{M} \times \{L, R\}$ the transition function (where L means left and R means right, and q_h the halting state).

The set of possible space-time diagrams of a machine (subset of $(A \times Q \times \{\leftrightarrow\} \cup A \times \{\leftarrow, \rightarrow\})^{\mathbb{Z}^2}$ where line n is the image of the line $n-1$ after one step of computation, and the arrows symbols are used so that there is a unique machine head in a line) is of finite type, with constraints on 3×2 patterns as follows :

1. If the first line of the pattern contains no head, it has to be as follows :

u	v	w
u	v	w

2. Else, if for instance the machine head is in the $(1, 3)$ position with state q_1 and data w and $\delta(q_1, w) = (q_2, x, L)$, we forbid another pattern than :

u	(q_2, v)	x
u	v	(q_1, w)

with similar rules for other local configurations.

3. Moreover, the incoming/outgoing arrows have to match (this guarantees that there is a unique machine working in a configuration).

Proposition 21. *There exists some $O(n)$-block gluing $\mathbb{Z}^2$-SFT with non decidable language.*

Idea of the proof. *We construct a structure subshift which consists of infinite and constantly growing areas for the computations of machines. These areas can be distorted by shifting its adjacent lines one with respect to the other. This can be done in any of the two possible directions. This allows the linear block gluing. Then we implement a universal Turing machine, and forbid it to stop. This implies the non decidability.*

Proof. Let X_{undec} be a subshift, product of two layers. Here is a description of these layers:

1. **Computation areas layer:**

   This layer has the following symbols:

   Its local rules are the following ones:

   - two horizontally adjacent non blank symbols have the same color.
   - two vertically adjacent non blank symbols verify the following rules:

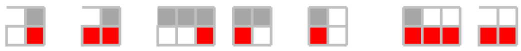
(a) if the bottom symbol is , the top symbol is or .
(b) if the bottom symbol is , the top symbol is .
(c) if the bottom symbol is , the top symbol is .
(d) if the bottom symbol is or , the top symbol is .

These rules allow to shift a row of the area from the one under, choosing the direction. Moreover the shifts happen each time by groups of two, so that the shift counterbalances the growth of the area.

- the patterns

are forbidden, where the gray symbol stands for any non blank symbol.

- the patterns



are forbidden, where the gray symbol stands for any non blank symbol. Similar rules are imposed, replacing the red symbol with a green one.

- the patterns

are forbidden, where the gray symbol stands for any non blank symbol. Similar rules replacing the red symbol with an orange one. These rules allow to control the shape of the areas.

These rules imply that:

- above ^{n} there is ^{n}, or ^{n}.
- above ^{n} there is ^{n}.
- above ^{n} there is ^{n}.
- above ^{n} there is ^{n}.
- above ^{n} there is ^{n}.

The computation areas consist of colored areas. They lie on a background of symbols. These areas are distorted infinite triangles: in two adjacent rows, the intersection of the area with the top row is larger than in the bottom row by one position on the right and one position on the left. Then this row is shifted or not, horizontally in one of the two directions, depending on the colors of the rows. See Figure 15 for an illustration.

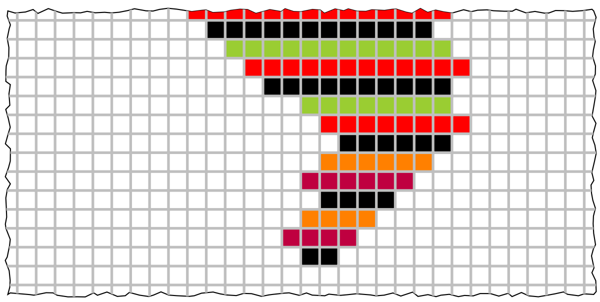


Figure 15: An example of configuration that respects the rules of the first layer of $X_{\texttt{Undec}}$.

2. **Machines layer:**

   The second layer consists in the implementation of a Turing machine over these areas.

   The symbols are the elements of $\mathcal{Q} \times \mathcal{A} \times \{\rightarrow, \leftarrow\}$, and the local rules are the following ones:

   (a) the blank symbols are superimposed with a blank symbol, and the Turing machine symbols are superimposed over non blank symbols.

   (b) Moreover, considering a 3×2 pattern whose projection on the first layer is fully colored and the bottom row is black, the rules of the space-time diagram of the machine apply. When the bottom row is not black, the symbols of this row are copied on the top row, on the shifted position according to the color of the bottom row. There rules of the space time diagram are adapted when on the border of the area.

   (c) Any halting state is forbidden.

   These rules imply that for two adjacent rows of an area:

   - if the bottom row is colored black, then the top row is the image of the bottom row by the machine process.
   - in the other cases, the top row is just the image of the bottom row by the shift in the direction corresponding to the color of the bottom row.

   We use a universal Turing machine, which has the following behavior when the initial tape is written with $w\#^\infty$ for w a word on $\{0, 1\}$: it reads the word w which codes for the number of a Turing machine, and then simulates this machine on empty tape.

3. **Properties of this subshift:**

   (a) **The language of this subshift is not decidable:** if it was, we would be able to decide which of the words $w\#^n$ can be written on the bottom of an area. This is impossible since the halting problem is not decidable.

   (b) **This subshift is sharp linearly block gluing:** the worst case for gluing two blocks is when these blocks are filled with colored symbols in the first layer. In order to glue them, we complete the projection of the two blocks in the first layer, as in example 10, into the bottom of a computation area, surrounding it with blank symbols except on the top. Then we complete the trajectory of the machine head if there is one. The two extended patterns can be glued horizontally without constraint on the distance, because lines can be shifted towards opposite directions. For vertical gluing, we extend one of this patterns shifting the area in one direction so that the columns above this patterns are blank. The number of rows and columns depends linearly on the size of this pattern. Then we glue the second pattern on the top, shifting the area in the opposite direction and filling all the positions of $\mathbb{Z}^2$ left undefined by blank symbols.

□

3.4 Existence of aperiodic linearly block gluing subshifts

In this section, we give a proof of the following theorem:

Theorem 22. *There exists a linearly block gluing aperiodic $\mathbb{Z}^2$-SFT.*

Idea of the proof. *This proof uses an operator on subshifts which transforms linearly net gluing subshifts of finite type into linearly block gluing subshifts of finite type. Moreover, it preserves the aperiodicity. The principle of this transformation is to distort $\mathbb{Z}^2$, as illustrated on Figure 16, multiple times and in different directions.*

We then apply this transformation on the Robinson subshift which is known to be aperiodic and that we proved to be linearly net gluing (Proposition 17).

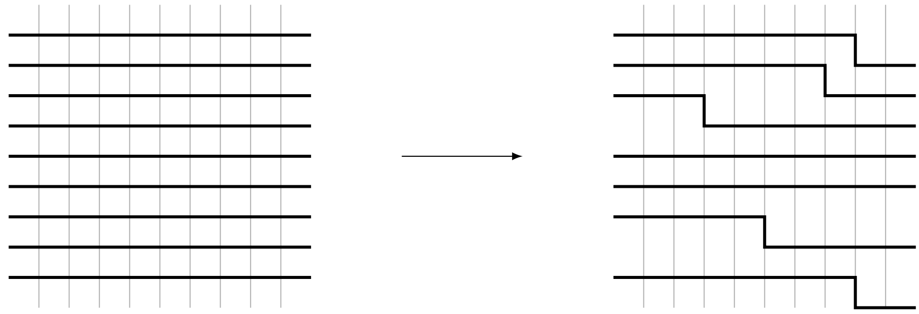


Figure 16: Illustration of the misshaping principle of the transformation for the proof of Theorem 22.

3.4.1 A subshift inducing pseudo-coverings by curves

Definition Let us denote Δ the $\mathbb{Z}^2$ SFT on alphabet $\{\rightarrow, \downarrow\}$, defined by the following forbidden patterns:

$$\begin{array}{c}\downarrow\\ \downarrow\end{array}, \quad \begin{array}{cc}\rightarrow & \downarrow\\ \rightarrow & \rightarrow\end{array}.$$

Pseudo-coverings by curves Let us introduce some words in order to talk about the global behavior induced by these rules:

- An (infinite) **curve** in $\mathbb{Z}^2$ is a set $\mathcal{C} = \varphi(\mathbb{Z})$ for some application $\varphi : \mathbb{Z} \to \mathbb{Z}^2$ such that for all $k \in \mathbb{Z}$, $\varphi(k+1) = \varphi(k) + (1,0)$ or $\varphi(k+1) = \varphi(k) + (1,-1)$.
- We say that a curve is **shifted downwards** at position $\mathbf{j} \in \mathbb{Z}^2$ when there exists some $k \in \mathbb{Z}$ such that $\varphi(k) = \mathbf{j}$ and $\varphi(k+1) = \mathbf{j} + (1,-1)$.
- A **pseudo-covering of $\mathbb{Z}^2$ by curves** is a sequence of curves $(\mathcal{C}_k)_{k\in\mathbb{Z}}$ such that for every $\mathbf{j} \in \mathbb{Z}^2$, there exists some $k \in \mathbb{Z}$ such that $\{\mathbf{j}, \mathbf{j}+(0,1)\} \bigcap \mathcal{C}_k \neq \emptyset$ (meaning that every element of $\mathbb{Z}^2$ is in a curve or the vector just above is), and for every $k \neq k'$, $\mathcal{C}_k \bigcap \mathcal{C}_{k'} = \emptyset$ (the curves do not intersect). We say that two curves in this pseudo-covering are **contiguous** when the area delimited by these two curves does not contain any third curve. The gap between two contiguous curves in some column is the distance between the intersection of these two curves with the column. This gap is 0 or 1 between two contiguous curves in a pseudo-covering.

A configuration in Δ induces a pseudo-covering by curves Let $\delta \in \Delta$. Let us consider the pseudo-covering of $\mathbb{Z}^2$ by curves $(\mathcal{C}_k(\delta))_{k\in\mathbb{Z}}$, such that $\mathcal{C}_k(\delta) = \varphi_{\delta,k}(\mathbb{Z})$ and where $\varphi_{\delta,k}$ is as follows.

If $\delta_{(0,0)} = \rightarrow$, then $(0,0) = \varphi_{\delta,0}(0)$. Else $(0,1) = \varphi_{\delta,0}(0)$. In addition, for m the biinfinite sequence of integers such that $\delta_{(m_k,0)}$ is the kth $\rightarrow$ in the column 0, counting from the previous considered one, then $(m_k,0) = \varphi_{\delta,k}(0)$.

For every $\mathbf{i} \in \mathbb{Z}^2$ such that $\mathbf{i} = \varphi_{\delta,k}(n)$ for some $k \in \mathbb{Z}, n \in \mathbb{Z}$:

- if $\delta_{\mathbf{i}} = \rightarrow$, and $\delta_{\mathbf{i}+(1,0)} = \downarrow$, then $\varphi_{\delta,k}(n+1) = \mathbf{i} + (1,-1)$ (the curve is shifted downwards in this column).
- else $\delta_{\mathbf{i}} = \rightarrow$ and $\delta_{\mathbf{i}+(1,0)} = \rightarrow$, then $\varphi_{\delta,k}(n+1) = \mathbf{i} + (1,0)$.

The first rule implies that all the curves of this pseudo-covering can not be shifted downwards multiple times in the same column. The second one implies that if a curve is shifted downwards at position $\mathbf{i} \in \mathbb{Z}^2$, then there is no curve going through position $\mathbf{i} - (1,1)$.

3.4.2 Distortion operators on subshifts of finite type

Let $\mathcal{A}$ be some alphabet. Denote $S_{\mathcal{A}}$ the set of SFT over $\mathcal{A}$. We introduce operators $d_{\mathcal{A}} : \mathcal{S}_{\mathcal{A}} \to \mathcal{S}_{\tilde{\mathcal{A}}}$, with $\tilde{\mathcal{A}} = (\mathcal{A} \cup \{\square\}) \times \{\rightarrow, \downarrow\}$.

Pseudo-projection Consider the subshift $\Delta_{\mathcal{A}} \subset (\mathcal{A} \cup \{\square\})^{\mathbb{Z}^2} \times \Delta$, where the forbidden patterns are the ones defining Δ and the patterns where a symbol in $\mathcal{A}$ is superimposed to a $\downarrow$ symbol or where $\square$ is superimposed to a $\rightarrow$ symbol.

Define a **pseudo-projection** $\mathcal{P} : \Delta_{\mathcal{A}} \to \mathcal{A}^{\mathbb{Z}^2}$, as follows: for $(y, \delta) \in \Delta_{\mathcal{A}}$,

$$(\mathcal{P}(y, \delta))_{i,j} = y_{\varphi_{\delta,j}(i)}.$$

Notice that the function $\mathcal{P}$ is continuous but not shift invariant.

We denote π_1 the projection on the first layer ($\pi_1(y, \delta) = y$), and π_2 the projection on the second layer.

Definition of the operators Let X be some SFT on the alphabet $\mathcal{A}$, and define $d_{\mathcal{A}}(X) = \mathcal{P}^{-1}(X)$. Denoting r the rank of the SFT X, $d_{\mathcal{A}}(X)$ can be defined by imposing that, considering the intersection of a set of r contiguous curves with r consecutive columns, the corresponding r-block is not a forbidden pattern in X. Because the gap between two contiguous curves is bounded, $d_{\mathcal{A}}(X)$ is defined by a finite set of forbidden patterns. Then this is a SFT.

One can think to $d_{\mathcal{A}}(X)$ as having two layers. The first one has alphabet $(\mathcal{A} \cup \{\square\})$ and called the X layer. The second one has alphabet $\{\rightarrow, \downarrow\}$ and is called the Δ layer. For X is the subshift X_R or X_{adR}, Figure 17 shows an example of pattern in the X layer of the subshift $d_{\mathcal{A}}(X)$ whose pseudo-projection is the supertile south west order two supertile.

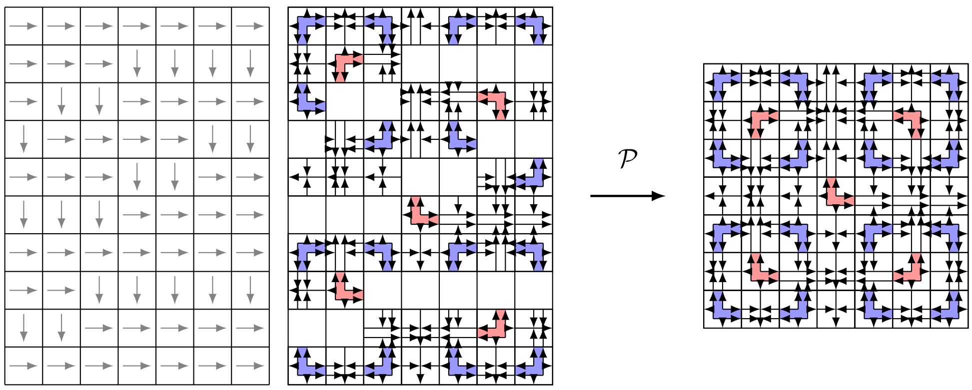


Figure 17: Example of a pattern which is sent to the south west two order supertile by pseudo-projection.

Properties of the operators $d_{\mathcal{A}}$ We use the following properties of the operators $d_{\mathcal{A}}$ in order to prove Theorem 22.

Proposition 23. *For an aperiodic SFT X on the alphabet $\mathcal{A}$, $d_{\mathcal{A}}(X)$ is also aperiodic.*

Idea of the proof. *The main argument of this proof is that if a configuration in $d_{\mathcal{A}}(X)$ is periodic, then the projection on the Δ layer is periodic. This means that although there is a distortion of the configuration in X, the distortion is done in a periodic way. From this, we deduce that the pseudo-projection on X of this configuration is periodic.*

Proof. Assume that there exists a configuration $z \in d_{\mathcal{A}}(X)$ which is periodic: there exists $n > 0$ such that for all i, j, $z_{i+n,j} = z_{i,j+n} = z_{i,j}$. We will prove that the pseudo-projection of z on X, $x = \mathcal{P}(z)$ is periodic.

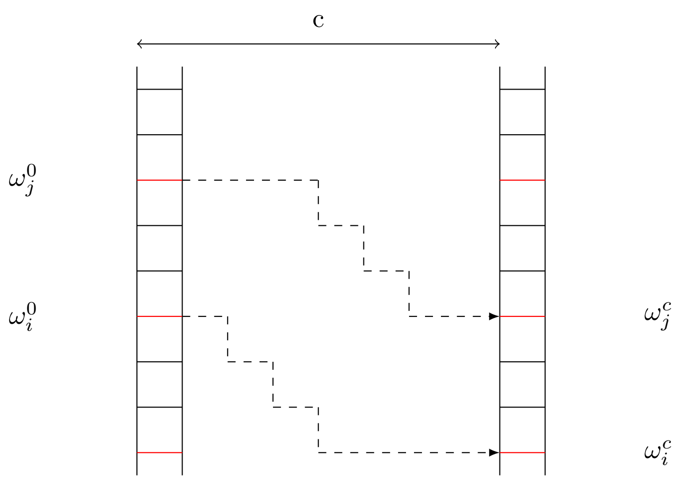


Figure 18: Schema of the proof of Proposition 23.

1. **Coding the positions of intersections of the curves with a column:**

   To each column k in z we associate the bi-infinite word ω^k in $(\mathbb{Z}/n\mathbb{Z})^{\mathbb{Z}}$ such that for all $i \in \mathbb{Z}$, ω_i^k is the element $\overline{m_i}$ of $\mathbb{Z}/n\mathbb{Z}$, class modulo n of m_i where $(0, m_i)$ is the intersection position of the ith curve of $\pi_2(z)$ with the column k.

2. **Function relating the codings of two columns:**

   Following a curve (see Figure 18) from the column 0 to the column n, we get an application ψ from the set of possible $\overline{m_i}$ into itself. This comes from the vertical periodicity of the projection of z on the second layer. The word ω^n is obtained from ω^0 applying ψ to all the letters in ω^n.

3. **Coding of the intersections and periodicity:**

   Since ψ is an invertible function from a finite set into itself (indeed, we have an inverse map following the curve backwards), there exists some $c > 0$ integer such that $\psi^c = \mathrm{Id}$. As a consequence, $\omega^{nc+j} = \omega^j$ for all integers j. That means that the column $cn+j$ is obtained by shifting kn times downwards the column j, for some $k \geq 0$. Using the horizontal periodicity of y, we then have that

   $$(x_{j,l})_{l\in\mathbb{Z}} = (x_{cn+j,l+kn})_{l\in\mathbb{Z}}.$$

   Using the vertical periodicity, that $(x_{j,l})_{l\in\mathbb{Z}} = (x_{cn+j,l})_{l\in\mathbb{Z}}$, hence the configuration $x \in X$ is periodic, which can not be true.

As a consequence, no configuration in $d_{\mathcal{A}}(X)$ can be periodic. Thus this subshift is aperiodic. □

The following proposition will be a useful tool in order to prove that the operators $d_{\mathcal{A}}$ transform linearly net gluing subshifts into block gluing ones.

Proposition 24 (Completing blocks)**.** *There exists an algorithm $\mathcal{T}$ that, taking as input some locally admissible n-block p of Δ, outputs a rectangular pattern $\mathcal{T}(p)$ which has p as a sub-pattern and such that:*

- *the number of curves in $\mathcal{T}(p)$ is equal to the number of its columns,*
- *the dimensions of $\mathcal{T}(p)$ are smaller than $5n$,*

- *the top and bottom rows of $\mathcal{T}(p)$ have only $\rightarrow$ symbols- this means that all the curves crossing $\mathcal{T}(p)$ comes from its left side and go to the right side.*

Remark 4. *The properties of the pattern $\mathcal{T}(p)$ ensure that this is a globally admissible pattern. Hence every locally admissible pattern of the subshift Δ is globally admissible.*

Idea of the proof. *The proof consists in extending the curves that cross a pattern from above and below. Then we add curves on the top and bottom that are straighter and straighter.*

Proof. If p is a 1-block, and p is a single $\rightarrow$, then the result is direct. If p is a single $\downarrow$, then it can be extended in

$$\begin{matrix} \rightarrow & \rightarrow & \rightarrow \\ \rightarrow & \downarrow & \downarrow \\ \downarrow & \rightarrow & \rightarrow \\ \rightarrow & \rightarrow & \rightarrow \end{matrix},$$

which verifies the previous assertion.

If p is a n-block with $n \geq 2$:

First step. Extending the curves that enter in the block upside/downside:

1. If in the top row of the block p there is the pattern $\downarrow\rightarrow$ - this means that there is an incoming curve (the position of the $\rightarrow$ is in this curve) - then we add symbols in the row just above. We extend each of the incoming curves, considering the curves from left to right, in this row. For this purpose we add a $\downarrow$ over the $\rightarrow$ for each of the patterns $\downarrow\rightarrow$. Then we add $\rightarrow$ symbols on the left of this one until meeting another $\downarrow$ or the left side of the block. If in the added row there are $\downarrow\rightarrow$ patterns, then return to the beginning of this step. Else, stop.

2. Do similar operations on the bottom of the block.

Since the number of $\downarrow\rightarrow$ patterns in the top row is strictly decreasing, this series of operations stops at some point.

Example 25. *If we take p the following 4-block*

$$\begin{matrix} \rightarrow & \rightarrow & \downarrow & \rightarrow \\ \rightarrow & \downarrow & \rightarrow & \rightarrow \\ \downarrow & \rightarrow & \rightarrow & \rightarrow \\ \rightarrow & \rightarrow & \rightarrow & \rightarrow \end{matrix}$$

at this point, we obtain:

$$\begin{matrix} \rightarrow & \rightarrow & \rightarrow & \downarrow \\ \rightarrow & \rightarrow & \downarrow & \rightarrow \\ \rightarrow & \downarrow & \rightarrow & \rightarrow \\ \downarrow & \rightarrow & \rightarrow & \rightarrow \\ \rightarrow & \rightarrow & \rightarrow & \rightarrow \end{matrix}$$

Second step. Completing the pattern on the top and bottom until the top row and bottom row are straight:

While the top curve of the pattern is not straight, apply the following procedure:

1. On the top of the last column, add a $\downarrow$ and keep adding $\downarrow$ on the left until meeting on the left an already defined symbol (there can be such symbols, introduced in the first step) or the left extremity.

2. Add another curve above by the following procedure. Add a $\rightarrow$ on the top of the last column, and then add $\rightarrow$ symbols on the left until meeting an already defined symbol on the left. When that happens, add a $\downarrow$ above and then add $\rightarrow$'s on the left until reaching a defined symbol. Repeat this operation until reaching the first column.

Since the number of times that the top curve is shifted downwards decreases at each step, this series of operations stops.

Do similar operations on the bottom.

Example 26. *At this point, we obtain:*

$$\begin{array}{cccc}
\rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow & \downarrow \\
\rightarrow & \rightarrow & \downarrow & \rightarrow \\
\rightarrow & \downarrow & \rightarrow & \rightarrow \\
\downarrow & \rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow & \rightarrow
\end{array}$$

Third step. Equalization of the number of curves and the number of columns:

If the number of columns is smaller than the number of curves, then add a number of columns equal to the difference, by adding copies of the last column on its right side. If the number of curves is smaller, then add lines of $\rightarrow$ symbols on the top.

Example 27. *After this last step we obtain:*

$$\begin{array}{ccccc}
\rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow & \downarrow & \downarrow \\
\rightarrow & \rightarrow & \downarrow & \rightarrow & \rightarrow \\
\rightarrow & \downarrow & \rightarrow & \rightarrow & \rightarrow \\
\downarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow
\end{array}$$

For p some n-block, the dimensions of $\mathcal{T}(p)$ are smaller than the sum of:

1. the dimension of p (equal to n)
2. two times the number of entering curves by the top and outgoing by the bottom (one for completing the curves (first step)
3. and one for reducing the shifts (second step)).

Each one of these numbers is smaller than n. The third step does not make this bound greater, because in this pattern the number of curves is smaller than the number of lines. As a consequence, the dimensions of $\mathcal{T}(p)$ are smaller than $5n$. $\square$

Proposition 28. *Let X be some linearly net gluing subshift on alphabet $\mathcal{A}$. There exists some vector function $\boldsymbol{u}'$ and a function g that:*

1. *take as arguments two n-blocks p, q for some n,*
2. *respectively associates to these blocks an element of $\mathbb{Z}^2$ and an element of $\mathbb{N}$*

These functions verify that for any couple of n-blocks p, q in the language of $d_{\mathcal{A}}(X)$, the gluing set of p relative to q contains infinite columns regularly displayed. Moreover, the gluing set contains regularly displayed positions in a central column:

$$g(p,q)(\mathbb{Z}\backslash\{0\})\boldsymbol{e}^2 + \boldsymbol{u}'(p,q) \subset \Delta_{d_{\mathcal{A}}(X)}(p,q),$$

$$g(p,q)(\mathbb{Z}\backslash\{-3,...,3\})\boldsymbol{e}^1 + \mathbb{Z}\boldsymbol{e}^2 + \boldsymbol{u}'(p,q) \subset \Delta_{d_{\mathcal{A}}(X)}(p,q),$$

where function g verifies that

$$\max_{p,q} g(p,q) = O(n),$$

where the maximum is over the n-blocks.

Idea of the proof. *This proof consists in analyzing how the operator acts on the gluing set of a n-block p relative to another pattern q. The operator allows perturbations to be introduced on these sets.*

Proof. Let X be some f net gluing subshift on alphabet $\mathcal{A}$, where $f(n) = O(n)$.

Formulation of the linear net gluing of X: This means that there exist two function $\mathbf{u} : \mathcal{L}_n(X)^2 \to \mathbb{Z}^2$ and $\tilde{f} : \mathcal{L}_n(X)^2 \to \mathbb{N}$ such that for all $n > 0$ and every couple of n-blocks r, s in the language of X, the gluing set of r relative to s in X contains

$$\mathbf{u}(r,s) + (n + \tilde{f}(r,s))(\mathbb{Z}^2 - (0,0)),$$

and for all r, s n-blocks,

$$\tilde{f}(r,s) \leq f(n).$$

We consider in this proof that $\mathbf{u} = \mathbf{0}$, since this proof can be adapted to a general function $\mathbf{u}$ without difficulty.

Sufficient conditions to verify: It is sufficient to prove the statement of the proposition for patterns whose projection on Δ are $\mathcal{T}(\pi_2(p))$ and $\mathcal{T}(\pi_2(q))$. Indeed, the size of these patterns is bounded by a linear function of the size of the patterns $p, q \in d_{\mathcal{A}}(X)$.

Let p, q two n-blocks in the language of the subshift $d_{\mathcal{A}}(X)$. Without loss of generality, we can consider that the patterns $\mathcal{T}(\pi_2(q))$ and $\mathcal{T}(\pi_2(p))$ have the same number of curves crossing them. We denote $\tilde{p}$ and $\tilde{q}$ some admissible patterns whose projections of Δ are respectively $\mathcal{T}(\pi_2(p))$ and $\mathcal{T}(\pi_2(q))$. The pseudo-projections of these patterns on X are m-block of X, where m is the number of curves in $\mathcal{T}(\pi_2(p))$ and $\mathcal{T}(\pi_2(q))$. This is due to the fact that the top and bottom rows of these patterns are straight. These pseudo-projections are denoted $\mathcal{P}(\tilde{p})$ and $\mathcal{P}(\tilde{q})$, according to previous notations.

We place the pattern $\mathcal{P}(\tilde{q})$ on position $(0,0)$.

The gluing sets of $d_{\mathcal{A}}(X)$ contain infinite columns periodically displayed: Let us show that the gluing set of $\tilde{p}$ relative to $\tilde{q}$ contains infinite columns periodically displayed.

Let $k \geq 4$ and l integers such that $l \neq 0$, and consider some vector

$$\mathbf{u} = (m + \tilde{f}(\mathcal{P}(\tilde{p}), \mathcal{P}(\tilde{q}))(k,l).$$

We will prove the following:

- When $l \geq 2$ and t is any integer such that

  $$0 \leq t \leq \tilde{f}(\mathcal{P}(\tilde{p}), \mathcal{P}(\tilde{q})) + m,$$

  the position $\mathbf{u} + t.\mathbf{e}^2$ is in the gluing set in $d_{\mathcal{A}}(X)$ of the pattern $\tilde{p}$ relative to $\tilde{q}$.
- When $l \leq -2$, this set contains the position $\mathbf{u} - t.\mathbf{e}^2$, for the same integers t.
- When $l = 1$ or $l = -1$, then this set contains $\mathbf{u} - t.\mathbf{e}^2$, for all t such that

  $$0 \leq t \leq 2(\tilde{f}(\mathcal{P}(\tilde{p}), \mathcal{P}(\tilde{q})) + m).$$

As a consequence, this gluing set contains the whole infinite column that contains

$$(m + \tilde{f}(\mathcal{P}(\tilde{p}), \mathcal{P}(\tilde{q}))(k,0)$$

for all $k \geq 8$. Indeed, it contains an infinity of segments which overlap only on their border. We have the same property for $k \leq -4$, by reversing p and q.

1. **When $|l| \geq 2$:**

   Let t be some integer such that

   $$0 \leq t \leq \tilde{f}(\mathcal{P}(\tilde{p}), \mathcal{P}(\tilde{q})) + m.$$

   In the case $l \geq 2$, we do the following operations. See a schema on Figure 19.

   (a) We extend the curves crossing the pattern $\tilde{q}$ in a straight way until infinity.

   (b) In the case $l \geq 2$, we add straight curves below the obtained pattern. We do that in such a way that these curves have gap 0 between them. On the top, we introduce t times a straight infinite curve with gap 1 with the curve below. Then we add

   $$l(\tilde{f}(\mathcal{P}(\tilde{p}), \mathcal{P}(\tilde{q})) + m) - t - m$$

   straight infinite curves with gap 0 with the curve below.

   (c) Then we add the pattern $\tilde{p}$ on position $\mathbf{u} + t.\mathbf{e}^2$.

   (d) We extend the curves crossing this pattern in a straight way until infinity.

   (e) We add straight lines on the top, without gaps between them.

   (f) Then we color the curves with elements of the curves with elements of $\mathcal{A}$ such that the configuration is admissible. This is possible from the net gluing property of the subshift X. Indeed, the position of the pattern $\mathcal{P}(\tilde{p})$ relatively to $\mathcal{P}(\tilde{q})$ in the pseudo-projection of this configuration is $\mathbf{u}$. Moreover, this vector is in the gluing set of the first pattern relatively to the second one.

   The case $l \leq -2$ is similar. The difference is that the pattern $\tilde{p}$ appears on position $\mathbf{u} - t.\mathbf{e}^2$.

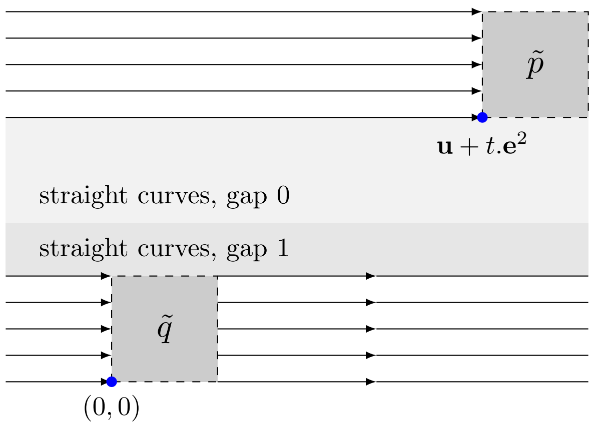


Figure 19: Illustration of the construction for the proof of Theorem 33 when $l \geq 2$.

2. **When $l = 1$ or -1:**

   Here we prove that the pattern $\tilde{p}$ can be glued relatively to $\tilde{q}$ on position $\mathbf{u} - t.\mathbf{e}^2$.

   The steps of a construction of a configuration that supports this gluing are as follows:

   (a) **Compactification of the outgoing curves:**

   We extend $\mathcal{T}(\pi_2(q))$ using the following procedure. While in the last column of the pattern there is some sub-pattern $\begin{smallmatrix}\rightarrow \\ \downarrow\end{smallmatrix}$ (meaning that there is a gap between two outgoing curves), do the following: on the right of the patterns $\begin{smallmatrix}\rightarrow \\ \downarrow\end{smallmatrix}$, write $\begin{smallmatrix}\downarrow \\ \rightarrow\end{smallmatrix}$ and write a copy of the other $\rightarrow$ symbols on their right side.

Example 29. *Taking the same example as in the proof of Proposition 24, the result is:*

$$\begin{array}{ccccccc}
\rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow \\
\rightarrow & \rightarrow & \rightarrow & \downarrow & \downarrow & \downarrow & \rightarrow \\
\rightarrow & \rightarrow & \downarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \downarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\downarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow
\end{array}$$

Since there are m curves the number of additional columns on the right for this step is smaller than m. Indeed, one column is sufficient to reduce the gap between a curve and the curve just below.

(b) **Making the curves shift:**
We add columns on the right of the extension of $\mathcal{T}(\pi_2(q))$. We follow the following procedure, in order to make all the curves in it shift t times:

i. Consider the right part of the pattern constituted with $\rightarrow$ symbols and add a triangle made of $\rightarrow$ symbols except on the diagonal part. On this part we write $\downarrow$ symbols (this is the first shift).

Example 30. *Taking the same example as previously, the result is:*

$$\begin{array}{cccccccccccc}
\rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & & & & & \\
\rightarrow & \rightarrow & \rightarrow & \downarrow & \downarrow & \downarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow \\
\rightarrow & \rightarrow & \downarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & \\
\rightarrow & \downarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & & \\
\downarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & & & \\
\rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & & & &
\end{array}$$

ii. Then repeat $t-1$ times the following operation: add under each $\downarrow$ on the right side a $\rightarrow$ under, and after that a $\downarrow$ on the right of the $\rightarrow$.

Example 31. *Taking the same example as previously, with* $t = 3$*, the result is:*

$$\begin{array}{cccccccccccccc}
\rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & & & & & & & \\
\rightarrow & \rightarrow & \rightarrow & \downarrow & \downarrow & \downarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & & \\
\rightarrow & \rightarrow & \downarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & \rightarrow & \downarrow & \\
\rightarrow & \downarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & \rightarrow & \downarrow & \rightarrow & \downarrow \\
\downarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & \rightarrow & \downarrow & \rightarrow & \downarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & \rightarrow & \downarrow & \rightarrow & \downarrow & \rightarrow & \\
 & & & & & & & \rightarrow & \downarrow & \rightarrow & \downarrow & \rightarrow & & \\
 & & & & & & & & \rightarrow & \downarrow & \rightarrow & & & \\
 & & & & & & & & & \rightarrow & & & &
\end{array}$$

There are at most $2(m + \tilde{f}(\mathcal{P}(\tilde{p}, \tilde{q})))$ additional columns for this step. Indeed, t is smaller than this number and only t columns are needed to shift a compact set of curves.

iii. Complete the curves with $\rightarrow$ symbols so that they end in the last column added.

Example 32. *Taking the same example as previously, with* $t = 3$*, the result is:*

$$\begin{array}{cccccccccccccc}
\rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & & & & & & & \\
\rightarrow & \rightarrow & \rightarrow & \downarrow & \downarrow & \downarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & & \\
\rightarrow & \rightarrow & \downarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & \rightarrow & \downarrow & \\
\rightarrow & \downarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & \rightarrow & \downarrow & \rightarrow & \downarrow \\
\downarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & \rightarrow & \downarrow & \rightarrow & \downarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & \rightarrow & \downarrow & \rightarrow & \downarrow & \rightarrow & \rightarrow \\
 & & & & & & & \rightarrow & \downarrow & \rightarrow & \downarrow & \rightarrow & \rightarrow & \rightarrow \\
 & & & & & & & & \rightarrow & \downarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\
 & & & & & & & & & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow
\end{array}$$

Here, no column is added.

iv. Then extend the curves straightly on a number of columns so that the total number of additional columns is equal to $\mathbf{u}_1 - m$. This is possible since the number of added columns at this point is smaller than $m + 2(m + \tilde{f}(\mathcal{P}(\tilde{p}), \mathcal{P}(\tilde{q})) \leq \mathbf{u}_1$, since $k \geq 4$.

(c) **Extension:**
Then, we extend these curves straightly until infinity on the east side and on the west side.

(d) **Additional curves:**
We add $\tilde{f}(\mathcal{P}(\tilde{p}), \mathcal{P}(\tilde{q}))$ curves on the top when $l = 1$ (resp. on the bottom when $l = -1$) of the obtained pattern at this point, without gap between them. This means that, from left to right, when the curve just below is shifted downwards the curve is also shifted immediately after.

(e) **Positioning the pattern $\mathcal{T}(\pi_2(p))$:**

i. In the last columns where symbols were added in the last step, we position the pattern $\mathcal{T}(\pi_2(p))$. We place it on the top when $l = 1$ (resp. on the bottom when $l = -1$) of the last added curves.

ii. After this we extend the curves on the west side straightly until infinity.

iii. On the east side, we extend the curve without introducing gaps. This step is possible since the minimal value of k is taken sufficiently large. This means that the shifts of the outgoing curves of $\tilde{q}$ do not affect the area where the pattern $\tilde{p}$ is supposed to be glued.

iv. On the top and bottom of the obtained pattern we add curves without introducing any gap in such a way that we fill $\mathbb{Z}^2$.

v. In the end we add $\mathcal{A}$ symbols over the curves, when not already determined. This is possible from the linear net gluing property of X. Indeed, there are $\tilde{f}(\mathcal{P}(\tilde{p}), \mathcal{P}(\tilde{q}))$ added between the patterns $\tilde{p}$ and $\tilde{q}$ and the pattern $\tilde{p}$ is on a position in the column containing $\mathbf{u}$.

See Figure 20 for an illustration of this case, when $l = 1$.

From this construction we deduce that

$$\left\{\mathbf{w} \in \mathbb{Z}^2 \mid \mathbf{w}_1 = \mathbf{v}_1(g(p, q)),\ |\mathbf{v}_1| \geq 4\right\} \subset \Delta_{d_{\mathcal{A}}(X)}(p, q),$$

where

$$g(p, q) = m + \tilde{f}(\mathcal{P}(\tilde{p}), \mathcal{P}(\tilde{q})).$$

Thus,

$$\max_{p,q} g(p, q) = O(n).$$

Indeed, because $m \leq 5n$ and f is non decreasing, this number is smaller than $4(f(5n)+5n) = O(n)$. This comes from the fact that $f(n) = O(n)$.

The gluing sets of $d_{\mathcal{A}}(X)$ contain periodic positions in the central column: Consider some $\mathbf{u} = (m + \tilde{f}(\mathcal{P}(\tilde{p}), \mathcal{P}(\tilde{q})))\mathbf{v}$ with $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$, $\mathbf{v}_2 \neq 0$, and $\mathbf{v}_1 = 0$. The pattern p can be glued relatively to q in $d_{\mathcal{A}}(X)$ in position $\mathbf{u}$ in a configuration $x^{\mathbf{u}}$.

Indeed, the pattern $\mathcal{T}(\pi_2(q))$ can be glued relatively to $\mathcal{T}(\pi_2(p))$ in Δ with relative position $\mathbf{u}$. In order to prove that one can glue the two patterns with this relative position. Then one completes straightly the curves that go through the two patterns and fulfill $\mathbb{Z}^2$ with straight curves. One shifts the configuration in such a way that $\pi_2(q)$ appears in position $(0, 0)$. Then one completes this configuration with letters in $\mathcal{A}$ so that the pseudo-projection is $x^{\mathbf{u}}$.

This means that

$$\left\{\mathbf{w} \in \mathbb{Z}^2 \mid \mathbf{w}_2 \in (m + \tilde{f}(\mathcal{P}(\tilde{p}), \mathcal{P}(\tilde{q})))(\mathbb{Z}\backslash\{0\}),\ \mathbf{w}_1 = 0\right\} \subset \Delta_{d_{\mathcal{A}}(X)}(p, q).$$

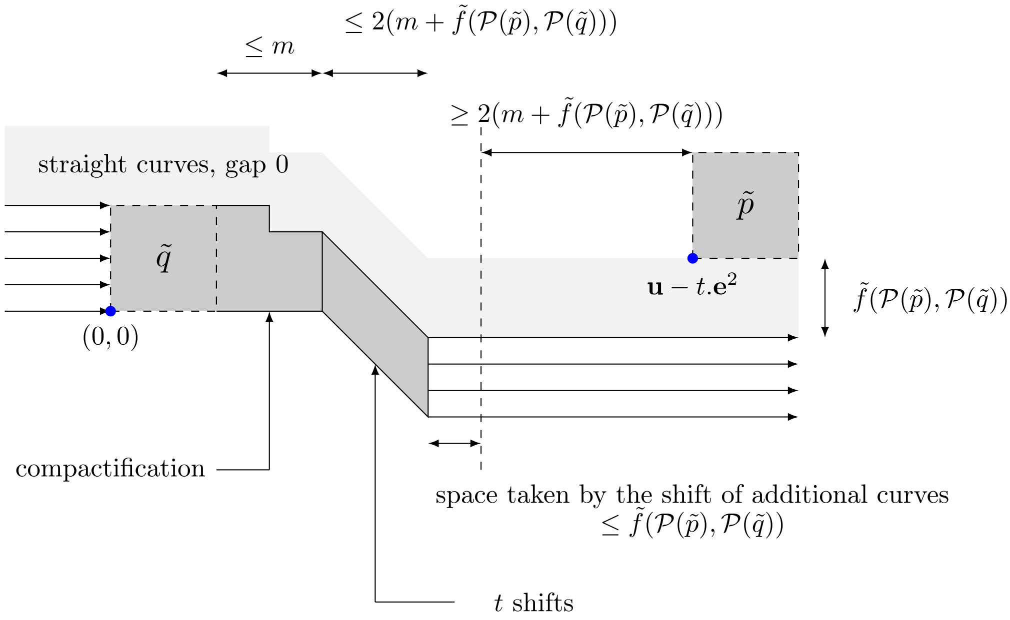


Figure 20: schema of the construction for the proof of Theorem 33 when $l = 1$.

The Figure 21 shows the set of positions that we proved to be in the gluing set of the pattern p relatively to q.

□

Let us denote ρ the transformation on subshifts that acts as a rotation by an angle $\pi/2$. Let X be a subshift on an alphabet $\mathcal{A}$ and defined by a set $\mathcal{F}$ of forbidden patterns. Then $\rho(X)$ is the subshift on alphabet $\mathcal{A}$ defined by the set of patterns that are image by rotation of the patterns in $\mathcal{F}$. Thus ρ transforms SFT into SFT.

Theorem 33. *The operator $d_{\tilde{\mathcal{A}}} \circ \rho \circ d_{\mathcal{A}}$ transforms linear net-gluing subshifts of finite type into linear block gluing ones.*

Idea of the proof. *The idea is to see how this operator acts on the gluing sets of n-blocks p relative to another q and see that it fulfills the part of $\mathbb{Z}^2$ outside of a box containing q whose size is $O(n)$.*

Proof. Since ρ acts as a $\pi/2$ rotation over patterns (and thus on configurations), the gluing set of some n-block p in the language of $\rho \circ d_{\mathcal{A}}(X)$ relatively to another one q contains the positions shown by Figure 22. Using the same procedure as in the proof of Proposition 28, we get that the gluing set of two n-blocks in the language of $d_{\tilde{\mathcal{A}}} \circ \rho \circ d_{\mathcal{A}}(X)$ contains some set of positions as in Figure 23. Indeed, this procedure introduced a vertical perturbation in the positions of the gluing sets. From the form of the sets included in the gluing sets on Figure 22, this perturbation transforms the gluing sets of the subshift $d_{\tilde{\mathcal{A}}} \circ \rho \circ d_{\mathcal{A}}(X)$ by fulfilling the plane outside the box having size $O(n)$.

This means that the subshift $d_{\tilde{\mathcal{A}}} \circ \rho \circ d_{\mathcal{A}}(X)$ is linearly block gluing.

□

Proof. of Theorem 22: We know that $X_{\mathtt{R}}$ is linearly net gluing. As a consequence, the subshift $d_{\tilde{\mathcal{A}}} \circ \rho \circ d_{\mathcal{A}}(X_{\mathtt{R}})$ is linearly block gluing. It is also aperiodic, since $X_{\mathtt{R}}$ is aperiodic. □

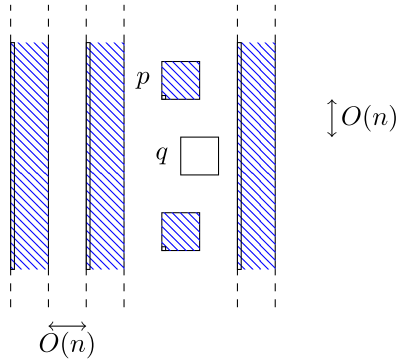


Figure 21: Schematic representation of a set of positions included in the gluing set of some couple of n-blocks in $d_{\mathcal{A}}(X)$.

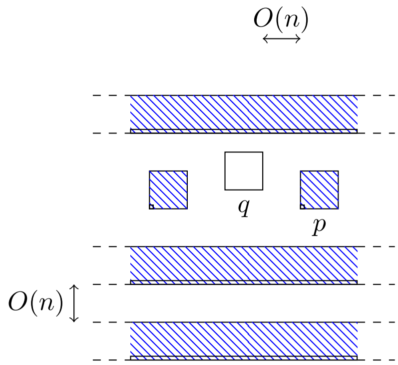


Figure 22: Schematic representation of a set of positions included in the gluing set of some couple of n-blocks in $\rho \circ d_{\mathcal{A}}(X)$.

4 Entropy of block gluing $\mathbb{Z}^2$-SFTs

In this section we present some results about the computability of entropy of block gluing $\mathbb{Z}^2$-SFT. The reader will find the proof of the main theorem in the next section.

4.1 Notion of computability of the entropy

Given a SFT defined by a finite set of forbidden patterns $\mathcal{F}$, a pattern p is **locally admissible** if no pattern of $\mathcal{F}$ appears in p and a pattern p is **globally admissible** if it appears in $X_{\mathcal{F}}$.

Let us recall that $N_n(X) = |\mathcal{L}_n(X)|$ denotes the number of n-blocks in the language of a subshift X. Its entropy is defined as:

$$h(X) = \inf_n \frac{\log_2(N_n(X))}{n^2}$$

At a real number it is possible to precise its computability following the **arithmetical hierarchy** introduced in [WZ01]. We need the two following definition.

Definition 34. *A real number x is **computable** when there exists an algorithm which given as input some integer n outputs some rational number r_n such that*

$$|x - r_n| \leq 2^{-n}.$$

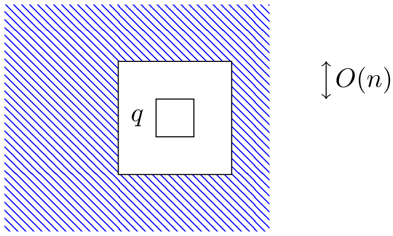


Figure 23: Schematic representation of a set of positions included in the gluing set of some couple of n-blocks in $d_{\tilde{\mathcal{A}}} \circ \rho \circ d_{\mathcal{A}}(X)$.

*A real number x is **right-recursively enumerable** or Π_1-**computable** when there exists an algorithm which given as input some integer n outputs some rational number r_n such that*

$$x = \inf_n r_n.$$

In fact, under using this notion of computability, it possible to characterize the class of numbers which are entropy of a multidimensional SFT:

Theorem 35 ([HM10])**.** *The possible entropies of multidimensional subshifts of finite type are exactly the Π_1-computable numbers.*

4.2 Computability of the entropy for the sub-logarithmic regime

In this section we show that the entropy of block gluing SFT with a sub-logarithmic gap function is computable. This generalizes Proposition 3.3 in [PS15] which states that a multidimensional SFT which is block gluing with a constant gap function is computable.

Proposition 36. *Let X be some f-block gluing $\mathbb{Z}^2$-SFT on some alphabet A, with f a non decreasing function that verifies for some $\epsilon > 0$:*

$$\forall n \in \mathbb{N}, f(n) \leq \frac{n^{1/\log_2(5)}}{\log(n)^{1+\epsilon}}.$$

If the complexity function $(N_n(X))_n$ is computable, then the entropy of X is computable.

Proof. : Consider $k \geq 1$, $n \geq 1$, and a number 4^k of 4^n-blocks in the language of X. We group them by two and glue the two elements of each group horizontally, at distance $f(4^n)$ (which is possible, from the block gluing property). Then make groups of two new formed patterns (see Figure 24) and glue them vertically with distance $f(2.4^n + f(4^n))$ (completing them into blocks before gluing). Repeat these two operations until there is a unique block left. Denote $l_k(n)$ and $h_k(n)$ its length and height, which verify :

$$\begin{array}{rcr} l_0 & = & 4^n \\ h_0 & = & 4^n \\ l_{k+1} & = & 2l_k + f(h_k) \\ h_{k+1} & = & 2h_k + f(l_{k+1}) \end{array}$$

This comes from the fact that $l_{k+1} \geq h_k$ and $h_k \geq l_k$, for all $k \geq 0$. This fact is true for $k = 0$ and if true for k, then $h_{k+1} \geq 2l_k + f(h_k) = l_{k+1}$ (for f is non decreasing), and $l_{k+2} = 2l_{k+1} + f(h_{k+1}) \geq 2h_k + f(l_{k+1}) = h_{k+1}$.

This construction leads to $N_{h_k(n)} \geq (N_{4^n})^{4^k}$ (we can choose the 4^k blocks independently).

Moreover, because $h_k \geq l_k$, and f is non decreasing, $h_{k+1} \leq (2id + f \circ (2id + f))(h_k)$, hence $h_k \leq (2id + f \circ (2id + f))^{\circ k}(4^n)$. Let us denote $d_k(n)$ this last number. We have

$$\frac{\log_2(N_{h_k(n)})}{(h_k(n))^2} \geq 4^k \frac{\log_2(N_{4^n})}{(h_k(n))^2} \geq \frac{4^{2n}2^{2k}}{(d_k(n))^2} \frac{\log_2(N_{4^n})}{4^{2n}} \quad (1)$$

Let us denote g the function defined for all integer n by $g(n) = f(2n + f(n))$. Hence, $d_k(n) = (2id + g)^{\circ k}(n)$, and by induction, using $(2id + g)^{\circ k} = g \circ (2id + g)^{\circ k-1} + 2(2id + g)^{\circ k-1}$:

$$d_k(n) = 2^k 4^n + \sum_{j=0}^{k-1} 2^{k-1-j} g((2id + g)^j (4^n))$$

Using the second condition of the statement, $f \leq id$ so $g \leq f \circ (3id) \leq 3id$. As a consequence we get :

$$\frac{d_k(n)}{2^k 4^n} \leq 1 + \sum_{j=0}^{k-1} 2^{-(j+1)} \frac{g(5^j 4^n)}{4^n}.$$

As a consequence (the first inequality coming from the definition of the entropy), the last sum converges and taking $k \to +\infty$ we get :

$$\frac{\log_2(N_{4^n})}{(4^n)^2} \geq h(X) \geq \frac{\log_2(N_{4^n})}{(4^n)^2} * \frac{1}{(1 + \sum_{j=0}^{\infty} 2^{-(j+1)} \frac{g(5^j 4^n)}{4^n})}$$

Using $g \leq f(3id)$ we have :

$$\frac{g(5^j 4^n)}{4^n} \leq \frac{3^{1/\log_2(5)} 2^j}{(\log_2(3) + j \log_2(5) + 2n)^{1+\epsilon} 4^{n(1-1/\log_2(5))}}$$

$$\frac{\log_2(N_{4^n})}{(4^n)^2} \geq h(X) \geq \frac{\log_2(N_{4^n})}{(4^n)^2} * \frac{1}{(1 + 2 * 4^{n(1-1/\log_2(5))} \sum_{j=1}^{\infty} \frac{1}{j^{(1+\epsilon)}})}$$

Thus, if $(N_n(X))_n$ is a computable sequence, the entropy is a computable number. □

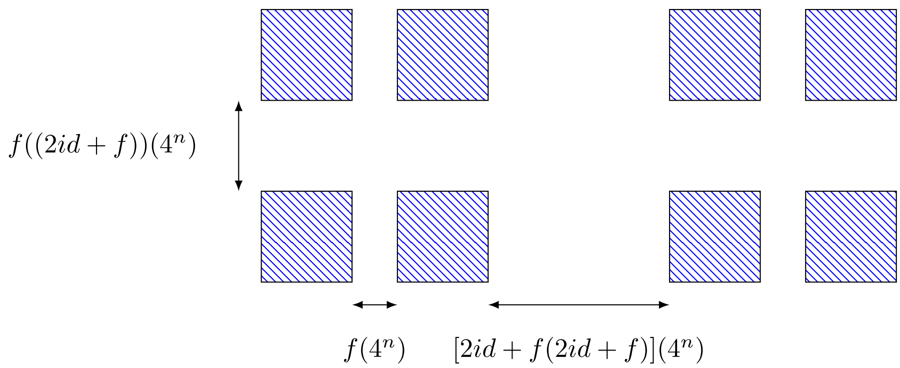


Figure 24: An illustration of the proof of Proposition 36. First three steps of the gluing process of 4^n-blocks.

Corollary 37. *For X an SFT which is $o(log(n))$ block gluing, the entropy $h(X)$ is computable.*

4.3 Characterization of the entropies of linearly block gluing $\mathbb{Z}^2$-SFT

There is a gap between two behaviors for the entropy regarding the gluing property (low and strong block gluing). We prove here a theorem of realization which characterize the possible entropies of sufficiently low (meaning linearly) block gluing subshifts of finite type.

Theorem 38. *The possible entropies of linearly block gluing $\mathbb{Z}^2$-SFT are exactly the non-negative Π_1-computable numbers.*

In [HM10], M. Hochman and T. Meyerovitch used the type of construction we make in this part to prove that every Π_1-computable number is the entropy of a $\mathbb{Z}^2$-SFT. They expressed the question about the realization of every Π_1-computable number as the entropy of a $\mathbb{Z}^2$-SFT that would be transitive (**Problem 9.1** of [HM10]). Like linearly block gluing imply transitivity, we answer here to this question.

Corollary 39. *The possible entropies of transitive $\mathbb{Z}^2$-SFT are exactly the non-negative Π_1-computable numbers.*

4.3.1 Outline of the proof of Theorem 38

The steps list of the proof of Theorem 38 is the following:

1. We first prove that any Π_1-computable number h is the entropy of a linearly net gluing $\mathbb{Z}^2$-SFT. It is sufficient to realize the numbers in $[0,1]$. Indeed, in order to realize the numbers in $]1,+\infty[$ one can take the product of a linearly net gluing SFT having entropy in $[0,1]$ with a full shift. When h is in $\{0,1/4,1/2,3/4,1\}$, this is easy to find a linearly net gluing $\mathbb{Z}^2$-SFT whose entropy is h. Indeed, this can be done by allowing random bits on a regularly displayed set of positions in $\mathbb{Z}^2$. Hence we only have to realize the numbers in $[0,1]\backslash\{0,1/4,1/2,3/4,1\}$.

   The steps of the realization for these numbers are as follows:

   (a) First, for any Π_1-computable sequence $s \in \{0,1\}^{\mathbb{N}}$ and any $N > 0$, we construct in Section 5 a linearly net gluing $\mathbb{Z}^2$-SFT $X_{s,N}$. The integer N corresponds to a threshold and the sequence s corresponds to a control on the entropy of the subshift.

   This part is an adaptation of the construction in [HM10]. This construction uses an implementation of Turing machines whose work is to control the frequency of apparition of some random bits. These bits generate the entropy through frequency bits specifying if a random bit can be present or not. In this construction, the obstacles for transitivity (and thus linear net gluing) are the following ones:

   i. the identification of frequency bits over areas that are not closely related to the structures

   ii. and the possibility of degenerated behaviors of the Turing machines dynamics in infinite areas.

   The first obstacle is solved by a modification of the identification areas (this mechanism is described in Section 5.3 and abstracted on Figure 25) corresponding to the insides of Robinson's structures defining computation units (they will be defined in a more precise way in the following). The second one is solved by the simulation of degenerated behaviors in all the computation units aside the intended behaviors. This ensures that the results of this simulation is not taken into account - this means that the frequency bits are not affected, using error signals propagating through the border of Robinson's structures (this is described in Section 5.4 and Section 5.7, and abstracted on Figure 26). The subshift is net gluing since any pattern can be completed, with control on the size, into a pattern over a simulation area.

   (b) In Section 6 we prove an explicit formula for the entropy of the subshifts $X_{s,N}$ which depends on the parameters s, N. Afterwards, for any $h \in [0,1]\backslash\{0,1/4,1/2,3/4,1\}$, we associate some $\mathbb{Z}^2$-SFT X_h by choosing some parameters s, N so that the entropy of X_h is h. Then we prove in Section 7 that this subshift is linearly net gluing.

2. We present in Section 8 adaptations $d_{\mathcal{A}}^{(r)}$ of the operator $d_{\mathcal{A}}$ presented in Section 3.4.1. These operators verify the equalities

$$h(d_{\mathcal{A}}^{(r)}(Z)) = \frac{\log_2(1+r)}{r} + h(Z)$$

for any subshift Z. Since the additional entropy $\frac{\log_2(1+r)}{r}$ induced by the operator is computable and that the operator $d_{\mathcal{A}}^{(r)} \circ \rho \circ d_{\mathcal{A}}^{(r)} \circ \rho \circ d_{\mathcal{A}}^{(r)}$ transforms linearly net gluing SFTs into linearly block gluing ones, this allows any Π_1-computable number in $]0,1]$ to be realized as the entropy of a block gluing SFT. Since the entropy zero is trivial to realize, this means that any Π_1-computable number in $[0,1]$ is the entropy of a linearly block gluing SFT.

4.3.2 Description of the layers in the construction of the subshifts $X_{s,N}$

Here is a more precise description of the construction of $X_{s,N}$ for a given $h \in [0,1]$.

First, we fix some parameters s and N, where s is a Π_1-computable sequence of $\{0,1\}^{\mathbb{N}}$, and $N > 0$ is an integer.

The construction of the subshift $X_{s,N}$ involves a hierarchy of computing units that we call cells. Each cell is divided into four parts and in each of these parts a machine works. The center (called nucleus) of each cell contains an information which codes for the behavior of the machines. Two of these machines compute and the other two simulate degenerated behaviors. This means in particular that the initial tape of the machine is left free and that machine heads can enter at any time and in any state on the two sides of the machine area. The aim of the computing machines is to control some random bits that generate the entropy, through frequency. These bits are grouped in infinite sets having frequency given by a formula. This allows the entropy to be expressed as the sum of three entropies $h_{\texttt{int}}$, $h_{\texttt{comp}}(s)$ and $h_{\texttt{sim}}(N)$. The first one is generated by random bits, and serve to place the entropy of the subshift in one of the quarters of the segment $[0,1]$. Each of the two other entropies is the sum of a series. The first one involves the sequence s and is the entropy due to random bits. The second one is generated by the symbols left free in the simulation area. We choose N so that the sum of $h_{\texttt{int}}$ and $h_{\texttt{sim}}(N)$ is smaller than $h_{\texttt{comp}}(s)$. By the choice of the sequence s we control the series whose sum is $h_{\texttt{comp}}(s)$. We choose this entropy so that the total entropy is h.

Here is a detailed description of the layers in this construction.

- *Structure layer [Section 5.1]:* This layer consists in the subshift $X_{\texttt{R}}$. Recall that any configuration of this subshift exhibits a cell hierarchy. In this setting, for all n the order n cells appear periodically in the vertical and horizontal directions. Some additional marks allow the decomposition of the cells into sub-structures that we describe in this section. In each of these sub-structures specific behaviors occur. In particular, each cell is decomposed into four parts called quarters.

- *Basis layer* [Section 5.2] The blue corners in the structure layer will be superimposed with random bits in $\{0,1\}$. This allows to an entropy in $[0,1/4]$ to be generated. Turing machines will control the frequency of positions where the random bits can be 1 through the use of frequency bits. This generates an entropy equal to the Π_1-computable number

$$h' = h - \frac{\lfloor 4h \rfloor}{4} \in [0,1/4[.$$

In order to generate the entropy h, we consider $i = \lfloor 4h \rfloor$. We impose that the other positions are superimposed with random bits such that the positions where the random bit is equal to 1 have with frequency $i/3$ in this set. Hence the total frequency of the positions where the random bit can be 1 in $\mathbb{Z}^2$ is

$$\frac{i}{3}\frac{3}{4} + h' = h.$$

- *Frequency bits layer* [Section 5.3]

  Each quarter is superimposed with a frequency bit. On positions with blue corners in the structure layer having frequency bit equal to 0, the random bit is equal to 0.

- *Cells coding layer* [Section 5.4]

  In this layer we superimpose to the center of the cells (that we call nuclei) of each cell a symbol which specifies two adjacent quarters of the cell when $n > N$. It represents all the quarters when $n \leq N$. This symbol is called the DNA of the cell.

  The quarters represented in the DNA are called computation quarters. The other ones are called simulation quarters. In these ones the function of the machines is to simulate any degenerated behavior of the computing machines.

  Since simulation induces parasitic entropy, we choose after the construction some N such that this entropy is smaller than h. This allows programming the machines in such a way that the entropy generated by the random bits in the other quarters complements this entropy so that the total entropy is h.

- *Synchronization layer* [Section 5.5] In this section, we synchronize the frequency bits of the computation quarters of all the cells having the same order.

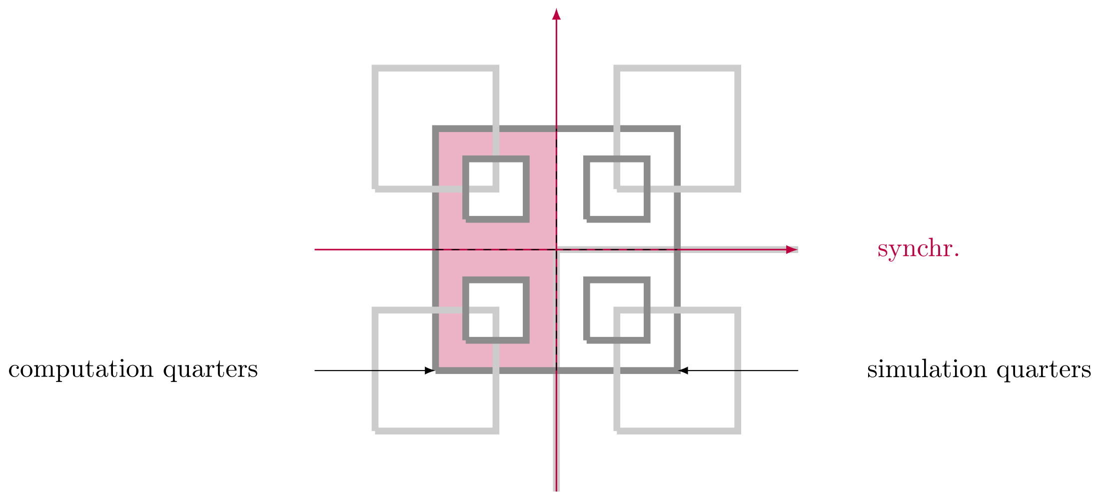


Figure 25: Illustration of the frequency bits identification areas (colored purple) in our construction. The squares designate some petal structures of the Robinson subshift.

- *Computation areas layer* [Section 5.6] In this layer are specified the areas supporting the computations of the machines. The function of each position in this area amongst the following ones are also specified:

  1. transfer of information, vertical or horizontal,
  2. or the execution of one step of computation.

  This is done using signals that detect the rows and columns in a cell that do not encounter a smaller cell. The intersections of such line and column are the positions where a machine executes one step of its computation. The other positions of these lines and columns are used to transmit information.

  Since the construction of these areas can have degenerated behaviors in infinite cells, these behaviors are also simulated in simulation quarters. We use error signals in order to impose that the computation areas are well constructed in computation quarters.

- *Machines layer [Section 5.7]:*

  Consider $(s_k^{(n)})_{n,k} \in \{0,1\}^{\mathbb{N}^2}$ some sequence such that for all k,

$$s_k = \inf_n s_k^{(n)}.$$

  This layer supports the computations of the machines in all the quarters of each cell. Each quarter of a cell has its proper direction of time and space. The machine is programmed so that when well initialized, it writes successively for all n the bits $s_k^{(n)}$, $k = 0...n$ on the 2^kth position of its tape corresponding to a column that is just one the left of order k cells. The sequence is chosen afterwards so that the total entropy of $X_{s,N}$ is h.

  The frequency bits of order k cells corresponding to the computation quarters is transported in the column just on the right of these cells. This is done in order for the machine to have access to this information.

  If at some point one of the frequency bits is greater than a bit written by the machine in this column, then the machine enters in halting state.

  We allow the machine heads to enter in error state. However, when this happens it transmits a signal through its trajectory back in time until initialization. We forbid the coexistence of this signal with the representation of this quarter in the DNA. As a consequence, the computations of a machine are taken into account if and only if in a computation quarter where these computations have some meaning.

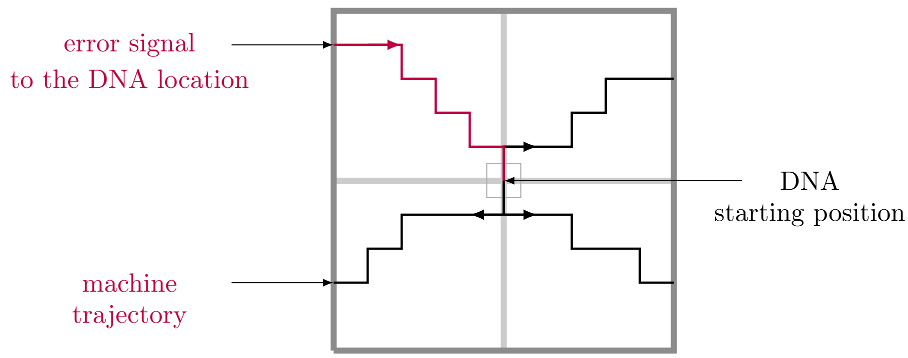


Figure 26: Illustration of the machines mechanisms in our construction. The gray square designates a computing unit splitted in four parts. In each of this parts evolves a computing machine departing from the center of the cell. If the machine reaches the border in error state, it triggers an error signal.

  The directions of time and space depend on the orientation of the quarter in the cell. In each quarter, these directions are given on Figure 27.

The next sections are devoted to make the proof of Theorem 38 more precise.

5 Construction of the subshifts $X_{s,N}$

5.1 Structure layer

In this section, we present the structure layer and the structures observable in this layer that will be used in the following.

This layer has three sub-layers, as follows.

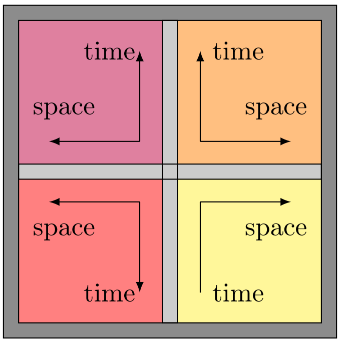


Figure 27: Schema of time and space directions in the four different quarters of a cell.

5.1.1 Specialized sub-structures of the cells

The first sublayer of the structure layer is the subshift $X_{\mathtt{R}}$.

Here we list some structures that appear in this layer and the designation that we use for them. Let x be a configuration in this layer.

- Recall that an order n cell is the part of $\mathbb{Z}^2$ enclosed in an order $2n+1$ petal.
- The center of a cell is a red corner which is valued with 0. The position where this symbol appears is called the **nucleus** of the cell.
- In a cell the union of the column and the line containing the nucleus is called the **reticle** of the cell. In particular, the nucleus is the intersection of the reticle's arms.
- We call the set of positions in the border of a cell (defined as the position which has a neighbor outside of the cell) the **wall**.
- The set of positions in a cell that are not in the wall, in the reticle, or in another (smaller) cell included to the considered one is called the **cytoplasm**.
- We call any of the four connected parts included in the cytoplasm and delimited by the reticle and the wall a **quarter** of the cell.

Figure 28 gives an example of pattern that can be superimposed over an order 1 cell in the structure layer. We illustrate on this figure the structures that we listed above.

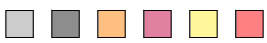
Recall that cells have the following properties:

- For all $m \geq 0$, and $i \in \{1, ..., m\}$, any order m cell contains properly 4.12^{i-1} order $m-i$ cells (meaning that these cells are not included in an order $< m$ cell). As a consequence, each order m cell contains properly 4.12^m blue corners.
- Moreover, in any configuration each order m cell (and in particular the nuclei) repeats periodically, in the horizontal and vertical directions, with period 4^{m+2}.

5.1.2 Coloring

In this section, we present the representation of the sub-structures of cells presented in Section 5.1.1. In particular, we color differently the four quarters of a cell, in order for the machines to have access to the direction of time and space in their quarter.

Symbols:



Each symbol corresponds to a part of the cell:

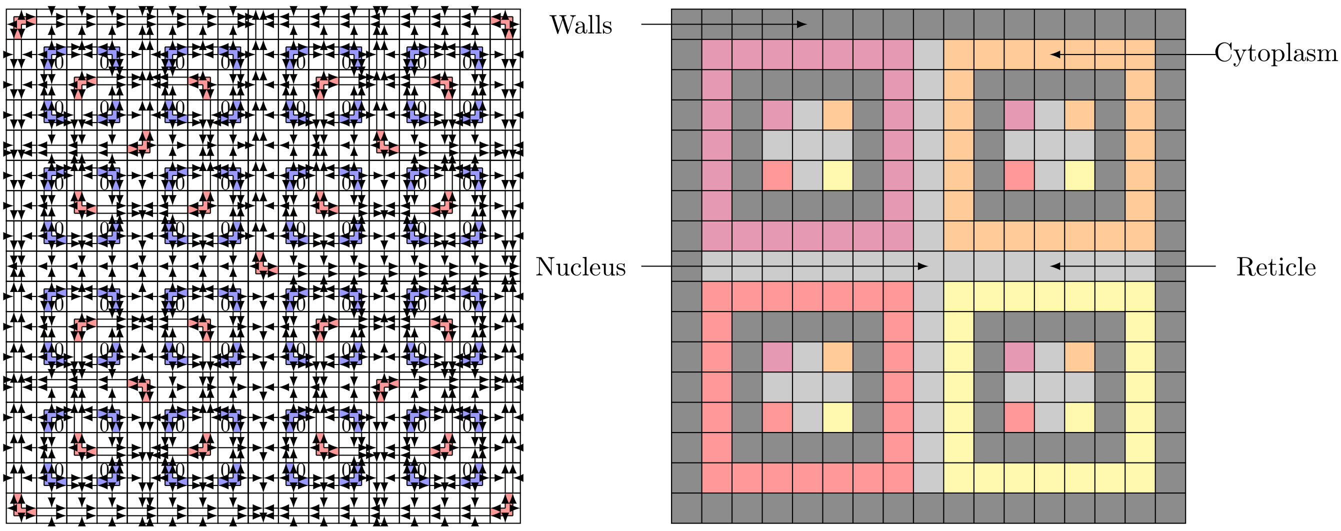


Figure 28: Example of the projection over the structure (first picture) and the synchronization (second one) layers of a pattern over an order 1 cell. This cell contains properly 4 order 0 cells, and 4.12 blue corners.

- corresponds to the reticle.
- corresponds to the walls.
- corresponds to the north east quarter.
- corresponds to the north west synchronization area.
- corresponds to the south east quarter.
- corresponds to the south west quarter.

Local rules:

1. **Localization:**
   - A position is colored with if and only if it lies in the walls of a cell.
   - the symbol can be superimposed only on red corners valued 0 or arrows symbols valued 0 for the direction of the long arrows.
   - the positions that are not amongst the type of positions specified by the two last rules are colored with else , , or .
2. **Transmission rules:**
   - Consider two vertically adjacent positions with vertical outgoing arrows (or two horizontally adjacent positions with horizontal ougoing arrows) such that the value corresponding to this direction in the two symbols is 0. Then if one of these positions is colored , then the other one is also colored .
   - Consider two adjacent positions which are not colored or . Then these two positions have the same color.

3. **Determination of the colors:**

   The idea of these rules is to determine the colors that appear in the neighborhood of particular positions in a cell. Together with the transmission rules, these rules will determine the color of any position in any configuration.

   In order to describe them, we use the vocabulary introduced in this text. However, one can translate these terms into symbols of the Robinson subshift (we do this for the second rule as an example).

   The rules are the following ones:

   - A south west (resp south east, north east, south west) corner of a cell induces the position on its north east (resp north west, south west, south east) to be colored (resp. , ,).
   - A position inside the west part of the north wall of a cell (meaning a four or six arrows symbol whose long arrows are horizontal, directed to the right, and valued 1 for the horizontal direction) (resp. east part) has its south neighbor position colored with (resp.). Its neighbor position on north is colored with the same color as the north east position and the north west position. We impose similar rules for positions in the other walls of cells.
   - A nucleus has respectively on its north west, north east, south east, south west positions the colors , , and , and this position is colored .
   - A position at the center of the south wall of a cell has its north west, north and north west neighbor positions colored with , and . Similar rules are imposed for the centers of the other walls.
   - On the west and east neighbor positions of a position colored , the possible couples of colors are:
     - on the west and on the east,
     - on the west and on the east.

     Similar rules are imposed for the vertical direction.

Global behavior:

The localization rules impose directly that the walls of the finite cells are colored . With the transmission rules combined with the rules determining colors, the reticle is colored . Moreover, for any cell the positions in the same quarter are colored with else , , or . The color is determined according to the orientation of the quarter in the cell: the first (resp. second, third, fourth) color is for the south west quarter (resp. south east, north east, north west).

These rules allow the same coloration for infinite cells.

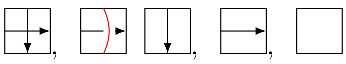
Figure 29 shows an example of coloration of an order 2 cell.

5.1.3 Synchronization net

In this section, we specify a network connecting the nuclei of order n cells for all n. This allows the synchronization of the frequency bits corresponding to computation quarters of the cells.

Symbols:

The symbols of this third sublayer are the following ones :



Local rules:

- The symbol is superimposed on and only on the nuclei, corners, and centers of the walls of cells. As a consequence, when a line of arrows crosses a column of arrows on another position, then the symbol superimposed on this position is .

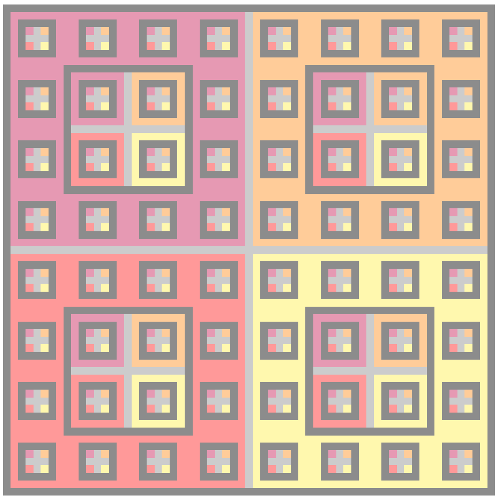


Figure 29: An example of pattern in the synchronization layer over an order 2 cell.

- The symbols $\boxed{\downarrow}$ and $\boxed{\rightarrow}$ are transmitted in the direction of the arrow: north and south for the first one, east and west for the second one.
- Consider a position $\mathbf{u}$. If it is superimposed with a vertical (resp. horizontal) arrow, then the positions $\mathbf{u} \pm \mathbf{e}^1$ (resp. $\mathbf{u} \pm \mathbf{e}^2$) are not superimposed with a vertical (resp. horizontal) arrow.

Global behavior:

The first two rules build **wires** of the net, defined to be infinite columns (resp. rows) whose positions are superimposed with a vertical (resp. horizontal) arrow. These are the columns (resp. rows) intersecting corners and nuclei of cells. The other columns (resp. rows) are not superimposed with a vertical (resp. horizontal arrow): this is a consequence of the last rule.

The **intersections of the wires** are superimposed with one of the two first symbols of the alphabet $\boxed{\downarrow\rightarrow}$, $\boxed{\rightarrow}$.

The first symbol is superimposed on nuclei and corners, where the frequency bits will be synchronized. The other intersections have the second symbol. On these positions, the frequency bits won't be synchronized. As a consequence, only cells having the same order are synchronized.

See on Figure 30 an example of a net superimposed on an order 1 cell.

5.2 Basis layer

The basis layer supports **random bits**. We recall that $i = \lfloor 4h \rfloor$.

Symbols:

The symbols of this layer are 0 and 1.

Local rules:

- if $i = 0$, then on any position $\mathbf{u}$ superimposed with a blue corner, the positions $\mathbf{u}+\mathbf{e}^1$, $\mathbf{u}+\mathbf{e}^1$ and $\mathbf{u}+\mathbf{e}^1+\mathbf{e}^2$ are superimposed with the bit 0.
- if $i = 1$, the positions $\mathbf{u}+\mathbf{e}^1$ and $\mathbf{u}+\mathbf{e}^1+\mathbf{e}^2$ are superimposed with 0, and there is no constraint on the bit on position $\mathbf{u}+\mathbf{e}^2$.
- if $i = 2$, the position $\mathbf{u}+\mathbf{e}^1$ is superimposed with 0.
- if $i = 3$, there is no constraint.

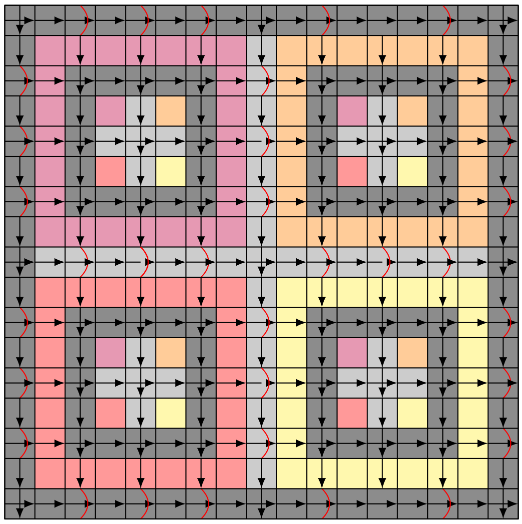


Figure 30: Example of a synchronization net in an order 2 cell

Global behavior:
In this layer are superimposed random bits in $\{0,1\}$ on any position. According to the value of i, we impose constraints on positions that are not superimposed with a blue corner in the structure layer. We dot this in such a way that the entropy produced by these bits is the maximal possible value smaller than h.

The positions with a blue corner are the ones where the random bits will be regulated by frequency bits. These bits are themselves controlled by the machines in order to complete the entropy generated by the random bits on the other positions.

5.3 Frequency bits layer

Symbols:
The symbols of this layer are $0, 1$ and a blank symbol.
Local rules:

- **Localization:** The non-blank symbols are superimposed to the cytoplasm positions.
- **Synchronization on quarters:** two adjacent positions in the cytoplasm have the same frequency bit.

Global behavior:
The bits $0, 1$ are called **frequency bits**. In a quarter, these bits are equal. Hence each quarter is attached with a unique frequency bit.

5.4 Cells coding layer

Symbols:

and a blank symbol.
Local rules:

- The non-blank symbols are superimposed to the nuclei, the blank symbols to other positions.

- The DNA symbol is and can only be over a nucleus of an order $\leq N$ cell.
- The others DNA symbols are over $> N$ order cells.

Global behavior:

On the nucleus of every cell is superimposed a symbol called the **DNA**. It rules the behavior of each of the machines working in the cytoplasm, telling which ones of the machines execute simulation and which ones compute.

When the order of the cell is smaller or equal to N, then all the machines compute. When the order is greater than N, two of the machines compute and the other two execute simulation.

5.5 Synchronization layer

This layer serves for the synchronization of the frequency bits corresponding to computation quarters.

Symbols:

Elements of $\{0,1\}$ and $\{0,1\}^2$, and a blank symbol.

Local rules:

1. **Localization:**
   - The non-blank symbols are superimposed on and only on positions having a non-blank symbol in the synchronization net sublayer.
   - Positions superimposed with , , or are superimposed in the present layer with an element of $\{0,1\}$. The positions superimposed with are superimposed with an element of $\{0,1\}^2$.
2. **Synchronization:**
   - On a nucleus, if the DNA symbol is , then the bit in this layer is equal to the frequency bit on north east, south east, south west and north west positions.
   - When the DNA symbol is not , then the bit in this layer is equal to the frequency bits corresponding to the colors represented in the DNA symbol.
3. **Information transfer rules:**
   - Considering two adjacent positions with , or , the bits superimposed on these two positions are equal.
   - On a position with symbol, the first bit of the couple is equal to the bit of positions on north and south. The second one to the bit of positions on west and east.
   - On a position superimposed with , the bit is equal to the bit on south, west, east and north positions.

Global behavior:

Using the information contained in the nucleus (the DNA), we synchronize the two frequency bits of the computation quarters. These bits are transmitted through the wires of the synchronization net. They are synchronized on the positions having the symbol . As a consequence, when $n \leq N$, all the frequency bits of order n cells are equal. When $n > N$, the frequency bits corresponding to computation quarters are synchronized. The other two are left free and are not synchronized between cells.

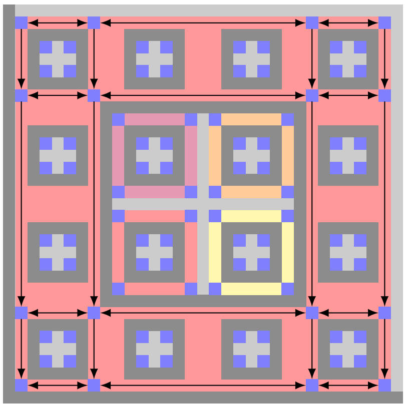


Figure 31: Example of a computation area in a computation quarter in an order 2 cell.

5.6 Computation areas layer

This layer specifies the function of each position of the cytoplasm relatively to the Turing machines: information transfer (vertical or horizontal) or execution of one computation step. This is done as in [Rob71]. However, the constitution of the computation areas in infinite cells is not well controlled. That is the reason why, in order to ensure the linear net gluing property, we simulate degenerated behaviors for the constitution of computation areas in the simulation quarters. We use error signals to ensure that the computation areas are well constituted in the computation quarters.

Symbols:

Elements of $\{\texttt{in}, \texttt{out}\}^2 \times \{\texttt{in}, \texttt{out}\}^2$, elements of

$$\{ \; , \; , \; , \; , \; , \; , \; \},$$

elements of

$$\{ \; , \; , \; , \; , \; , \; , \; , \; , \; , \; , \; , \; \},$$

and a blank symbol.

The first set corresponds to signals that propagate horizontally and vertically in the cytoplasm. The second set corresponds to error signals propagating on the reticle and to the nucleus. The last ones correspond to the propagation of the error signals through the walls.

Local rules:

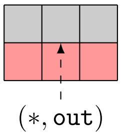
- **Localization:**
  - The elements of $\{\texttt{in}, \texttt{out}\}^2 \times \{\texttt{in}, \texttt{out}\}^2$ are superimposed on and only on positions in the cytoplasm. The first two coordinates are associated to the horizontal direction, and the second two to the vertical direction.
  - The elements of $\{ \; , \; , \; , \; , \; , \; , \; \}$ can be superimposed only on reticle positions.
  - All the other positions are superimposed with the blank symbol.
- **Transmission of the cytoplasm signals:** on a cytoplasm position $\mathbf{u}$, the two first coordinates of the symbol are transmitted to the positions $\mathbf{u} \pm \mathbf{e}^1$ and the second two are transmitted to the positions $\mathbf{u} \pm \mathbf{e}^2$, when these positions are in the cytoplasm.

  In a computation quarter, these are signals allowing the lines and columns of the cells which do not intersect a smaller cell to be specified. In the first (resp. second) couple, the symbol $\texttt{in}$ for the first coordinate corresponds to the fact that the position is in a segment of row (resp.

column) originating from the inside of the cell wall. The symbol `out` corresponds to the fact that the position is in a segment of row (resp. column) originating from the outside of the cell wall. For the second coordinate, these symbols have the same signification concerning the end of the segment instead. The next rules impose that when near the pertinent parts of a cell and inside it, if a symbol $\texttt{in} - \texttt{out}$ does not correspond to the nature of the origin or end at this position, this triggers an *error signal*. When outside the cell, the origin or end is imposed - thus not triggering an error signal. These rules are presented for positions in the **red quarter**: similar rules are imposed for the other ones.

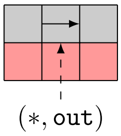
- **Triggering error signals (inside the wall and reticle):** On a position $\mathbf{u}$ in the horizontal arm of the reticle (specified by having a reticle symbol different from the nucleus and having a reticle position on the right and on the left), if the position $\mathbf{u} - \mathbf{e}^2$ has its second couple having second coordinate equal to `out`, then the red quarter is represented in the symbol superimposed on position $\mathbf{u}$.

  For instance, the pattern



  ,

  where $*$ means any symbol, and the couple represented in $\{\texttt{in}, \texttt{out}\}^2$ is the second one, implies the following:



  .

  On a position $\mathbf{u}$ in the vertical arm of the reticle (specified by having a reticle symbol different from the nucleus and having a reticle position on the top and bottom), if the position $\mathbf{u} - \mathbf{e}^1$ has its first couple having second coordinate equal to `out`, the red quarter is represented in the symbol superimposed on position $\mathbf{u}$.

- On a position $\mathbf{u}$ in the horizontal part of the wall (specified by having a wall symbol different from the corner, and having a wall position on left and right), if the position $\mathbf{u} + \mathbf{e}^2$ has its second couple having first coordinate equal to `out`, then the symbol on the position $\mathbf{u}$ is an arrow symbol

   or .

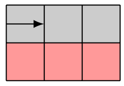
  On a position $\mathbf{u}$ in the vertical part of the wall (specified by having a wall symbol different from the corner, and having a wall position on top and bottom), if the position $\mathbf{u} + \mathbf{e}^1$ has its second couple having first coordinate equal to `out`, then the symbol on the position $\mathbf{u}$ is an arrow symbol

   or .

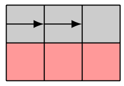
- **Enforcing cytoplasm signals (outside the wall):**

  Considering a wall position $\mathbf{u}$ which is on the west (resp. east, north, south) wall of a cell, the position $\mathbf{u} - \mathbf{e}^1$ (resp. $\mathbf{u} + \mathbf{e}^1$, $\mathbf{u} + \mathbf{e}^2$, $\mathbf{u} - \mathbf{e}^2$ has the second coordinate of its second couple (resp. first coordinate and second couple, first coordinate first couple, second coordinate first couple) equal to `out`.

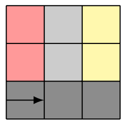
- **Propagation of error signals**. An arrow symbol propagates in the direction pointed by the arrow on the wall, while the next position in this direction is not near a reticle position, as in the following pattern:



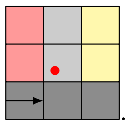
implies the following one:



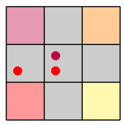
- On a position $\mathbf{u}$ of the north (resp. east, south, west) arm of the reticle, specified by the colors on the sides, if the position $\mathbf{u} - \mathbf{e}^1$ (resp. $\mathbf{u} - \mathbf{e}^1$, $\mathbf{u} + \mathbf{e}^2$, $\mathbf{u} + \mathbf{e}^1$) is not the nucleus, then the symbol on this position contains the symbol on position $\mathbf{u}$.
- **Connection between error signals:** when on a position $\mathbf{u}$ on the wall which is near a position on the reticle, if one of the wall symbols aside contains an error symbol, then the reticle position has an error symbol where the corresponding quarter is represented. For instance, the pattern



implies the following:



- **Forbidding wrong error signals**. On any of the four reticle positions around the nucleus, there can not be a symbol that contains a color which is in the DNA. For instance, the following pattern is forbidden:



Global behavior:

In any quarter of a cell, the segment of rows and columns are colored with a couple of symbols in $\{\mathtt{in}, \mathtt{out}\}$. One for the origin of the segment, the other one for the end of it. If it originates from or ends at the outside of a cell, then the corresponding symbol is forced to be `out`.

Moreover, when near the walls or the reticle and inside the corresponding cell, if the corresponding symbol is `out` then an error signal is triggered and propagates to the nucleus. On the walls, a propagation direction is chosen. In the reticle, the error signals contain the information about the quarter where the error was detected. Around the nucleus, we forbid an error signal to come from a computation quarter.

In a *computation quarter* of a cell, each row which does not intersect a smaller cell has first couple equal to $(\mathtt{in}, \mathtt{in})$, since it originates inside the cell, and ends inside, on the reticle. Each column which does not intersect a smaller cell has second couple equal to $(\mathtt{in}, \mathtt{in})$. The couples on other segments of rows or columns are determined in a similar way, according to their origin and end. This is enforced by the propagation of error signals to the nucleus.

The positions marked with $((\mathtt{in}, \mathtt{in}), (\mathtt{in}, \mathtt{in}))$ are called **computation positions**. The ones that have first (resp. second) couple equal to $(\mathtt{in}, \mathtt{in})$ and second (resp. first) not equal to $(\mathtt{in}, \mathtt{in})$ are **horizontal transfer positions** (resp. vertical transfer positions). See Figure 31 for a representation of a computation area in the red quarter of an order two cell. On this figure, computation positions are represented by a blue square. Vertical and horizontal transfer positions by arrows in this direction.

Remark 5. *These mechanisms can not be easily simplified, since an infinite row or a column can not "know" if it is a free row or column of its infinite cell. Moreover, from the division of the cells it is difficult to code this with a hierarchical process.*

Remark 6. *In the literature, most of the constructions using substitutions include the construction of the computation areas [HM10] with substitution rules. However, in order to get the net gluing property, and furthermore the block gluing property, we need a more flexible construction of the computation areas. The method presented above was used initially in the construction of Robinson for his undecidability result [Rob71].*

5.7 The machines (RNA)

In this section, we present the implementation of Turing machines that will check that the frequency bit of level n cells are equal to s_n, for all n.

In order to have the linear block gluing property, we have to adapt the Turing machine model in order to simulate each possible degenerated behavior of the machines. This is done as follows: in each of the quarters of a cell, a machine is implemented. For this machine, the directions of space and time are as on Figure 32: the rules of the machine will depend on the color of the quarter. Moreover, for each of the quarters, we initialize the tape with elements of $\mathcal{A} \times \mathcal{Q}$. The set $\mathcal{Q}$ is the state set of the machine and $\mathcal{A}$ its alphabet. Machine heads can enter on the two sides of the computation area. Signals will be used to verify that in the computation quarters the machine is well initialized. This means that no head enters on the sides, and on the initial row there is a unique machine head on the position near the nucleus in initial state. Moreover, all the letters in $\mathcal{A}$ are blank.

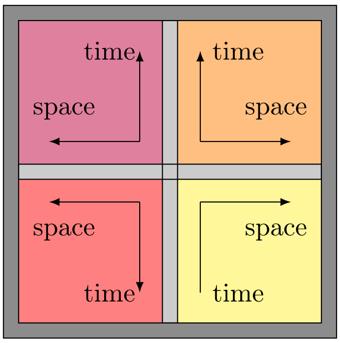


Figure 32: Schema of time and space direction in the four different quarters of a cell.

As usually in this type of constructions, the tape is not connected. Between two computation positions, the information is transported. In our model, each computation position takes as input up to four symbols coming from bottom and the sides. It outputs up to two symbols to the top and sides. Moreover, we add special states to the definition of Turing machine. We do this in order to manage the presence of multiple machine heads. We describe this model in Section 5.7.1, and then show how to implement it with local rules in Section 5.7.2.

If a machine head enters an error state, this triggers an error signal that propagates through the trajectory of the machine. This signal is taken into account only for computation quarters.

5.7.1 Adaptation of computing machines model to linear block gluing property

In this section we present the way computing machines work in our construction. The model that we use is adapted in order to have the linear block gluing property, and is defined as follows:

Definition 40. *A* ***computing machine*** $\mathcal{M}$ *is some tuple* $= (\mathcal{Q}, \mathcal{A}, \delta, q_0, q_e, q_s, \#)$. *The set* $\mathcal{Q}$ *is* *the state set, $\mathcal{A}$ the alphabet, q_0 the initial state, and $\#$ is the blank symbol, and*

$$\delta : \mathcal{A} \times \mathcal{Q} \rightarrow \mathcal{A} \times \mathcal{Q} \times \{\leftarrow, \rightarrow, \uparrow\}.$$

The other elements q_e, q_s are states in $\mathcal{Q}$. They are such that for all $q \in \{q_e, q_s\}$ and for all a in $\mathcal{A}$, $\delta(a, q) = (a, q, \uparrow)$.

The special states q_e, q_s in this definition have the following meaning:

- error state q_e: a machine head enters this state when it detects an error or when it collides with another machine head.

  This state is not forbidden in the subshift, but this is replaced by the sending of an error signal. We forbid the coexistence of the error signal with a well initialized tape. The machine stops moving when it enters this state.

- shadow state q_s: this state corresponds to the absence of head. We need to introduce this state so that the number of possible space-time diagrams in finite cells has a closed form.

Any Turing machine can be transformed in such a machine by adding some state q_s verifying the properties listed above.

When the machine is well initialized, none of these states and letters will be reached. Hence this machine behave as the initial one. As a consequence, one can consider that the machine we used has these properties.

In this section, we use a machine which successively for all $n \geq 0$ writes the bits $s_k^{(n)}$, $k = 1...n$, on positions $p_n = 2^n$ (which is a computable function). This position corresponds to the number of the first active column from left to right which is just on the right of an order n two dimensional cell on a face amongst active columns

Recall that s is the Π_1-computable sequence defined at the beginning of the construction. The sequence $(s_k^{(n)})$ is a computable sequence such that for all k, $s_n = \inf_k s_k^{(n)}$.

5.7.2 Implementation of the machines

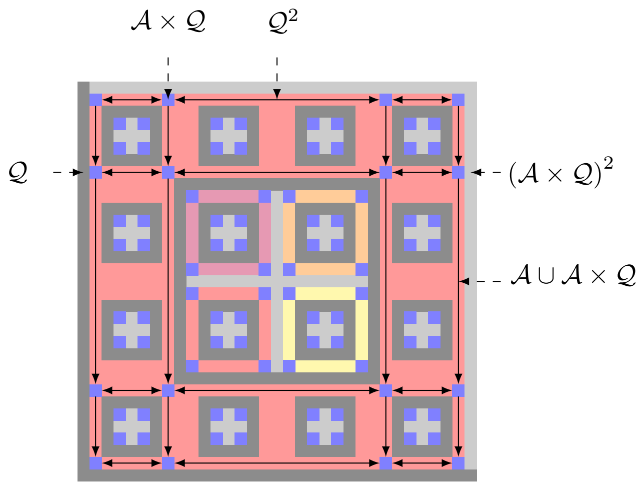


Figure 33: Localization of the machine symbols in the red quarter of an order two cell.

In this section, we describe the second sublayer of this layer.

Symbols:

The symbols are elements of the sets $\mathcal{A} \times \mathcal{Q}$, $\mathcal{A}$, $\mathcal{Q}^2$, $\mathcal{Q}$, $(\mathcal{A} \times \mathcal{Q})^2$, and a blank symbol.

Local rules:

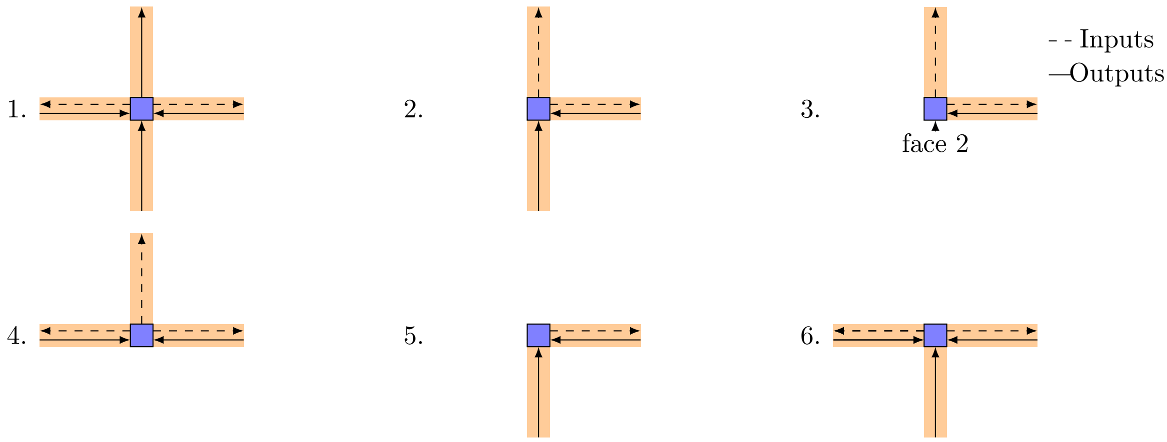


Figure 34: Schema of the inputs and outputs directions when inside the area (1) and on the border of the area (2,3,4,5,6).

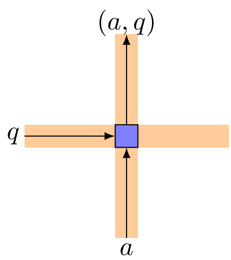


Figure 35: Illustration of the standard rules (1).

- **Localization:** the non-blank symbols are superimposed on information transfer rows and columns, as well as positions corresponding to information transfer rows and columns on the arms of the reticle and the east and west walls. More precisely:
  - the possible symbols for information transfer columns are elements of the sets $\mathcal{A}$ and $\mathcal{A} \times \mathcal{Q}$. The elements of $\mathcal{A} \times \mathcal{Q}$ are on computation positions. The other ones on the other positions of these lines and columns.
  - the positions on the vertical (resp. horizontal) arms of the reticle corresponding to an information transfer line are colored with an element of $\mathcal{Q}^2$ (resp. $(\mathcal{A} \times \mathcal{Q})^2$). The first coordinate corresponds to the machine heads entering in the west quarter. The second one corresponds to machine heads entering in the east one (resp. machine head and letter entering in the north and south ones).
  - on the west and east walls, the symbols are in $\mathcal{Q}$. They correspond to machine heads entering in the adjacent quarter. See an illustration on Figure 33.
- **Transmission:**

  Along the rows and columns, the symbol is transmitted while not on computation positions.
- **Computation positions rules:**

  Consider some computation position. These rules depend on the orientation of the quarter in the cell. We describe them in the north east quarter. The rules in the other quarters are obtained by symmetry, respecting the orientation of time and space given on Figure 32.

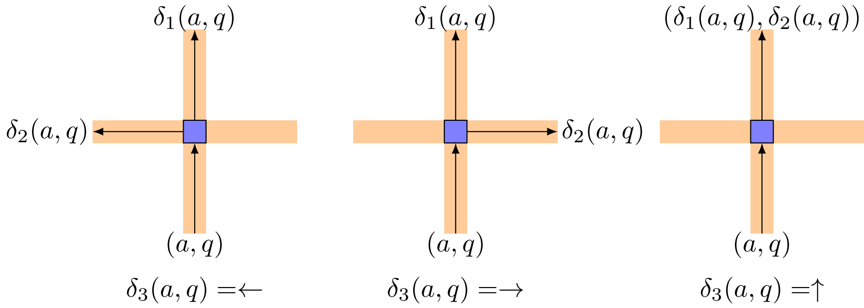
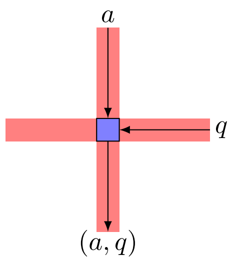


Figure 36: Illustration of the standard rules (2).



For such a position, the **inputs** include:

1. the symbols written on the south position,
2. the first symbol written on the west position (except in the leftmost column, where the input is the second symbol of the west position),
3. and the second symbol on the east position (except when in the rightmost column, where the input is the unique symbol written on east position).

The **outputs** include:

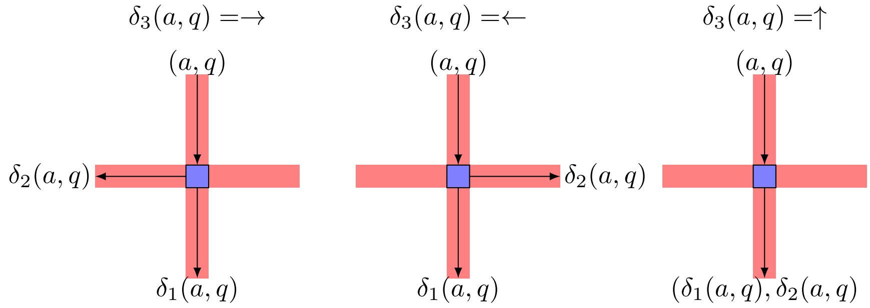
1. the symbols written on the north position when not in the topmost row,
2. the second symbol of the west position (when not in the leftmost column),
3. and the first symbol on the east position (when not on the rightmost column).

Moreover, on the row near the reticle, the inputs from inside the area are always the shadow state q_s. The input from the bottom is free. As a consequence the couple written on the the position is also free. This is also true for the elements of $\mathcal{Q}$ on the computation positions in the leftmost and rightmost columns and the triple of symbols written on the position near the nucleus.

See Figure 34 for an illustration.

On the first row, all the inputs are determined by the counter and by the above rule. Then each row is determined from the adjacent one on the bottom and the inputs on the sides. This is due to the following rules, which on each computation position determine the outputs from the inputs:

1. **Collision between machine heads:** if there are at least two elements of $\mathcal{Q}\backslash\{q_s\}$ in the inputs, then the computation position is superimposed with (a, q_e). The output on the top (when this exists) is (a, q_e), where a is the letter input below. The outputs on the sides are q_s. When there is a unique symbol in $\mathcal{Q}\backslash\{q_s\}$ in the inputs, this symbol is



called the machine head state (the symbol q_s is not considered as representing a machine head).

2. **Standard rule:**
   (a) when the head input comes from a side, then the functional position is superimposed with (a,q). It outputs the couple (a,q) above, where a is the letter input under, and q the head input. The other outputs are q_s. See Figure 35 for an illustration of this rule.
   (b) when the head input comes from under, the output is $\delta_1(a,q)$ above when the $\delta_3(a,q)$ is in $\{\rightarrow, \leftarrow\}$ and $(\delta_1(a,q), \delta_2(a,q))$ when $\delta_3(a,q) = \uparrow$. The head output is in the direction of $\delta_3(a,q)$ when this output direction exists, and equal to $\delta_2(a,q)$ when this direction is in $\{\rightarrow, \leftarrow\}$. The other output is q_s. See Figure 36 for an illustration.
   The computation positions rules in the other quarters are similar. These rules in a purple quarter are obtained by reversing west and east, in the red one by reversing west and east and moreover north and south, and in the yellow one by reversing north and south. For instance the previous schemata are changed to the following one in the red quarter.
   This corresponds to changing the direction of time and space of evolution of the Turing machine, as abstracted on Figure 32.
3. **Collision with border:** When the output direction does not exist, the output is (a, q_e) on the top. The output on the side is q_s. The computation position is superimposed with (a,q).
4. **No machine head:** when all the inputs in $\mathcal{Q}$ are q_s, then the output above is in $\mathcal{A}$ and equal to a.

Global behavior:

In each of the quarters of any cell is implemented a computing machine according to our model, with multiple machine heads on the initial tape and entering in each row. In the next section, we will impose that in the computation quarters there are no machine heads entering on the sides. We also impose that the tape is well initialized. This is done using signals. As a consequence, in these quarters the computations are as intended. This means that a Turing machine writes successively the bits $s_k^{(n)}$ on the $p_k = 2^k$th column of its tape (in order to impose the value of the frequency bits). It enters in the error state q_e when it detects an error - meaning that the corresponding frequency bit in the column just on the right is greater than the written bit.

When this is not the case, the computations are determined by the rules giving the outputs on computation positions from the inputs. When there is a collision of a machine head with the border it enters into state q_e and when heads collide, they fusion into a unique head in state q_e.

5.7.3 Empty tape and sides signals

This sublayer serves for the propagation of a signal which detects if the initial tape of a machine is well initialized, and if a machine head enters on a side.

Symbols:

Elements of $\{$🟥,🟩$\}^2$, elements of $\{$🟥,🟩$\}$ and a blank symbol.

Local rules:

- **Localization:** the non-blank symbols are superimposed on and only on the arms of the reticle, and the west and east walls. The east and west walls are colored with elements of $\{$🟥,🟩$\}$, and the reticle with element of $\{$🟥,🟩$\}^2$.
- **Triggering the signal:** the topmost and bottommost positions of the two walls are superimposed with 🟩.
- **Transmission rules:**
  1. **In the walls:**
     - On the north (resp. south) part of one of the walls, the symbol 🟩 propagates upwards while in the wall. It propagates downwards (resp. upwards) while not encountering a symbol in $\mathcal{Q}\backslash\{q_s\}$. When this is is the case, the color becomes 🟥.
     - The symbol 🟥 propagates downwards (resp. upwards) in the north (resp. south) part of the wall, and upwards (resp. downwards) while not encountering a symbol in $\mathcal{Q}\backslash\{q_s\}$.
     - The center of the wall is colored with a couple of colors. The first one is equal to the color of the north position. The second one is equal to the color of the south position.

     With words, a signal propagates through the north (resp. south) part of the wall. This signal is triggered in state 🟩. When it detects the first symbol in $\mathcal{Q}\backslash\{q_s\}$, it changes its state which becomes 🟥. This information is transmitted to the center of the wall.
  2. **In the reticle:** The rules for the reticle are similar, except that:
     - there are two signals for each arm, one for each adjacent quarter.
     - the propagation direction is to the east for the west arm, and to the west for the east arm. The case of vertical arms is similar as the case of walls.
     - when the signal starts in state 🟩: on the west arm (resp. east one) each signal detects the first symbol from left to right (resp right to left) different from $(\#, q_s)$ when not on the rightmost position (resp. leftmost one), and different from $(\#, q_0)$ when on this position. These symbols correspond to the quarter associated with this signal.
     - when it starts in state 🟥, the arm just transmits this information to the nucleus.
     - In the vertical arms, the signals detect the first symbol different from q_s, from south to north for the south arm, and from north to south for the north arm.
- **Computation quarters are initialized with empty tape and sides:** considering a nucleus $\mathbf{u}$, the symbol on position $\mathbf{u}+\mathbf{e}^1$ has first (resp. second) coordinate equal to 🟩 if the orange (resp. yellow) quarter is represented in the DNA.
- the symbol on position $\mathbf{u}-\mathbf{e}^1$ has first (resp. second) coordinate equal to 🟩 if the purple (resp. red) quarter is represented in the DNA.
- There are similar conditions for the symbols on positions $\mathbf{u} \pm \mathbf{e}^2$.

Global behavior:

These rules induce the propagation of a signal triggered in state which detects, for each of the quarter, if the sides and the tape are well initialized: if this is not the case, then the signal detects an error. It sends this information, which corresponds to state , to the nucleus through the reticle. We forbid this signal to come from a quarter which is represented in the DNA. As a consequence, the computation quarters are well initialized. The simulation quarters are left free.

5.7.4 Error signals

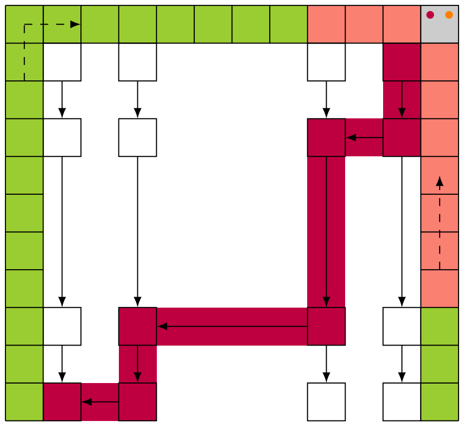


Figure 37: Schema of a possible trajectory of a machine in a simulation quarter. The empty tape and sides signals represented correspond to the south west quarter. The dashed arrows represent the propagation direction of these signals.

Symbols:

This sub-layer has the following symbols: and .

Local rules:

- **Localization:** the symbol can be superimposed only on positions having in its machine symbol a part in $\mathcal{Q}\backslash\{q_s\}$.
- **Transmission:** for two adjacent vertical transfer or horizontal transfer positions, the symbols in this sublayer are the same.
- when on a computation position $\mathbf{u}$, if two of the positions $\mathbf{u} \pm \mathbf{e}^1$ and $\mathbf{u} \pm \mathbf{e}^2$ have a part in $\mathcal{Q}\backslash\{q_s\}$, these two positions have the same symbol in this layer.
- **Triggering the error signal when on halting state:** a position with a symbol having a part equal to q_h is superimposed with .
- **Machine heads can not enter in error state:** if $\mathbf{u}$ is a nucleus position and the red quarter (resp. yellow, orange, purple ones) is represented in the DNA, then the position $\mathbf{u} - \mathbf{e}^2 - \mathbf{e}^1$ (resp. $\mathbf{u} - \mathbf{e}^2 + \mathbf{e}^1$, $\mathbf{u} + \mathbf{e}^2 + \mathbf{e}^1$, $\mathbf{u} + \mathbf{e}^2 - \mathbf{e}^1$) can not be superimposed with

Global behavior:

When a machine enters in error state, then it sends through its trajectory an error signal (represented by the symbol). See Figure 37 for a schematic example of a possible trajectory of a machine. In a computation quarter, since the tape is well initialized and the error signal is forbidden, this means that the machines in such a quarter effectively forbid the frequency bits f_n to be different from s_n.

6 Entropy formula for the entropy of $X_{s,N}$ and choices of the parameters.

In this section, we prove a formula for the entropy of the subshifts $X_{s,N}$. Using this formula, we describe how to choose N so that the entropy generated by simulation is small enough. Then s is chosen in order to complete this entropy so that the entropy of $X_{s,N}$ is equal to h.

The formula relies on the density of the observable structures in the subshift $X_{\mathtt{R}}$.

6.1 Density properties of the subshift $X_{\mathtt{R}}$

In this section we define the density of a subset of $\mathbb{Z}^2$ and compute the density of some subsets related to the subshift $X_{\mathtt{R}}$.

Definition 41. *Let Λ be a subset of $\mathbb{Z}^2$. Denote for $n \geq 1$:*

$$\mu_n(\Lambda) = \left|\Lambda \cap [\![-n,n]\!]^2\right|.$$

The upper and lower densities of Λ in $\mathbb{Z}^2$ are defined as respectively

$$\overline{\mu}(\Lambda) = \limsup_n \frac{\mu_n(\Lambda)}{(2n+1)^2}$$

and

$$\underline{\mu}(\Lambda) = \liminf_n \frac{\mu_n(\Lambda)}{(2n+1)^2}.$$

When the limit exists, it is called the density of Λ, and denoted $\mu(\Lambda)$.

Lemma 42. *1. Let Λ be a subset of $\mathbb{Z}^2$ having a density. Then $\Lambda^c = \mathbb{Z}^2 \backslash \Lambda$ has a density and*

$$\mu(\Lambda^c) = 1 - \mu(\Lambda).$$

2. Let $(\Lambda_k)_{k=1..m}$ be a finite sequence of subsets of $\mathbb{Z}^2$ such that for all $j \neq k$, $\Lambda_j \cap \Lambda_k = \emptyset$. Then the set $\bigcup_{k=1}^m \Lambda_k$ has density equal to

$$\mu\left(\bigcup_{k=1}^m \Lambda_k\right) = \sum_{k=1}^m \mu(\Lambda_k).$$

Proof. 1. For all n, the number of elements of Λ^c in $[\![-n,n]\!]^2$ is $(2n+1)^2 - |(\Lambda \cap [\![-n,n]\!]^2|$. Hence we have that

$$\mu_n(\Lambda^c) = 1 - \mu_n(\Lambda).$$

This means that Λ^c has a density equal to $1 - \mu(\Lambda)$.

2. For all n, the number of elements of $\bigcup_{k=1}^m \Lambda_k$ in $[\![-n,n]\!]^2$ is the sum of the numbers of elements in Λ_k, $k = 1...m$. An immediate consequence is that

$$\mu_n\left(\bigcup_{k=1}^m \Lambda_k\right) = \sum_{k=1}^m \mu_n(\Lambda_k).$$

Hence the density of the set $\bigcup_{k=1}^m \Lambda_k$ is indeed the sum of the densities of the sets Λ_k, $k = 1..m$.

□

Let x be some configuration in the subshift $X_{\mathtt{R}}$. For all $k \geq 0$, $\Lambda_k(x)$ denotes the set of positions that are included in an order k cell, not included in any smaller cell, and on which a blue corner is superimposed. Moreover, denote the set of positions on which is not superimposed a blue corner $\Lambda_*(x)$.

Lemma 43. *For all x in the subshift $X_{\mathtt{R}}$, we have the following:*

1. *the density $\mu(\Lambda_k(x))$ exists for all $k \geq 0$ and*
$$\mu(\Lambda_k(x)) = \frac{3^k}{4^{k+2}}$$
and the convergence of the functions $x \mapsto \mu_n(\Lambda_k(x))$ is uniform.

2. *the set $\Lambda_*(x)$ has a density equal to*
$$\mu(\Lambda_*(x)) = \frac{3}{4}$$
and the convergence of the functions $x \mapsto \mu_n(\Lambda_*(x))$ is uniform.

3. *for all $m \geq 0$, the set $\left(\bigcup_{k=0}^m \Lambda_k(x)\right)^c$ has a density equal to*
$$\mu\left(\left(\bigcup_{k=0}^{m} \Lambda_k(x)\right)^c\right) = \frac{1}{4}\frac{3^{m+1}}{4^{m+1}}.$$

Proof. 1. From the form of the subshift $X_{\mathtt{R}}$, for any configuration x and for all n the set $[\![-n,n]\!]^2$ can be covered with a number smaller than $(\lceil \frac{2n+1}{2.4^{k+1}} \rceil + 1)^2$ of 2.4^{k+1}-blocks centered on an order k cell in the configuration x. In each of these blocks there are exactly 4.12^k positions in Λ_k (see the properties listed in Section 5.1.1). Moreover, such a pattern contains a number of translates of $[\![0, 2.4^{k+1}]\!]^2$ which is at least $(\lfloor \frac{2n+1}{2.4^{k+1}+1} \rfloor - 1)^2$. As a consequence, for all n, k,
$$4.12^k . \frac{(\lfloor \frac{2n+1}{2.4^{k+1}} \rfloor - 1)^2}{(2n+1)^2} \leq \frac{\mu_n(\Lambda_k)}{(2n+1)^2} \leq 4.12^k \frac{(\lceil \frac{2n+1}{2.4^{k+1}} \rceil + 1)^2}{(2n+1)^2}.$$
This implies that
$$\frac{\mu_n(\Lambda_k(x))}{(2n+1)^2} \to \frac{12^k}{16^{k+1}} = \frac{3^k}{4^{k+2}}.$$

2. Moreover, the set $[\![-n,n]\!]^2$ is covered by at most $(\lceil \frac{2n+1}{2} \rceil + 1)^2$ blocks on $[\![0,1]\!]^2$ such that the symbol on position $(0,0)$ is a blue corner. This set contains at least a number $(\lfloor \frac{2n+1}{2} \rfloor - 1)^2$ of translates of $[\![0,1]\!]^2$. In each of these squares the number of positions in $\Lambda_*(x)$ is 3. This implies that
$$3\frac{(\lfloor \frac{2n+1}{2} \rfloor - 1)^2}{(2n+1)^2} \leq \frac{\mu_n(\Lambda_*(x))}{n} \leq 3\frac{(\lfloor \frac{2n+1}{2} \rfloor - 1)^2}{(2n+1)^2},$$
and we deduce that
$$\mu(\Lambda_*(x)) = \frac{3}{4}.$$

3. From the second point of Lemma 42,
$$\mu\left(\left(\bigcup_{k=0}^{m} \Lambda_k(x)\right)^c\right) + \mu\left(\left(\bigcup_{k=0}^{m} \Lambda_k(x)\right)^c\right) + \mu\left(\Lambda_*(x)\right) = 1.$$
As a consequence, using the two first points in the statement of the lemma and the second point of Lemma 42,
$$\mu\left(\left(\bigcup_{k=0}^{m} \Lambda_k(x)\right)^c\right) = \frac{1}{4} - \sum_{k=0}^{m} \frac{3^k}{4^{k+2}} = \sum_{k=0}^{+\infty} \frac{3^k}{4^{k+2}} - \sum_{k=0}^{m} \frac{3^k}{4^{k+2}} = \sum_{k=m+1}^{+\infty} \frac{3^k}{4^{k+2}} = \frac{3^{m+1}}{4^{m+2}}.$$
$\square$

6.2 A formula for the entropy depending on the parameters

In this section we prove a formula for the entropy depending on s and N.

Definition 44. *Let $(a_n)_n$ be a sequence of non-negative numbers. The series $\sum a_n$ converges at a computable rate when there is a computable function $n : \mathbb{N} \to \mathbb{N}$ (the rate) such that for all $t \in \mathbb{N}$,*

$$\left| \sum_{n \geq n(t)} a_n \right| \leq 2^{-t}$$

Remark 7. *Let $(a_n)_n$ and $(b_n)_n$ be two sequence of non-negative numbers such that for all n, $a_n \leq b_n$. If the series $\sum b_n$ converges at a computable rate, then the series $\sum a_n$ also converges at computable rate.*

Remark 8. *Let $(a_n)_n$ be some sequence of real numbers. If the series $\sum a_n$ converges at computable rate, then $\sum_{n=0}^{+\infty}$ is a computable number.*

Lemma 45. *There exists a sequence $(\kappa_k)_{k\geq 0}$ of non-negative real numbers such that the series $\sum_k \kappa_k$ converges at computable rate, and the entropy of the subshift $X_{s,N}$ is*

$$\boxed{h(X_{s,N}) = \frac{\lfloor 4h \rfloor}{4} + \sum_{k=0}^{N} \frac{3^k}{4^{k+2}} s_k + \frac{1}{2} \sum_{k=N+1}^{+\infty} \frac{3^k}{4^{k+2}} s_k + \sum_{k=N+1}^{+\infty} \kappa_k}.$$

Proof. Let us prove that

$$\begin{array}{rcl} h(X_{s,N}) & = & \sum_{k=N+1}^{+\infty} \left(\frac{1}{32.16^k} + \frac{1}{32.8^k} \log_2(|\mathcal{A}|) + \frac{3}{32.8^k} \log_2(|\mathcal{Q}|) + \frac{1}{8.4^k} + \frac{\log_2(\kappa_k^*)}{4.16^{k+1}} \right) \\ & & + \frac{1}{2} \sum_{k=N+1}^{+\infty} \frac{3^k}{4^{k+2}} (1 + s_k) \\ & & + \sum_{k=0}^{N} \frac{3^k}{4^{k+2}} s_k \\ & & + \frac{\lfloor 4h \rfloor}{4} \end{array},$$

where $(\kappa_k^*)_k$ is a computable sequence of integers.

Number of pattern on the proper blue corners positions in a cell:

For all $k > N$, the number of globally admissible patterns on the set of proper positions of an order k cell is equal to

$$4.|\mathcal{A}|^{2.2^k} |\mathcal{Q}|^{6.2^k}.(2^{12^k} + 1)^2.2^{2.12^k s_k}.2^{2.4.(4^k - 1)}.\kappa'(k).$$

Indeed:

1. The factor 4 corresponds to the number of possibilities for the DNA symbol on the nucleus of this cell [*See Section 5.4*].
2. The factor $|\mathcal{A}|^{2.2^k}|\mathcal{Q}|^{6.2^k}$ corresponds to the number of possibilities for filling the initial tapes of the two simulation quarters and the set of states of the machine heads entering on the two sides. Let us recall that the number of columns (resp. lines) in of quarter, that do not intersect a smaller cell, is equal to 2^k [*See Section 5.7*].
3. The factor $(2^{12^k} + 1)^2$ corresponds to the possibilities for the random bits on blue corner positions in these two quarters. Let us recall that the number of such positions in a quarter is equal to 12^k. Thus 2^{12^k} is the number of possibilities for the random bits when the frequency bit is 1, and 1 is the number of possibilities when the frequency bit is 0. [*See Section 5.2 and Section 5.7*].
4. The factor $2^{2.12^k s_k}$ corresponds to the possibilities for random bits in the computation quarters, since the frequency bit is determined to be s_k [*See Section 5.2 and Section 5.7*].

5. The last factor $2^{2.4.(4^k-1)}$ corresponds to the number of possibilities for the undetermined `in`, `out` symbols in the two simulation quarters. Let us recall that 4^k-1 is the number of lines (resp. columns) intersecting a quarter of an order k cell (since the number of non determined symbols correspond the the number of possible contacts a segment can have with the reticle or the walls of the cell). [*See Section 5.6*].

6. The factor 4 is the number of sides of a quarter and 2 is the number of simulation quarters.

7. κ_k^* denotes the number of possibilities for the set of symbols in the error signals sublayer. This number is not simple to express, since it depends on the states of machine heads on the sides of the area after computation. However, for this reason it can be computed. We have $\kappa_k^* \leq 4^{6.(4^{k+1}+1)}$ for the reason that the possible sets of error symbols correspond to the choices of at most four symbols on every position of the walls and the reticle. [*See Section 5.7*].

When $k \leq N$, this number is

$$2^{4.12^k s_k},$$

which corresponds to the number of possible sets of random bits.

Upper and lower bounds:

Let $m > N$. We shall give an upper bound and a lower bound on the number of $(2n+1)$-blocks in the language of $X_{s,N}$, for all n. This depends on the integer m.

The **lower bound** is obtained as follows:

1. we give a lower bound on the number of blue corners in an order $k \leq m$ cells, included in some translate of $[\![-n,n]\!]^2$.

2. we give a lower bound on the number of non-blue corners in this set.

3. taking the product of the possible patterns over the union of these cells, we get a lower bound.

Consider some configuration x. The set $[\![-n,n]\!]^2$ contains at least $\left(\lfloor \frac{2n+1}{2.4^{m+1}} \rfloor - 1\right)^2$ translates of $[\![1, 2.4^{m+1}]\!]^2$ centered on an order m cell. Moreover, there are at least a number $\left(\lfloor \frac{2n+1}{2} \rfloor - 1\right)^2$ of translates of $[\![0,1]\!]^2$ such that the $(0,0)$ symbol is a blue corner. On each of these squares, the number of possible patterns on the set $[\![0,1]\!]^2 \backslash (0,0)$ is $2^{\lfloor 4h \rfloor}$.

$$\begin{array}{rcl} N_{2n+1}(X_{s,N}) & \geq & \prod_{k=N+1}^m \left(4.|\mathcal{A}|^{2.2^k}|\mathcal{Q}|^{6.2^k}.(2^{12^k}+1)^2.2^{2.12^k s_k}.2^{2.4.(4^k-1)}.\kappa_k^*\right)^{\left(\lfloor \frac{2n+1}{2.4^{k+1}} \rfloor -1\right)^2} \\ & & .\prod_{k=1}^N \left(2^{\left(\lfloor \frac{2n+1}{2.4^{k+1}} \rfloor -1\right)^2 .4.12^k s_k}\right).2^{\left(\lfloor \frac{2n+1}{2} \rfloor -1\right)^2 \lfloor 4h \rfloor} \end{array}.$$

As a consequence,

$$\begin{array}{rcl} \frac{\log_2(N_{2n+1}(X_{s,N}))}{(2n+1)^2} & \geq & \frac{1}{(2n+1)^2}\sum_{k=N+1}^m \left(\lfloor \frac{2n+1}{2.4^{k+1}} \rfloor - 1\right)^2 (2+2.2^k.\log_2(|\mathcal{A}|)+6.2^k.\log_2(|\mathcal{Q}|)+8.(4^k-1) \\ & & +\log_2(\kappa_k^*)) + \frac{1}{(2n+1)^2}\sum_{k=N+1}^m \left(\lfloor \frac{2n+1}{2.4^{k+1}} \rfloor - 1\right)^2 \left(2.12^k(1+s_k)\right) \\ & & +\frac{1}{(2n+1)^2}\sum_{k=1}^N \left(\lfloor \frac{2n+1}{2.4^{k+1}} \rfloor - 1\right)^2 .4.12^k s_k \\ & & +\frac{1}{(2n+1)^2}.\left(\lfloor \frac{2n+1}{2} \rfloor - 1\right)^2 \lfloor 4h \rfloor \end{array}.$$

Taking n tending towards infinity, we obtain:

$$\begin{array}{rcl} h(X_{s,N}) & \geq & \sum_{k=N+1}^m \left(\frac{1}{32.16^k} + \frac{1}{32.8^k}\log_2(|\mathcal{A}|) + \frac{3}{32.8^k}\log_2(|\mathcal{Q}|) + \frac{1}{8.4^k} + \frac{\log_2(\kappa_k^*)}{4.16^{k+1}}\right) \\ & & +\frac{1}{2}\sum_{k=N+1}^m \frac{3^k}{4^{k+2}}(1+s_k) \\ & & +\sum_{k=0}^N \frac{3^k}{4^{k+2}} s_k \\ & & +\frac{\lfloor 4h \rfloor}{4} \end{array}.$$

Indeed, from the upper bound on $\kappa'(k)$, $k > N$ the series corresponding to this sequence in the formula above converges.

As this is true for all $m > N$, taking $m \to +\infty$, we obtain

$$\begin{array}{rcl} h(X_{s,N}) & \geq & \sum_{k=N+1}^{+\infty} \left(\frac{1}{32.16^k} + \frac{1}{32.8^k} \log_2(|\mathcal{A}|) + \frac{3}{32.8^k} \log_2(|\mathcal{Q}|) + \frac{1}{8.4^k} + \frac{\log_2(\kappa_k^*)}{4.16^{k+1}} \right) \\ & & + \frac{1}{2} \sum_{k=N+1}^{+\infty} \frac{3^k}{4^{k+2}} (1 + s_k) \\ & & + \sum_{k=0}^{N} \frac{3^k}{4^{k+2}} s_k \\ & & + \frac{\lfloor 4h \rfloor}{4} \end{array}.$$

On the other hand, the **upper bound** is obtained as follows:

1. Since any n-block of $X_{\mathtt{R}}$ can be extended into an order $\lceil \log_2(n) \rceil + 4$ supertile, there exists some $K > 0$ such that for all n the number of n-blocks in the structure layer is smaller than $(2^{K(\lceil \log_2(n) \rceil + 4)})^2 \leq 2^{9K} n^{2K}$.

2. For all $m \leq n$ and x, the set $[\![-n, n]\!]^2$ is covered by at most $\left(\lceil \frac{2n+1}{2.4^{m+1}} \rceil + 1\right)^2$ translates of $[\![1, 2.4^{m+1}]\!]^2$ centered on an order m cell.

3. The set $[\![-n, n]\!]^2$ is covered by at most a number $\left(\lceil \frac{2n+1}{2} \rceil + 1\right)^2$ of translates of $[\![0, 1]\!]^2$ such that the $(0, 0)$ symbol is a blue corner. On each of these squares, the number of possible patterns on the set $[\![0, 1]\!]^2 \backslash (0, 0)$ is $2^{\lfloor 4h \rfloor}$.

4. $\Lambda'_m(x)$ denotes the set of positions that are not in an order $\leq m$ cell in x. Then the number of possibilities for these positions in $[\![-n, n]\!]^2$ is smaller than $c^{|\Lambda'_n(x) \cap [\![-n,n]\!]^2|}$, where c is the cardinality of the alphabet of the subshift $X_{s,N}$.

Hence we have the following inequality:

$$\begin{array}{rcl} N_{2n+1}(X_{s,N}) & \leq & 2^{9K} n^{2K} . \prod_{k=N+1}^{m} \left(4.|\mathcal{A}|^{2.2^k} |\mathcal{Q}|^{6.2^k} .(2^{12^k} + 1)^2 .2^{2.12^k s_k} .2^{2.4.(4^k - 1)} .\kappa_k^* \right)^{\left(\lceil \frac{2n+1}{2.4^{k+1}} \rceil + 1\right)^2} \\ & & . \prod_{k=0}^{N} \left(2^{\left(\lceil \frac{2n+1}{2.4^{k+1}} \rceil + 1\right)^2 .4.12^k s_k} \right) .2^{\left(\lceil \frac{2n+1}{2} \rceil + 1\right)^2 \lfloor 4h \rfloor} .c^{|\Lambda'_m(x) \cap [\![-n,n]\!]^2|} \end{array}.$$

This implies, as for the lower bound, and since $\log_2(2^{9K} n^{2K})/(2n+1)^2 \to 0$ when n tends to infinity, and from the third point of Lemma 43, that

$$\begin{array}{rcl} h(X_{s,N}) & \leq & \sum_{k=N+1}^{+\infty} \left(\frac{1}{32.16^k} + \frac{1}{32.8^k} \log_2(|\mathcal{A}|) + \frac{3}{32.8^k} \log_2(|\mathcal{Q}|) + \frac{1}{8.4^k} + \frac{\log_2(\kappa_k^*)}{4.16^{k+1}} \right) \\ & & + \frac{1}{2} \sum_{k=N+1}^{+\infty} \frac{3^k}{4^{k+2}} (1 + s_k) \\ & & + \sum_{k=0}^{N} \frac{3^k}{4^{k+2}} s_k \\ & & + \frac{\lfloor 4h \rfloor}{4} + \log_2(c) \frac{3^{m+1}}{4^{m+2}} \end{array}.$$

Taking $m \to +\infty$, as $3^m/4^m \to 0$, we have the upper bound:

$$\begin{array}{rcl} h(X_{s,N}) & \leq & \sum_{k=N+1}^{+\infty} \left(\frac{1}{32.16^k} + \frac{1}{32.8^k} \log_2(|\mathcal{A}|) + \frac{3}{32.8^k} \log_2(|\mathcal{Q}|) + \frac{1}{8.4^k} + \frac{\log_2(\kappa_k^*)}{4.16^{k+1}} \right) \\ & & + \frac{1}{2} \sum_{k=N+1}^{+\infty} \frac{3^k}{4^{k+2}} (1 + s_k) \\ & & + \sum_{k=0}^{N} \frac{3^k}{4^{k+2}} s_k \\ & & + \frac{\lfloor 4h \rfloor}{4} \end{array}.$$

□

6.3 Choosing the parameters values

In this section we explain how to choose s, N and M such that the entropy of $X_{s,N}$ is equal to h and there exists some M such that for all $k \geq 1$ integer, $s_{2kM} = 1$ and $s_{(2k+1)M} = 0$. This constraint on the sequence s will serve for the linear net gluing property. Here is the process that we follow for these choices:

1. we choose $N > 0$ and $M > 0$ such that
$$\begin{array}{rl} h > & \sum_{k=N+1}^{+\infty} \left(\frac{1}{32.16^k} + \frac{1}{32.8^k}\log_2(|\mathcal{A}|) + \frac{3}{32.8^k}\log_2(|\mathcal{Q}|) + \frac{1}{8.4^k}\right) + \frac{1}{2}\sum_{k=N+1}^{+\infty} \frac{3^k}{4^{k+2}} \\ + & \sum_{k=1}^{+\infty} \frac{3^{2kM}}{4^{2(kM+1)}} + \frac{\lfloor 4h \rfloor}{4} \end{array}.$$
This is possible since $\frac{\lfloor 4h \rfloor}{4} < h$. The other terms of the sum in the left member of this inequality tends to 0 when N, M tends to $+\infty$. For this value of M, we impose the constraint on s that for all $k \geq 1$ integer, $s_{2kM} = 1$ and $s_{(2k+1)M} = 0$.

2. Since
$$\sum_{k=0} \frac{3^k}{4^{k+2}} + \frac{\lfloor 4h \rfloor}{4} = \frac{1}{4} + \frac{\lfloor 4h \rfloor}{4} > h,$$
the maximal value for the entropy $h(X_{s,N})$ for s verifying the constraint is greater than h. Moreover, the minimal value is smaller than h. The number
$$\begin{array}{rl} z' = & \sum_{k=N+1}^{+\infty} \left(\frac{1}{32.16^k} + \frac{1}{32.8^k}\log_2(|\mathcal{A}|) + \frac{3}{32.8^k}\log_2(|\mathcal{Q}|) + \frac{1}{8.4^k}\right) \\ & + \frac{1}{2}\sum_{k=N+1}^{+\infty} \frac{3^k}{4^{k+2}} + \sum_{k=1}^{+\infty} \frac{3^{2kM}}{4^{2(kM+1)}} + \frac{\lfloor 4h \rfloor}{4} \end{array}$$
is computable, and as a consequence the number
$$z = h - z'$$
is a Π_1-computable number. Hence we now have to choose s a Π_1-computable sequence in $\{0,1\}^{\mathbb{N}}$ such that the number
$$\frac{1}{2}\sum_{k=N+1}^{+\infty} \frac{3^k}{4^{k+2}} s_k + \sum_{k=0}^{N} \frac{3^k}{4^{k+2}} s_k$$
is equal to z. This is possible since z is a Π_1-computable number.

For these values of s and N and given the choice of M, the subshift $X_{s,N}$ is denoted X_h.

7 Linear net gluing of X_h

In this section we prove that the subshift X_h is linearly net gluing. This proof consists of two steps: first proving that any block can be extended into a cell, with control over the size of this cell. Then we prove the gluing property on cells having the same order. For reading this section, the reader should have some familiarity with the construction of the Robinson subshift [Rob71].

7.1 Completing blocks

The point of this section is to prove the following lemma:

Lemma 46. *Let $n \geq 0$ an integer, and P some $2^{n+1}-1$-block. This pattern can be completed into an admissible pattern in X_h over an order*
$$\left(2\left(\left\lceil \frac{n+1}{2.2M} \right\rceil + 1\right) + 3\right) M$$
cell.

Proof. Let $n \geq 0$ an integer and P some $2^{n+1}-1$-block which appears in some configuration x of the subshift X_h.

1. **Intersecting four order $2n$ supertiles:**

   This part of the proof is similar to the beginning of the proof of Lemma 16. The pattern P can be extended in the configuration x into a pattern over some pattern over one of the structure layer patterns on Figure 45, composed with four $2n$ order supertiles with a cross separating them.

2. **Completing the structure, frequency bits and basis:**

   All these patterns intersect non-trivially at most two different cells in the configuration x, one included into the other. The intersection with the small one is included into the union of two quarters of the cell with the separating segment. The intersection with the great one is included into a quarter of this cell. Similarly as in the proof of Lemma 16, the new formed pattern can be completed into an order $\left\lceil \frac{\lceil \log_2(2^{n+1}-1) \rceil}{2} \right\rceil + 2 + k$ cell, for any $k \geq 0$ in the Robinson sub-layer. However, in the present proof, we have to take care about the frequency bits of these two cells. Indeed, the order of the cell into which the pattern is completed has to be coherent with the frequency bit.

   That is why we slightly modify the way we complete the structure layer. In the cases when the pattern intersects non-trivially two cells (the schemata corresponding to the case of an intersecting with two cells are the numbers $1, 2, 3$, the second $5'$, $6, 7$, the third 8, and the first 9 on Figure 45), we first complete the smallest cell into a cell having minimal order greater than $\left\lceil \frac{\lceil \log_2(2^{n+1}-1) \rceil}{2} \right\rceil + 2$. This is done in such a way that the corresponding bit is imposed to be equal to the actual bit attached to the part of the cell intersecting the pattern at this point. This means that we extend this part into an order $k.M$ cell, with k equal to

   $$2\left(\left\lceil \frac{\lceil \log_2(2^{n+1}-1) \rceil}{2.2M} \right\rceil + 1 \right)$$

   or

   $$2\left(\left\lceil \frac{\lceil \log_2(2^{n+1}-1) \rceil}{2.2M} \right\rceil + 1 \right) + 1$$

   Then the part of the second cell is completed into an order $k'.M$ cell, with k' equal to

   $$2\left(\left\lceil \frac{\lceil \log_2(2^{n+1}-1) \rceil}{2.2M} \right\rceil + 1 \right) + 2$$

   or

   $$2\left(\left\lceil \frac{\lceil \log_2(2^{n+1}-1) \rceil}{2.2M} \right\rceil + 1 \right) + 3.$$

   Thus, in any case, the initial pattern can be completed into an order

   $$\left(2\left(\left\lceil \frac{n+1}{2.2M} \right\rceil + 1 \right) + 3 \right) M$$

   cell.

   The case when the pattern intersects non-trivially only one cell, this completion is done similarly.

After this, one can complete the frequency bits layer according to the values in the initial pattern. One can also complete the synchronization net sublayer and the synchronization layer, since these are determined by the Robinson sublayer and the frequency bits. Then the random bits are chosed according to the frequency bits.

3. **Completing the computation areas, machines trajectories, and error signals:**

   In this paragraph, we describe how to complete the other layers (computation areas and machines layers) over this completed pattern.

   For each of the two non-trivially intersected cells, the lines and columns that do not intersect the initial pattern and that are between this pattern and the nucleus are colored $(\texttt{out}, \texttt{out})$. This allows the extension of the pattern in the machine layer simply by transport information between the nucleus and this part of the area. Indeed, there is not computation position outside the initial pattern. This is illustrated on Figure 39. When completing the machine's trajectories in the direction of time, we simply apply the computation rules of the machine.

   Error signals for the computation areas are triggered by these choices. We choose the propagation direction according to the presence or the absence of an error signal in the initial pattern. This is illustrated on Figure 38. The empty tape and sides signals and error signals of the machines are completed in a similar way.

   If the nucleus was in the initial pattern, then all these layers can be completed according to the configuration x. If this was not the case, then we choose the DNA such that the quarters present in the initial pattern - there are at most two since the nucleus is not in the initial pattern - are not represented in the DNA.

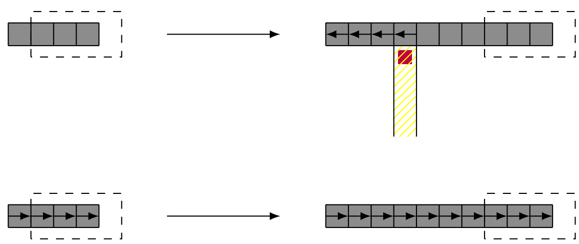


Figure 38: Illustration of the completion of the arrows according to the error signal in the known part of the area, designated by a dashed rectangle. The yellow color designates some $(\texttt{out}, \texttt{out})$ column.

□

7.2 Linear net gluing of X_h

Theorem 47. *The subshift X_h is linearly net gluing.*

Proof. Let P and Q be two $(2^{n+1}-1)$-blocks in the language of X_h. These two patterns can be completed into admissible patterns over order

$$\left(2\left(\left\lceil\frac{n+1}{2.2M}\right\rceil+1\right)+3\right)M$$

cells. The gluing set of these two cells is

$$4^{\left(2\left(\left\lceil\frac{n+1}{2.2M}\right\rceil+1\right)+3\right)M+2}.\mathbb{Z}^2\backslash(0,0).$$

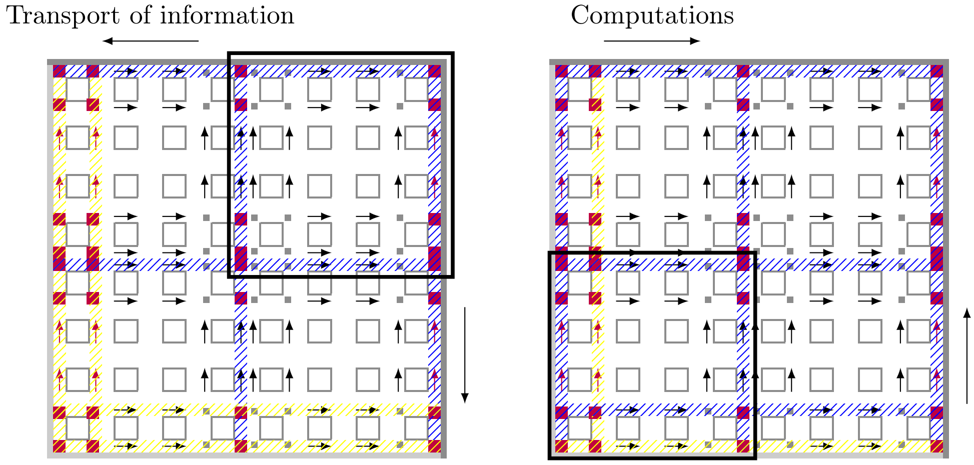


Figure 39: Illustration of the completion of the $\texttt{in}, \texttt{out}$ signals and the space-time diagram of the machine. The known part of the cell is surrounded by a black square. The yellow lines and columns are colored with a symbol different from $(\texttt{in}, \texttt{in})$ in the computation areas layer, while the blue columns and lines are.

This means that there exists some vector $\mathbf{u}$ such that the gluing set of the pattern P contains

$$\mathbf{u} + (2^{n+1} - 1 + f(2^{n+1} - 1)).\mathbb{Z}^2 \backslash (0, 0),$$

where

$$f(2^{n+1} - 1) = 4^{\left(2\left(\left\lceil \frac{n+1}{2.2M} \right\rceil + 1\right) + 3\right) M+2} - 2^{n+1} + 1 = O(2^{n+1} - 1).$$

As a consequence of Proposition 5, the subshift X_h is linearly net gluing. □

8 Transformation of X_h into a linearly block gluing SFT

In this section, we prove that every Π_1-computable non-negative real number is the entropy of some linearly block gluing $\mathbb{Z}^2$-SFT.

In order to prove this assertion, we use modified versions of the operator $d_{\mathcal{A}}$, depending on an integer parameter $r \geq 1$, that we denote $d_{\mathcal{A}}^{(r)}$. The definition of the operators $d_{\mathcal{A}}^{(r)}$ consists in imposing that the curves appearing in the definition of $d_{\mathcal{A}}$ are composed by length r straight segments, as illustrated on Figure 40.

$$\begin{array}{ccccccccc} \rightarrow & \rightarrow & \rightarrow & \downarrow & \downarrow & \downarrow & \rightarrow & \rightarrow & \rightarrow \\ \downarrow & \downarrow & \downarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \end{array}$$

Figure 40: Making the curves more rigid. In this example, $r = 3$.

Each of these segments can be superimposed with random colors defined to be a length r word amongst the words $0^r, 10^{r-1}, ...1^r$. This sequence is imposed to be 0^r when the segment is not surrounded with other segments, as illustrated on Figure 41.

The idea behind this definition is that with these random colors, the patterns crossed only by straight curves are the most numerous ones, and thus the entropy is easier to compute, with the cost of a parasitic entropy. The parameter r serves to control the parasitic entropy.

$\tilde{A}_r$ denotes the alphabet of $d_{\mathcal{A}}^{(r)}$. The operators $d_{\tilde{\mathcal{A}}_r}^{(r)} \circ \rho \circ d_{\mathcal{A}}^{(r)}$ still transform linearly net gluing subshifts into linearly block gluing ones, and the entropy of $d_{\mathcal{A}}^{(r)}(Z)$ for a subshift Z is a function of $h(Z)$:

$$\begin{array}{ccccccccc}
\rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow & \color{red}{\rightarrow} & \color{red}{\rightarrow} & \color{red}{\rightarrow} & \rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow
\end{array}$$

Figure 41: Segment (colored red) surrounded by other ones. In this example, $r = 3$.

Theorem 48. *For any Z on $\mathbb{Z}^2$ and on alphabet $\mathcal{A}$, the entropy of $d_{\mathcal{A}}^{(r)}(Z)$ is*

$$h(d_{\mathcal{A}}^{(r)}(Z)) = h(Z) + \frac{\log_2(1+r)}{r}.$$

Since any non-negative Π_1-computable number is the entropy of a linearly net gluing SFT, for all $r \geq 1$, all the numbers in $\left[2\frac{\log_2(1+r)}{r}, +\infty\right[$ (the factor 2 here comes from the fact that we apply two operators) are entropy of a linearly block gluing SFT. As a consequence, since 0 is the entropy of the full shift on alphabet $\{0\}$, which is linearly block gluing, all the non-negative Π_1-computable numbers are entropy of a linearly block gluing SFT.

Question 1. *In this proof, the entropy 0 is obtained in a different way than other Π_1-computable numbers. Is there some non-trivial SFT which is linearly block and have entropy 0 ?*

8.1 Definition of the operators $d_{\mathcal{A}}^{(r)}(Z)$

Let Z be a $\mathbb{Z}^2$-SFT on alphabet $\mathcal{A}$, and $r \geq 1$. The subshift $d_{\mathcal{A}}^{(r)}(Z)$ is defined as a product of four layers:

1. the first layer is Δ,
2. the second one is Z, with similar rules with respect to Δ as in the definition of $d_{\mathcal{A}}$.
3. the counter layer [Section 8.1.1]: in this layer we impose, using a counter, the curves to be composed of length r segments.
4. the random colors layer [Section 8.1.2]: here we superimpose random colors to the length r segments of curves.

8.1.1 Counter layer

Symbols:
The elements of $\mathbb{Z}/r\mathbb{Z}$ and a blank symbol.
Local rules:

- **Localization:** the non blank-symbols are superimposed on and only on the positions having a $\rightarrow$ symbol in the Δ layer.
- **Incrementing the counter:** over a pattern

  $$\rightarrow\rightarrow$$

  or

  $$\begin{array}{cc} \rightarrow & \downarrow \\ \downarrow & \rightarrow \end{array}$$

  in the Δ layer, if the value of the counter on the left position with $\rightarrow$ is $\bar{\imath}$, then the value on the right position is $\bar{\imath} + \bar{1}$.

- **The curves can shift downwards only on position with maximal counter value:** on the pattern

$$\begin{array}{cc} \rightarrow & \downarrow \\ \downarrow & \rightarrow \end{array},$$

the left position with $\rightarrow$ has counter value equal to $\overline{r-1}$.

Global behavior:

On each curve induced by the restriction to the Δ layer we superimpose independent counters that are incremented along the curves. A curve can shift downwards only when the counter has maximal value. This implies that the curves are composed of length r segments. A segment is defined to be a part of the curve between two consecutive positions where the counter has value $\overline{0}$ and $\overline{r-1}$.

8.1.2 Random colors layer

Symbols:

The symbols of this layer are $0, 1$ and a blank symbol.

Local rules:

- **Localization:** the bits $0, 1$ are superimposed on and only on positions with symbol $\rightarrow$.
- **Restriction of the possible colors:** Along a length r segment of curve, the symbol 1 propagates to the left. The symbol 0 propagates to the right.
- **Isolation rule:** if a segment is not surrounded by other segments, its color is 0^r.

Global behavior:

Each length r segment is attached with a word in $\{0^r, 10^{r-1}, ..., 1^r\}$. Moreover, if the segment is not surrounded by other segments, its color is 0^r.

8.2 Transformation of linearly net gluing subshifts into linearly block gluing ones

We have a result similar to Theorem 33 for the operators $d_{\mathcal{A}}^{(r)}$:

Theorem 49. *For all $r \geq 0$, the operator $d_{\tilde{\mathcal{A}}_r}^{(r)} \circ \rho \circ d_{\mathcal{A}}^{(r)}$ transforms linear net-gluing subshifts of finite type into linear block gluing ones.*

Proof. The steps of the proof of this theorem are exactly the same ones as for the proof of Theorem 33. However, the gap function of the image subshift is r times the gap function of the image subshift obtained when applying $d_{\tilde{\mathcal{A}}} \circ \rho \circ d_{\mathcal{A}}$. The presence of the counters do not have any impact since when we proved that two patterns can be glued we don't connect the curves of the two patterns. □

Δ'_r denotes the subshift that consists in the product of the Δ layer with the counter layer and random colors layer, with rules relating these layers. Moreover, Δ_r denotes the product of the Δ layer with the counter layer, with rules relating these layers.

The following sections are devoted to the proof of Theorem 48.

8.3 Lower bound

Lemma 50. *For any subshift Z on alphabet $\mathcal{A}$, we have the following lower bound on the entropy of $d_{\mathcal{A}}^{(r)}(Z)$:*

$$h(d_{\mathcal{A}}^{(r)}(Z)) \geq h(Z) + \frac{\log_2(1+r)}{r}.$$

Proof. The language of $d_{\mathcal{A}}^{(r)}(Z)$ contains all the kr-blocks whose symbols in the Δ layer are all equal to $\rightarrow$ and such that in the first column, the value of the counter is $\overline{0}$ in each line.

The number of such patterns is $N_{kr}(Z).(r+1)^{rk^2}$ (the first factor is the number of choices in the Z layer and the other one is the number of choices in the random colors layer). Hence

$$N_{kr}(d_{\mathcal{A}}^{(r)}(Z)) \geq (r+1)^{rk^2}.N_{kr}(Z).$$

Which implies

$$\frac{\log_2(N_{kr}(d_{\mathcal{A}}^{(r)}(Z)))}{kr.kr} \geq \frac{\log_2(1+r)}{r} + \frac{\log_2(N_{kr}(Z))}{kr.kr}.$$

We deduce, taking $k \rightarrow +\infty$, that

$$h(d_{\mathcal{A}}^{(r)}(Z)) \geq \frac{\log_2(1+r)}{r} + h(Z).$$

□

8.4 Upper bound

We prove an upper bound for $h(d_{\mathcal{A}}^{(r)}(Z))$ in two steps:

1. In Section 8.4.1, we prove a bound on the number of possible pseudo-projections of a kr-block onto the subshift Z. This is done ussing an upper bound on the number of curves crossing a n-block.
2. In Section 8.4.2, we give an upper bound on the number of kr-blocks in Δ'_r. We do this by analyzing the possible ways to extend a kr-block in the language of this subshift into a $(k+1)r$-block which stays admissible.

8.4.1 Upper bound on the number of pseudo-projections

Lemma 51. *Let $n \geq 1$ and P be some n-block in the language of Δ_r. The number of curves crossing P is equal to $k_n + k'_n$, where k_n is the number of $\rightarrow$ symbols on the south west - north east diagonal and k'_n is the number of patterns*

$$\begin{array}{cc} \rightarrow & \downarrow \\ & \rightarrow \end{array}$$

such that the $\downarrow$ symbol is on the south west - north east diagonal.

Proof. Each curve crossing P crosses the diagonal. This is due to the fact that in each column, the curve goes straightly onto the right or is shifted downwards. When it crosses the diagonal, there are two possibilities: either it is shifted downwards, and it corresponds to the pattern

$$\begin{array}{cc} \rightarrow & \downarrow \\ & \rightarrow \end{array},$$

or it is not and this corresponds to the symbol $\rightarrow$ on the diagonal. Hence counting this patterns gives the number of curves crossing P. On Figure 42, the pattern has three times the symbol $\rightarrow$ on the diagonal and once the pattern

$$\begin{array}{cc} \rightarrow & \downarrow \\ & \rightarrow \end{array}.$$

One can see that the total is equal to the number of curves crossing this pattern.

□

Lemma 52. *The number of pseudo-projections of a kr-block pattern in the language of $d_{\mathcal{A}}^{(r)}(Z))$ is smaller or equal to $N_{kr}(Z)$.*

Proof. A a consequence of Lemma 51, a kr-block contains at most kr curves and the number of positions of a curve in a kr block is smaller than kr. Hence the number of pseudo-projections of a kr-block on Z is smaller than $N_{kr}(Z)$. □

$$\begin{array}{cccccc}
\rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & \downarrow \\
\rightarrow & \downarrow & \downarrow & \downarrow & \rightarrow & \rightarrow \\
\downarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & \downarrow \\
\rightarrow & \rightarrow & \rightarrow & \downarrow & \rightarrow & \rightarrow \\
\downarrow & \downarrow & \downarrow & \rightarrow & \rightarrow & \rightarrow
\end{array}$$

Figure 42: Example of pattern P in the Δ layer. The patterns on the diagonal allow the number of curves to be counted. In this example, $r = 3$.

8.4.2 Upper bound on the number of colored curves patterns

In this section, we give an upper bound on the number of kr-block in the language of Δ'_r, for $k \geq 1$. It relies on an upper bound on the number of patterns in specific sets, defined as follows.

Consider some kr-block P in the language of Δ_r. Define $\mathbb{U}_P$ to be the minimal set containing $[\![0, kr-1]\!]^2$ such that there exists some pattern P' - which is unique - on support $\mathbb{U}_P$:

- whose restriction on $[\![0, kr-1]\!]^2$ is P,
- and such that the leftmost (resp. rightmost) position in any curve in P' crossing the left (resp. right) side of P has counter $\overline{0}$ (resp. $\overline{r-1}$).

On Figure 43, one can find some example of such completion of a block P into the pattern P'.

$$\begin{array}{cccccccccc}
\rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & \downarrow & & \\
 & \rightarrow & \rightarrow & \rightarrow & \downarrow & \downarrow & \rightarrow & \rightarrow & \rightarrow & \\
 & & \downarrow & \downarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & & \\
\rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & \rightarrow & \rightarrow & \rightarrow \\
 & & \downarrow & \downarrow & \downarrow & \downarrow & \rightarrow & \rightarrow & \rightarrow & \\
 & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & &
\end{array}$$

$$\begin{array}{cccccccccc}
\overline{0} & \overline{1} & \overline{2} & \overline{0} & \overline{1} & \overline{2} & & & & \\
 & \overline{0} & \overline{1} & \overline{2} & & & \overline{0} & \overline{1} & \overline{2} & \\
 & & & & \overline{0} & \overline{1} & \overline{2} & & & \\
\overline{0} & \overline{1} & \overline{2} & \overline{0} & \overline{1} & \overline{2} & & \overline{0} & \overline{1} & \overline{2} \\
 & & & & & & \overline{0} & \overline{1} & \overline{2} & \\
 & \overline{0} & \overline{1} & \overline{2} & \overline{0} & \overline{1} & \overline{2} & & &
\end{array}$$

Figure 43: Illustration of the extension of a block by completing the segments, $r = 3$. The pattern P is represented by dark symbols, and is completed into the projection of P' by red ones.

Let us denote $\mathcal{T}_{kr}$ the set of patterns in the language of Δ'_r whose projection on Δ_r is P' for some kr-block P in the language of Δ_r.

Lemma 53. *For all $k \geq 1$, we have the following upper bound on the number of kr-blocks in the language of Δ'_r:*

$$|N_{kr}(\Delta'_r)| \leq \alpha(r).2^{\lambda(r).k}(r+1)^{rk^2},$$

where $\alpha(r), \lambda(r) > 0$ depend only on r.

Idea of the proof. *The idea of the proof is to get an upper bound on the cardinality of $\mathcal{T}_{kr}$ for all $k \geq 1$ considering the possible extensions of a pattern in $\mathcal{T}_{kr}$ into a pattern of $\mathcal{T}_{(k+1)r}$. We derive then an upper bound for the number of kr-blocks in the language of Δ'_r.*

Proof. 1. **Upper bound on the extensions of patterns in $\mathcal{T}_{kr}$ into patterns of $\mathcal{T}_{(k+1)r}$:** Consider some pattern Q in $\mathcal{T}_{kr}$. We will first consider the number of possibilities to extend this pattern on the right side and then on the top, as illustrated on Figure 44.

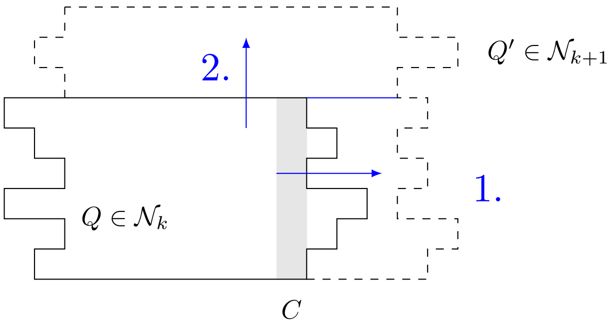


Figure 44: Illustration of the considered order for completion in order to give an upper bound on $|\mathcal{T}_{(k+1)r}|/|\mathcal{T}_{kr}|$.

(a) **Extensions on the right side:**
The restriction C of Q on the rightmost complete column $\{kr-1\} \times [\![0, kr-1]\!]$ (colored gray on Figure 44) is sub-pattern of a pattern C' over $\{kr-1\} \times [\![-1, kr]\!]$ or $\{kr-1\} \times [\![0, kr]\!]$ such that this pattern is the (vertical) concatenation of patterns

$$\begin{matrix}\rightarrow \\ \vdots \\ \rightarrow \\ \downarrow\end{matrix}.$$

One can see this by adding a symbol $\rightarrow$ on the top, and a symbol $\downarrow$ on the bottom if the bottommost symbol in this column is $\rightarrow$ (we do not complete C into C', which is just an artifact allowing an upper bound on the number of ways to extend Q on its right side). Each of these patterns corresponds to a set of vertically consecutive curves going out of the pattern Q through its right side.

A completion of Q on the right side is determined by the following choices:

- for each of the outgoing curves, choose if this curve is shifted downwards in the set of additional positions or not.
- for each of the added segments of curve, choose a color to superimpose over it.

Moreover, for each set of consecutive curves, if one of these curves is shifted downwards, this forces the curves in this set to be shifted downwards as well in the set additional positions. This shift is realized in a column on the left of the column where the first curve is shifted. Since the additional shift positions lie in a set of r consecutive columns, this means that only the r bottommost curves in this set can be shifted downwards in the additional columns. The number of possible choices for these shifts is $\min(j, r)$ (since the position of the shift is determined by the counter), where j is the number of curves in this set.

For this set of curves the number of choices for the colors is $(r+1)^j$. As a consequence, for a set of consecutive curves represented by a pattern

$$\begin{matrix}\rightarrow \\ \vdots \\ \rightarrow \\ \downarrow\end{matrix},$$

with j symbols $\rightarrow$, the number of possible extensions on the right of this pattern is smaller than $(r+1)^{j+1}$. Since in this formula, $j+1$ is the height of the pattern

$$\begin{array}{c}\rightarrow\\ \vdots\\ \rightarrow\\ \downarrow\end{array},$$

the number of possible extensions of Q on the right side is smaller than the product of these numbers. This is equal to

$$(r+1)^{kr+2},$$

kr being the height of the pattern C.

(b) **Extensions on the top:**

Let us consider a possible completion $Q^{(1)}$ of Q on the right side. We give an upper bound on the number of possible ways to complete this pattern on the row $[\![0,(k+1)r-1]\!] \times \{kr\}$ just above this pattern. This depends only on the restriction of $Q^{(1)}$ on $[\![0,(k+1)r-1]\!] \times \{kr-1\}$. This way, taking the power r of this bound will provide a bound for the possible completions of $Q^{(1)}$ into a pattern $Q^{(2)}$ of $\mathcal{T}_{(k+1)r}$.

The restriction of $Q^{(1)}$ on its topmost row $[\![0,(k+1)r-1]\!] \times \{kr-1\}$ can be decomposed into a (possibly empty) concatenation of patterns

$$\rightarrow \ldots \rightarrow\downarrow \ldots \downarrow,$$

possibly followed on the right by a pattern

$$\rightarrow \ldots \rightarrow,$$

and possibly preceded on the left by a pattern

$$\downarrow \ldots \downarrow.$$

For instance, on the following pattern, we represent the decomposition with parentheses:

$$(\downarrow\downarrow\downarrow)\,(\rightarrow\rightarrow\rightarrow\downarrow\downarrow\downarrow)\,(\rightarrow\rightarrow\rightarrow\downarrow\downarrow\downarrow)\,(\rightarrow\rightarrow\rightarrow)\,.$$

According to this decomposition, the possibilities for completing Q' on the top are as follows:

- If the pattern on the top row of $Q^{(1)}$ is equal to

  $$\rightarrow \ldots \rightarrow,$$

  then the pattern can extended on top with some pattern which consists in the concatenation of

  $$\downarrow \ldots \downarrow,$$

  with

  $$\rightarrow \ldots \rightarrow$$

  on the right, one of which can be empty. For instance, if the pattern on the top row is

  $$\rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow$$

  and $r=3$, one can extend the pattern in the following ways:

  $$\begin{array}{cccccccc}\downarrow & \downarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow\\ \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & & \end{array},$$

$$\begin{array}{cccccc} \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\ \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \end{array},$$

or

$$\begin{array}{cccccc} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \end{array}.$$

- When we extend with the pattern

$$\rightarrow \dots \rightarrow,$$

there are r possibilities for the positions of the counter symbols, and at most $(r+1)^{k+2}$ possibilities for the random colors of the added segments (there are at most $(k+2)$ ones). As a consequence, in this case there are at most $r(r+1)^{k+2} \leq (r+1)^{k+3}$ possible extensions.

- When we extend with some pattern

$$\downarrow \dots \downarrow \rightarrow \dots \rightarrow,$$

there are r possibilities for the rightmost position of the row where the counter has value $\overline{0}$. For each j such that the added curve shifts downwards in a column between the jrth and the $(j+1)r-1$th one, the number of possible colorings of this curve is at most $(r+1)^{k-j+1}$ (since in this case, the number of added segments is at most $(k-j+1)$). As a consequence, the number of completions in this case is at most

$$r\sum_{j=1}^{k+1}(r+1)^j = r(r+1)\frac{(r+1)^{k+1}-1}{r+1-1} \leq (r+1)^{k+2}.$$

- When the pattern on the top row of $Q^{(1)}$ is

$$\downarrow \dots \downarrow,$$

the only possibility for completion in the Δ layer is by

$$\rightarrow \dots \rightarrow.$$

The number of possibilities in the other layers is given by the choice of the counter position and the colors, and is smaller than $(r+1)^{k+3}$.

- **Mixed cases:** For the same reasons as in the two previous cases, the number of possible extensions on the top of the $\rightarrow \dots \rightarrow$ pattern in the decomposition of the top row pattern and the $\downarrow \dots \downarrow$ pattern, and the leftmost (resp. rightmost) pattern

$$\rightarrow \dots \rightarrow \downarrow \dots \downarrow$$

when there is no pattern $\downarrow \dots \downarrow$ (resp. $\rightarrow \dots \rightarrow$) in the decomposition, are at most

$$(r+1)^{\left\lceil \frac{l}{r} \right\rceil + 3},$$

where l is the length of this pattern.
The possible extensions over the other patterns

$$\rightarrow \dots \rightarrow \downarrow \dots \downarrow$$

in the decomposition are as follows:

– the $r\left\lceil \frac{m}{r} \right\rceil$ rightmost symbols in the extending row are equal to $\rightarrow$, where m is the length of the sub-pattern $\downarrow \dots \downarrow$: this is imposed by the presence on the right of $\rightarrow$ symbols. This corresponds to a shifted curve, which has to go straight while there are symbols $\downarrow$ below it during a number rj of columns, for some j. In this step, the colors of the added segments is 0^r, since they are not surrounded by other segments. At this point, the completion over this part of the top row looks as follows:

$$\begin{array}{cccccccc} & & & & & \rightarrow & \rightarrow & \rightarrow \\ \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \downarrow & \downarrow \end{array}.$$

– then we have to choose the position of the shift of this curve in the row that we are adding. Indeed, the top row pattern is preceded by $\downarrow$ symbols on the left. This means that this is shifted downwards there. The added curve has thus to be shifted in a column on the right of this one. For these added segments, we have to choose a color.

For these patterns, the number of possible extensions is thus at most

$$\sum_{k=0}^{\left\lfloor \frac{l}{r} \right\rfloor - 1} (r+1)^k \leq (r+1)^{\left\lfloor \frac{l}{r} \right\rfloor},$$

where l is the length of this pattern. Indeed, there are at most $\left\lfloor \frac{l}{r} \right\rfloor$ added segment over this part of the top row, and that at least one of them has trivial color 0^r.
As a consequence, in these cases, the number of possible extensions over the top row is at most $(r+1)^{\frac{L}{r}+8}$, where L is the total length of the top row. As a consequence, since $L = rk$, this number is equal to

$$(r+1)^{k+9}.$$

In any of these cases the number of possible extensions is smaller than $(r+1)^{k+9}$. Hence the total number of possible extensions of a pattern in $\mathcal{T}_{kr}$ into a pattern in $\mathcal{T}_{(k+1)r}$ is at most $3(r+1)^{k+9}$, since there are three cases.

2. **Upper bound on the cardinality of $\mathcal{T}_{kr}$:**

As a consequence, the number of possible extensions of a pattern in $\mathcal{T}_{kr}$ into a pattern in $\mathcal{T}_{(k+1)r}$ is at most

$$(r+1)^{kr+2}\left(3(r+1)^{k+9}\right)^r = 3^r(r+1)^{2kr+9r+2}.$$

It follows, using inductively this inequality, that the number of pattern in $\mathcal{T}_{(k+1)r}$ is smaller than

$$|\mathcal{T}_r|.(r+1)^{k.(2+9r)}.3^{k.r}.(r+1)^{2r\sum_{i=1}^{k} i} = |\mathcal{T}_r|.(r+1)^{k.(2+9r)}.3^{kr}.(r+1)^{rk(k+1)}.$$

3. **Upper bound on $N_{kr}(\Delta'_r)$:**

As a consequence, the number of kr-blocks in the language of Δ'_r is smaller than

$$|\mathcal{T}_r|.(r+1)^{k.(2+10r)}.3^{kr}.(r+1)^{rk^2}.(r+1)^{2(k+1)r},$$

since any kr-block is sub-pattern of a pattern in $\mathcal{T}_{(k+1)r}$.

$\square$

Proof. of Theorem 48: As a consequence of Lemma 52, and Lemma 53, we get that the number of kr-blocks in the language of $d_{\mathcal{A}}^{(r)}(Z)$ smaller than

$$N_{kr}(Z).\alpha(r).2^{\lambda(r).k}.(r+1)^{rk^2}.$$

Hence,

$$h(d_{\mathcal{A}}^{(r)}(Z)) \leq h(Z) + \frac{\log_2(1+r)}{r}$$

Using Lemma 50, we have the equality:

$$h(d_{\mathcal{A}}^{(r)}(Z)) = h(Z) + \frac{\log_2(1+r)}{r}.$$

□

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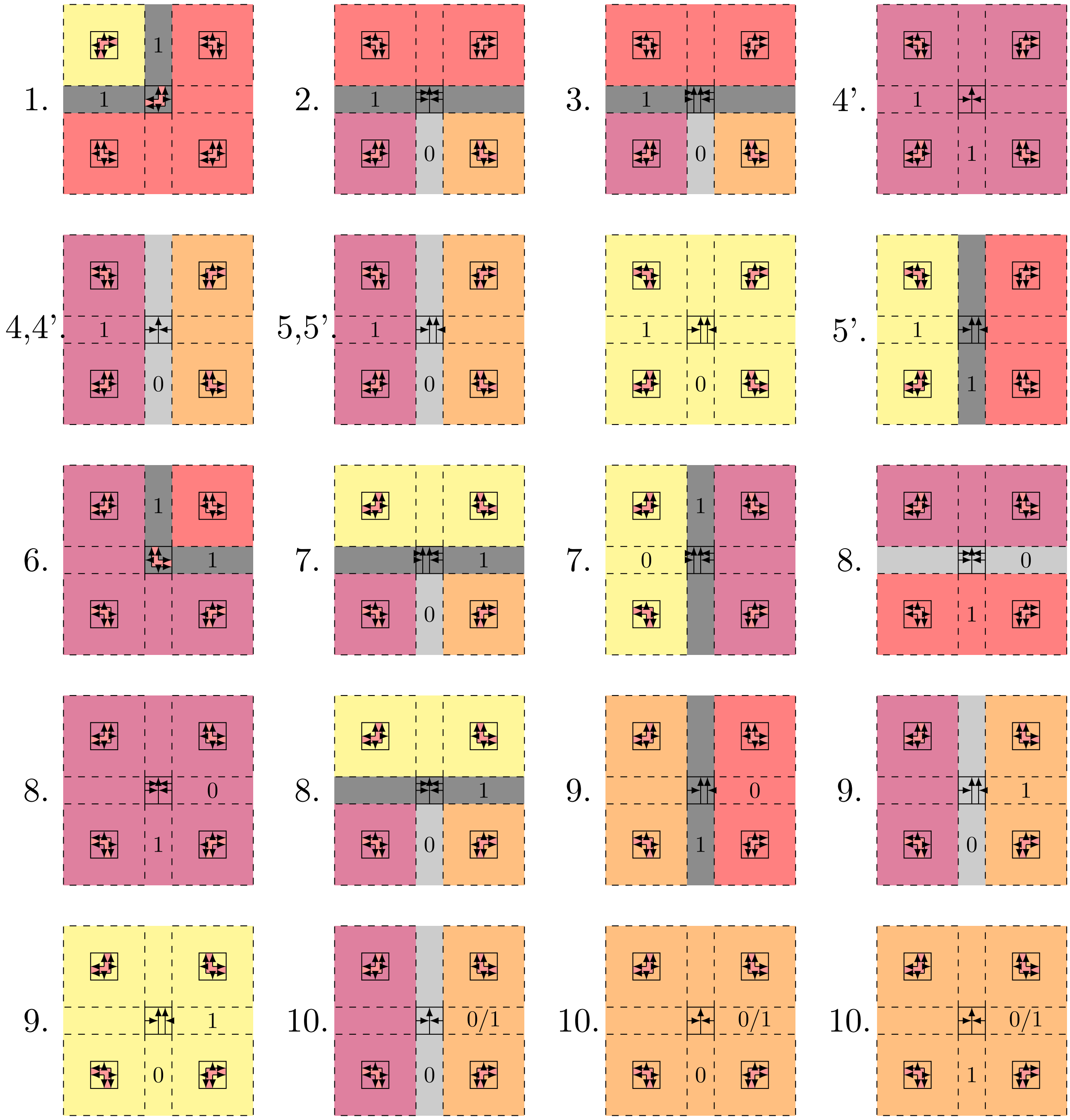

Figure 45: Possible orientations of four neighbor supertiles having the same order.