TWO POINTS OF VIEW TO STUDY THE ITERATES OF A RANDOM CONFIGURATION BY A CELLULAR AUTOMATON

M. SABLIK

Laboratoire d'Analyse, Topologie, Probabilités UMR 6632
Centre de Mathmatiques et Informatique, 39, rue F. Joliot Curie, 13453 Marseille Cedex 13
E-mail address: `sablik@latp.univ-mrs.fr`

Abstract. We study the dynamics of the action of cellular automata on the set of shift-invariant probability measures according two points of view. First, the robustness of the simulation of a cellular automaton on a random configuration can be viewed considering the sensitivity to initial condition in the space of shift-invariant probability measures. Secondly we consider the evolution of the quantity of information in the orbit of a random initial state.

Introduction

Despite the apparent simplicity of their definition, cellular automata can have very complex behaviours which are observed on space time diagrams. To try to understand this complexity, they can be considered as dynamical systems. Generally, one studies the dynamics of the $\mathbb{N}$-action of a cellular automaton on the set of configurations $\mathcal{A}^{\mathbb{Z}}$, where $\mathcal{A}$ is a finite alphabet, endowed with the product topology. However, very simple cellular automata as the power of the shift, denoted σ^m for $m \in \mathbb{Z}$, are sensitive. That is to say, they are considered with a highly chaotic behaviour. This does not correspond to the intuitive idea which give the space-time diagram.

The shortcoming of the Cantor distance is to privilege the central coordinates whereas there may be no reason to give more importance to coordinates around the origin. Moreover, one considers only the action of the cellular automaton without considering the shift action. Indeed, space-time diagrams of a cellular automata $(\mathcal{A}^{\mathbb{Z}}, F)$ are not so different from that of $(\mathcal{A}^{\mathbb{Z}}, \sigma^m \circ F)$ for $m \in \mathbb{Z}$. However, if F is not nilpotent, $\sigma^m \circ F$ is sensitive for m taken quite far from the origin. The reason is that Cantor topology is non-homogeneous, thus a simple transport of information is enough to obtain sensitivity.

One point of view can be to address the $\mathbb{Z} \times \mathbb{N}$-action (σ, F) in order to emphasize the spatiotemporal structure. In [Sab06], one gives general definitions to talk about directional dynamics; the purpose is to study the sets of directions which have a certain kind of dynamics. Another point of view is to kill the $\mathbb{Z}$-action of σ and consider the $\mathbb{N}$-action of F on a σ-invariant object. In this direction, G. Cattaneo, E. Formenti, L. Margara et J. Mazoyer [CFMM97] introduce another topology defined by the Besicovitch pseudo distance which measures the upper density of the differences between two configurations in order

Key words and phrases: Cellular automata, shift invariant topology, entropy of a configuration.

to give the same importance at all cells. For this distance, the shift map is an isometry. However, with this topology, we lose the compactness of the space which is the traditional framework of topological dynamics.

Another natural σ-invariant object is the set of σ-invariant probability measures, denoted $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$. Indeed, a cellular automaton acts on the set of configurations and canonically transforms $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ into another σ-invariant measure denoted $F_*\mu$. Hence, cellular automata also have a natural action on the set of shift-invariant measures. In this space the shift has the same behavior as the identity and a sensitive cellular automaton in this space is not only capable of *"transporting"* information but it is also able to *"create"* new information outwardly.

The study of the action of a cellular automaton on $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ could be interesting when we use cellular automata to simulate. Indeed, generally we observe the action of a cellular automaton on a random initial configuration, that is to say a configuration chosen according to a probability measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$. But it is more natural to study the action of a cellular automaton directly on $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ instead the set of configurations chosen according a given probability measure.

In this article we exhibit two problematics:

• The more natural for a dynamical system is the study of sensitivity to initial conditions of the map $F_* : \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z}) \to \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$. This approach can be interesting when we use cellular automata to simulate. Indeed, generally we start the simulation with a random configuration choosen according to a distribution $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$, this approach may help to evaluate the impact of a mistake when we choose the initial configuration with a distribution $\nu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ near to the expected distribution μ.

• The information contained in a random configuration according to a measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ can be expressed by the entropy of the shift $h_\mu(\sigma)$. It would be natural to obseve the evolution of this quantity when a cellular automaton acts on μ. It is easy to see that $(h_{F_*^n\mu}(\sigma))_{n\in\mathbb{N}}$ decreases towards $h_\mu^\infty(F)$, but for some cellular automata, like linear cellular automata, it appears a phenomenon of gap between $h_\mu^\infty(F)$ and the apparent entropy $h_\mu^a(F)$, defined by [Mar00], which coresponds to the entropy observed in the central window. This can explain why linear cellular automata have complex space-time diagrams.

We illustrate these two points of view by the study of two important classes: cellular automata with directional equicontinuous points which have very regular space-time diagrams and linear cellular automata which have complex space-time diagrams.

1. Action of a cellular automaton on $\mathcal{A}^\mathbb{Z}$ (Background)

1.1. Space of configurations

1.1.1. *Cantor topology.* Let $\mathcal{A}$ be a finite set. We consider $\mathcal{A}^\mathbb{Z}$, the *configuration space* of $\mathbb{Z}$-indexed sequences in $\mathcal{A}$. If $\mathcal{A}$ is endowed with the discrete topology, $\mathcal{A}^\mathbb{Z}$ is compact, perfect and totally disconnected in the product topology. Moreover one can define a metric on $\mathcal{A}^\mathbb{Z}$ compatible with this topology:

$$\forall x, y \in \mathcal{A}^\mathbb{Z}, \quad d_C(x,y) = 2^{-\min\{|i| : x_i \neq y_i \ i\in\mathbb{Z}\}}.$$

Let $\mathbb{U} \subset \mathbb{Z}$. For $x \in \mathcal{A}^\mathbb{Z}$, the restriction of x to $\mathbb{U}$ is denoted by $x_\mathbb{U} \in \mathcal{A}^\mathbb{U}$. For a pattern $w \in \mathcal{A}^\mathbb{U}$, define $[w]_\mathbb{U} = \{x \in \mathcal{A}^\mathbb{Z} : x_\mathbb{U} = w\}$ the cylinder centered on w.

The shift map $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by $(x_i)_{i\in\mathbb{Z}} \mapsto (x_{i+1})_{i\in\mathbb{Z}}$. A *subshift* $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ is a closed σ-invariant subset of $\mathcal{A}^{\mathbb{Z}}$. Denote $\mathcal{L}_n(\Sigma)$, the set of patterns $u \in \mathcal{A}^{[0,n-1]}$ such that there exists $x \in \Sigma$ which verifies $x_{[0,n-1]} = u$. The *language* of Σ is $\mathcal{L}(\Sigma) = \cup_{n\in\mathbb{N}}\mathcal{L}_n(\Sigma)$.

1.1.2. *Besicovitch topology.* G. Cattaneo, E. Formenti, L. Margara and J. Mazoyer [CFMM97] introduce the Besicovitch pseudo distance which is σ-invariant. For $x, y \in \mathcal{A}^{\mathbb{Z}}$, it is defined by:

$$d_B(x,y) = \limsup_{n\to\infty} \frac{\mathrm{Card}(\{m \in [-n,n] : x_m \neq y_m\})}{2n+1}.$$

The topology induced by this pseudo-distance has good properties for studying dynamical systems except the compacity.

1.2. Action of a cellular automaton on $\mathcal{A}^{\mathbb{Z}}$

1.2.1. *Definition of CA.* A *cellular automaton* (CA) is a pair $(\mathcal{A}^{\mathbb{Z}}, F)$ where $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by $F(x)_m = \overline{F}((x_{m+u})_{u\in\mathbb{U}})$ for all $x \in \mathcal{A}^{\mathbb{Z}}$ and $m \in \mathbb{Z}$ where $\mathbb{U} \subset \mathbb{Z}$ is a finite set named *neighborhood* and $\overline{F} : \mathcal{A}^{\mathbb{U}} \to \mathcal{A}$ is a *local rule.* By Hedlund's theorem [Hed69], it is equivalent to say that it is a continuous function which commutes with the shift ($\sigma^m \circ F = F \circ \sigma^m$ for all $m \in \mathbb{Z}$).

1.2.2. *General definitions about dynamical systems.* Let (X, d) be a metric space and $F : X \to X$ be a continuous function. There is a lot of definitions to precise the sensitivity to initial conditions of the dynamical system generated by the $\mathbb{N}$-action of F on X. We recall here some of them:

• $x \in X$ is an *equicontinuous point* if for all $\varepsilon > 0$, there exists $\delta > 0$, such that for all $y \in X$, if $d(x,y) < \delta$ then $d(F^n(x), F^n(y)) < \varepsilon$ for all $n \in \mathbb{N}$. Denote $Eq_d(F)$ the set of equicontinuous points. If $x \notin Eq_d(F)$, it is a sensitive point.

• (X, F) is *equicontinuous* if for all $\varepsilon > 0$, there exists $\delta > 0$, such that for all $x, y \in X$, if $d(x,y) < \delta$ then $d(F^n(x), F^n(y)) < \varepsilon$ for all $n \in \mathbb{N}$.

• (X, F) is *sensitive* if there exists $\varepsilon > 0$ such that for all $\delta > 0$ and $x \in X$, there exists $y \in X$ and $n \in \mathbb{N}$ such that $d(x,y) < \delta$ and $d(F^n(x), F^n(y)) > \varepsilon$

• (X, F) is $\mathbb{N}$-*expansive* if there exists $\varepsilon > 0$ such that for all $x \neq y$ there exists $n \in \mathbb{N}$ such that $d(F^n(x), F^n(y)) > \varepsilon$.

In an intuitive sense, sensitivity and expansivity denote a certain complexity of the dynamical system whereas equicontinuity denotes a strong regularity.

1.2.3. *Directional dynamics.* In [Sab06], one adapts the dynamical classification of [Kůr97] according to a direction $\alpha \in \mathbb{R}$. The study of such dynamics make appear some discrete geometry in space time diagrams. We recall here the notion of equicontinuous points. In this case the information is blocked between walls of slope α generated by blocking words.

Proposition 1.1 ([Sab06])**.** *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA, let $\mathbb{U} = [r,s]$ be a neighborhood of F, and let $\alpha \in \mathbb{R}$. A point $x \in \mathcal{A}^{\mathbb{Z}}$ is* equicontinuous of slope α *if $\forall\varepsilon > 0, \exists\delta > 0$ such that $\forall y \in \mathcal{A}^{\mathbb{Z}}$, if $d_C(x,y) < \delta$ then $d_C(F^n \circ \sigma^{\lfloor\alpha n\rfloor}(x), F^n \circ \sigma^{\lfloor\alpha n\rfloor}(y)) < \varepsilon$ for all $n \in \mathbb{N}$.*

The existence of equicontinuous points is equivalent to the existence of a blocking word u of slope α and width $e \geq \max(\lfloor\alpha\rfloor + 1 + s, -\lfloor\alpha\rfloor + 1 - r)$. *That is to say, there exists* $p \in [0, |u| - e]$ *such that:*

$$\forall x, y \in [u]_{[0,|u|-1]}, \forall n \in \mathbb{N}, \qquad \sigma^{\lfloor n\alpha\rfloor} \circ F^n(x)_{[p,p+e]} = \sigma^{\lfloor n\alpha\rfloor} \circ F^n(y)_{[p,p+e]}.$$

A CA is *equicontinuous of slope* α if every $x \in \mathcal{A}^{\mathbb{Z}}$ is equicontinuous of slope α.

A CA is $\mathbb{N}$-*expansive of slope* α if there exists $\varepsilon > 0$ such that for all $x, y \in \mathcal{A}^{\mathbb{Z}}$, there exists $n \in \mathbb{N}$ which verifies $d_C(F^n \circ \sigma^{\lfloor\alpha n\rfloor}(x), F^n \circ \sigma^{\lfloor\alpha n\rfloor}(y)) > \varepsilon$.

2. Action of a cellular automaton on $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$

A natural σ-invariant object on which cellular automata can act, is the set of σ-invariant probability measures $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$. This approach is not only theoretical. Indeed, in simulations, the action of a CA is observed on a random configuration chosen according to a probability measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$, but it is more easy to study directly the action of the CA on the probability measure μ.

2.1. Measures on $\mathcal{A}^{\mathbb{Z}}$

Let $\mathfrak{B}$ be the Borel sigma-algebra of $\mathcal{A}^{\mathbb{Z}}$. Denote by $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ the set of probability measures on $\mathcal{A}^{\mathbb{Z}}$ defined on the sigma-algebra $\mathfrak{B}$. Usually $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ is endowed with the weak* topology: a sequence $(\mu_n)_{n\in\mathbb{N}}$ of $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ converges to $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ if and only if for all finite subset $\mathbb{U} \subset \mathbb{Z}$ and for all pattern $u \in \mathcal{A}^{\mathbb{U}}$, one has $\lim_{n\to\infty} \mu_n([u]_{\mathbb{U}}) = \mu([u]_{\mathbb{U}})$. In the weak* topology, the set $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ is compact and metrizable. A metric is defined by:

$$\forall \mu, \nu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}}), \qquad d_*^{\mathcal{M}}(\mu, \nu) = \sum_{n\in\mathbb{N}} \frac{1}{|\mathcal{A}|^n} \sum_{u\in\mathcal{A}^{[0,n]}} \left|\mu([u]_{[-n,n]}) - \nu([u]_{[-n,n]})\right|.$$

Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a mesurable function. It is possible to consider the action of F on $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ defined by:

$$F_*\mu(B) = \mu(F^{-1}(B)), \text{ for all } \mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}}) \text{ and } B \in \mathfrak{B}.$$

A probability measure $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ is said σ-*invariant* if $\sigma_*\mu = \mu$. Denote $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ the set of σ-invariant probability measures. It is a compact convex subset of $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ (see [DGS76] for more details). A probability measure $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ is σ-*ergodic* if for all σ-invariant subset $B \in \mathfrak{B}$ (i.e. $\sigma^{-1}(B) = B$ μ-almost everywhere) are trivial (i.e. $\mu(B) = 0$ or 1). The set of σ-ergodic probability measures is denoted by $\mathcal{M}_\sigma^{\mathrm{erg}}(\mathcal{A}^{\mathbb{Z}})$.

Example 2.1. Let $x \in \mathcal{A}^{\mathbb{Z}}$. Define the *Dirac measure in* x by $\delta_x(A) = 1$ if $x \in A$ and 0 if not, where $A \in \mathfrak{B}$. The set of Dirac measures is dense in $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ for the weak* topology.

One remarks that if the configuration is not σ-uniform, the Dirac measure associated is not σ-invariant. However, if we take a σ-periodic configuration x of period $p \in \mathbb{N}$, one constructs a σ-ergodic measure by taking the mean of the Dirac's measures of the σ-orbit: $\widetilde{\delta_x} = \frac{1}{p}\sum_{m\in[0,p-1]} \delta_{\sigma^m(x)}$.

Example 2.2. For all $a \in \mathcal{A}$, put $p_a \in [0,1]$ a real such that $\sum_{a\in\mathcal{A}} p_a = 1$. Define *the Bernoulli measure according to the probability vector* $(p_a)_{a\in\mathcal{A}}$ by $\lambda_{(p_a)_{a\in\mathcal{A}}}([u]_{\mathbb{U}}) = \prod_{m\in\mathbb{U}} p_{u_m}$ for all $u \in \mathcal{L}_{\mathcal{A}^{\mathbb{Z}}}([u]_{\mathbb{U}})$. If all p_a are equal to $\frac{1}{\mathrm{Card}(\mathcal{A})}$, one obtains the *uniform Bernoulli measure* which is just denoted by $\lambda_{\mathcal{A}^{\mathbb{Z}}}$.

2.2. Generic points

For some σ-invariant probability measures, there exist special points of $\mathcal{A}^{\mathbb{Z}}$ which represent the measure. That is to say the frequency of apparition of a pattern corresponds to the measure of the cylinder centered on this pattern. This allows to give a symbolic interpretation of the distance $d_*^{\mathcal{M}}$.

A point $x \in \mathcal{A}^{\mathbb{Z}}$ is *generic* if for all $\mathbb{U} \subset \mathbb{Z}$ finite and for every pattern $u \in \mathcal{A}^{\mathbb{U}}$ the sequence $(f(u,x,n))_{n\in\mathbb{N}}$ converges where

$$f(u,x,n) = \frac{1}{2n+1} \sum_{m\in[-n,n]} \mathbf{1}_{[u]_{\mathbb{U}}}(\sigma^m(x)),$$

is the *frequency of apparition of the pattern u in x at the order n*. The limit of this sequence is denoted $f(u,x)$, this is the *frequency of apparition of the pattern u in x*. Denote $\mathcal{G}$ the set of generic points.

Let $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$. Denote $\mathcal{G}(\mu)$ the set of *generic points of* μ, this is the set of points $x \in \mathcal{G}$ such that for every pattern, the frequency of this pattern in x is equal to the measure of the cylinder centered on this pattern. When μ is σ-ergodic, the Birkhoff Theorem says that $\mu(\mathcal{G}(\mu)) = 1$.

Consider the function $\phi : (\mathcal{G}, d_B) \to (\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}), d_*^{\mathcal{M}})$ which associates a generic point $x \in \mathcal{G}$ to the measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ defined by $\mu([u]_{\mathbb{U}}) = f(u,x)$ for all pattern $u \in \mathcal{A}^{\mathbb{U}}$. It is easy to verify that ϕ is surjective and continuous. Moreover, the image of a generic point by F is also a generic point. Thus the correspondence between $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ and $\mathcal{G}$ is summed up in the following commutative diagram:

$$\begin{array}{ccc}
(\mathcal{G}, d_B) & \xrightarrow{F} & (\mathcal{G}, d_B) \\
\big\downarrow \phi & & \big\downarrow \phi \\
(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}), d_*^{\mathcal{M}}) & \xrightarrow{F_*} & (\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}), d_*^{\mathcal{M}})
\end{array}$$

2.3. Action of a CA on $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$

Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA. The map $\mu \mapsto F_*\mu$ is continuous for the *weak** topology and preserves convex combinations. Thus, $F_* : \mathcal{M}(\mathcal{A}^{\mathbb{Z}}) \to \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ defines a dynamical system. However, for all $x \in \mathcal{A}^{\mathbb{Z}}$ one has $F_*\delta_x = \delta_{F(x)}$. Thus the map $x \mapsto \delta_x$ allows to consider $(\mathcal{A}^{\mathbb{Z}}, F)$ as a sub-system of $(\mathcal{M}(\mathcal{A}^{\mathbb{Z}}), F_*)$, so the dynamics of $F_* : \mathcal{M}(\mathcal{A}^{\mathbb{Z}}) \to \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ contains the dynamics of $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$. Moreover, the weak$*$ topology on $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ privileges the origin. Thus it is preferable to restrict the initial space.

Since F commutes with the shift, if $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ then $F_*\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$. Thus one can study the dynamical system defined by:

$$\begin{array}{rccc}
F_* : & \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) & \longrightarrow & \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \\
 & \mu & \longmapsto & F_*\mu \qquad \text{such that } F_*\mu(B) = \mu(F^{-1}(B)) \ \forall B \in \mathfrak{B}.
\end{array}$$

Remark 2.3. If $(\mathcal{A}^{\mathbb{Z}}, F)$ is a surjective CA, $\lambda_{\mathcal{A}^{\mathbb{Z}}}$ is F-invariant (see [Hed69]). One deduces that $\lambda_{\mathcal{A}^{\mathbb{Z}}}$ is a fixed point of F_*.

3. Sensitivity to initial conditions of $F_* : \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z}) \to \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$

A natural question in the study of dynamical systems is the sensitivity to initial conditions. This point of view could be interesting when we use cellular automata to simulate in order to characterize distributions which are unstable.

3.1. Expansivity of F_*

The function F_* preserves convex combinations in $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$. In [BS07], we use this property to show that there are not CA which acts $\mathbb{N}$-expansively on $(\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z}), d_*^\mathcal{M})$. This shows that in $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$, the information cannot be transporting to distinguish initial points.

Theorem 3.1. [BS07] *F_* cannot act $\mathbb{N}$-expansively on $(\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z}), d_*^\mathcal{M})$.*

3.2. Equicontinuity

If $(\mathcal{A}^\mathbb{Z}, F)$ is equicontinuous of slope α, according to [Sab06], it is periodic in this direction. Thus it is the same for F_*. One deduces the next proposition.

Proposition 3.2. *Let $(\mathcal{A}^\mathbb{Z}, F)$ be an equicontinuous CA of slope α. Then F_* is equicontinous in $(\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z}), d_*^\mathcal{M})$.*

3.3. Equicontinuous points

For some CA it is possible to characterize the set of equicontinuous points. These measures are stable for perturbations. The next example shows that there exist cellular automata with equicontinuous and sensitive points in $(\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z}), d_*^\mathcal{M})$.

Example 3.3. Consider the CA defined on $\mathcal{A} = \mathbb{Z}/2\mathbb{Z}$ by $F(x)_i = x_{i-1} \cdot x_i \cdot x_{i+1}$ for all $x \in \mathcal{A}^\mathbb{Z}$ and $i \in \mathbb{Z}$. It is easy to see that for all σ-ergodic probability measure μ which verifies $\mu([0]) > 0$, the sequence $(F_*^n \mu)_{n\in\mathbb{N}}$ converges toward $\delta_{\infty 0^\infty}$ in $(\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z}), d^\mathcal{M})$. So, one has:

$$Eq_{d_*^\mathcal{M}}(F_*) \cap \mathcal{M}_\sigma^{\mathrm{erg}}(\mathcal{A}^\mathbb{Z}) = \mathcal{M}_\sigma^{\mathrm{erg}}(\mathcal{A}^\mathbb{Z}) \setminus \{\delta_{\infty 1^\infty}\}.$$

For CA with equicontinuous points of slope α, we characterize a large class of measures which are equicontinuous points in the space $(\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z}), d_*^\mathcal{M})$.

Theorem 3.4. *Let $(\mathcal{A}^\mathbb{Z}, F)$ be a CA and let $B \in \mathcal{A}^*$ be a blocking word of slope $\alpha \in \mathbb{R}$. Then every σ-ergodic probability measure $\mu \in \mathcal{M}_\sigma^{erg}(\mathcal{A}^\mathbb{Z})$ such that $\mu([B]) > 0$ is an equicontinuous point of $F_* : (\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z}), d_*^\mathcal{M}) \to (\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z}), d_*^\mathcal{M})$.*

Proof. Let $\varepsilon > 0$, let μ be a σ-ergodic probability measure which charges B and let ν be a σ-invariant measure. For $n \in \mathbb{N}$, one defines $X_{i,n}^k$, the set of points $x \in \mathcal{A}^\mathbb{Z}$ such that there is an occurrence of B in $[-k - \lfloor n\alpha \rfloor, -\lfloor n\alpha \rfloor]$ and another in $[i - 1 - \lfloor n\alpha \rfloor, k + i - 1 - \lfloor n\alpha \rfloor]$.

Let i_0 be such that $\sum_{i=i_0+1}^{\infty} \frac{1}{|\mathcal{A}|^i} \leq \varepsilon$ and let $n \in \mathbb{N}$. Since B is charged by μ, by σ-ergodicity, there exists $k \in \mathbb{N}$ such that $\mu(X_{i,n}^k) \geq 1 - \varepsilon$ for all $i \leq i_0$. Moreover $X_{i,n}^k$ can be written as an union of cylinders centered on $[-k - \lfloor n\alpha \rfloor, k + i - 1 - \lfloor n\alpha \rfloor]$ of words of $\mathcal{A}^{i+2k}$. By σ-invariance, one deduces that $|\mu(X_{i,n}^k) - \nu(X_{i,n}^k)| \leq |\mathcal{A}|^{i+2k} d(\mu, \nu)$, so:

$$\nu(X_{i,n}^k) \geq 1 - \varepsilon - |\mathcal{A}|^{i+2k} d(\mu, \nu).$$

Let $i \leq i_0$ and let $u \in \mathcal{A}^i$. Put $X_{u,n}^k = F^{-n} \circ \sigma^{-\lfloor \alpha n \rfloor}([u]_{[0,i-1]}) \cap X_{i,n}^k$. Taking the lower bounds of $\mu(X_{i,n}^k)$ and $\nu(X_{i,n}^k)$, one deduces:

$$\begin{aligned} |F_*^n\mu([u]_{[0,i-1]}) - F_*^n\nu([u]_{[0,i-1]})| &\leq |F_*^n\mu([u]_{[0,i-1]}) - \mu(X_{u,n}^k)| + |\mu(X_{u,n}^k) - \nu(X_{u,n}^k)| \\ &\quad + |F_*^n\nu([u]_{[0,i-1]}) - \nu(X_{u,n}^k)| \\ &\leq \varepsilon + |\mu(X_{u,n}^k) - \nu(X_{u,n}^k)| + \varepsilon + |\mathcal{A}|^{|u|+2k} d(\mu,\nu). \end{aligned}$$

A summation gives for all $i \leq i_0$ the following inequality:

$$\begin{aligned} \sum_{u\in\mathcal{A}^i} |F_*^n\mu([u]_{[0,i-1]}) - F_*^n\nu([u]_{[0,i-1]})| &\leq 2\varepsilon|\mathcal{A}|^i + |\mathcal{A}|^{2i+2k} d(\mu,\nu) \\ &\quad + \sum_{u\in\mathcal{A}^i} |\mu(X_{u,n}^k) - \nu(X_{u,n}^k)|. \end{aligned}$$

Let $Y_{u,n}^k$ be the set of words $v \in \mathcal{A}^{|u|+2k}$ such that there exists $y \in F^{-n} \circ \sigma^{-\lfloor \alpha n \rfloor}[u]_{[0,i-1]} \cap [v]_{[-k-\lfloor n\alpha \rfloor, k+|u|-\lfloor n\alpha \rfloor]} \cap X_{|u|,n}^k$. Since B is a blocking word of slope α, for all $v \in Y_{u,n}^k$, for all $x \in [v]_{[-k-\lfloor n\alpha \rfloor, k+|u|-\lfloor n\alpha \rfloor]}$, one has $F^n(x) \circ \sigma^{\lfloor \alpha n \rfloor}(x)_{[0,|u|-1]} = u$. One deduces that:

$$X_{u,n}^k = F^{-n} \circ \sigma^{-\lfloor \alpha n \rfloor}([u]_{[0,i-1]}) \cap X_{|u|,n}^k = \bigcup_{Y_{u,n}^k} [v]_{[-k-\lfloor n\alpha \rfloor, k+|u|-\lfloor n\alpha \rfloor]}.$$

Thus, it is possible to decompose the sets $X_{u,n}^k$ to obtain a sum of measures of cylinders centered on $\mathbb{V} = [-k - \lfloor n\alpha \rfloor, k + |u| - \lfloor n\alpha \rfloor]$ with words in $\mathcal{A}^{i+2k}$:

$$\begin{aligned} \sum_{u\in\mathcal{A}^i} |\mu(X_{u,n}^k) - \nu(X_{u,n}^k)| &= \sum_{u\in A^i} \left| \sum_{v\in Y_{u,n}^k} \mu([v]_{\mathbb{V}}) - \nu([v]_{\mathbb{V}}) \right| \\ &\leq \sum_{v\in\mathcal{A}^{i+2k}} |\mu([v]_{[0,i-1]}) - \nu([v]_{[0,i-1]})|. \end{aligned}$$

By summation of previous inequalities, it follows that:

$$\begin{aligned} d_*^{\mathcal{M}}(F_*^n\mu, F_*^n\nu) &= \sum_{i\leq i_0} \frac{1}{|\mathcal{A}|^{2i}} \sum_{u\in\mathcal{A}^i} |F_*^n\mu([u]_{[0,i-1]}) - F_*^n\nu([u]_{[0,i-1]})| \\ &\quad + \sum_{i>i_0} \frac{1}{|\mathcal{A}|^{2i}} \sum_{u\in\mathcal{A}^i} |F_*^n\mu([u]_{[0,i-1]}) - F_*^n\nu([u]_{[0,i-1]})| \\ &\leq \frac{2\varepsilon}{|\mathcal{A}|-1} + |\mathcal{A}|^{2k}(1+|\mathcal{A}|^{i_0}) d(\mu,\nu) + \varepsilon. \end{aligned}$$

This shows that the orbits $(F_*^n\mu)_{n\in\mathbb{N}}$ and $(F_*^n\nu)_{n\in\mathbb{N}}$ stay close to each other when μ and ν are close enough. ∎

3.4. The case of algebraic CA

The uniform Bernoulli measure (see example 2.2) has an important role in the study of σ-invariant measures. It is known that for a large class of algebraic CA and a large class of measures, the Cesàro mean of the iterates of a measure by the CA converges to the uniform Bernoulli measure. This result was proved with tools from stochastic processes in

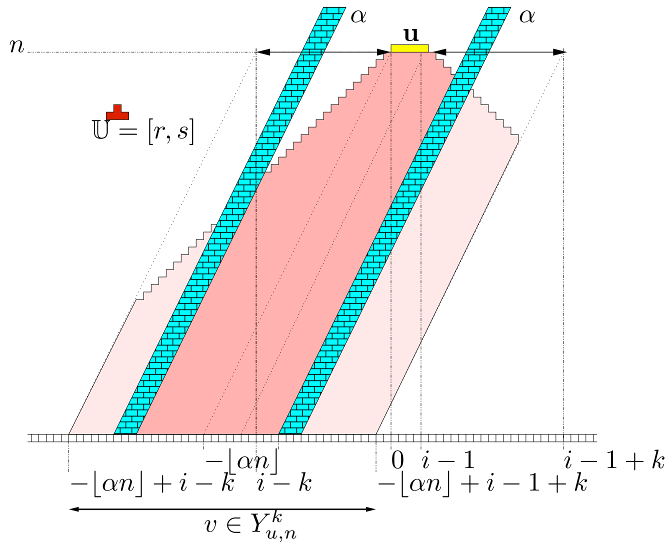

Figure 1: Blocking word of slope α and $d_*^{\mathcal{M}}$-equicontinuity.

[FMMN00], and with harmonic analysis tools in [PY02] and [PY04]. We use this result to show the $d_*^{\mathcal{M}}$-sensitivity of linear CA.

Theorem 3.5. *Let $\mathcal{A}$ be an Abelian finite group and let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a non trivial linear CA (that is to say a group endomorphism on the product group $\mathcal{A}^{\mathbb{Z}}$ which is not a product of shifts). Then $Eq_{d_*^{\mathcal{M}}}(F_*) = \emptyset$.*

Proof. Since $(\mathcal{A}^{\mathbb{Z}}, F)$ is a non trivial linear CA, there exist p prime, a surjective endomorphism $\pi : \mathcal{A} \to \mathbb{Z}/p\mathbb{Z}$ and a factor CA $(\mathbb{Z}/p\mathbb{Z}, \widehat{F})$ such that $\pi \circ F = \widehat{F} \circ \pi$ (where π is extended coordinate to coordinate). It is easy to verify that $\widehat{F}$ is a non-trivial linear CA on $\mathbb{Z}/p\mathbb{Z}$. Since π_* is open, it is sufficient to prove the sensitivity for this factor of F_*. Thus we assume that we are in this case.

Let $((\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}}, F)$ be a nontrivial linear CA with p prime. In [PY02], they show that there is a weak* dense set in $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$, the harmonic measures set denoted $\mathcal{H}$, such that every measure $\mu \in \mathcal{H}$ verifies

$$\lim_{n \in \mathbb{J} \to \infty} d_*^{\mathcal{M}}(F_*^n \mu, \lambda_{\mathcal{A}^{\mathbb{Z}}}) = 0,$$

where $\lambda_{\mathcal{A}^{\mathbb{Z}}}$ is the uniform Bernoulli measure and $\mathbb{J}$ is a subset of $\mathbb{N}$ of upper density 1.

Let P be the set of (σ, F)-periodic points, it is a dense subset of $\mathcal{A}^{\mathbb{Z}}$ [BK99]. According to example 2.1, it is easy to see that the set $\{\widetilde{\delta_x} : x \in P\}$ is weak* dense in $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$.

Let $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ such that $\mu \neq \lambda_{\mathcal{A}^{\mathbb{Z}}}$ and let $\varepsilon < \frac{1}{2} d_*^{\mathcal{M}}(\lambda_{\mathcal{A}^{\mathbb{Z}}}, \mu)$. For all $\delta < \varepsilon$. There exists $\mu' \in \mathcal{H}$ and $x \in P$ such that $d_*^{\mathcal{M}}(\mu, \mu') < \delta$ and $d_*^{\mathcal{M}}(\mu, \widetilde{\delta_x}) < \delta$. Thus one has $\lim_{n \in \mathbb{J} \to \infty} d_*^{\mathcal{M}}(F_*^n \mu', \lambda_{\mathcal{A}^{\mathbb{Z}}}) = 0$, where $\mathbb{J}$ is a subset of $\mathbb{N}$ of density 1. Moreover, if p is the (σ, F)-period of x, one has $F_*^{pn} \widetilde{\delta_x} = \widetilde{\delta_x}$ for all $n \in \mathbb{N}$. Since $\mathbb{J}$ has upper density 1, there exists $n \in \mathbb{N}$ such that $pn \in \mathbb{J}$ and $d_*^{\mathcal{M}}(F_*^{pn} \mu', \lambda_{\mathcal{A}^{\mathbb{Z}}}) < \varepsilon$. Thus one has $d_*^{\mathcal{M}}(F_*^{pn} \widetilde{\delta_x}, F_*^{pn} \mu') > \varepsilon$. One deduces that $\mu \notin Eq_{d_*^{\mathcal{M}}}(F_*)$.

If $\mu = \lambda_{\mathcal{A}^{\mathbb{Z}}}$ there exists μ' such that the sequence $(F_*^n \mu')_{n \in \mathcal{A}^{\mathbb{Z}}}$ has at least two adherence points: one of them is $\mu = \lambda_{\mathcal{A}^{\mathbb{Z}}}$ [PY02], denote μ'' another. Since $\mu = \lambda_{\mathcal{A}^{\mathbb{Z}}}$ is F_*-invariant, it is possible to find in the sequence $(F_*^n\mu')_{n\in\mathcal{A}^\mathbb{Z}}$ a point close to μ but μ'' is a closure point of the orbit of this point. So $\mu \notin Eq_{d_*^\mathcal{M}}(F_*)$.

Thus $Eq_{d_*^\mathcal{M}}(F_*) = \emptyset$, but this method do not allow to obtain an uniform sensitive constant. There is a problem around of $\lambda_{\mathcal{A}^\mathbb{Z}}$. ■

4. Quantity of information in a μ-generic configuration

4.1. Problematic

Let $(\mathcal{A}^\mathbb{Z}, F)$ be a CA. A configuration $x \in \mathcal{A}^\mathbb{Z}$ converges generally to the *limit set* $\Lambda_F = \bigcap_{n\in\mathbb{N}} F^n(\mathcal{A}^\mathbb{Z})$. However, when you look the simulation of a CA on points chosen according to a σ-invariant probability measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$, the limit set capture so many points. This does not correspond to the observation. That is why P. Kůrka and A. Maass [KM00] introduce the concept of *μ-limit set*, denoted $\Lambda_F(\mu)$, defined by:

$$u \notin \mathcal{L}(\Lambda_F(\mu)) \Longleftrightarrow \lim_{n\to\infty} F_*([u]_{[0,|u|-1]}) = 0.$$

Naturally, one has $\Lambda_F(\mu) \subset \Lambda_F$. In symbolic dynamics, the complexity of a subshift Σ is mesured thanks to the topological entropy:

$$h_{\text{top}}(\Sigma) = \lim_{n\to\infty} \frac{\log(\text{Card}(\mathcal{L}_n(\Sigma)))}{n}.$$

Let $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$. It is also possible to define the metric entropy of σ relative to μ by:

$$h_\mu(\sigma) = \lim_{n\to\infty} -\frac{1}{n} \sum_{u\in\mathcal{A}^n} \mu([u]) \log(\mu([u])).$$

We refer to [DGS76] for more general definitions and main properties.

In fact, the information contained in a μ-generic configuration can be expressed by the entropy of the shift. A comparative study of the entropy of the shift and the Kolmogorov complexity was carried out by A. Brudno [Bru82]. It could be interesting to study the sequence $(h_{F_*^n\mu}(\sigma))_{n\in\mathbb{N}}$ which corresponds to the evolution of the quantity of information when we start with a random configuration chosen according to μ.

Let $\mathbb{U} \subset \mathbb{Z}$ be finite, put $\mathcal{P}_\mathbb{U} = \{[u]_\mathbb{U} : u \in \mathcal{A}^\mathbb{U}\}$. The *quantity of information* in the partition $\mathcal{P}_\mathbb{U}$ is defined by $H_{F_*^n\mu}(\mathcal{P}_\mathbb{U}) = -\sum_{u\in\mathcal{A}^\mathbb{U}} F_*^n\mu([u]_\mathbb{U}) \log(F_*^n\mu([u]_\mathbb{U}))$.

To symplify the notation, denote $\mathcal{P}_k = \mathcal{P}_{[0,k-1]}$, thus $h_{F_*^n\mu}(\sigma) = \lim_{k\to\infty} \frac{1}{k} H_{F_*^n\mu}(\mathcal{P}_k)$ corresponds to the metric entropy after n iterations of F. If $\mathbb{U} = [r,s]$ is a neighborhood of F, it is easy to see that $H_{F_*^{n+1}\mu}(\mathcal{P}_{0,k-1}) \le H_{F_*^n\mu}(\mathcal{P}_{[r,s+k-1]})$. One deduces:

Proposition 4.1. *Let $(\mathcal{A}^\mathbb{Z}, F)$ be a CA and $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$. If $n_1 \le n_2$ then $h_{F_*^{n_1}\mu}(\sigma) \ge h_{F_*^{n_2}\mu}(\sigma)$.*

Thus the quantity of information in a random configuration decreases under the action of a CA. It is natural since a CA does not create explicitly information. One deduces that $(h_{F_*^n\mu}(\sigma))_{n\in\mathbb{N}}$ decrease, so it is possible to define the *infinite entropy* of $(\mathcal{A}^\mathbb{Z}, F)$ by:

$$h_\mu^\infty(F) = \lim_{n\to\infty} h_{F_*^n\mu}(\sigma) = \inf_{n\in\mathbb{N}} h_{F_*^n\mu}(\sigma) = \lim_{n\to\infty} \lim_{k\to\infty} \frac{1}{k} H_{F_*^n\mu}(\mathcal{P}_k).$$

One problem of this definition is that at each time, we consider arbitrary long patterns: we take the limit according to the parameter k before the time parameter n. So it is difficult to detect the correlations between paterns of the same length under the action of F. However, if we observe the evolution of a CA, we just look a fixed window of the space-time diagram. Thus, the entropy observed would be naturally defined as the inversion of limits in the formula of $h_\mu^\infty(F)$. That is why B.Martin defines in [Mar00] *the apparent entropy*:

$$h_\mu^a(F) = \lim_{k\to\infty} \limsup_{n\to\infty} \frac{1}{k} H_{F_*^n\mu}(\mathcal{P}_k).$$

In [Mar00], the author exhibits the link between $h_\mu^a(F)$ and the rate of compression by gzip of space-time diagrams where the initial configuration are μ-generic configurations.

4.2. Problem of gap between $h_\mu^\infty(F)$ and $h_\mu^a(F)$

We propose to exhibit links between the different notions of complexity: $h_\mu^\infty(F)$, $h_\mu^a(F)$ and $h_{\text{top}}(\Lambda_F(\mu))$. Theorem 4.2 shows a natural inequality between the different values, in particular it can appear a phenomenon of gap between $h_\mu^\infty(F)$ and $h_\mu^a(F)$. This means that for a CA F and an initial distribution μ, the complexity which is observed in the space-time diagram (i.e. $h_\mu^a(F)$) is different from the expected value (i.e. $h_\mu^\infty(F)$). Theorem 4.3 proves that for each CA there exists a dististribution such that the complexity observed in the space time diagram corresponds to the disorder spreaded by the CA.

Theorem 4.2. *Let $(\mathcal{A}^\mathbb{Z}, F)$ be a CA and $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$, one has:*

$$h_{top}(\Lambda_F) \geq h_{top}(\Lambda_F(\mu)) \geq h_\mu^a(F) \geq \sup_{\nu\in\mathcal{V}} h_\nu(\sigma) \geq \inf_{\nu\in\mathcal{V}} h_\nu(\sigma) \geq h_\mu^\infty(F),$$

where $\mathcal{V}$ is the set of closure points of $(F_*^n\mu)_{n\in\mathbb{N}}$.

Proof. Let k and n in $\mathbb{N}$, one has:

$$\begin{aligned} H_{F_*^n\mu}(\mathcal{P}_k) &= -\sum_{u\in\mathcal{L}_k(\mathcal{A}^\mathbb{Z})} F_*^n\mu([u]_{[0,k-1]}) \log(F_*^n\mu([u]_{[0,k-1]})) \\ &= -\sum_{u\in\mathcal{L}_k(\Lambda_F(\mu))} F_*^n\mu([u]_{[0,k-1]}) \log(F_*^n\mu([u]_{[0,k-1]})) \\ &\qquad - \sum_{u\in\mathcal{L}_k(\mathcal{A}^\mathbb{Z})\setminus\mathcal{L}(\Lambda_F(\mu))} F_*^n\mu([u]_{[0,k-1]}) \log(F_*^n\mu([u]_{[0,k-1]})). \end{aligned}$$

By definition of $\Lambda_F(\mu)$, one has $\sum_{u\in\mathcal{L}_k(\mathcal{A}^\mathbb{Z})\setminus\mathcal{L}(\Lambda_F(\mu))} F_*^n\mu([u]_{[0,k-1]}) \log(F_*^n\mu([u]_{[0,k-1]})) \to 0$ when $n \to 0$. We deduce that $h_{\text{top}}(\Lambda_F(\mu)) \geq h_\mu^a(F)$.

Let $\nu \in \mathcal{V}$ be a limit of a subsequence $(F^{n_i}\mu)_{i\in\mathbb{N}}$ (there exist such subsequences since $(\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z}), d_*^\mathcal{M})$ is compact). By continuity of $\mu \mapsto H_\mu(\mathcal{P}_k)$, one has:

$$\limsup_{n\to\infty} H_{F_*^n\mu}(\mathcal{P}_k) \geq \limsup_{i\to\infty} H_{F^{n_i}\mu}(\mathcal{P}_k) = H_\nu(\mathcal{P}_k).$$

So one obtains $h_\mu^a(F) \geq h_\nu(\sigma)$.

Let $\nu \in \mathcal{V}$ be a limit of a subsequence $(F^{n_i}\mu)_{i\in\mathbb{N}}$, by upper semi-continuity of $\mu \mapsto h_\mu(\sigma)$ (see [DGS76]), one has $h_\nu(X,\sigma) \geq \limsup_{i\to\infty} h_{F^{n_i}\mu}(X,\sigma) = h_\mu^\infty(F)$. ∎

Theorem 4.3. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA. There exists $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ such that:*
$$h^\infty_\mu(F) = h^a_\mu(F) = h_{top}(\Lambda_F(\mu)) = h_{top}(\Lambda(F)).$$

Proof. Let $n \in \mathbb{N}$, since the subshift $F^n(\mathcal{A}^{\mathbb{Z}})$ is intrinsically ergodic (that is to say, there exists an unique σ-ergodic measure of maximal entropy, see [DGS76]), there exists ν_n such that $h_{\nu_n}(\sigma) = h_{\text{top}}(F^n(\mathcal{A}^{\mathbb{Z}}))$. Moreover the operator $F^n_* : \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \to \mathcal{M}_\sigma(F^n(\mathcal{A}^{\mathbb{Z}}))$ defined by $\mu \mapsto F^n_*\mu$ is surjective, thus there exists $\mu_n \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ such that $\nu_n = F^n_*\mu_n$ for all $n \in \mathbb{N}$.

Let μ be a limit of a subsequence $(\mu_{n_i})_{i\in\mathbb{N}}$ of $(\mu_n)_{n\in\mathbb{N}}$. Let $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $h^\infty_\mu(F) + \varepsilon \geq h_{F^n_*\mu}(\sigma)$. Moreover by upper semi-continuity of the entropy, there exists $i_0 \in \mathbb{N}$ such that $h_{F^n_*\mu}(\sigma) \geq h_{F^n_*\mu_{n_i}}(\sigma) - \varepsilon$ for all $i \geq i_0$. So if we choose i such that $n_i \geq n$ one has:
$$h^\infty_\mu(F) + 2\varepsilon \geq h_{F^n_*\mu_{n_i}}(\sigma) \geq h_{F^{n_i}_*\mu_{n_i}}(\sigma) = h_{\text{top}}(F^{n_i}(\mathcal{A}^{\mathbb{Z}})) \geq h_{\text{top}}(\Lambda(F)).$$

The inequality is true for every $\varepsilon > 0$, we deduce that $h^\infty_\mu(F) \geq h_{\text{top}}(\Lambda(F))$. The equality follows by the previous theorem. ∎

Remark 4.4. If $(\mathcal{A}^{\mathbb{Z}}, F)$ is surjective, $\lambda_{\mathcal{A}^{\mathbb{Z}}}$ is a fixed point of F_*, but it is also the unique measure of maximal entropy, so $\Lambda_F(\lambda_{\mathcal{A}^{\mathbb{Z}}}) = \mathcal{A}^{\mathbb{Z}}$. The measure $\lambda_{\mathcal{A}^{\mathbb{Z}}}$ verifies the case of equality in Theorem 4.2.

4.3. Illustration for some class of cellular automata

By Theorem 4.3, for all CA there exists a measure μ such that $h^\infty_\mu(F) = h^a_\mu(F)$. In this subsection we search to link between the dynamic of a cellular automaton and the case of equality in Theorem 4.2.

Proposition 4.5. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be an equicontinuous CA of slope α and $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$. Then there exists $m \in \mathbb{N}$ such that $h^\infty_\mu(F) = h^a_\mu(F) = h_{F^m_*\mu}(\sigma)$.*

Proof. By Theorem 4.3 of [Sab06], there exist a period $p \in \mathbb{N}$ and a preperiod $m \in \mathbb{N}$ such that $\sigma^{\lfloor (m+p)\alpha \rfloor} \circ F^{m+p}(x) = \sigma^{\lfloor m\alpha \rfloor} \circ F^m$. One deduces that $\mathcal{V}$, in Theorem 4.2, is $\{F^{m+n}_*\mu : n \in [0, p-1]\}$. Thus $h^\infty_\mu(F) = h^a_\mu(F)$. ∎

Proposition 4.6. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA which has two blocking word B' and B'' of slope respectively α' and α''. Let $\mu \in \mathcal{M}^{erg}_\sigma(\mathcal{A}^{\mathbb{Z}})$ such that $\mu(B') > 0$ and $\mu(B'') > 0$. Then $h^\infty_\mu(F) = h^a_\mu(F) = h_{top}(\Lambda_F(\mu)) = 0$.*

Proof. Since μ charges two blocking words of different slope, $(\mathcal{A}^{\mathbb{Z}}, F)$ has two directions of μ-almost equicontinuity. According to Theorem 4.8 of [Sab06], one deduces that there exists $\mathcal{A}_\infty \subset \mathcal{A}$ such that $\Lambda_F(\mu) = \{{}^\infty a^\infty : a \in \mathcal{A}_\infty\}$. One deduces the result. ∎

Proposition 4.7. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be an algebraic CA. There is a dense set of measures $\mathcal{H}$ such that for all $\mu \in \mathcal{H}$ one has $h^\infty_\mu(F) \neq h^a_\mu(F) = \log(\text{Card}(\mathcal{A}))$.*

Proof. In [PY02], they show that there is a weak* dense set in $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$, the harmonic measures set denoted $\mathcal{H}$, such that every measure $\mu \in \mathcal{H}$ verifies $\lim_{n\in\mathbb{J}\to\infty} d^{\mathcal{M}}_*(F^n_*\mu, \lambda_{\mathcal{A}^{\mathbb{Z}}}) = 0$, where $\lambda_{\mathcal{A}^{\mathbb{Z}}}$ is the uniform Bernoulli measure and $\mathbb{J}$ is a subset of $\mathbb{N}$ of upper density 1. By Theorem 4.2, one deduces that $h^a_\mu(F) = h_{\lambda_{\mathcal{A}^{\mathbb{Z}}}}(\sigma) = \log(\text{Card}(\mathcal{A}))$. However, if $\mu \neq \lambda_{\mathcal{A}^{\mathbb{Z}}}$, one has $h^\infty_\mu(F) \leq h_\mu(\sigma) < \log(\text{Card}(\mathcal{A}))$. ∎

Thus there is a phenomenon of gap between $h_\mu^\infty(F)$ and $h_\mu^a(F)$. This means that the space-time diagram of algebraic CA looks more complex than the initial configuration. In fact, the combinatory involved by the local rule is so important and it seems to appear information. On the contrary, when $h_\mu^\infty(F) = h_\mu^a(F)$, this means that the CA cannot mix suffisently the informations contained in a μ-random configuration.

References

[BK99] Mike Boyle and Bruce Kitchens. Periodic points for onto cellular automata. *Indag. Math. (N.S.)*, 10(4):483–493, 1999.

[Bru82] A. A. Brudno. Entropy and the complexity of the trajectories of a dynamic system. *Trudy Moskov. Mat. Obshch.*, 44:124–149, 1982.

[BS07] Laurent Bienvenu and Mathieu Sablik. The dynamics of cellular automata in shift-invariant topologies. *In Proc. of Development in Language Theory*, 2007.

[CFMM97] Gianpiero Cattaneo, Enrico Formenti, Luciano Margara, and Jacques Mazoyer. A shift-invariant metric on $S^{\mathbf{Z}}$ inducing a non-trivial topology. In *Mathematical foundations of computer science 1997 (Bratislava)*, volume 1295 of *Lecture Notes in Comput. Sci.*, pages 179–188. Springer, Berlin, 1997.

[DGS76] Manfred Denker, Christian Grillenberger, and Karl Sigmund. *Ergodic theory on compact spaces*. Springer-Verlag, Berlin, 1976. Lecture Notes in Mathematics, Vol. 527.

[FMMN00] Pablo A. Ferrari, Alejandro Maass, Servet Martínez, and Peter Ney. Cesàro mean distribution of group automata starting from measures with summable decay. *Ergodic Theory Dynam. Systems*, 20(6):1657–1670, 2000.

[Hed69] Gustav A. Hedlund. Endormorphisms and automorphisms of the shift dynamical system. *Math. Systems Theory*, 3:320–375, 1969.

[KM00] Petr Kůrka and Alejandro Maass. Limit sets of cellular automata associated to probability measures. *J. Statist. Phys.*, 100(5-6):1031–1047, 2000.

[Kůr97] Petr Kůrka. Languages, equicontinuity and attractors in cellular automata. *Ergodic Theory Dynam. Systems*, 17(2):417–433, 1997.

[Mar00] Bruno Martin. Apparent entropy of cellular automata. *Complex Systems*, 12(2):135–155, 2000.

[PY02] Marcus Pivato and Reem Yassawi. Limit measures for affine cellular automata. *Ergodic Theory Dynam. Systems*, 22(4):1269–1287, 2002.

[PY04] Marcus Pivato and Reem Yassawi. Limit measures for affine cellular automata. II. *Ergodic Theory Dynam. Systems*, 24(6):1961–1980, 2004.

[Sab06] Mathieu Sablik. Directional dynamics for cellular automata: A sensitivity to initial condition approach. to appear in *Theoretical Computer Science*, September 2006.