Entry times in automata with simple defect dynamics.

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In this paper, we consider a simple cellular automaton with two particles of different speeds that annihilate on contact. Following a previous work by Kůrka et al., we study the asymptotic distribution, starting from a random configuration, of the waiting time before a particle crosses the central column after time n. Drawing a parallel between the behaviour of this automata on a random initial configuration and a certain random walk, we approximate this walk using a Brownian motion, and we obtain explicit results for a wide class of initial measures and other automata with similar dynamics.

1 Introduction

Self-organization in cellular automata is the emergence of structures when one iterates a cellular automaton on a random initial configuration. For some cellular automata, self-organization takes the form of the emergence and the persistence of homogeneous regions separated by boundaries which propagate and sometimes collide over time like particles. These particles and their dynamics under the action of the CA have been studied empirically ([1], [3]), and Pivato proposed a general formalism to describe this phenomenon [11]. Under some assumptions on the dynamics of particles, it is possible to get information on the asymptotic behaviour of the automaton. See for example [2], [7], or [4], where we proved the following: when particles have constant speeds and destructive interactions, and under some assumptions on the initial measure, the probability for particles to appear in a central cylinder tends to 0 as time tends to infinity except possibly for one particular speed.

For some automata with simple dynamics, this kind of results can be refined with a quantitative approach: that is, to determine the asymptotic distribution of some random variable related to the particles. In [5], Kůrka, Formenti and Dennunzio considered $T_n(a)$, the entry time after time n on an initial configuration a, which is the waiting time before a particle appears in a given position after time n. They restricted their study to a gliders automaton, which is a cellular automaton on 3 states: a background state and two particles evolving at speeds 0 and -1 that annihilate on contact. Thus, we have one entry time for each type of particle ($T_n^+(a)$ and $T_n^-(a)$). When the initial configuration is drawn according to the Bernoulli measure of parameters $(\frac{1}{2}, 0, \frac{1}{2})$, which means that each cell contains, independently, a particle of each type with probability $\frac{1}{2}$, they proved that

$$\forall x \in \mathbb{R}^+,\ \mu\left(\frac{T_n^-(a)}{n} \leq x\right) \underset{n\to\infty}{\longrightarrow} \frac{2}{\pi}\arctan\sqrt{x}.$$

They also called to develop formal tools in order to be able to handle more complex automata, starting with the $(-1,1)$ symmetric case.

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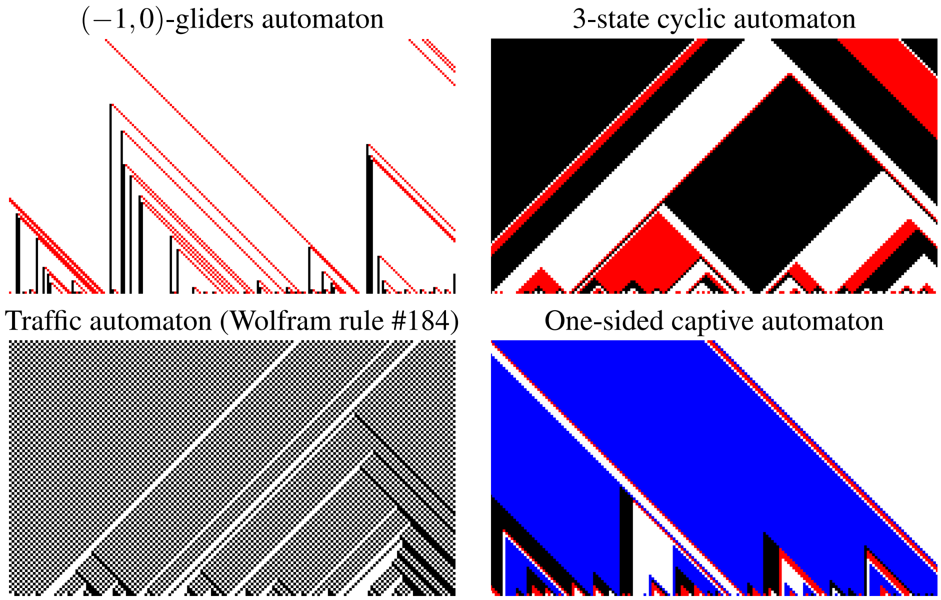

Figure 1: Some automata with simple defect dynamics.
In all diagrams, time goes from bottom to top.

In section 3, we extend in some sense this result to allow arbitrary values for the speeds v_- and v_+, and relax the conditions on the initial measure to some α-mixing conditions. Then, when $v_- < 0$ and $v_+ \geq 0$, we have:

$$\forall x \in \mathbb{R}^+,\ \mu\left(\frac{T_n^-(a)}{n} \leq x\right) \underset{n\to\infty}{\longrightarrow} \frac{2}{\pi}\arctan\left(\sqrt{\frac{-v_- x}{v_+ - v_- + v_+ x}}\right),$$

and symetrically if we exchange $+$ and $-$. The proof relies on the fact that the behaviour of gliders automata can be characterized by some random walk process; his general idea was introduced by Kůrka & Maass in [6] and was already used in [5]. In our case, a particle appearing in a position corresponds to a minima between two concurrent random walks. Under α-mixing conditions, we rescale this process and approximate it with a Brownian motion. Thus we obtain the explicit asymptotic distribution of entry times.

Furthermore, this result can be extended to other automata with similar behaviour, such as those in Fig. 1, by factorizing them onto a gliders automaton. This point is discussed in section 4.

2 Definitions

2.1 Cellular automata

Let $\mathscr{A}$ be a finite alphabet. We consider the spaces $\mathscr{A}^* = \cup_{n\in\mathbb{N}}\mathscr{A}^{[0,n]}$ of finite **words** and $\mathscr{A}^{\mathbb{Z}}$ of bi-infinite **configurations**. We note $|u|$ the **length** of any word u. For $a \in \mathscr{A}^{\mathbb{Z}}$, define its **subwords** $a_{[x,y]} = u_x \dots u_y \in \mathscr{A}^{y-x+1}$ for $x, y \in \mathbb{Z}$. $\mathscr{A}^{\mathbb{Z}}$ is compact in the product topology, and the **cylinders**

$[u]_m = \{a \in \mathscr{A}^{\mathbb{Z}} : a_{[m,m+|u|-1]} = u\}$ for $u \in \mathscr{A}^*$ and $m \in \mathbb{Z}$ are clopen sets and form a base for this topology.

The **shift** function $\sigma : \mathscr{A}^{\mathbb{Z}} \to \mathscr{A}^{\mathbb{Z}}$ is defined by $(\sigma(a))_v = a_{v+1}$ for all $a \in \mathscr{A}^{\mathbb{Z}}$ and $v \in \mathbb{Z}$. A **cellular automaton** or CA is a continuous function $F : \mathscr{A}^{\mathbb{Z}} \to \mathscr{A}^{\mathbb{Z}}$ which commutes with σ. Equivalently, F is a CA iff there exists a **local rule** $f : \mathscr{A}^{\mathbb{U}} \to \mathscr{A}$, where $\mathbb{U}$ is a neighbourhood of the origin, such that $F(a)_z = f(a_{z+\mathbb{U}})$. We study the action of F on $\mathscr{A}^{\mathbb{Z}}$, and especially the sequence $(F^n(a))_{n\in\mathbb{N}}$ for some initial configuration a. For a (partial) graphical representation, we associate a color with each state and represent some finite window of $(F^n(a)_k)_{k\in\mathbb{Z},n\in\mathbb{N}}$ (the **space-time diagram**).

Definition 1 *Let $v_-, v_+ \in \mathbb{Z}$ such that $v_- < v_+$. The (v_-,v_+)-**gliders automaton** (or GA) is the CA of neigbourhood $[-|v_+|, -|v_-|]$ defined on the alphabet $\mathscr{A} = \{-1, 0, +1\}$ by the local rule:*

$$f(a_{-r}\dots a_r) = \begin{cases} +1 & \text{if } a_{-v_+} = +1 \text{ and } \forall N \leq -v_-, \sum_{t=-v_++1}^{N} a_t \geq 0 \\ -1 & \text{if } a_{-v_-} = -1 \text{ and } \forall N \geq -v_+, \sum_{t=N}^{-v_--1} a_t \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In the space-time diagrams of gliders automata, we adopt the convention $\square = 0$, $\blacksquare = +1$, $\color{red}\blacksquare = -1$.

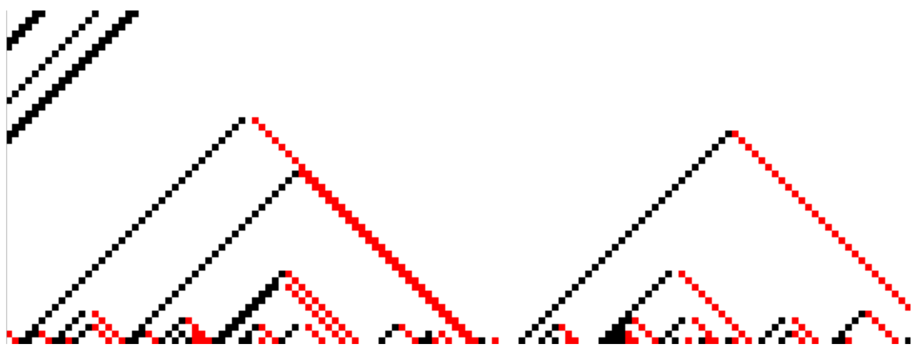

Figure 2: Space-time diagram of the $(-1,1)$-gliders automaton on a random initial configuration.

2.2 Measures on $\mathscr{A}^{\mathbb{Z}}$

Let A be a finite alphabet. We define $\mathscr{M}(\mathscr{A}^{\mathbb{Z}})$ the set of probability measures on the borelians of $\mathscr{A}^{\mathbb{Z}}$. In all the following, we write $\forall_\mu a$ for "for μ-almost all a".

Definition 2 *Let $\pi : \mathscr{A}^{\mathbb{Z}} \to \mathscr{B}^{\mathbb{Z}}$ a measurable application. It induces an action $\pi_* : \mathscr{M}(\mathscr{A}^{\mathbb{Z}}) \to \mathscr{M}(\mathscr{B}^{\mathbb{Z}})$ by defining $\pi_*\mu(U) = \mu(\pi^{-1}U)$ for any borelian U. We write $\pi\mu$ instead of $\pi_*\mu$ to simplify notations.*

σ-action: We note $\mathscr{M}_\sigma(\mathscr{A}^{\mathbb{Z}})$ the **σ-invariant probability measures** on the borelians of $\mathscr{A}^{\mathbb{Z}}$, i.e. the measures such that $\sigma\mu = \mu$. In this case we note $\mu([u])$ for $\mu([u]_0)$. For any $k \in \mathbb{Z}$, let γ_k be the projection $\gamma_k(a) = a_k$. By σ-invariance, if a is drawn according to μ, then $(a_k)_{k\in\mathbb{Z}}$ is a sequence of stationary, non necessarily independent random variables, each drawn according to $\gamma_0\mu$.

Examples:

Dirac measure Let $a \in \mathscr{A}^{\mathbb{Z}}$ be a periodic configuration of minimal period n. We define $\delta_a(U) = \frac{k}{n}$, where $k = \left|\{0 \leq i \leq n-1 \ : \ \sigma^i(a) \in U\}\right|$.

Bernoulli measure Let $(p_a)_{a\in\mathscr{A}}$ be a sequence satisfying $\sum_{a\in\mathscr{A}} p_a = 1$. Then, define μ as $\mu([u]) = p_{u_0}p_{u_1}\cdots p_{u_{|u|-1}}$.

2-step Markov measure Let $(p_{ij})_{i,j\in\mathscr{A}}$ be a matrix satisfying $\sum_j p_{ij} = 1$ for all i, and let (μ_i) an eigenvector associated with the eigenvalue 1 (which is unique if the matrix is irreducible). The associated Markov measure is defined as $\mu([u]) = \mu_{u_0}p_{u_0u_1}\cdots p_{u_{|u|-2}u_{|u|-1}}$. Define similarly an n-step Markov measure.

We introduce the notion of **α-mixing** for a measure $\mu \in \mathscr{M}_\sigma(\mathscr{A}^{\mathbb{Z}})$. Let $\mathscr{R}_n^{+\infty}$, resp. $\mathscr{R}_{-\infty}^0$, be the σ-algebra generated by the cylinders $\{[u]_n \mid u \in \mathscr{A}^*\}$, resp. $\{[u]_{-k} \mid k \in \mathbb{N}, u \in \mathscr{A}^k\}$. The **$\alpha$-mixing coefficients** of μ are

$$\alpha_\mu(n) = \sup\{|\mu(A\cap B) - \mu(A)\mu(B)| : A \in \mathscr{R}_{-\infty}^0, B \in \mathscr{R}_n^\infty\}$$

***F*-action:** Consider the sequence $(F^k\mu)_{k\in\mathbb{N}}$. Putting on $\mathscr{M}_\sigma(\mathscr{A}^{\mathbb{Z}})$ the weak-* topology, we consider the set of limit points $\Gamma_F(\mu)$ of this sequence.

The μ-limit set $\Lambda_F(\mu) = \bigcup_{\nu\in\Gamma_F(\mu)} supp(\nu)$ is of particular interest for self-organization, as discussed in [6]. Indeed, consider words that appear arbitrarily far in space-time diagrams, i.e., such that $F^n\mu([u]) \not\to 0$ (**μ-persistent** words). Then, one can show that a configuration appears in the μ-limit set iff all its subwords are μ-persistent.

In all the following, $\mathscr{A} = \{-1,0,1\}$. We consider two particular subclasses of $\mathscr{M}_\sigma(\mathscr{A}^{\mathbb{Z}})$:

- $\mathscr{B}er$ the set of Bernoulli measures of parameters $(p, 1-2p, p)$ for some $0 < p \le \frac{1}{2}$;
- $\mathscr{M}ix$ the set of measures satisfying:
  - $\mathbb{E}(\gamma_0\mu) = 0$;
  - $\sigma_\mu^2 = \mathbb{E}(\gamma_0\mu^2) + \sum_{k=1}^\infty \mathbb{E}(\gamma_0\mu\cdot\gamma_k\mu) > 0$ (asymptotic variance);
  - $\exists\varepsilon > 0, \sum_{n\ge0} \alpha_\mu(n)^{\frac{1}{2}-\varepsilon} < \infty$.

In particular, $\mathscr{B}er \subset \mathscr{M}ix$. The hypotheses for $\mathscr{M}ix$ are chosen so that the large-scale behaviour of the partial sums $M_a(k)$, defined by $M_a(0) = 0$ and $\forall k \in \mathbb{Z}, M_a(k+1) - M_a(k) = a_k$, can be approximated by a Brownian motion. This invariance principle is the core of our proofs.

Definition 3 *A* **Brownian motion** *(or* **Wiener process***) B of mean μ and variance σ^2 is a a continuous time stochastic process taking values in $\mathbb{R}$ such that:*

- *$B(0) = 0$,*
- *$t \mapsto B(t)$ is almost surely continuous,*
- *$B(t_2) - B(t_1)$ follow the normal law of mean 0 and variance $(t_2 - t_1)\sigma^2$;*
- *For $t_1 < t_2 \le t_1' < t_2'$, increments $B(t_2) - B(t_1)$ and $B(t_2') - B(t_1')$ are independent.*

See [10] for a general introduction to Brownian motion.

Theorem 1 ([12]) *Let $\mu \in \mathscr{M}ix$. Then, $\sigma_\mu^2 < \infty$ and it is possible to construct a sequence $(Z_i)_{i\in\mathbb{Z}}$ of centered Gaussian variables of variance σ_μ^2 such that*

$$\forall\alpha > 0, \forall_\mu a \in \mathscr{A}^{\mathbb{Z}}, \sup_{-n\le k\le n} \left| M_a(k) - \sum_{i=0}^k Z_i \right| = o\left(n^{1/2}\right).$$

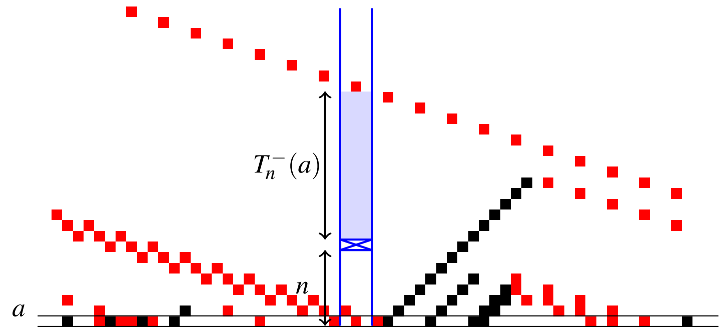

Figure 3: An entry time for the (-3,1)-gliders automaton.

In other words, if we rescale the process:

$$S_a : \begin{array}{l} \mathbb{R} \to \mathbb{R} \\ t \mapsto (t - \lfloor t \rfloor) M_a(\lfloor t \rfloor) + (\lfloor t \rfloor + 1 - t) M_a(\lfloor t \rfloor + 1) \end{array} \qquad \textit{and} \qquad S_a^k : t \mapsto \frac{S_a(kt)}{\sqrt{k}}.$$

Then, if $x < y \in \mathbb{R}$ are any fixed constants, we can construct a Brownian process B_a of parameters $(0, \sigma > 0)$ on $[x, y]$ such that:

$$\forall_\mu a \in \mathscr{A}^{\mathbb{Z}}, \sup_{[x,y]} \left| S_a^n(t) - B_a(t) \right| \underset{n \to \infty}{\longrightarrow} 0$$

For a survey of invariant principles under different assumptions, see [9].

3 Main result

3.1 Entry times

The main result of [4] implies that, for any σ-ergodic initial measure μ (this includes Bernoulli and 2-step Markov measures; see op.cit. for exact definitions), $\Lambda_F(\mu)$ contains at most one kind of particle, which one depending on whether $\mu([+1]) > \mu([-1])$ or the opposite. If $\mu([+1]) = \mu([-1])$, for example when $\mu \in \mathscr{M}ix$, $\Lambda_F(\mu)$ only contains the particleless configuration. This implies that $F^n\mu$ converges to the Dirac measure on this configuration, which means that the probability of seeing a particle in any given column tends to 0 as $t \to \infty$.

Definition 4 (Entry times) *Let $v_- < 0 \le v_+ \in \mathbb{Z}$ and $a \in \mathscr{A}^{\mathbb{Z}}$. We define:*

$$T_n^-(a) = \min\{k \in \mathbb{N} \mid \exists i \in [0, |v_-| - 1], F^{k+n}(a)_i = -1\},$$

*with $T_n^-(a) = \infty$ if this set is empty. This is the **entry time** of a into the set $\{b \in \mathscr{A}^{\mathbb{Z}} \mid \exists i \in [0, |v_-| - 1], b_i = -1\}$ after time n at position 0. We define $T_n^+(a)$ in a similar manner.*

The size of the considered window is such that any particle "passing through" the position 0 will appear in this window exactly once (see Fig. 3). Of course entry times for particles of speed 0 make no sense. From now on, we will only consider T^- for simplicity, all the results being valid for T^+.

As a direct consequence of Birkhoff's ergodic theorem, we see that when $\mu([-1]) > \mu([+1])$:

- $\mu(T_n^+(a)=\infty) \underset{n\to\infty}{\longrightarrow} 1$;
- $\mu\left(\frac{T_n^-(a)}{n} \le x\right) \underset{n\to\infty}{\longrightarrow} 1$.

Kurka & al. proved the following result:

Theorem 2 (cf. [5]) *For the $(-1,0)$-GA ("Asymmetric gliders"), if the initial measure μ is the Bernoulli one of parameters $(\frac{1}{2},0,\frac{1}{2})$, then we have asymptotically:*

$$\forall x \in \mathbb{R}^+,\ \mu\left(\frac{T_n^-(a)}{n} \le x\right) \underset{n\to\infty}{\longrightarrow} \frac{2}{\pi}\arctan\sqrt{x}.$$

In the same article, they conjectured that this result could be extended to any initial Bernoulli measure of parameters $(p,1-2p,p)$ by replacing the right-hand term by $\frac{2}{\pi}\arctan\sqrt{2px}$.

Theorem 3 (Main result) *For any (v_-,v_+)-GA with $v_-<0$ and $v_+\ge 0$, with the initial measure $\mu \in \mathscr{M}ix$, we have asymptotically:*

$$\forall x \in \mathbb{R}^+,\ \mu\left(\frac{T_n^-(a)}{n} \le x\right) \underset{n\to\infty}{\longrightarrow} \frac{2}{\pi}\arctan\left(\sqrt{\frac{-v_-x}{v_+-v_-+v_+x}}\right).$$

Notice that this limit is independent of the variance of $\pi_0\mu$, disproving the conjecture when $\mu \in \mathscr{B}er$.

3.2 Technical lemmas

Lemma 1 $\forall j \in \mathbb{Z}, \forall n \ge 1,$

$$M_{F(a)}(j) < \min_{\{j+1,\dots,j+n\}} M_{F(a)} \Leftrightarrow M_a(j-v_+) < \min_{\{j+1-v_+,\dots,j+n-v_-\}} M_a,$$
$$M_{F(a)}(j) < \min_{\{j-n,\dots,j-1\}} M_{F(a)} \Leftrightarrow M_a(j-v_-) < \min_{\{j-n-v_+,\dots,j-1-v_-\}} M_a.$$

and those inequalities still hold if we replace < by $\le$.

Proof. We prove those four inequalities by induction on n. At each step, we will prove only the first inequality with $<$, the other cases being symmetric.

Base case.

$$\begin{aligned} M_{F(a)}(j) < M_{F(a)}(j+1) &\Leftrightarrow F(a)_j = +1 \\ &\Leftrightarrow a_{j-v_+} = +1 \text{ and } \forall N \le -v_-, \sum_{t=-v_++1}^{N} a_{j+t} \ge 0 \\ &\Leftrightarrow M_a(j-v_+) < \min_{\{j+1-v_+,\dots,j+1-v_-\}} M_a \end{aligned}$$

Induction. Assume the four inequalities hold for some n. We distinguish two cases:

- if $F(a)_j \ne 1$, then $a_{j-v_+} \ne +1$, or $a_{j-v_+} = +1$ and $\exists N \le v_-, \sum_{t=j-v_+}^{j+N} a_t < 0$. In any case,

  $$M_{F(a)}(j) \ge \min_{\{j+1,\dots,j+n+1\}} M_{F(a)} \text{ and } M_a(j-v_+) \ge \min_{\{j+1-v_+,\dots,j+n-v_-\}} M_a$$

  and so the inequality holds.

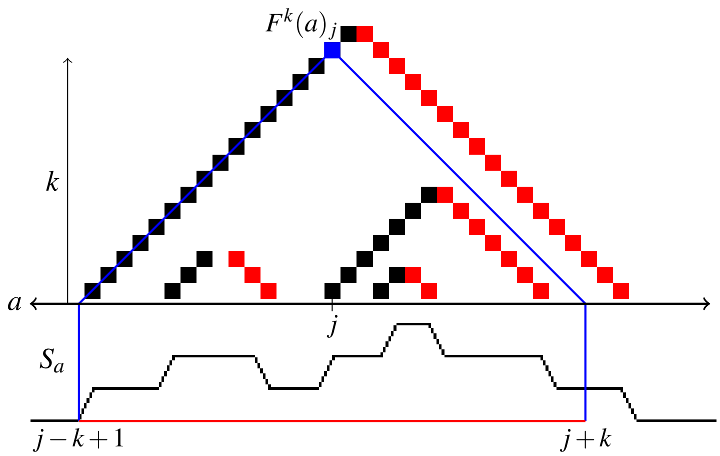

Figure 4: Illustration of lemma 2. A strict minimum is reached on $j-k+1$.

- if $F(a)_j = 1$, then $a_{j-v_+} = 1$. We again distinguish two cases:
  - If $M_{F(a)}(j+1) \leq \min\limits_{\{j+2,\dots,j+n+1\}} M_{F(a)}$, then $M_a(j-v_+ + 1) \leq \min\limits_{\{j-v_+ +2,\dots,j+n-v_- +1\}} M_a$ by induction hypothesis, and since $M_{F(a)}(j) = M_{F(a)}(j+1)-1$ and $M_a(j-v_+) = M_a(j-v_+ + 1)-1$, the inequality holds.
  - Otherwise, $M_{F(a)}(j+1) > \min\limits_{\{j+2,\dots,j+n+1\}} M_{F(a)}$ and $M_a(j-v_+ + 1) > \min\limits_{\{j+2-v_+,\dots,j+n-v_-\}} M_a$ by induction hypothesis, and the inequality holds for the same reason. $\square$

Lemma 2 $\forall j \in \mathbb{Z}, \forall k \geq 0,$

$$F^k(a)_j = -1 \Leftrightarrow M_a(j-v_-k+1) < \min_{\{j-v_+k,\dots,j-v_-k\}} M_a$$
$$F^k(a)_j = +1 \Leftrightarrow M_a(j-v_+k) < \min_{\{j-v_+k+1,\dots,j-v_-k+1\}} M_a$$

This is illustrated in Fig. 4, where M_a is replaced by its piecewise affine interpolation S_a.

Proof. By induction on k, proving only the first equality at each step:

Base case. Obviously, $M_a(j+1) < M_a(j) \Leftrightarrow a_j = -1$.

Induction. Now suppose those equalities hold for a given rank k. By induction hypothesis, $F^{k+1}(a)_j = -1 \Leftrightarrow M_{F(a)}(j-v_-k+1) < \min\limits_{\{j-v_+k,\dots,j-v_-k\}} M_{F(a)}$ and we conclude by lemma 1. $\square$

3.3 Proof of the theorem

For any $a \in \mathscr{A}^{\mathbb{Z}}$, the lemma 2 on the column 0 gives:

$$\begin{aligned} T_n^-(a) &= \min\left\{k \geq 0 \mid \exists j \in [0, -v_-[\ , M_a(-v_-(n+k)+j+1) < \min_{\{-v_+(n+k)+j,\dots,-v_-(n+k)-j\}} M_a\right\} \\ &= \min\left\{k \geq 0 \mid \exists j \in [0, -v_-[\ , M_a(-v_-(n+k)+j+1) < \min_{\{-v_+(n+k)+j,\dots,-v_-n\}} M_a\right\} \end{aligned}$$

We keep notations from Thm. 1: S_a is the piecewise affine interpolation of M_a and S_a^n is a rescaling of S_a. Note that when the previous condition is attained on $M_a(x)$, then it is attained for $S_a(y)$ as soon as $y > x-1$, and reciprocally. Thus:

$$\begin{aligned} T_n^-(a) &= \inf\left\{t \geq 0 \mid \exists j \in [0, -v_-[\ , S_a(-v_-(n+t)+j+2) < \min_{[-v_+(n+t)+j+1, -v_-n]} S_a\right\} \\ &= \left\lceil \inf\left\{t \geq 0 \mid S_a(-v_-(n+t)+2) < \min_{[-v_+(n+t)+1, -v_-n]} S_a\right\}\right\rceil \\ &= \left\lceil \inf\left\{t \geq 0 \mid S_a^n\left(-v_-\left(1+\frac{2}{n}\right)+\frac{1}{n}\right) < \min_{[-v_+(1+\frac{t}{n})+\frac{1}{n}, -v_-]} S_a^n\right\}\right\rceil \\ &= \left\lceil n \cdot \inf\left\{t \geq 0 \mid S_a^n\left(-v_-(1+t)+\frac{2}{n}\right) < \min_{[-v_+(1+t)+\frac{1}{n}, -v_-]} S_a^n\right\}\right\rceil \end{aligned}$$

And finally, since S_a^n is $\sqrt{n}$-Lipschitz and $\forall x, n \in \mathbb{R}\times\mathbb{N}$, $x \leq \frac{\lceil nx \rceil}{n} \leq x + \frac{1}{n}$:

$$\begin{aligned} &\mu\left(\min_{[-v_-, -v_-(1+x)]} S_a^n + \frac{3}{\sqrt{n}} < \min_{[-v_+(1+x), -v_-]} S_a^n\right) \leq \mu\left(\frac{T_n^-(a)}{n} \leq x\right) \\ &\mu\left(\frac{T_n^-(a)}{n} \leq x\right) \leq \mu\left(\min_{[-v_-, -v_-(1+x)]} S_a^n - \frac{4}{\sqrt{n}} < \min_{[-v_+(1+x), -v_-]} S_a^n\right) \end{aligned} \tag{1}$$

Using Thm. 1, we can construct a Brownian motion B_a such that $S_a^n \to B_a$ for the $||.||_\infty$ norm on $L^\infty([-v_+(1+x), -v_-(1+x)])$. By symmetry, $B_a^l(t) = B_a(-v_- - t) - B_a(-v_-)$ and $B_a^r(t) = B_a(-v_- + t) - B_a(-v_-)$ are two independent Brownian motion satisfying $B_a^l(0) = B_a^r(0) = 0$. Consequently, for any $\varepsilon > 0$ and n large enough:

$$\begin{aligned} \mu\left(\min_{[-v_-, -v_-(1+x)]} S_a^n - \varepsilon < \min_{[-v_+(1+x), -v_-]} S_a^n\right) &\leq \mu\left(\min_{[-v_-, -v_-(1+x)]} B_a - 2\varepsilon < \min_{[-v_+(1+x), -v_-]} B_a\right) \\ &\leq \mu\left(\min_{[0, -v_-x]} B_a^l - 2\varepsilon < \min_{[0, -v_-+v_+(1+x)]} B_a^r\right) \end{aligned} \tag{2}$$

and similarly for the left-hand term of (1). For a brownian motion B and any $b > 0$, we have by rescaling $\mu\left(\min_{[0,b]} B \geq m\right) = \mu\left(\min_{[0,1]} B \geq \frac{m}{\sqrt{b}}\right)$. Furthermore, since B_a^l and B_a^r are independent, so are $\min_{[0,1]} B_a^l$ and $\min_{[0,1]} B_a^r$. That means that for any $y, z > 0$:

$$\begin{aligned}
\mu\left(\min_{[0,y]} B_a^l < \min_{[0,z]} B_a^r\right) &= \int_{-\infty}^{0}\int_{-\infty}^{0} 1_{\{\sqrt{y}\cdot m_1 \leq \sqrt{z}\cdot m_2\}} d\mathbb{P}_{\{\min B_a^r\}}(m_2) d\mathbb{P}_{\{\min B_a^l\}}(m_1) \\
&\underset{(i)}{=} \frac{4}{2\pi}\int_{-\infty}^{0}\int_{-\infty}^{\frac{\sqrt{z}\cdot m_2}{\sqrt{y}}} e^{\frac{-m_1^2}{2}} e^{\frac{-m_2^2}{2}} dm_1 dm_2 \\
&\underset{(ii)}{=} \frac{2}{\pi}\int_{\pi}^{\pi+\arctan\left(\sqrt{\frac{y}{z}}\right)} \int_{0}^{+\infty} re^{\frac{-r^2}{2}} dr d\theta \\
&= \frac{2}{\pi}\arctan\left(\sqrt{\frac{y}{z}}\right) \qquad (3)
\end{aligned}$$

(i) by using the law of the minimum of a Brownian motion (see [10]), (ii) by passing in polar variables. For $\varepsilon > 0$, a similar calculation gives:

$$\begin{aligned}
\left|\mu\left(\min_{[0,y]} B_a^l - 2\varepsilon < \min_{[0,z]} B_a^r\right) - \mu\left(\min_{[0,y]} B_a^l < \min_{[0,z]} B_a^r\right)\right| &\leq \frac{4}{2\pi}\int_{-\infty}^{0}\int_{\frac{\sqrt{z}\cdot m_2}{\sqrt{y}}}^{\frac{\sqrt{z}\cdot m_2+2\varepsilon}{\sqrt{y}}} e^{\frac{-m_1^2}{2}} e^{\frac{-m_2^2}{2}} dm_1 dm_2 \\
&\leq \frac{8\varepsilon}{2\pi\sqrt{y}}\int_{-\infty}^{0} e^{\frac{-ym_2^2}{2z}} e^{\frac{-m_2^2}{2}} dm_2 \\
&\underset{\varepsilon\to 0}{\longrightarrow} 0 \qquad (4)
\end{aligned}$$

And similarly for the left-hand term. Combining (1), (2), (3) and (4), the theorem follows. □

4 Extension to other automata

Definition 5 *Let F_1, F_2 be two CA on $\mathscr{A}^{\mathbb{Z}}$ and $\mathscr{B}^{\mathbb{Z}}$, respectively. We say that F_1* ***factorizes onto*** *F_2 if there exists a* ***factor*** *$\pi : \mathscr{A}^{\mathbb{Z}} \to \mathscr{B}^{\mathbb{Z}}$, i.e. a continuous transformation that commutes with σ, such that $\pi \circ F_1 = F_2 \circ \pi$.*

Similarly to a CA, a factor is entirely defined by a neighbourhood $\mathbb{U}$ and a local rule $p : \mathscr{A}^{\mathbb{U}} \to \mathscr{B}$. See [8] for a description of the role of factors in symbolic dynamics.

In this section, we will extend the Thm. 3 to automata that factorize onto a gliders automaton, starting by showing how to find such a factor. In the examples given in Fig. 1, a similar behaviour is observed: starting from a random configuration, strips constituted of periodic patterns appear and persist, and the boundaries between these strips behave as particles of constant speed. Intuitively, these automata exhibit the same behaviour as a gliders automaton, if we see the regular patterns as the background and the boundaries as particles.

Definition 6 *Let $\mathscr{A}$ a finite alphabet. A* **subshift of finite type** *(SFT) $\Sigma \subset \mathscr{A}^{\mathbb{Z}}$ is a set of configurations defined by a set of finite* **forbidden patterns** *$U \subset \mathscr{A}^r$, where $r > 0$ is the* **order** *of the SFT. Precisely,*

$$a \in \Sigma \Leftrightarrow \forall i \in \mathbb{Z}, a_{[i,i+r-1]} \notin U.$$

The set of **defects** of $a \in \mathscr{A}^{\mathbb{Z}}$ with regard to Σ is $\mathbb{D}_\Sigma(a) = \{i \in \mathbb{Z} : a_{[i,i+r-1]} \in U\}$. Even when Σ is F-invariant (which means that the strips are persistent), there is no general relationship between defects of a and defects of $F(a)$. However, some automata have defects (relatively to some SFT, the choice being usually obvious) behave as particles of constant speed with only two possible speeds v_+ and v_-; in other words, we can separate the forbidden patterns into two sets U^+ and U^-, with corresponding defects $\mathbb{D}^+_\Sigma(a)$ and $\mathbb{D}^-_\Sigma(a)$, and we have:

$$\begin{aligned}
i \in \mathbb{D}^+_\Sigma(F(a)) \Leftrightarrow \quad & i - v_+ \in \mathbb{D}^+_\Sigma(a) \\
& \text{and } \forall N \le -v_-, \left|[-v_+ + 1, N] \cap \mathbb{D}^+_\Sigma(a)\right| \ge \left|[-v_+ + 1, N] \cap \mathbb{D}^-_\Sigma(a)\right| \\
i \in \mathbb{D}^-_\Sigma(F(a)) \Leftrightarrow \quad & i - v_- \in \mathbb{D}^-_\Sigma(a) \\
& \text{and } \forall N \le -v_-, \left|[N, -v_- - 1] \cap \mathbb{D}^-_\Sigma(a)\right| \ge \left|[N, -v_- - 1] \cap \mathbb{D}^+_\Sigma(a)\right|
\end{aligned}$$

That is, the defects behave as the particles of the gliders automaton. Now it is easy to see that such an automaton factorizes onto the (v_-, v_+)-GA with the factor $\pi : \mathscr{A}^{\mathbb{Z}} \to \{-1,0,1\}^{\mathbb{Z}}$ of neighbourhood $[-r, r]$, where r is the order of Σ and π is defined by the local rule $p : \mathscr{A}^r \to \{-1,0,1\}$ defined as:

$$p(u) = \begin{cases} +1 \text{ if } u \in U^+; \\ -1 \text{ if } u \in U^-; \\ 0 \text{ if } u \notin U. \end{cases}$$

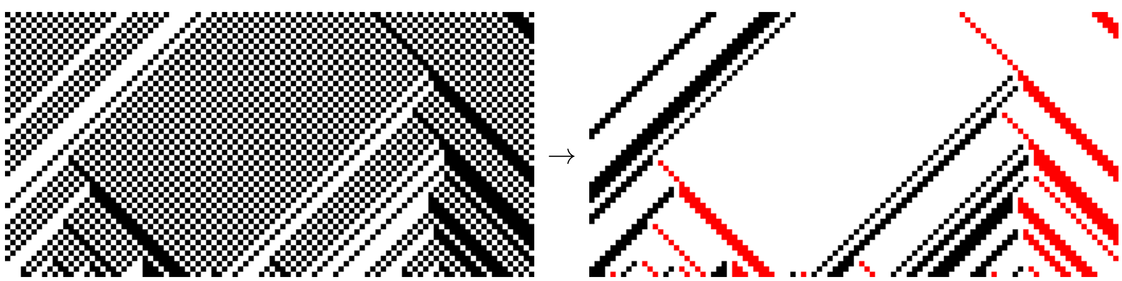

Figure 5: Factor π: □□ → ■ ■■ → ■ □■, ■□ → □.

Example:

Traffic automaton: Let $\mathscr{A} = \{0,1\}$ and F be the CA of neighbourhood $\{-1,0,1\}$ defined by the local rule:

$$f(u_{-1}, u_0, u_1) = \begin{cases} 1 & \text{if } u_{-1} = 1 \text{ and } u_0 = 0 \\ 1 & \text{if } u_0 = 1 \text{ and } u_1 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The behaviour of F can be observed in Fig. 5 with the convention $0 = \square, 1 = \blacksquare$. It is apparent that the relevant SFT corresponds to the set of forbidden patterns $U^+ = \{\square\square\}, U^- = \{\blacksquare\blacksquare\}$ ("checkerboard SFT") and the corresponding defects have the dynamics of a $(-1,1)$-GA. The induced factor is indicated in Fig. 5.

In order to extend the Thm. 3 to such CA with initial measure μ, we must first ensure that $\pi\mu \in \mathscr{M}ix$.

Proposition 1 *Let $\pi : \mathscr{A}^{\mathbb{Z}} \to \mathscr{A}^{\mathbb{Z}}$ be a factor, $\mu \in \mathscr{M}_\sigma(\mathscr{A}^{\mathbb{Z}})$ and $k > 0$ any real such that $\sum_{n \ge 0} \alpha_\mu(n)^k < \infty$. Then, $\sum_{n \ge 0} \alpha_{\pi\mu}(n)^k < \infty$.*

Proof. We keep the notations from the definition of $\alpha_\mu(n)$. Since π is continuous, there exists $r>0$ such that $\mathbb{U}\subset[-r,r]$ and $\pi(a)_k$ only depends on $a_{[k-r,k+r]}$. Then, $\pi^{-1}\mathscr{R}^0_{-\infty}\subset\mathscr{R}^r_{-\infty}$ and $\pi^{-1}\mathscr{R}^\infty_n\subset\mathscr{R}^{+\infty}_{n-r}$. By σ-invariance, we have for all n $\alpha_{\pi(\mu)}(n)<\alpha_\mu(n-2r)$, and the theorem follows. $\square$

This is true in particular if μ is a Bernoulli measure or a 2-step Markov measure associated with a irreducible, aperiodic matrix. Hence, in that case, we only have to prove that μ weighs evenly the sets of particles -1 and $+1$, and that the corresponding asymptotic variance is not zero. Under those assumptions, we can extend the previous theorem with the forbidden patterns playing the role of the particles.

Corollary 1 *Let F be a CA on the alphabet $\mathscr{A}$ and $\mu\in\mathscr{M}_\sigma(\mathscr{A}^\mathbb{Z})$. Suppose that F factorizes onto a (v_-,v_+)-GA via a factor π, defined on a neighbourhood $[-r,r]$ by the local rule p, so that $\pi\mu\in\mathscr{M}ix$.*
Then, Thm. 3 holds if we replace "$a_k=-1$" by "$p(a_{[k,k+r-1]})=-1$", and similarly for $+1$.

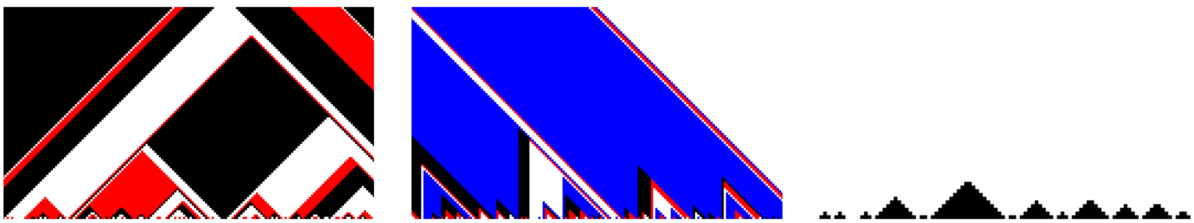

Figure 6: The 3-state cyclic CA, a one-sided captive CA and the product CA.

Examples: (In all the following, we use the convention $\square=0, \blacksquare=1, {\color{red}\blacksquare}=2, {\color{blue}\blacksquare}=3$.)

Traffic automaton: Consider the factor defined earlier. If μ is a measure such that $\pi\mu\in\mathscr{M}ix$, then Thm. 3 applies. For example, this is true for the 2-step Markov measure defined by the matrix $\begin{pmatrix} p & 1-p \\ 1-p & p\end{pmatrix}$ and the eigenvector $\begin{pmatrix}1/2\\1/2\end{pmatrix}$ with $p>0$. A particular case is the Bernoulli measure of parameters $(\frac{1}{2},\frac{1}{2})$.

Cyclic automaton: Let $\mathscr{A}=\mathbb{Z}/3\mathbb{Z}$ and F the CA of neighbourhood $\{-1,0,1\}$ defined by the local rule

$$f(u_{-1},u_0,u_1)=\begin{cases} u_0+1 & \text{if } u_{-1}=u_0+1 \text{ or } u_1=u_0+1,\\ u_0 & \text{otherwise.}\end{cases}$$

The relevant SFT is the SFT of order 2 obtained by forbidding $U^+=\{\blacksquare\square, \square{\color{red}\blacksquare}, {\color{red}\blacksquare}\blacksquare\}$, U^- being symmetric ("monochromatic SFT"). The corresponding defects obviously behave as particles of speed +1 or -1 and F factorizes onto the $(-1,1)$-GA. If μ is such that $\pi\mu\in\mathscr{M}ix$, then Thm. 3 applies. This is true in particular when μ is any 2-step Markov measure defined by a matrix $(p_{ij})_{1\leq i,j\leq 3}$ satisfying $p_{01}+p_{12}+p_{20}=p_{10}+p_{21}+p_{02}$, all of these values being >0, with $(\mu_i)_{1\leq i\leq 3}$ its only eigenvector. This includes any nondegenerate Bernoulli measure.

One-sided captive automata: Let F be a CA of neighbourhood $\{0,1\}$ on any alphabet, such that $f(u_{-1},u_0)\in\{u_{-1},u_0\}$. We consider the monochromatic SFT, that is, the SFT of order 2 obtained by forbidding words with two different letters. For $a\neq b\in\mathscr{A}$, if $f(a,b)=a$, we consider that $ab\in U^+$; otherwise, $ab\in U^-$. In this way, F factorizes onto the $(-1,0)$-GA, and if $\pi\mu\in\mathscr{M}ix$, then Thm. 3 applies.

Notice that this class of automata contain the identity ($\forall a,b, f(a,b)=b$) and the shift σ ($\forall a,b, f(a,b)=a$). However, since we have in each case $U^+=\emptyset$ or $U^-=\emptyset$, it is impossible to find a measure that weighs evenly each kind of particle, and so $\pi\mu$ cannot belong in $\mathscr{M}ix$.

Counter-example:

Product automaton: Let $\mathscr{A}=\mathbb{Z}/2\mathbb{Z}$ and F be the CA of neighbourhood $\{-1,0,1\}$ defined by the local rule $f(a_{-1},a_0,a_1)=a_{-1}\cdot a_0\cdot a_1$. The relevant SFT corresponds to the forbidden patterns $U^+=\{\blacksquare\square\}, U^-=\{\square\blacksquare\}$. Then, F factorizes onto the $(-1,1)$-GA. If μ is a Bernoulli measure, then $\pi\mu$ satisfies all conditions of $\mathscr{M}ix$ except that $\sigma_\mu=0$; indeed, we can check that for $\pi\mu$-almost all configurations, the particles $+1$ and -1 altern. Hence, only one particle can cross any given column after time 0, and therefore $\mu\left(\frac{T_n^-(a)}{n}\leq x\right)\underset{n\to\infty}{\longrightarrow}0$.

5 Conclusion

Even though we showed that the asymptotic distributions of entry times are known for some class of cellular automata and a large class of measures, this covers only very specific dynamics. It is not known how these results extend for more than 2 particles and/or other kind of particle interaction. In particular, there is no obvious stochastic process characterizing the behaviour of such automata that would play the role of M_a in our proofs.

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References

[1] N. Boccara, J. Nasser & M. Roger (1991): *Particlelike structures and their interactions in spatiotemporal patterns generated by one-dimensional deterministic cellular-automaton rules*. Phys. Rev. A 44(2), pp. 866–875, doi:10.1103/PhysRevA.44.866.

[2] Kari Eloranta (1994): *The dynamics of defect ensembles in one-dimensional cellular automata*. Journal of Statistical Physics 76, pp. 1377–1398. Available at `http://dx.doi.org/10.1007/BF02187067`. 10.1007/BF02187067.

[3] J.E. Hanson & J.P. Crutchfield (1997): *Computational mechanics of cellular automata: An example*. Physica D: Nonlinear Phenomena 103(1-4), pp. 169–189.

[4] B. Hellouin de Menibus & M. Sablik (2011): *Self-organization in cellular automata: a particle-based approach*. In: Developments in Language Theory, Lecture Notes in Computer Science 6795, pp. 251–263. Available at `http://dx.doi.org/10.1007/978-3-642-22321-1_22`.

[5] P. Kůrka, E. Formenti & A. Dennunzio (2011): *Asymptotic distribution of entry times in a cellular automaton with annihilating particles*. Proceedings of AUTOMATA, pp. 89–100.

[6] P. Kůrka & A. Maass (2000): *Limit sets of cellular automata associated to probability measures*. Journal of Statistical Physics 100(5), pp. 1031–1047.

[7] Petr Kůrka (2003): *Cellular automata with vanishing particles*. Fundam. Inform. 58(2003), pp. 203–221. Available at `http://iospress.metapress.com/content/yv0nneqckhn5x7eu/`.

[8] D.A. Lind & B. Marcus (1995): *An introduction to symbolic dynamics and coding*. Cambridge Univ Pr.

[9] F. Merlévède & E. Rio (2012): *Strong approximations of partial sums under dependence conditions with applications to dynamical systems*. Stochastic Processes and their applications 122, pp. 386–417.

[10] P. Mörters & Y. Peres (2010): *Brownian motion. Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge.

[11] M. Pivato (2007): *Defect particle kinematics in one-dimensional cellular automata*. Theoretical Computer Science 377(1-3), pp. 205–228.

[12] Q.M. Shao & C. Lu (1987): *Strong approximations for partial sums of weakly dependent random variables.* Sci. Sin. Ser. A. 30(6), pp. 575–587.

[13] W. A. Stein et al. (2012): *Sage Mathematics Software (Version 4.2.1)*. The Sage Development Team. `http://www.sagemath.org`.

6 Annex non-published: Density of particles and convergence rate

6.1 Density of particles

Definition 7 *The* density $d_-(a)$ *of particles* -1 *in* $a \in \mathscr{A}^{\mathbb{Z}}$ *is defined as:*

$$d_-(a) = \limsup_{n\to\infty} \frac{\#\{i \in [-n,n] \mid a_i = -1\}}{2n+1}$$

We define $d_+(a)$ *in a similar manner.*

In all the following, any result on d_- will also be true for d_+ by symmetry.

Theorem 4 *Let* F *be a* (v_-, v_+)*-GA with initial measure* $\mu \in \mathscr{M}ix$*. Then:*

$$\forall_\mu a \in \mathscr{A}^{\mathbb{Z}}, \forall \varepsilon > 0, d_-(F^n(a)) = O\left(n^{-\frac{1}{4}+\varepsilon}\right)$$

If furthermore $\mu \in \mathscr{B}er$*:*

$$\forall_\mu a \in \mathscr{A}^{\mathbb{Z}}, d_-(F^n(a)) = \Theta\left(n^{-\frac{1}{2}}\right)$$

Proof. When $\mu \in \mathscr{M}ix$, it is in particular σ-ergodic, and so are its images $F^k\mu$. By Birkhoff's ergodic theorem, one has $\forall_\mu a \in \mathscr{A}^{\mathbb{Z}}, \forall k, d_-(F^k(a)) = F^k\mu([-1])$.

We first prove the theorem when $v_- < 0$ and $v_+ \geq 0$. By lemma 2,

$$\mu(F^k(a)_0 = -1) = \mu\left(M_a(-v_-k+1) < \min_{\{-v_+k,\ldots,-v_-k\}} M_a\right)$$

which is the probability that the random walk starting from 0 remains stricly positive for a time $(v_+ - v_-)k$, also known as its survival time.

Minoration ($\mu \in \mathscr{B}er$): When the random walk is symmetric and the steps are independant, we have the equivalent $\mu(F^k(a)_0 = -1) \sim \frac{1}{\sqrt{(v_+-v_-)k}}$. Since by definition $F^n\mu([-1]) = \mu(F^k(a)_0 = -1)$, we obtain the minoration.

Majoration: Using Theorem 1, we have $\mu\left(\min_{[0,1]} S_k^a(x) = 0\right) < \mu\left(\min_{[0,1]} B_a(x) > -S\right)$ where $S = \sup_{[0,1]} \left|S_a^k(x) - B_a(x)\right| = O\left(n^{-\frac{1}{4}+\varepsilon}\right)$.

Furthemore $\mu\left(\min_{[0,1]} B_a(x) > -S\right) = \int_{-S}^{0} e^{-x^2/2} dx < S = O\left(n^{-\frac{1}{4}+\varepsilon}\right)$.

General case (any $v_- < v_+$): Let F be the (v_-, v_+)-GA. Then F^2 is the $(2v_-, 2v_+)$-GA and $F^2 \circ \sigma^{-(v_++v_-)}$ is the $(v_- - v_+, v_+ - v_-)$-GA. Applying the previous case on this automaton, we obtain asymptotic bounds. To conclude, it is enough see that the density of particles is σ-invariant and decreasing under the action of $F(a)$.

6.2 Rate of convergence

Definition 8 *The* Lévy-Prohorov distance *on $\mathscr{M}(\mathscr{A}^{\mathbb{Z}})$ is defined as:*

$$\forall \mu_1, \mu_2 \in \mathscr{M}_\sigma(\mathscr{A}^{\mathbb{Z}}),\ d(\mu_1, \mu_2) = \sum_{u \in \mathscr{A}^*}^{\infty} \frac{\mu_1([u]) - \mu_2([u])}{|\mathscr{A}|^n}.$$

The Lévy-Prohorov distance induces the weak-* topology, and therefore it is natural to use it to measure the rate of convergence of $F^n\mu$ to its limit measure.

Theorem 5 *Let F be the (v_-, v_+)-GA with initial measure $\mu \in \mathscr{M}ix$. Then:*

$$\forall \alpha > 0,\ d(F^k\mu, \delta_{\infty 0^\infty}) = O\left(k^{-1/8+\alpha}\right)$$

If furthermore $\mu \in \mathscr{B}er$:

$$d(F^k\mu, \delta_{\infty 0^\infty}) = \Omega\left(k^{-1/2}\right)$$

We first prove the theorem for $v_- < 0$ and $v_+ > 0$. The distance can be re-written:

$$\forall k \in \mathbb{N}, d(F^k\mu, \delta_{\infty 0^\infty}) = 2\sum_{n=1}^{\infty} \frac{\mu(F^k(a)_{[0,n]} \neq [0, \ldots 0])}{3^n}.$$

Minoration ($\mu \in \mathscr{B}er$): Obviously, $d(F^k\mu, \delta_{\infty 0^\infty}) > 2\mu(F^k(a)_0 \neq 0)$. We conclude with Theorem 4.

Majoration: Let $N \in \mathbb{N}$. We have:

$$d(F^k\mu, \delta_{\infty 0^\infty}) < 2\sum_{n=1}^{N} \frac{\mu(F^k(a)_{[0,N]} \neq [0, \ldots 0])}{3^n} + O\left(\frac{1}{3^N}\right) < 3\mu(F^k(a)_{[0,N]} \neq [0, \ldots 0]) + O\left(\frac{1}{3^N}\right)$$

Since this is a non-increasing function of N, we can assume w.l.o.g that N is a multiple of v_+ and v_-, and we prove by straightforward induction that

$$\begin{aligned} F^k(a)_{[0,N]} \neq [0, \ldots 0] &\Rightarrow \exists n \in \left[k - \frac{N}{v_+}, k - \frac{N}{v_-}\right], F^n(a)_0 \neq 0 \\ &\Rightarrow T^+_{k-\frac{N}{v_+}}(a) < \frac{N}{v_+} - \frac{N}{v_-} \end{aligned}$$

We now suppose that k has the form $(K+1)\frac{N}{v_+}$. In all the following, asymptotic bounds hold when K and N tends to infinity.

$$\begin{aligned} F^k(a)_{[0,N]} \neq [0, \ldots 0] &\Rightarrow \frac{T^+_{\frac{KN}{v_+}}(a)}{KN/v_+} < \frac{(v_- - v_+)}{Kv_-} \\ &\Rightarrow \left(\min_{[0,y]} B^l_a + \frac{1}{\sqrt{n}} < \min_{[0,z]} B^r_a\right) \end{aligned}$$

where $n = \frac{KN}{v_+} = \Theta(KN), x = \frac{(v_+ - v_-)}{K}$ and $y = (v_+ - v_-)(1 - \frac{v_+}{Kv_-})$. According to Thm.1, we have for all $\alpha > 0$:

$$\begin{aligned}
\mu\left(\min_{[0,x]} P_a^n + \frac{1}{\sqrt{n}} < \min_{[0,y]} Q_a^n\right) &\leq \mu\left(\min_{[0,x]} B_a + \frac{1}{\sqrt{n}} + O\left((KN)^{-1/4+\alpha}\right) < \min_{[0,y]} B'_a\right) \\
&\leq \frac{2}{\pi}\arctan\sqrt{\frac{x}{y}} + O\left((KN)^{-1/4+\alpha}\right)\left[\frac{2}{\pi\sqrt{a}}\int_{-\infty}^{0} e^{\frac{-xm_2^2}{2y}} e^{\frac{-m_2^2}{2}} dm_2\right] \\
&\leq \frac{2}{\pi}\arctan\sqrt{\frac{1}{K - v_+/v_-}} + O\left((KN)^{-1/4+\alpha}\right)\left[\frac{\sqrt{2K}}{\sqrt{\pi(v_+ - v_-)}}\right] \\
&= O\left(\sqrt{\frac{1}{K}}\right) + O\left((KN)^{-1/4+\alpha}\right) O\left(\sqrt{K}\right)
\end{aligned}$$

Since $k = \Theta(KN)$, it is possible to have K and N grow in such a way that $K = \Theta(k^{1/8+\alpha})$ and $N = \Theta(k^{7/8-\alpha})$. Then this last term is $O\left(k^{-1/(8+\alpha)}\right)$ and the term $O\left(\frac{1}{3^N}\right)$ is negligible, so the majoration ensues.

General case: Apply the same method as in the previous section, considered that d and all considered measures are σ-invariant and that any CA is Lipschitz w.r.t d. $\square$