The domino problem for self-similar structures

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Abstract. We define the domino problem for tilings over self-similar structures of $\mathbb{Z}^d$ given by forbidden patterns. In this setting we exhibit non-trivial families of subsets with decidable and undecidable domino problem.

Introduction

In its original form, the domino problem was introduced by Wang [10] in 1961. It consists of deciding if copies of a finite set of Wang's tiles (square tiles of equal size, not subject to rotation and with colored edges) can tile the plane subject to the condition that two adjacent tiles possess the same color in the edge they share. Wang's student Berger showed undecidability for the domino problem on the plane in 1964 [3] by using a reduction to the halting problem. In 1971, Robinson [8] simplified Berger's proof.

Symbolic dynamics classically studies sets of colorings of $\mathbb{Z}^d$ from a finite set of colors which are closed in the product topology and invariant by translation, such sets are called subshifts. Given a finite set of patterns $\mathcal{F}$ (a pattern is a coloring of a finite part of $\mathbb{Z}^d$), we associate a subshift of finite type $X(\mathcal{F})$ which corresponds to the set of colorings which does not contain any occurrence of patterns in $\mathcal{F}$. The domino problem can therefore be expressed in this setting: given a finite set of forbidden patterns $\mathcal{F}$, it is possible to decide whether the subshift of finite type $X(\mathcal{F})$ is not empty?

It is well known that there exists an algorithm deciding if a subshift of finite type is empty in dimension one [5] and that there is no such algorithm in higher dimensions. The natural question that comes next is: *What is the frontier between decidability and undecidability in the domino problem?*

One way to explore this question is to consider subshifts defined over more general structures, such as finitely generated groups or monoids and ask for which groups the domino problem is decidable. This approach has yielded various result in different structures: Some examples are the hyperbolic plane [6], confirming a conjecture of Robinson [9] and Baumslag-Solitar groups [1]. The conjecture in this direction is that the domino problem is decidable if and only if the group is virtually free. The conjecture is known to hold in the case of virtually nilpotent groups [2]. The main idea of the proof of this result is to construct a grid by local rules in order to use the classical result in $\mathbb{Z}^2$.

In this paper we explore another way to delimit the frontier between decidability and undecidability of this problem. In geometry the structures which lie between the line and the plane can have Hausdorff dimension strictly between one and two. In this article we propose a way to define the domino problem in a digitalization of such fractal structures. In Section 1 we use self-similar substitutions to define a "fractal" structure where a natural version of the domino problem can be defined. We exhibit a large class of substitutions (including the one which represents the Sierpiński triangle) where the domino problem is decidable (Section 2), another class (including the Sierpiński carpet) where the problem is undecidable (Section 4) and an intermediate class where the question is still open (see Figure 1 for an example of each of these classes).

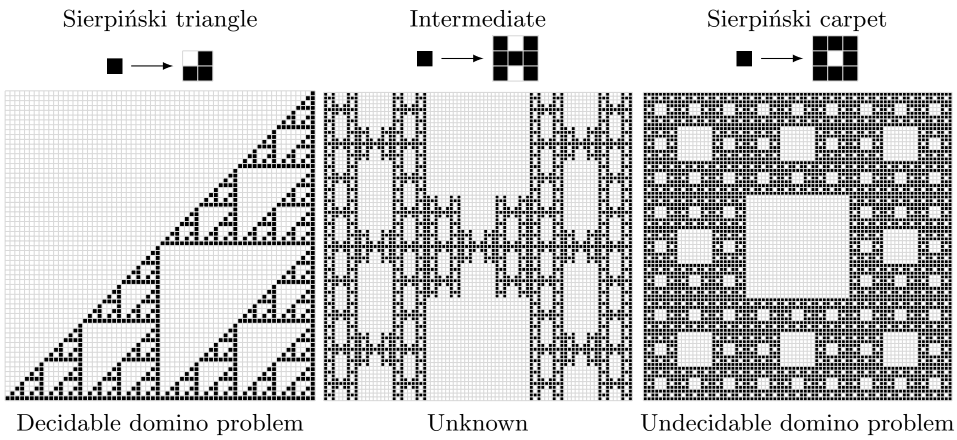


Fig. 1. Some digitalizations of fractal structures and the status of their domino problem

1 Position of the problem

1.1 Coloring of $\mathbb{Z}^d$ and local rules

Given a finite alphabet $\mathcal{A}$, a coloring of $\mathbb{Z}^d$ is called a *configuration*. The set of configurations, denoted $\mathcal{A}^{\mathbb{Z}^d}$, is a compact set according to the usual product topology. A *subshift* is a closed set of configurations which is invariant by the shift action. Given a finite subset $S \subset \mathbb{Z}^d$, a *pattern with support* S is an element p of $\mathcal{A}^S$. A pattern $p \in \mathcal{A}^S$ *appears* in a configuration $x \in \mathcal{A}^G$ if there exists $z \in \mathbb{Z}^d$ such that $x_{z+S} = p$. In this case we write $p \sqsubset x$.

Equivalently, a subshift can be defined with a set of forbidden patterns $\mathcal{F}$ as the set of configurations where no patterns of $\mathcal{F}$ appear. We denote it by $X(\mathcal{F})$. If $\mathcal{F}$ is finite, $X(\mathcal{F})$ is called *subshift of finite type* which can be considered as the set of tilings defined by the local contraints given by $\mathcal{F}$.

1.2 Self similar structure

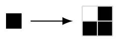
We want to extend the condition of coloring to self similar structures of $\mathbb{Z}^d$. This means that only some cells can be decorated by elements of $\mathcal{A}$. To formalize that, a structure is coded as a subset of $\{0,1\}^{\mathbb{Z}^d}$ and self similarity is obtained by a substitution.

Let $\mathcal{A}$ be a finite alphabet. A *substitution* is a function $s : \mathcal{A} \to \mathcal{A}^R$ where $R = [1, l_1] \times \cdots \times [1, l_d]$ is a d-dimensional rectangle. It is naturally extended to act over patterns which have rectangles as support by concatenation. We denote the successive iterations of s over a symbol by s, s^2, s^3 and so on. The subshift generated by a substitution s is the set

$$X_s := \{x \in \mathcal{A}^{\mathbb{Z}^d} | \forall p \sqsubset x, \exists n \in \mathbb{N}, a \in \mathcal{A}, p \sqsubset s^n(a)\}.$$

To obtain self-similar structures, we restrict the notion of substitution to $\{0,1\}$ imposing that the image of 0 consists on a block of 0s. These substitutions are called said *self-similar*. Self-similar substitutions represent digitalizations of the iterations of the following procedure: start with the hypercube $[0,1]^d$, subdivide it in a $l_1 \times \cdots \times l_d$ grid and remove the blocks in the positions z of the grid where $s(1)_z = 0$. Then repeat the same procedure with every sub-block.

Example 1. Consider $\mathcal{A} = \{\square, \blacksquare\}$. The self-similar substitution s such that:

 and 

is called the Sierpiński triangle substitution and is extended by concatenation as shown in Figure 2.

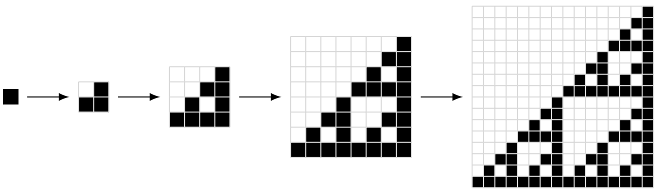


Fig. 2. First four iterations of the Sierpiński triangle substitution.

1.3 Coloring of a self similar structure and local rules

Let $\mathcal{A}$ be a finite alphabet where $0 \in \mathcal{A}$ and s be a self-similar substitution. Consider $X_s \subset \{0,1\}^{\mathbb{Z}^d}$ the associated self-similar structure. A configuration $x \in \mathcal{A}^{\mathbb{Z}^d}$ is *compatible* with s if $\pi(x) \in X_s$ where π is a map which send all elements of $\mathcal{A} \setminus \{0\}$ onto 1 and 0 onto 0. Given a finite set of patterns $\mathcal{F}$ we define the *set of configurations on* X_s *defined by local rules* as

$$X_s(\mathcal{F}) = \left\{x \in \mathcal{A}^{\mathbb{Z}^d} : \pi(x) \in X_s \text{ and no pattern of } \mathcal{F} \text{ appears in } x\right\}.$$

1.4 The domino problem on self-similar structures

The domino problem for on $\mathbb{Z}^d$ is defined as the language

$$\mathtt{DP}(\mathbb{Z}^d) = \{\mathcal{F} \text{ finite set of patterns} : X(\mathcal{F}) \neq \emptyset\}.$$

It is the language of all finite sets of patterns over a finite alphabet such that it is possible to construct a configuration without patterns of $\mathcal{F}$.

Classical results which can be found in [5] show that the domino problem for $\mathbb{Z}$ is decidable. In the other hand, we know that for $d > 1$ the domino problem for $G = \mathbb{Z}^d$ is undecidable (see [3, 8]). This gap of decidability when the dimension increases motivates us to define the domino problem for structures which lay between those groups. Thus given a self-similar substitution s we introduce the *s-based domino problem* as the language

$$\mathtt{DP}(s) := \{\mathcal{F} \text{ finite set of patterns} : X_s(\mathcal{F}) \neq \{0^{\mathbb{Z}^d}\}\}.$$

That is, $\mathtt{DP}(s)$ is the set of all finite sets of forbidden patterns such that there is at least a configuration containing a non-zero symbol. We assume implicitly that $\mathcal{F}$ does not contain any pattern consisting only of 0.

2 Self-similar structures with decidable domino problem

In this section we present a family of self-similar substitutions such that the domino problem associated is decidable. In order to present this result in the most general setting, we introduce the channel number of a self-similar substitution.

Let $\mathbb{H} = \{-1, 0, 1\}^d$ and consider the set $\Lambda \subset \{0, 1\}^{\{1,2,3\}^d}$ consisting of all d-dimensional hypercube patterns of side 3 which appear in X_s and that have a 1 in the center. Let $\Lambda_n = s^n(\Lambda)$ be the set of the images of each $q \in \Lambda$ under s^n by concatenation and S_n be the support corresponding to the image of position $(2, \dots, 2)$ of q under s^n. We define the *n-channel number* $\chi(s, n)$ of s as follows:

$$\chi(s, n) = \max_{p \in \Lambda_n} |\{z \in S_n \mid \exists h \in (z + \mathbb{H}) \cap (supp(p) \setminus S_n), p_z = p_h = 1\}|$$

In other words, it is the maximum number of positions in the support of the pattern $s^n(1)$ such that if we surround it either by blocks of 0 or copies of $s^n(1)$ appearing in X_s there might be two symbols 1, one appearing in $s^n(1)$ and another outside, at distance smaller than 1. We say that s is *channel bounded* if there exist $K \in \mathbb{N}$ such that for all n then $\chi(s, n) \leq K$. The Sierpiński triangle substitution is an example of a channel bounded substitution.

Theorem 1. *For every channel bounded self-similar substitution s the domino problem $\mathtt{DP}(s)$ is decidable.*

Proof. Let $\mathcal{F}$ be set of forbidden patterns over the alphabet $\mathcal{A}$ for which we are deciding the emptiness of $X_s(\mathcal{F})$. By compacity of the space of configurations, $X_s(\mathcal{F}) \neq \{0^{\mathbb{Z}^d}\}$ if and only if for every $n \in \mathbb{N}$ there exists a pattern p over $\mathcal{A}$

which does not contain any pattern from $\mathcal{F}$ and satisfying $\pi(p) = s^n(1)$. In consequence, it suffices to show that if s is channel bounded, it is possible to calculate N such that if it is possible to cover $s^N(1)$ with symbols from $\mathcal{A}$ without any pattern from $\mathcal{F}$ appearing then $X_s(\mathcal{F}) \neq \{0^{\mathbb{Z}^d}\}$. Indeed, an algorithm could calculate N and try every possible pattern p such that $\pi(p) = s^N(1)$. If there exists one which does not violate any rule from $\mathcal{F}$ it returns that $X_s(\mathcal{F}) \neq \{0^{\mathbb{Z}^d}\}$, otherwise it returns that $X_s(\mathcal{F}) = \{0^{\mathbb{Z}^d}\}$.

For simplicity, suppose that $\forall p \in \mathcal{F}, supp(p) \subseteq \mathbb{H}$ and let K be a bound for $\chi(s,n)$ (If the support if $\{-m,\dots,m\}^d$ we can recalculate K). We claim that $N = 2^{|\mathcal{A}|^{(3^d-1)K}}$ suffices. Consider a pattern p such that $\pi(p) = s^n(1)$. For each $q \in \Lambda$ consider the set of positions from the definition of $\chi(s,n)$ (that is, the set of positions which matter when considering only q) and store the symbols appearing in p in those positions (order the positions lexicographically) as a tuple $(q, a_1, a_2, \dots a_K) \in \Lambda \times \mathcal{A}^K$. Therefore all the information concerning the dependency of p with its possible surroundings can be stored on $|\Lambda|$ tuples. Now, given the set of all patterns $(p_j)_{j\in J}$ such that $\pi(p_j) = s^n(1)$ and which do not contain any forbidden patterns we can extract the $|\Lambda|$ tuples from each one of them. All this information for the level n is represented as a subset of $(\mathcal{A}^K)^\Lambda$. By definition of these tuples this is the only information needed in order to construct the patterns which contain no pattern from $\mathcal{F}$ and project under π to $s^{n+1}(1)$, moreover, the tuples representing those patterns can be obtained from the ones of $s^n(1)$ because the positions from the definition of $\chi(s,n+1)$ necessarily appear in the patterns of $s^n(1)$. This means it is possible to extract pasting rules which can be codified in a function $\mu_s : 2^{(\mathcal{A}^K)^\Lambda} \to 2^{(\mathcal{A}^K)^\Lambda}$.

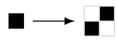
This function gives therefore all the useful information concerning how to construct the tuples of level $n+1$ from the tuples of level n. Obviously, $\mu_s(\emptyset) = \emptyset$, therefore there are two possibilities: either this function arrives eventually at $\emptyset$ and there are no patterns p such that $\pi(p) = s^m(1)$ for some $m \in \mathbb{N}$ or μ_s cycles and thus it's possible to construct patterns projecting to $s^m(1)$ for arbitrarily big m. Anyway, by pigeonhole principle this behavior must occur before $|2^{(\mathcal{A}^K)^\Lambda}| \leq 2^{|\mathcal{A}|^{(3^d-1)K}}$ iterations of μ_s.

3 The Mozes property for self-similar structures

Most of the proofs of the undecidability of the domino problem on $\mathbb{Z}^2$ are based on the construction of a self similar structure. A Theorem proven by Mozes [7] shows that every $\mathbb{Z}^d$-substitutive subshift is a sofic subshift for $d \geq 2$. This theorem fails for the case $d = 1$. The importance of this result is the fact that multidimensional substitutions can be realized by local rules. In order to present a family of self-similar substitutions with undecidable domino problem we will make use of an analogue of the theorem shown by Mozes.

Definition 1. *A self-similar substitution s satisfies the Mozes property if for every substitution s' defined over the same rectangle and over an alphabet $\mathcal{A}$*

containing 0 *and such that* $\forall a \in \mathcal{A} \setminus \{0\}$, $\pi(s'(a)) = s(1)$ *and* $s'(0) = 0^R$ *there exists an alphabet* $\mathcal{B}$ *containing the symbol* 0*, a finite set of forbidden patterns* $\mathcal{F} \subseteq \mathcal{B}^*_{\mathbb{Z}^d}$ *and a local function* $\Phi : \mathcal{B} \to \mathcal{A}$ *such that* $\Phi(0) = 0$ *and the function* $\phi : \mathcal{B}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ *given by* $\phi(x)_z = \Phi(x_z)$ *is surjective from* $\Phi(X_s(\mathcal{F}))$ *to* $X_{s'}$*.*

Said otherwise, it's the analogous of saying that $X_{s'}$ is a sofic subshift, except that the SFT extension has to be an X_s-based subshift. Currently, we have been able to prove that several substitutions satisfy the Mozes property but we have not found a characterization. The channel bounded property is not relevant as it is possible to show that the Sierpiński triangle substitution satisfies the Mozes property, while the channel bounded substitution given by  does not.

Unfortunately, the proof that a self-similar substitution satisfies the Mozes property consists on a technical construction which has a lot of details. For the sake of the 10 page limit we skip the proof and just state that it is valid in an important example, the Sierpiński carpet substitution shown in Figure 3.

Theorem 2. *The Sierpiński carpet substitution satisfies the Mozes property.*

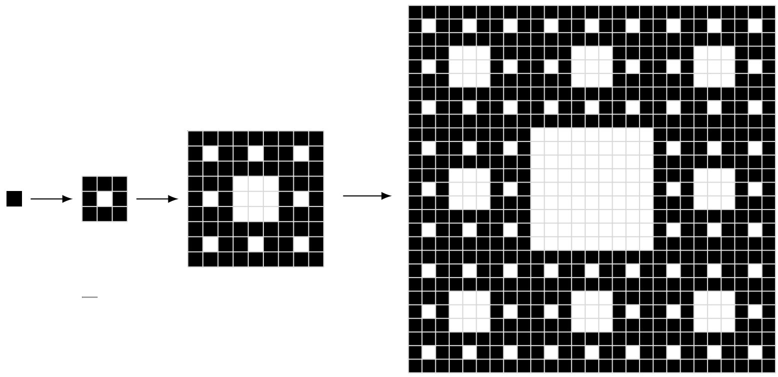


Fig. 3. The first iterations of the Sierpiński carpet substitution.

4 Self-similar structures where the domino problem is undecidable

In this section we present a family of self-similar substitutions s such that the s-based domino problem is undecidable. The definition of this class follows.

Definition 2. *A self-similar substitution* s *defined on* $[1, l_1] \times [1, l_2]$ contains a grid *if there are integers* $1 \leq i_1 < i_2 <= l_1$ *and* $1 \leq j_1 < j_2 <= l_2$ *such that* $j \in \{j_1, j_2\}$ *or* $i \in \{i_1, i_2\}$ *implies that* $s(1)_{(i,j)} = 1$*.*

One example of a self-similar substitution that contains a grid is the Sierpiński carpet. In what follows of this section we show the following theorem:

Theorem 3. *Let s be a self-similar substitution which satisfies the Mozes property and contains a grid. Then the domino problem* $\mathtt{DP}(s)$ *is undecidable.*

Proof. We claim that an oracle for $\mathtt{DP}(s)$ can be used to decide $\mathtt{DP}(\mathbb{Z}^2)$. This is enough to conclude, as $\mathtt{DP}(\mathbb{Z}^2)$ is undecidable.

Let s be defined on $[1, l_1] \times [1, l_2]$, some values satisfying the grid condition (i_1, i_2) and (j_1, j_2) and consider a substitution s' over the alphabet $\mathcal{A}(s') = \{\bullet, \updownarrow, \leftrightarrow, 0\}$ given by the following rules: Let $C = \{(i_1, j_1), (i_1, j_2), (i_2, j_1), (i_2, j_2)\}$, $H = \{(i, j) | i \in \{i_1, i_2\}\} \setminus C$ and $V = \{(i, j) | j \in \{j_1, j_2\}\} \setminus C$.

$$s'(\bullet)_z = \begin{cases} 0, & \text{if } s(1)_z = 0 \\ \leftrightarrow, & \text{if } z \in H \\ \updownarrow, & \text{if } z \in V \\ \bullet, & \text{else} \end{cases} \quad s'(\updownarrow)_z = \begin{cases} 0, & \text{if } s(1)_z = 0 \\ \updownarrow, & \text{if } z \in V \cup C \\ \bullet, & \text{else} \end{cases} \quad s'(\leftrightarrow)_z = \begin{cases} 0, & \text{if } s(1)_z = 0 \\ \leftrightarrow, & \text{if } z \in H \cup C \\ \bullet, & \text{else} \end{cases}$$

For example, in the case where s is the Sierpiński carpet we get:

$$\bullet \xrightarrow{s'} \begin{matrix} \bullet & \leftrightarrow & \bullet \\ \updownarrow & 0 & \updownarrow \\ \bullet & \leftrightarrow & \bullet \end{matrix} \qquad \updownarrow \xrightarrow{s'} \begin{matrix} \updownarrow & \bullet & \updownarrow \\ \updownarrow & 0 & \updownarrow \\ \updownarrow & \bullet & \updownarrow \end{matrix} \qquad \leftrightarrow \xrightarrow{s'} \begin{matrix} \leftrightarrow & \leftrightarrow & \leftrightarrow \\ \bullet & 0 & \bullet \\ \leftrightarrow & \leftrightarrow & \leftrightarrow \end{matrix}$$

For any $y \in X_{s'} \setminus \{0^{\mathbb{Z}^2}\}$ and $n \in \mathbb{N}$ we have $s'^n(\bullet) \sqsubset y$. Indeed, $\bullet$ appears in the image of every symbol $a \in \mathcal{A}(s') \setminus \{0\}$, then for every positive integer n, $\bullet$ appears at a bounded distance of every non-zero symbol in $s'^n(a)$. The same argument extends by induction by replacing $\bullet$ by $s'^n(\bullet)$ and using the fact that $s'^{n-1}(\bullet)$ appears at a bounded distance in every $s'^k(a)$ with $k > n$ necessarily implies that $s'^n(\bullet)$ appears at bounded distance in $s'^{k+1}(a)$.

As s satisfies the Mozes property there exists an alphabet $\mathcal{B}(s')$, a finite set $\mathcal{F}(s') \subset \mathcal{B}(s')^*_{\mathbb{Z}^2}$ and $\Phi : \mathcal{B}(s') \to \mathcal{A}(s')$ such that $\Phi(0) = 0$ and $\forall x \in X_s(\mathcal{F}(s'))$ $y_z := \Phi(x_z)$ satisfies that $y \in X_{s'}$.

Consider a finite set of forbidden patterns $\mathcal{F}$ over an alphabet $\mathcal{A}$ defining a $\mathbb{Z}^2$ subshift $X(\mathcal{F})$. Without loss of generality $\mathcal{F}$ contains only patterns with supports $\{(0, 0), (1, 0)\}$ and $\{(0, 0), (0, 1)\}$ (one can choose a conjugated version of $X(\mathcal{F})$ satisfying this property by using a higher block code. See [5]).

Finally, consider the alphabet $\mathcal{B} := \mathcal{B}(s') \times (\mathcal{A} \cup \{0\})$ along with the set of forbidden patterns $\mathcal{G}$ given by the union of the following sets:

- Zeros correspond: $\{(0, a) \mid a \in \mathcal{A}\} \cup \{(b, 0) \mid b \in \mathcal{B}(s') \setminus \{0\}\}$.
- First layer forbidden patterns: $\{p \times q \mid p \in \mathcal{F}(s'), q \in \mathcal{A}^{supp(p)}\}$. These forbidden patterns make sure that configurations belonging to the first layer of $X_s(\mathcal{G})$ belong to $X_s(\mathcal{F}(s'))$.

– Horizontal forbidden patterns: let $p \in \mathcal{B}^{\{(0,0),(1,0)\}}$ be denoted by (a,b,c,d) if $p(0,0) = (a,c)$ and $p(1,0) = (b,d)$ and $q \in \mathcal{A}^{\{(0,0),(1,0)\}}$ be denoted by (c,d) if $q(0,0) = c$ and $q(1,0) = d$. The set of horizontal forbidden patterns is $\{(a,b,c,d) \mid (a = \leftrightarrow, b \in \{\leftrightarrow, \bullet\}$ and $c \neq d)$ or $(a = \bullet, b = \leftrightarrow$ and $(c,d) \in \mathcal{F})\}$.
– Vertical forbidden patterns: let $p \in \mathcal{B}^{\{(0,0),(0,1)\}}$ be denoted by (a,b,c,d) if $p(0,0) = (a,c)$ and $p(0,1) = (b,d)$ and $q \in \mathcal{A}^{\{(0,0),(0,1)\}}$ be denoted by (c,d) if $q(0,0) = c$ and $q(0,1) = d$. The set of vertical forbidden patterns is given by $\{(a,b,c,d) \mid (a = \updownarrow, b \in \{\updownarrow, \bullet\}$ and $c \neq d)$ or $(a = \bullet, b = \updownarrow$ and $(c,d) \in \mathcal{F})\}$.

These rules codify the following idea: $\bullet$ carry arbitrary symbols from $\mathcal{A}$ in the second layer and the arrows send this information left and up respecting the rules from $\mathcal{F}$, see Figure 4. By iterating the substitution s it is easy to see that $s^n(1)$ actually contains 2^n vertical and horizontal lines. This means that the intersection of these lines contain symbols of $\mathcal{A}$ which represent a $2^n \times 2^n$ pattern which contains no forbidden patterns from $\mathcal{F}$ inside. Therefore if $X_s(\mathcal{G}) \neq \{0^{\mathbb{Z}^2}\}$ then $X(\mathcal{F}) \neq \emptyset$ by compacity. Conversely if $X(\mathcal{F}) \neq \emptyset$ it is possible to always build the second layer of a point having $s'^n(1)$ in the first layer.

$$\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{array} \sqsubset X(\mathcal{F}) \longleftrightarrow \begin{array}{ccccccccc} \bullet & \leftrightarrow & \bullet & \leftrightarrow & \leftrightarrow & \leftrightarrow & \bullet & \leftrightarrow & \bullet \\ \updownarrow & & \updownarrow & \bullet & & \bullet & \updownarrow & & \updownarrow \\ \bullet & \leftrightarrow & \bullet & \leftrightarrow & \leftrightarrow & \leftrightarrow & \bullet & \leftrightarrow & \bullet \\ \updownarrow & \bullet & \updownarrow & & & & \updownarrow & \bullet & \updownarrow \\ \updownarrow & & \updownarrow & & & & \updownarrow & & \updownarrow \\ \updownarrow & \bullet & \updownarrow & & & & \updownarrow & \bullet & \updownarrow \\ \bullet & \leftrightarrow & \bullet & \leftrightarrow & \leftrightarrow & \leftrightarrow & \bullet & \leftrightarrow & \bullet \\ \updownarrow & & \updownarrow & \bullet & & \bullet & \updownarrow & & \updownarrow \\ \bullet & \leftrightarrow & \bullet & \leftrightarrow & \leftrightarrow & \leftrightarrow & \bullet & \leftrightarrow & \bullet \end{array} \quad \begin{array}{ccccccccc} a_1 & a_2 & a_2 & a_3 & a_3 & a_3 & a_3 & a_4 & a_4 \\ b_1 & & b_2 & \star & & \star & b_3 & & b_4 \\ b_1 & b_2 & b_2 & b_3 & b_3 & b_3 & b_3 & b_4 & b_4 \\ c_1 & \star & c_2 & & & & c_3 & \star & c_4 \\ c_1 & & c_2 & & & & c_3 & & c_4 \\ c_1 & \star & c_2 & & & & c_3 & \star & c_4 \\ c_1 & c_2 & c_2 & c_3 & c_3 & c_3 & c_3 & c_4 & c_4 \\ d_1 & & d_2 & \star & & \star & d_3 & & d_4 \\ d_1 & d_2 & d_2 & d_3 & d_3 & d_3 & d_3 & d_4 & d_4 \end{array}$$

Fig. 4. In the left a pattern from $X(\mathcal{F})$. In the right its coding in $X_s(\mathcal{G})$. The $\star$s represent any symbol from $\mathcal{A}$.

Suppose there is an algorithm for deciding $\mathtt{DP}(s)$. Then for any $\mathcal{F}$ defining a $\mathbb{Z}^2$ subshift the alphabet $\mathcal{B}$ and the rules $\mathcal{G}$ can be built in order to decide if $X_s(\mathcal{G}) \neq \{0^{\mathbb{Z}^2}\}$. This is equivalent to deciding if $X(\mathcal{F}) \neq \emptyset$, therefore $\mathtt{DP}(\mathbb{Z}^2)$ can be decided. This yields the desired contradiction.

Adapting the proof of Theorem 5 we can prove that all self-similar substitutions which contain a grid satisfy the Mozes property, therefore Theorem 3 can also be stated without that assumption.

5 Generalizations and perspectives

Here we present some ideas to generalize previous results in order to advance towards a characterization of the self-similar structures where the domino problem is decidable. In the previous sections the information which allows to simulate grids is transfered through straight lines. We can imagine less rigid possibilities.

5.1 Connectivity

We propose a way to define the directions in which the information can be transfered in a substitution in $\mathbb{Z}^2$. Given a self-similar substitution defined over $[1, l_1] \times [1, l_2]$ we denote by $\mathbb{X}$ the set of coordinates z such that $s(1)_z = 1$. Let $\mathbb{S} = \{(0,-1), (0,1), (-1,0), (0,1)\}$ and $\mathbb{W}$ contains $\{(-1,-1), (1,1)\}$ if $\{(1,1), (l_1, l_2)\} \in \mathbb{X}$ and $\{(-1,1), (1,-1)\}$ if $\{(1, l_2), (l_1, 1)\} \in \mathbb{X}$. We say s admits a rigid (respectively flexible) vertical line at $1 \leq v \leq l_1$ if there is a non-repeating sequence of vertices $(v, 1) = x_1, \ldots x_n = (v, l_2)$ such that the differences $x_j - x_{j-1}$ belong to $\mathbb{S}$ (respectively $\mathbb{W} \cup \mathbb{S}$). We define rigid and flexible horizontal lines for $1 \leq h \leq l_2$ analogously. We also say that two lines are weakly disjoint if they share no consecutive pair of vertices in their path.

According to these notions, we distinguish the following four subclasses:

- s has *bounded connectivity* if s has at most one flexible horizontal and vertical line;
- s has *a isthmus* if $s(1)$ has at least two weakly disjoint flexible lines in one direction and at most one weakly disjoint flexible line in the other direction;
- s has *a weak grid* if $s(1)$ has at least two flexible horizontal lines and two flexible vertical lines which are pairwise weakly disjoint.
- s has *a strong grid* if $s(1)$ has at least two rigid horizontal lines and two rigid vertical lines which are pairwise weakly disjoint.

If s has bounded connectivity the proof of Theorem 1 can be adapted to show decidability. If s has a strong grid it is possible to adapt the proof of Theorem 3 to show the undecidability of the domino problem associated to such substitution, moreover, a generalization of that proof works even in the case of weak grids. Nevertheless we still have no results supporting either direction in the isthmus case. We believe that the Mozes property does not hold in the isthmus case, which would be evidence towards decidability. Figure 5 presents the domino problem of different substitutions according to this classification.

B. Connectivity	Isthmus	Weak grid	Strong grid
DP decidable	Unknown	DP undecidable	DP undecidable

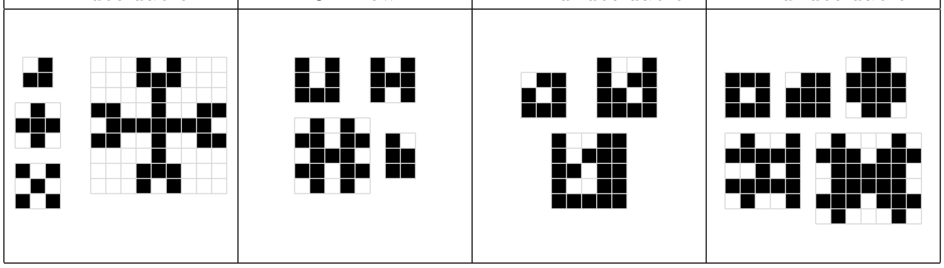


Fig. 5. Some examples of substitution according to this classification

5.2 Concluding remarks

In this article we introduced a version of the domino problem on self-similar structures in order to understand the frontier between decidability and undecidability in the domino problem when we go from the line (dimension 1) to the plane (dimension 2). In fact it does not depend on the Hausdorff dimension of the self-similar structure considered. Indeed, using the obtained results it is possible to obtain self similar structures with decidable domino problem and Hausdorff dimension arbitrary near to 2 (obtained by s_n) and self similar structures with undecidable domino problem and Hausdorff dimension arbitrary near to 1 (obtained by s'_n) .

Thus, the frontier between decidability and undecidability seems more likely to be based on the presence of a grid where it is possible to implement a computation. To confirm this hypothesis, it remains to study self similar structures with an isthmus. In the case of an isthmus the substitution presents an unique bridge which links different zones. This prevents the possibility of a Mozes-like [7] or Goodman-Strauss-like [4] proof of the Mozes property and therefore of the implementation of a computation. The main problem is that in order to simulate a substitution there is the need to transfer arbitrarily big amounts of information by that isthmus. We believe the study of this class of substitutions will certainly provide new tools to the study of how information can be transfered.

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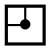
6 Annexes

Theorem 4. *The Sierpiński triangle substitution satisfies the Mozes property.*

Proof. Let the substitution s' satisfying the hypothesis of the Mozes property be defined over an alphabet $\mathcal{A}$. For the sake of simplicity during this proof we refer to positions $(1,1), (1,2)$ and $(2,2)$ in $R[2,2]$ as 1, 2 and 3 respectively. We proceed by constructing explicitly the alphabet $\mathcal{A}'$ and the set of finite rules $\mathcal{F}'$ which satisfy the requirements.

Consider the alphabet $\mathcal{A}'$ given by the following three types of tiles:

a)  b)  c) 

where each tile carries some extra information which is not shown in the picture. The dot and each of the two segments in each tile carry a tuple belonging to

$$\mathcal{A} \times \{1,2,3\} \times \mathcal{A}^3.$$

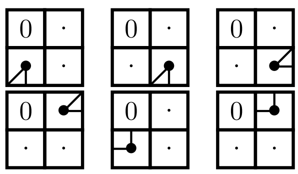
Where the triple in $\mathcal{A}^3$ is the image under s' of the element in $\mathcal{A}$ (ordered according to the previous simplification of the positions in $R[2,2]$). It will be represented as:

$$a, i \rightarrow \begin{matrix} 0 & a_3 \\ a_1 & a_2 \end{matrix}. \qquad \text{where } i, j \in \{1,2,3\} \text{ and } a, a_1, a_2, a_3 \in \mathcal{A}$$

The meaning of such a tuple is the following: it represents the substitution rule $s'(a)$ coming from s' with the additional information that a appears in position i in a previous substitution rule.

Now we proceed to describe the finite set of forbidden patterns $\mathcal{F}'$. We do so by first describing a set of rules and later showing how they can be obtained by forbidden rules. In this description, a single dot $\cdot$ means an arbitrary tile from $\mathcal{A}'$, # means a number in $\{1,2,3\}$ and $*$ means a symbol from $\mathcal{A}$.

1. **Structure rule:** The only admissible tilings of $s(1)$ are the ones where tiles of the type a), b) and c) are on positions 1, 2 and 3 respectively. Furthermore, in every such tiling of $s(1)$ the tuple in the dots must be the same for each dot. Also the tuples from lines that match in the border of a tile must coincide. It is possible to generate these rules in the following way: The first part of the rule can be easily enforced by checking the neighbors, that is, we forbid every pattern of size 2×2 which has any of the following configurations:



The rest of the rules described here can be obtained by further forbidding any 2×2 pattern which does not satisfy the conditions.

2. **Base rule:** Any tiling of $s(1)$ which carries in their dots a rule of the form:

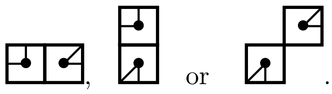
$$a, i \to \begin{matrix} & a_3 \\ a_1 & a_2 \end{matrix}$$

must satisfy the following rule: if second coordinate of the tuple is the number i=1 (respectively 2, 3) then the vertical (resp diagonal, horizontal) line in the triangle must carry a tuple where a must appear in the right hand side in position i, that is:

$$*, \# \to \begin{matrix} & a \\ * & * \end{matrix} \quad \text{supposing } i = 3\text{, for example}$$

This rule can be obtained in the same way as the rules before.

3. **Pasting rule:** Whenever patterns of the following form appear:



We demand that the two lines which have the same orientation must carry the same tuple. Also, the other pair of lines forming an angle must also carry the same tuple between them.
This rule can obviously be enforced by forbidding every pattern with the shapes shown above which does not satisfy the property.

4. **Extension rule:** When encountering patterns as in the last rule: if the tuple shared by the lines forming an angle carries in the second coordinate the number 1 (respectively 2, 3), then if the other two lines which share the orientation are vertical (respectively diagonal, horizontal) then they must carry a tuple which originates from the tuple shared by the lines which make an angle in the same way as in the base rule.

With these rules, we claim that by projecting each tile to the third coordinate in the position given by the type of tile (that is: a) goes to 1, b) to 2 and c) to 3) of the tuple which is held by the black $\bullet$, (this is the local rule Φ) then for every $n \in \mathbb{N}$, every tiling of $s^n(1)$ projects onto $s'^n(a)$ for an $a \in \mathcal{A}$.

Before starting, let $n \in \mathbb{N}$. We will refer to the set of positions of $R[2^n, 2^n]$ given by

$$B_n = \{(1, j), (j, j), (j, 2^n) \text{ for } j \in \{0, \ldots, 2^n\}\}$$

as the n-border and we will call n-skeleton to the set (See figure 7)

$$S_n = \{(2^{n-1} + 1, j), (j + 2^{n-1}, j), (j + 2^{n-1}, 2^{n-1} + 1) \text{ for } j \in \{1, \ldots, 2^{n-1}\}\}.$$

We proceed by induction. We show the previous claim along with two extra invariants: for any $n \in \mathbb{N}$ the tuple carried by horizontal lines in the lowest part

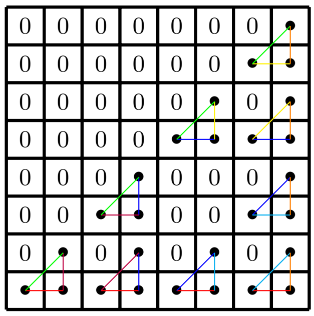


Fig. 6. A tiling of $s^3(1)$. Different colors represent different tuples in lines.

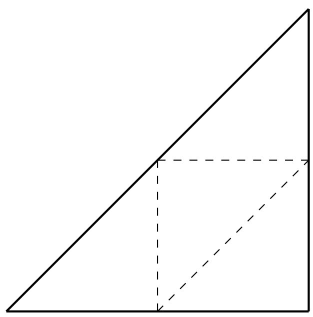


Fig. 7. The n-border is given by the black lines and the n-skeleton by the dashed lines.

of the border of $s^n(1)$ (respectively by the diagonals or the vertical lines) is the same. Also, for $n \geq 2$, the tuple in the n-skeleton is the same everywhere.

The structure rule ensures that every tiling of $s(1)$ satisfies both the claim and the invariants. The case for $n = 2$ is implied by the fact that any tiling of $s(2)$ is made by pasting together three tilings of $s(1)$, thus, by using the pasting rule we obtain the invariant over the border B_2. The pasting rule also ensures that the rules in the 2-skeleton S_2 must match and hence we have the two invariants. Using the base rule we obtain the claim for $n = 2$.

Suppose both the invariants and the claim are true for $n - 1$, using the structure of X_s, that is, that every tiling of $s^n(1)$ is formed by pasting 3 tilings of the previous level, the pasting rules again ensure that the invariants are satisfied for both B_n and S_n. Using the extension rule in the same fashion as the basic rule, we are ensured that each side of the skeleton carries the symbol which generated each one of the three tilings of $s^{n-1}(1)$, and thus the claim is also satisfied by n.

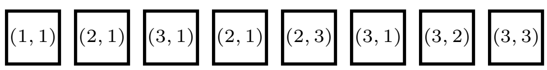
Now consider a configuration $x \in X_s(\mathcal{F})$. As the projection via ϕ defined by $\phi(x)_z = \Phi(x)_z$ of any tiling of $s^n(1)$ yields $s'^n(a)$ for $a \in \mathcal{A}$, then we conclude that $\phi(x) \in X_{s'}$ (every tiling of $X_s(\mathcal{F})$ can be partitioned in either tilings of $s^n(1)$ or patterns consisting only on zeros). In the other sense, it is clear that the construction allows every $s'^n(a)$ for $a \in \mathcal{A}$ which appears in the right side of a substitution rule to appear as the projection of a tiling of $s^n(1)$. As this are the only patterns that appear in $X_{s'}$ for an arbitrary size and they can always be constructed, we conclude that for each $y \in X_{s'}$ there exists a preimage which can be easily obtained by a compacity argument.

Therefore we conclude that $\phi : X_s(\mathcal{F}) \twoheadrightarrow X_{s'}$ satisfies the properties demanded by Definition 1.

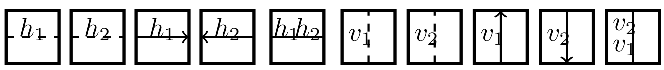
Theorem 5. *The Sierpiński carpet substitution satisfies the Mozes property.*

Proof. Let s' be a substitution over an alphabet $\mathcal{A}$ satisfying the requirements of the Mozes property. We construct an alphabet $\mathcal{B}$ and local rules which codify the transfer of information from the substitution s' between the different levels of s.

We start by defining the alphabet $\mathcal{B}$. Consider the set of tuples $T := R[3,3] \setminus \{(2,2)\}$, that is, the positions of $s(1)$ which carry a 1. We add each one of these tuples to $\mathcal{B}$ and represent them via the following tiles.

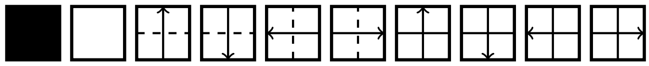


We say that a pair of tuples $(t_1, t_2) \in T^2$ are *horizontally adjacent* if $t_2 = t_1 + (1, 0)$ and we say they are *vertically adjacent* if $t_2 = t_1 + (0, 1)$. For every pair of horizontally adjacent tuples (h_1, h_2) and vertically adjacent tuples (v_1, v_2) we add to $\mathcal{B}$ the following tiles:



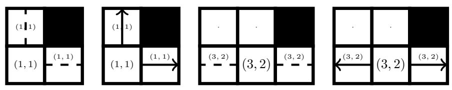
The tiles which have dashed lines are called *basic lines*, the ones which carry arrows are called *arrow lines*, and the tiles with a segment and two tuples are called a *middle line*.

For the construction, we will also need a *white tile*, a *black tile* (represents the 0) and a variety of *intersection tiles* where each intersection is between an arrow line and either a basic line or a middle line. Each of the lines in these tiles will carry tuples in the way shown above (these are not shown in the picture).

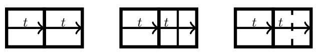


This defines the alphabet $\mathcal{B}$. Now we define the rules that every coloring of $\mathbb{Z}^2$ by these tiles must satisfy. These are all obtainable by using a finite number of forbidden patterns.

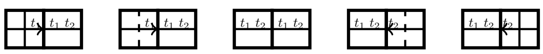
1. Whenever a black tile is accompanied by two tiles which are not black in the positions which are below and to the left then the position diagonally to the bottom and left of the 0 must carry a tuple in $t \in T$, conversely, every tuple tile must have a black tile in the position which is diagonally up and to the right and non-black tiles to the right and up:
2. Each tuple must be continued in the directions of the two adjacent tuples either by basic lines or by arrow lines carrying the same tuple. We show examples of this rule for tuples $(1, 1)$ and $(3, 2)$.



3. A basic horizontal (respectively vertical) tile can only be continued by another horizontal basic tile carrying the adjacent horizontal tuple. (Either of these basic lines could be part of an intersection tile). Thus the tuples connected by basic lines are at distance 3, see figure 8.
4. A horizontal (respectively vertical) arrow tile (with no intersections) can only be continued by another horizontal arrow tile carrying the same tuple, or by an intersection tile where the arrow carries the same tuple and matches.



5. A middle tile can only be continued in the direction of the segment either by an identical middle tile, or by an intersection tile. If the connection is with an intersection tile, then the arrow head from the intersection tile must match with the end of the middle tile so that their tuples coincide.



6. An intersection line can only be continued in the direction of the arrow by a middle line, and in the opposite direction by an arrow line, all carrying the same tuple as above. the other line (basic or middle) must follow rules 3 and 4.

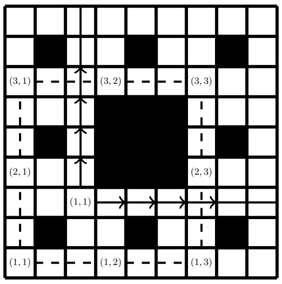


Fig. 8. A tiling of $s^2(1)$ using $\mathcal{B}$. The tuples in the lines aren't shown in order to make the picture readable.

Now we proceed to add extra information to the symbols (tiles) of $\mathcal{B}$. This information is given in the form of 4-tuples $(a, t_1, (a_t)_{t\in T}, t_2) \in S$ where $S := \mathcal{A} \times T \times \mathcal{A}^T \times T$ and such that they satisfy $s'(a) = (a_t)_{t\in T}$. This 4-tuple codes the following information: "I carry the symbol a_{t_2} which is $s'(a)_{t_2}$ and the symbol a appears in position t_1 of another substitution rule".

To black tiles we add no 4-tuples. To the rest of the tiles we add a 4-tuple in the background. Additionally, each tuple tile or line carries an extra 4-tuple of S. Intersection tiles carry three 4-tuples in total (one in the background and two associated to each line).

Finally we add the following rules:

1. **Structure rule:** Each tiling of $s(1)$ must carry the same 4-tuple in the background except for the last coordinate, where each one of them must match to the position of $s(1)$ it is holding (this can be done using the black tile as reference), also, in position (1,1) the second coordinate of the background 4-tuple must be the same as the tuple tile.
2. **Base rule:** In each tiling of $s(1)$, if the 4-tuples of S in the background are of the form $(a, t_1, (a_t)_{t\in T}, \cdot)$, then the 4-tuple of S which goes with the tuple tile t_1 in position (1,1) is of the form: $(b, t_2, (b_t)_{t\in T}, t_1)$ where $b_{t_1} = a$.
3. **Pasting rule:** Any set of 8 tuples which are connected by lines, and the lines that connect them must carry the same tuple from S, except by the last coordinate, which must coincide with the tuple they are carrying (in the case of middle tiles, it must be the smallest one lexicographically)
4. **Extension rule:** When two lines meet in an intersection tile, if the basic or middle lines carries the tuple $(a, t_1, (a_t)_{t\in T}, \cdot)$, then the tuple of the arrow must be of the form $(b, t_2, (b_t)_{t\in T}, t_1)$ where $b_{t_1} = a$.

This finishes the description of $\mathcal{B}$ and the local rules. Now we proceed to prove the result. Consider the function Φ which projects every tile by considering the 4-tuple $(a, t_1, (a_t)_{t\in T}, t_2)$ in the background and projecting it to a_{t_2}. We claim that this function satisfies the requirements of the Mozes property.

In order to prove that, it suffices to show that for any $n \in \mathbb{N}$ the projection by Φ of any tiling of $s^n(1)$ is $s'^n(a)$ for $a \in \mathcal{A}$ which appears at the right hand side of a substitution rule of s' and conversely that every tiling as such can be obtained as a projection of a pattern in the construction. These two facts are easily obtained simultaneously from the fact that for a tiling of $s^n(1)$ if the tuple tile from the first layer in position $(3^{n-1}, 3^{n-1})$, carries the 4-tuple $(a, t_1, (a_t)_{t \in T}, t_2)$ then the image via Φ of the whole block corresponds to $s'^n(a_{t_2})$.

We proceed to show the previous fact by induction, when $n = 1$ the result follows from the structure and base rule. Now suppose the property holds for $n-1$ and consider a valid tiling of $s^n(1)$ such that the 4-tuple in the center position described above is $(a, t_1, (a_t)_{t \in T}, t_2)$. The first set of rules imply that in any valid tiling of $s^n(1)$ the tuples originating in the centers of the 8 blocks of shape $s^{n-1}(1)$ composing $s^n(1)$ must be connected by arrows and middle tiles, and the pasting rule imposes the condition that all these tuples and lines must carry the same 4-tuple from S except for the last coordinate. As the tuple in position $(3^{n-1}, 3^{n-1})$ is not at distance 2 of any other tuple, then arrow tiles must extend from it eventually intersecting the connected structure formed by the 8 sub-blocks of level $n-1$. Finally, using the extension rule we obtain that the structure formed by the 8 tuple tiles must carry a 4-tuple of the form: $(a_{t_2}, t_2, (a_t)_{t \in T}, t')$ where t' depends on the 4-tuple carried by the structure of the 8 tuple tiles. using the induction hypothesis we obtain that the image of each of these blocks is $s^{n-1}(s(a_{t_2})_{t'})$ and thus the image of the whole block is $s'^{n-1}((s'(a_{t_2})_{t'})_{t' \in T}) = s'^n(a_{t_2})$.