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Fundamental study

Directional dynamics for cellular automata: A sensitivity to initial condition approach

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ABSTRACT

A cellular automaton is a continuous function F defined on a full-shift $\mathcal{A}^{\mathbb{Z}}$ which commutes with the shift σ. Often, to study the dynamics of F one only considers implicitly σ. However, it is possible to emphasize the spatio-temporal structure produced by considering the dynamics of the $\mathbb{Z} \times \mathbb{N}$-action induced by (σ, F).

In this purpose we study the notion of directional dynamics. In particular, we are interested in directions of equicontinuity and expansivity, which generalize the concepts introduced by Gilman [Robert H. Gilman, Classes of linear automata, Ergodic Theory Dynam. Systems 7 (1) (1987) 105–118] and P. Kůrka [Petr Kůrka, Languages, equicontinuity and attractors in cellular automata, Ergodic Theory Dynam. Systems 17 (2) (1997) 417–433]. We study the sets of directions which exhibit this special kind of dynamics showing that they induce a discrete geometry in space-time diagrams.

0. Introduction

A one-dimensional cellular automaton is a complex system defined by a local rule which acts on a configuration space $\mathcal{A}^{\mathbb{Z}}$ synchronously and uniformly. These simple models exhibit a wide variety of dynamical behaviors. During the last twenty years there have been several attempts to classify them according to their complex behavior. This problem starts with Wolfram [20] who proposed an empiric classification. He suggested four different types of behavior according to the observation of the space-time diagrams produced by the evolution of cellular automata.

Although the work of Wolfram was informal, this classification gave a determining impulse to the study of cellular automata and a lot of authors started to give new formal classifications. From the point of view of topological dynamics, Gilman [9] proposed a classification according to the sensitivity to initial conditions with respect to a Bernoulli measure and Hurley [12] classified according to their attractors. Afterwards, Kůrka [15] refined these two ideas and proposed a third one based on the possible language theoretical properties of codings of orbits in relation to a clopen partition. However, these classifications remain unsatisfactory from the qualitative point of view since the (considered) product topology privileges the central coordinates, while there is no reason to give more importance to coordinates around 0 than others. Thus, simple cellular automata like the powers of the shift map are sensitive to the initial conditions. This is not in accordance with the intuitive idea that appears when observing the extremely regular space-time diagrams of these cellular automata.

Previous classifications only take into account the action of the cellular automaton, without looking explicitly at the shift map denoted by σ. In order to capture the complexity of the structures observed in the space-time diagrams produced by a cellular automaton F, it is natural to study the combined action of F and σ. That is, the $\mathbb{Z} \times \mathbb{N}$-action given by the family of maps $\{\sigma^m \circ F^n : n \in \mathbb{N}, m \in \mathbb{Z}\}$ (in contrast with the $\mathbb{N}$-action defined by F and the $\mathbb{Z}$-action defined by σ). A shortcoming is

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that the $\mathbb{Z}\times\mathbb{N}$-action inherits expansivity from the shift map, independently of F. Thus we must restrict to a sub-semi-group of $\mathbb{Z}\times\mathbb{N}$. In fact, since $\mathbb{Z}\times\mathbb{N}$ can be embedded into $\mathbb{R}^2$, one can define a suspension of the action and consider directional dynamics according to a subspace of $\mathbb{R}^2$. This allows us to study irrational directions.

Inspired by the work of Boyle and Lind [1] on directional expansivity for general $\mathbb{Z}^d$-actions, in this paper we study directional dynamical properties of cellular automata from a topological and measure-theoretical point of view. This study allows us to generalize classifications of [9] and [15] but according to a given direction. The main result of this article is the characterization of the set of equicontinuous directions, directions containing equicontinuous points, expansive directions and μ-almost equicontinuous directions for an invariant measure μ. These results reveal a discrete geometry in the space-time diagrams.

1. Different actions on the configuration space $\mathcal{A}^{\mathbb{Z}}$

1.1. General semi-group actions

Let $\mathbb{M}$ be a semi-group and X a compact metric space. One says $\mathbb{M}$ acts on X if for any $m\in\mathbb{M}$ there is a continuous map $T^m : X\to X$ and $T^{m+n}=T^m\circ T^n$ for all $m,n\in\mathbb{M}$. The pair (X,T) is called a *dynamical system*. If $\mathbb{M}=\mathbb{N}$ or $\mathbb{Z}$ then $T=T^1$ is a generator and one speaks about the $\mathbb{M}$-action T. If $\mathbb{M}=\mathbb{Z}\times\mathbb{N}$ or $\mathbb{Z}\times\mathbb{Z}$, $S=T^{(1,0)}$ and $T=T^{(0,1)}$ are generators and one speaks about the $\mathbb{M}$-action (S,T).

A *morphism* between dynamical systems (X,T) and (X',T') is a continuous function $\pi : X\to X'$ such that $\pi\circ T^m=T'^m\circ\pi$ for all $m\in\mathbb{M}$. If π is surjective (X',T') is a *factor* of (X,T); if π is injective (X,T) is a *sub-system* of (X',T'); if π is bijective (X,T) and (X',T') are *conjugate*.

The following are the main dynamical properties of an action we will need (for more details see [10]). Let d be a metric in X.

- $(X,\mathbb{M})$ is $\mathbb{M}$-*equicontinuous* if for all $\varepsilon>0$ there exists $\delta>0$ such that if $d(x,y)<\delta$ then $d(T^m(x),T^m(y))<\varepsilon$ for all $m\in\mathbb{M}$.
- $x\in X$ is *a* $\mathbb{M}$-*equicontinuous point* if for all $\varepsilon>0$ there exists $\delta>0$ such that for all $y\in X$ with $d(x,y)<\delta$ one has $d(T^m(x),T^m(y))<\varepsilon$ for all $m\in\mathbb{M}$. Denote $Eq^{\mathbb{M}}(X,T)$ the set of $\mathbb{M}$-equicontinuous points of (X,T).
- $(X,\mathbb{M})$ is $\mathbb{M}$-*sensitive* if there exists $\varepsilon>0$ such that for all $x\in X$ and $\delta>0$ there exists $y\in X$ and $m\in\mathbb{M}$ such that $d(x,y)<\delta$ and $d(T^m(x),T^m(y))>\varepsilon$.
- $(X,\mathbb{M})$ is $\mathbb{M}$-*expansive* if there exists $\varepsilon>0$ such that for all $x\neq y$ there exists $m\in\mathbb{M}$ which verifies $d(T^m(x),T^m(y))>\varepsilon$.

Let us remark that in the literature a $\mathbb{N}$-expansive action is frequently called *positively expansive* and a $\mathbb{Z}$-expansive action is called expansive. In this article we prefer to precise the nature of the action.

1.2. The space dimension: $\mathbb{Z}$-action of the shift on $\mathcal{A}^{\mathbb{Z}}$

Configuration space. Let $\mathcal{A}$ be a finite set and $\mathcal{A}^{\mathbb{Z}}$ the *configuration space* of $\mathbb{Z}$-indexed sequences in $\mathcal{A}$. If $\mathcal{A}$ is endowed with the discrete topology, $\mathcal{A}^{\mathbb{Z}}$ is metrizable, compact and totally disconnected in the product topology. A compatible metric is given by:

$$\forall x,y\in\mathcal{A}^{\mathbb{Z}},\quad d_C(x,y)=2^{-\min\{|i|:x_i\neq y_i\ i\in\mathbb{Z}\}}.$$

Consider a not necessarily convex subset $\mathbb{U}\subset\mathbb{Z}$. For $x\in\mathcal{A}^{\mathbb{Z}}$, denote $x_{\mathbb{U}}\in\mathcal{A}^{\mathbb{U}}$ the restriction of x to $\mathbb{U}$. Given $w\in\mathcal{A}^{\mathbb{U}}$, one defines the cylinder centered at w by $[w]_{\mathbb{U}}=\{x\in\mathcal{A}^{\mathbb{Z}} : x_{\mathbb{U}}=w\}$. Denote by $\mathcal{A}^*$ the set of all finite sequences or *finite words* $w=w_0\ldots w_{n-1}$ with letters in $\mathcal{A}$; $|w|=n$ is the *length* of w. When there is no ambiguity, denote $[w]_i=[w]_{[i,i+|w|-1]}$.

Shift action. The *shift* map $\sigma:\mathcal{A}^{\mathbb{Z}}\to\mathcal{A}^{\mathbb{Z}}$ is defined by $\sigma(x)_i=x_{i+1}$ for $x=(x_m)_{m\in\mathbb{Z}}\in\mathcal{A}^{\mathbb{Z}}$ and $i\in\mathbb{Z}$. It is an homeomorphism of $\mathcal{A}^{\mathbb{Z}}$.

A closed and σ-invariant subset Σ of $\mathcal{A}^{\mathbb{Z}}$ is called a *subshift*. For $\mathbb{U}\subset\mathbb{Z}$ denote $\mathcal{L}_\Sigma(\mathbb{U})=\{x_{\mathbb{U}} : x\in\Sigma\}$ the set of patterns centered at $\mathbb{U}$. Since Σ is σ-invariant, it is sufficient to consider the words of length $n\in\mathbb{N}$, so we denote $\mathcal{L}_\Sigma(n)=\{x_{[0,n-1]} : x\in\Sigma\}$. The *language* of a subshift Σ is defined by $\mathcal{L}_\Sigma=\cup_{n\in\mathbb{N}}\mathcal{L}_\Sigma(n)$. By compacity, the language characterizes the subshift.

A subshift $\Sigma\subseteq\mathcal{A}^{\mathbb{Z}}$ is *transitive* if given words $u,v\in\mathcal{L}_\Sigma$ there is $w\in\mathcal{L}_\Sigma$ such that $uwv\in\mathcal{L}_\Sigma$. It is *mixing* if given $u,v\in\mathcal{L}_\Sigma$ there is $N\in\mathbb{N}$ such that $uwv\in\mathcal{L}_\Sigma$ for any $n\geq N$ and some $w\in\mathcal{L}_\Sigma(n)$.

A subshift $\Sigma\subset\mathcal{A}^{\mathbb{Z}}$ is *specified* if there exists $N\in\mathbb{N}$ such that for all $u,v\in\mathcal{L}_\Sigma$ and for all $n\geq N$ there exists a σ-periodic point $x\in\Sigma$ such that $x_{[0,|u|-1]}=u$ and $x_{[n+|u|,n+|u|+|v|-1]}=v$ (see [8] for more details).

A subshift $\Sigma\subset\mathcal{A}^{\mathbb{Z}}$ is *weakly-specified* if there exists $N\in\mathbb{N}$ such that for all $u,v\in\mathcal{L}_\Sigma$ there exist $n\leq N$ and a σ-periodic point $x\in\Sigma$ such that $x_{[0,|u|-1]}=u$ and $x_{[n+|u|,n+|u|+|v|-1]}=v$.

Specification (resp. weakly-specification) implies mixing (resp. transitivity) and density of σ-periodic points. Let Σ be a weakly-specified mixing subshift. By compacity there exists $N\in\mathbb{N}$ such that for any $x,y\in\Sigma$ and $i\in\mathbb{N}$ there exist $w\in\mathcal{L}_\Sigma$, $|w|\leq N$, and $j\in\mathbb{Z}$ such that $x_{(-\infty,i]}w\sigma^j(y)_{[i+|w|,\infty)}\in\Sigma$. If Σ is specified this property is true with $|w|=n$ and $n\geq N$.

Subshifts of finite type and sofic subshifts. A subshift Σ is of *finite type* if there exist a finite subset $\mathbb{U} \subset \mathbb{Z}$ and $\mathcal{F} \subset \mathcal{A}^{\mathbb{U}}$ such that $x \in \Sigma$ if and only if $\sigma^m(x)_{\mathbb{U}} \in \mathcal{F}$ for all $m \in \mathbb{Z}$. The diameter of $\mathbb{U}$ is called an *order* of Σ.

A subshift $\Sigma' \subset \mathcal{B}^{\mathbb{Z}}$ is *sofic* if it is the image of a subshift of finite type $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ by a map $\Pi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$, $\Pi((x_i)_{i\in\mathbb{Z}}) = (\pi(x_i))_{i\in\mathbb{Z}}$, where $\pi : \mathcal{A} \to \mathcal{B}$.

A transitive sofic is weakly-specified and a mixing sofic is specified. For precise statements and proofs about sofic subshifts and subshifts of finite type see [17] or [13].

1.3. The time dimension: $\mathbb{N}$-action of a cellular automaton on $\mathcal{A}^{\mathbb{Z}}$

Cellular automata. A *cellular automaton* (CA) $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is given by a local rule which acts uniformly and synchronously on the configuration space. That is, there are a finite segment or *neighborhood* $\mathbb{U} \subset \mathbb{Z}$ and a *local rule* $\overline{F} : \mathcal{A}^{\mathbb{U}} \to \mathcal{A}$ such that $F(x)_m = \overline{F}((x_{m+u})_{u\in\mathbb{U}})$ for all $x \in \mathcal{A}^{\mathbb{Z}}$ and $m \in \mathbb{Z}$. The *radius* of F is $r(F) = \max\{|u| : u \in \mathbb{U}\}$. By Hedlund's theorem [11], a *cellular automaton* is a pair $(\mathcal{A}^{\mathbb{Z}}, F)$ where $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is a continuous function which commutes with the shift.

Consideration of the past: Bijective CA. When the CA is bijective, since $\mathcal{A}^{\mathbb{Z}}$ is compact, F^{-1} is also a continuous function which commutes with σ. By Hedlund's theorem, $(\mathcal{A}^{\mathbb{Z}}, F^{-1})$ is a CA. We remark that the radius of F^{-1} can be arbitrary large compared to the radius of F. In this case one can study the $\mathbb{Z}$-action F on $\mathcal{A}^{\mathbb{Z}}$ and not only F as an $\mathbb{N}$-action. This means that we can consider positive (future) and negative (past) iterates of a configuration.

1.4. The space-time view

Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA. Since σ and F commute, they generate the $\mathbb{Z} \times \mathbb{N}$-action (σ, F) on $\mathcal{A}^{\mathbb{Z}}$. Similarly, if F is bijective, it is also possible to consider the $\mathbb{Z}\times\mathbb{Z}$-action (σ, F) on $\mathcal{A}^{\mathbb{Z}}$. This point of view is used to study the space-time diagrams produced by cellular automata. Nevertheless, the strong influence of the shift in the dynamics of the action (σ, F) forces us to restrict our study to some sub-semi-groups of $\mathbb{Z} \times \mathbb{N}$.

Let $\mathbb{K} = \mathbb{N}$ or $\mathbb{Z}$ (when $\mathbb{K} = \mathbb{Z}$ the CA is considered to be bijective). Let $\mathbb{M}$ be a sub-semi-group of $\mathbb{Z} \times \mathbb{K}$. If $\mathbb{M}$ itself contains a sub-semi-group of $\mathbb{Z} \times \{0\}$ then the dynamics of the restriction of (σ, F) to $\mathbb{M}$ contains the dynamics of a power of σ which is still meaningful. So we are interested in sub-semi-groups of rank one $\mathbb{M} = (p, q)\mathbb{K}$ where $(p, q) \in \mathbb{Z} \times \mathbb{K} \setminus \{0\}$. In this case it is equivalent to consider a direction $\alpha = \frac{p}{q} \in \mathbb{Q}$ and study the directional dynamics of slope α.

In view of considering the dynamics of a CA in irrational directions, the semi-group $\mathbb{Z} \times \mathbb{K}$ is embedded into $\mathbb{R}^2$. Put $\mathbb{S} = \mathbb{R}^+$ if $\mathbb{K} = \mathbb{N}$ and $\mathbb{S} = \mathbb{R}$ if $\mathbb{K} = \mathbb{Z}$. One defines the suspension of the $\mathbb{Z} \times \mathbb{N}$-action (σ, F) as the $\mathbb{R} \times \mathbb{S}$-action on $\mathcal{A}^{\mathbb{Z}} \times \mathbb{T} \times \mathbb{T}$ (where $\mathbb{T}$ is the 1-dimensional torus) defined for all $(m, n) \in \mathbb{R} \times \mathbb{S}$ by:

$$\begin{array}{llll} T^{(m,n)}: & \mathcal{A}^{\mathbb{Z}} \times \mathbb{T} \times \mathbb{T} & \longrightarrow & \mathcal{A}^{\mathbb{Z}} \times \mathbb{T} \times \mathbb{T} \\ & (x, \beta_1, \beta_2) & \longmapsto & (\sigma^{\lfloor m+\beta_1 \rfloor} \circ F^{\lfloor n+\beta_2 \rfloor}(x), \{m + \beta_1\}, \{n + \beta_2\}) \end{array}$$

where $\lfloor\cdot\rfloor$ and $\{\cdot\}$ are the integer and fractional parts respectively. In the next section we define directional dynamics for every slope $\alpha \in \mathbb{R}$ without using the notion of suspension. However the process is equivalent.

2. Directional sensitivity

In this section we define directional sensitivity to initial conditions. To compare the orbits of close points we consider two points of view:

(i) *A topological point of view:* points are chosen in a subshift $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$, this follows ideas in [15];

(ii) *A measure-theoretical point of view:* points are chosen according to a σ-invariant probability measure μ, this follows ideas in [9].

Let Σ be a subshift of $\mathcal{A}^{\mathbb{Z}}$ and assume $\mathbb{K} = \mathbb{N}$ or $\mathbb{Z}$. Let $x \in \mathcal{A}^{\mathbb{Z}}$, $\alpha \in \mathbb{R}$ and $\varepsilon > 0$. The ball (relative to Σ) centered at $x \in \Sigma$ of radius ε is given by $B_\Sigma(x, \varepsilon) = \{y \in \Sigma : d_C(x, y) < \varepsilon\}$ and the tube of slope α centered at x of width ε is:

$$D^{\alpha}_{\Sigma}(x, \varepsilon, \mathbb{K}) = \{y \in \Sigma : d_C(\sigma^{\lfloor n\alpha \rfloor} \circ F^n(x), \sigma^{\lfloor n\alpha \rfloor} \circ F^n(y)) < \varepsilon, \forall n \in \mathbb{K}\}.$$

One assumes $\mathbb{K} = \mathbb{Z}$ whenever the CA is bijective. If $\Sigma = \mathcal{A}^{\mathbb{Z}}$ one omits in the notation the subscript Σ.

2.1. A topological point of view

2.1.1. Topological definitions

Definition 2.1. Assume $\mathbb{K} = \mathbb{N}$ or $\mathbb{Z}$. Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA, $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ be a subshift and $\alpha \in \mathbb{R}$.

- The set $Eq^{\alpha}_{\mathbb{K}}(\Sigma, F)$ of $(\mathbb{K}, \Sigma)$-*equicontinuous points of slope* α is defined by

$$x \in Eq^{\alpha}_{\mathbb{K}}(\Sigma, F) \iff \forall \varepsilon > 0, \exists \delta > 0, \quad B_\Sigma(x, \delta) \subset D^{\alpha}_{\Sigma}(x, \varepsilon, \mathbb{K}).$$

- $(\mathcal{A}^{\mathbb{Z}}, F)$ *has* $(\mathbb{K}, \Sigma)$-*equicontinuous points of slope* α if $Eq^{\alpha}_{\mathbb{K}}(\Sigma, F) \neq \emptyset$.

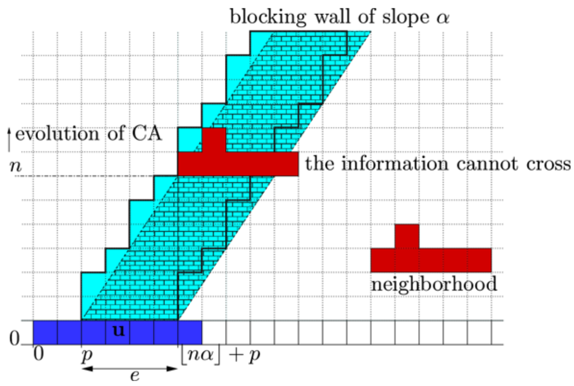

Fig. 1. u is a $\mathbb{N}$-blocking word of slope α.

- $(\mathcal{A}^{\mathbb{Z}}, F)$ is $(\mathbb{K}, \Sigma)$-*equicontinuous of slope* α if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in \Sigma, \quad B_{\Sigma}(x, \delta) \subset D^{\alpha}_{\Sigma}(x, \varepsilon, \mathbb{K}).$$

- $(\mathcal{A}^{\mathbb{Z}}, F)$ is $(\mathbb{K}, \Sigma)$-*sensitive of slope* α if

$$\exists \varepsilon > 0, \forall \delta > 0, \forall x \in \Sigma, \quad \exists y \in B_{\Sigma}(x, \delta) \setminus D^{\alpha}_{\Sigma}(x, \varepsilon, \mathbb{K}).$$

- $(\mathcal{A}^{\mathbb{Z}}, F)$ is $(\mathbb{K}, \Sigma)$-*expansive of slope* α if

$$\exists \varepsilon > 0, \forall x \in \Sigma, \quad D^{\alpha}_{\Sigma}(x, \varepsilon, \mathbb{K}) = \{x\}.$$

Since the domain of a CA is a two sided fullshift, it is possible to break up the concept of expansivity into right-expansivity and left-expansivity. The intuitive idea is that *information* can move by the action of a CA to the right and to the left in a two sided fullshift.

- $(\mathcal{A}^{\mathbb{Z}}, F)$ is $(\mathbb{K}, \Sigma)$-*right-expansive of slope* α if there exists $\varepsilon > 0$ such that $E^{\alpha}_{\Sigma}(x, \varepsilon, \mathbb{K}) \cap E^{\alpha}_{\Sigma}(y, \varepsilon, \mathbb{K}) = \emptyset$ for all $x, y \in \Sigma$ such that $x_{[0,+\infty)} \neq y_{[0,+\infty)}$.
- $(\mathcal{A}^{\mathbb{Z}}, F)$ is $(\mathbb{K}, \Sigma)$-*left-expansive of slope* α if there exists $\varepsilon > 0$ such that $E^{\alpha}_{\Sigma}(x, \varepsilon, \mathbb{K}) \cap E^{\alpha}_{\Sigma}(y, \varepsilon, \mathbb{K}) = \emptyset$ for all $x, y \in \Sigma$ such that $x_{(-\infty,0]} \neq y_{(-\infty,0]}$.

Thus the CA $(\mathcal{A}^{\mathbb{Z}}, F)$ is $(\mathbb{K}, \Sigma)$-expansive of slope α if it is $(\mathbb{K}, \Sigma)$-left-expansive and $(\mathbb{K}, \Sigma)$-right-expansive of slope α.

Remark 2.1. If $\Sigma \subset \Sigma'$ then $\Sigma \cap Eq^{\alpha}_{\mathbb{K}}(\Sigma', F) \subset Eq^{\alpha}_{\mathbb{K}}(\Sigma, F)$. Moreover, one has $Eq^{\alpha}_{\mathbb{Z}}(\Sigma, F) \subset Eq^{\alpha}_{\mathbb{N}}(\Sigma, F)$.

Remark 2.2. To make the concept of expansivity relevant one supposes that Σ is infinite. If not, one needs a finite number of cells to distinguish all points of Σ.

Remark 2.3. For CA on $\mathcal{A}^{\mathbb{N}}$ we just consider right-expansivity of slope α.

2.1.2. Directional equicontinuity and blocking particles

To translate equicontinuity concepts into space-time diagrams properties, we need the notion of a *blocking word* of slope α. The *wall* generated by a blocking word can be interpreted as a particle which has the direction α and kills any information coming from the right or the left. The notion of *particle* is recurrent in the study of CA (for instance see [3]).

Definition 2.2. Assume $\mathbb{K} = \mathbb{N}$ or $\mathbb{Z}$. Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA with neighborhood $\mathbb{U} = [r, s]$ (also of F^{-1} if $\mathbb{K} = \mathbb{Z}$). Let $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ be a subshift, $\alpha \in \mathbb{R}$, $e \in \mathbb{N}$ with $e \geq \max(\lfloor \alpha \rfloor + 1 + s, -\lfloor \alpha \rfloor + 1 - r)$ and $u \in \mathcal{L}_{\Sigma}$ with $|u| \geq e$. The word u is a $(\mathbb{K}, \Sigma)$-*blocking word of slope* α *and width* e if there exists $p \in [0, |u| - e]$ such that:

$$\forall x, y \in [u]_0 \cap \Sigma, \forall n \in \mathbb{K}, \sigma^{\lfloor n\alpha \rfloor} \circ F^n(x)_{[p,p+e-1]} = \sigma^{\lfloor n\alpha \rfloor} \circ F^n(y)_{[p,p+e-1]}.$$

The evolution of a cell $i \in \mathbb{Z}$ depends on the cells $[i + r, i + s]$. Thus, due to condition on e, it is easy to deduce that if u is a $(\mathbb{K}, \Sigma)$-blocking word of slope α and width e, then for all $j \in \mathbb{Z}$, $x, y \in [u]_j \cap \Sigma$ such that $x_{]-\infty,j]} = y_{]-\infty,j]}$ and $n \in \mathbb{K}$ one has $F^n(x)_i = F^n(y)_i$ for $i \leq \lfloor \alpha n \rfloor + p + e + j$. Similarly for all $x, y \in [u]_j \cap \Sigma$ such that $x_{[j,\infty)} = y_{[j,\infty)}$, one has $F^n(x)_i = F^n(y)_i$ for all $i \geq \lfloor \alpha n \rfloor + p$. Intuitively, no information can cross the wall of slope α and width e generated by the $(\mathbb{K}, \Sigma)$-blocking word (see Fig. 1).

The proof of the classification of CA given in [15] can be easily adapted to obtain a characterization of CA which have equicontinuous points of slope α.

Proposition 2.1. *Assume $\mathbb{K} = \mathbb{N}$ or $\mathbb{Z}$. Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA, $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ be a transitive subshift and $\alpha \in \mathbb{R}$. The following properties are equivalent:*

1. $(\mathcal{A}^{\mathbb{Z}}, F)$ *is not* $(\mathbb{K}, \Sigma)$*-sensitive of slope* α*;*
2. $(\mathcal{A}^{\mathbb{Z}}, F)$ *has a* $(\mathbb{K}, \Sigma)$*-blocking word of slope* α*;*
3. $Eq^{\alpha}_{\mathbb{K}}(\Sigma, F) \neq \emptyset$ *is a* σ*-invariant dense* G_δ *set.*

Proof. Let $\mathbb{U} = [r, s]$ be a neighborhood of F (and also of F^{-1} if $\mathbb{K} = \mathbb{Z}$).

(1) $\Rightarrow$ (2) Let $e \geq \max(\lfloor\alpha\rfloor + 1 + s, -\lfloor\alpha\rfloor + 1 - r)$. If $(\mathcal{A}^{\mathbb{Z}}, F)$ is not $(\mathbb{K}, \Sigma)$-sensitive of slope α, then there exist $x \in \Sigma$ and $k, p \in \mathbb{N}$ such that for all $y \in \Sigma$ verifying $x_{[0,k]} = y_{[0,k]}$ one has:

$$\forall n \in \mathbb{K},\ \sigma^{\lfloor n\alpha\rfloor} \circ F^n(x)_{[p,p+e-1]} = \sigma^{\lfloor n\alpha\rfloor} \circ F^n(y)_{[p,p+e-1]}.$$

Thus $x_{[0,k]}$ is a $(\mathbb{K}, \Sigma)$-blocking word of slope α and width e.

(2) $\Rightarrow$ (3) Let u be a $(\mathbb{K}, \Sigma)$-blocking word of slope α. Since (Σ, σ) is transitive, then there exists $x \in \Sigma$ containing an infinitely many occurrences of u in positive and negative coordinates. Let $k \in \mathbb{N}$. There exists $k_1 \geq k$ and $k_2 \geq k$ such that $x_{[-k_1,-k_1+|u|-1]} = x_{[k_2,k_2+|u|-1]} = u$. Since u is a $(\mathbb{K}, \Sigma)$-blocking word of slope α relatively to Σ, for all $y \in \Sigma$ such that $y_{[-k_1,k_2+|u|-1]} = x_{[-k_1,k_2+|u|-1]}$ one has

$$\sigma^{\lfloor n\alpha\rfloor} \circ F^n(x)_{[-k,k]} = \sigma^{\lfloor n\alpha\rfloor} \circ F^n(y)_{[-k,k]} \quad \forall n \in \mathbb{K}.$$

One deduces that $x \in Eq^{\alpha}_{\mathbb{K}}(\Sigma, F)$.

Moreover, since Σ is transitive, the subset of points in Σ containing infinitely many occurrences of u in positive and negative coordinates is a σ-invariant dense G_δ set of Σ.

(3) $\Rightarrow$ (1) Follows directly from definitions. $\square$

Remark 2.4. When Σ is not transitive one can show that any $(\mathbb{K}, \Sigma)$-equicontinuous point of slope α contains a $(\mathbb{K}, \Sigma)$-blocking word of slope α. Reciprocally, a point $x \in \Sigma$ containing infinitely many occurrences of a $(\mathbb{K}, \Sigma)$-blocking word of slope α in positive and negative coordinates is a $(\mathbb{K}, \Sigma)$-equicontinuous point of slope α. However, if Σ is not transitive, the existence of a $(\mathbb{K}, \Sigma)$-blocking word does not imply that one can repeat it infinitely many times.

2.1.3. A directional classification

Thanks to Proposition 2.1 it is possible to establish a classification as in [15], but following a given direction.

Theorem 2.2. *Assume* $\mathbb{K} = \mathbb{N}$ *or* $\mathbb{Z}$*. Let* $(\mathcal{A}^{\mathbb{Z}}, F)$ *be a CA,* $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ *be a transitive subshift and* $\alpha \in \mathbb{R}$*. One of the following cases holds:*

1. $Eq^{\alpha}_{\mathbb{K}}(\Sigma, F) = \Sigma \iff (\mathcal{A}^{\mathbb{Z}}, F)$ *is* $(\mathbb{K}, \Sigma)$*-equicontinuous of slope* α*;*
2. $\emptyset \neq Eq^{\alpha}_{\mathbb{K}}(\Sigma, F) \neq \Sigma \iff (\mathcal{A}^{\mathbb{Z}}, F)$ *is not* $(\mathbb{K}, \Sigma)$*-sensitive of slope* $\alpha \iff (\Sigma, F)$ *has a* $(\mathbb{K}, \Sigma)$*-blocking word of slope* α*;*
3. $(\mathcal{A}^{\mathbb{Z}}, F)$ *is* $(\mathbb{K}, \Sigma)$*-sensitive of slope* α *but is not* $(\mathbb{K}, \Sigma)$*-expansive of slope* α*;*
4. $(\mathcal{A}^{\mathbb{Z}}, F)$ *is* $(\mathbb{K}, \Sigma)$*-expansive of slope* α*.*

Proof. First we prove the first equivalence. From definitions we deduce that if $(\mathcal{A}^{\mathbb{Z}}, F)$ is $(\mathbb{K}, \Sigma)$-equicontinuous of slope α then $Eq^{\alpha}_{\mathbb{K}}(\Sigma, F) = \Sigma$. In the other direction, put $D(x, y) = \sup(\{d_C(\sigma^{\lfloor n\alpha\rfloor} \circ F^n(x), \sigma^{\lfloor n\alpha\rfloor} \circ F^n(y)) : \forall n \in \mathbb{N}\})$ for all $x, y \in \Sigma$. The function $D : \Sigma^2 \to \mathbb{R}$ is a distance and $Eq^{\alpha}_{\mathbb{K}}(\Sigma, F)$ is the set of continuous points of the function $\mathrm{Id} : (\Sigma, d_C) \to (\Sigma, D)$. By compacity, if this function is continuous on Σ, then it is uniformly continuous. One deduces that $(\mathcal{A}^{\mathbb{Z}}, F)$ is $(\mathbb{K}, \Sigma)$-equicontinuous of slope α.

The second equivalence and the classification follow directly from Proposition 2.1. $\square$

2.2. A measure-theoretical point of view

Gilman in [9] defines the notion of μ-almost sensibility to the initial conditions for any Bernoulli measure μ. It is possible to naturally extend this definition for all σ-invariant measures. Furthermore, as in the topological case, the μ-almost sensitivity to initial conditions can be defined for a slope $\alpha \in \mathbb{R}$.

2.2.1. Measure-theoretical definitions

The definition of an equicontinuous point x is that the tube of slope α centered at x has nonempty interior. To adapt this notion to the measurable case, we replace the nonempty interior condition by a positive measure condition. The next lemma justifies this point of view. It was adapted to any probability measure from [9] where only Bernoulli measures are considered.

Denote $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ the set of probability measures on $\mathcal{A}^{\mathbb{Z}}$. Let $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ be the set of σ-invariant probability measures on $\mathcal{A}^{\mathbb{Z}}$ (that is, $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ if $\mu(\sigma^{-1}(B)) = \mu(B)$ for all measurable subsets B of $\mathcal{A}^{\mathbb{Z}}$) and put $\mathcal{M}^{\mathrm{erg}}_\sigma(\mathcal{A}^{\mathbb{Z}})$ the set of σ-ergodic probability measures on $\mathcal{A}^{\mathbb{Z}}$ (a measure is σ-ergodic if all σ-invariant measurable subsets are trivial). Of course $\mathcal{M}^{\mathrm{erg}}_\sigma(\mathcal{A}^{\mathbb{Z}}) \subset \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \subset \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$. A measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ is σ-strongly mixing if for all measurable sets A and B $\mu(A \cap \sigma^{-n}B) \to \mu(A)\mu(B)$ as $n \to +\infty$.

Lemma 2.3. *Let $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ and E be a measurable set with $\mu(E) > 0$. For μ-almost all $x \in E$, one has:*

$$\lim_{n\to\infty} \frac{\mu(E \cap B(x, 2^{-n}))}{\mu(B(x, 2^{-n}))} = 1.$$

Proof. For $m \in \mathbb{N}$, we consider the set:

$$E_m = \left\{x \in E : \liminf_{n\to\infty} \frac{\mu(E \cap B(x, 2^{-n}))}{\mu(B(x, 2^{-n}))} \geq 1 - \frac{1}{m}\right\}.$$

If $\mu(E_m) = \mu(E)$ for all $m \in \mathbb{N}$, the property is shown. Now, assume that there exists $m \in \mathbb{N}$ such that $\mu(E_m) \neq \mu(E)$. Replacing E by $E \setminus E_m$ one can assume that $\mu(E) > 0$ and that for infinitely many $n \in \mathbb{N}$ and all $x \in E$ one has:

$$(\dagger) \qquad \left(1 - \frac{1}{m}\right)\mu(B(x, 2^{-n})) > \mu(E \cap B(x, 2^{-n})).$$

The measure μ is regular, then, since $\mu(E) > 0$, it is possible to assume that E is closed and thus compact. So there exist $j \in \mathbb{Z}$, $k, l \in \mathbb{N}$ and $l + 1$ different words $u_i \in \mathcal{A}^k$ for $i \in [0, l]$ such that the cylinders $[u_i]_j$ cover E and verify:

$$(\ddagger) \qquad \mu(E) > \left(1 - \frac{1}{m}\right)\sum_{i=0}^{l} \mu([u_i]_j).$$

Let $n > \max(|j|, |j + k - 1|)$ such that (†) holds. For all $i \in [0, l]$, it is possible to decompose $[u_i]_j$ as a disjoint union of longer cylinders $[u_i]_j = \cup_h [v_{i,h}]_{-n}$ where $v_{i,h} \in \mathcal{A}^{2n+1}$. Using (†) and (‡) one obtains the following contradiction:

$$\mu(E) > \left(1 - \frac{1}{m}\right)\sum_{i,h} \mu([v_{i,h}]_{-n}) > \mu(E). \quad \square$$

Definition 2.3. Assume $\mathbb{K} = \mathbb{N}$ or $\mathbb{Z}$. Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA and $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$.

- The set $Eq^{\alpha}_{\mathbb{K}}(F, \mu)$ of $(\mathbb{K}, \mu)$-*almost equicontinuous points of slope* α is defined by

$$x \in Eq^{\alpha}_{\mathbb{K}}(F, \mu) \iff \forall \varepsilon > 0, \quad \mu(D^{\alpha}(x, \varepsilon, \mathbb{K})) > 0.$$

- $(\mathcal{A}^{\mathbb{Z}}, F)$ is $(\mathbb{K}, \mu)$-*equicontinuous of slope* α if $\mu(Eq^{\alpha}_{\mathbb{K}}(\mathcal{A}^{\mathbb{Z}}, F)) > 0$;
- $(\mathcal{A}^{\mathbb{Z}}, F)$ is $(\mathbb{K}, \mu)$-*almost equicontinuous of slope* α if $\mu(Eq^{\alpha}_{\mathbb{K}}(F, \mu)) > 0$;
- $(\mathcal{A}^{\mathbb{Z}}, F)$ is $(\mathbb{K}, \mu)$-*almost expansive of slope* α if there exists $\varepsilon > 0$ such that $\mu(D^{\alpha}(x, \varepsilon, \mathbb{K})) = 0$ for all $x \in \mathcal{A}^{\mathbb{Z}}$.

Remark 2.5. It is not necessary to specify the set of definition of the CA since this information is contained in the support of the measure $\mathrm{supp}(\mu)$. Moreover $Eq^{\alpha}_{\mathbb{K}}(F, \mu) \subset \mathrm{supp}(\mu)$.

Remark 2.6. Since $Eq^{\alpha}_{\mathbb{K}}(\Sigma, F)$ and $Eq^{\alpha}_{\mathbb{K}}(F, \mu)$ are σ-invariant, if μ is σ-ergodic and they have positive measure then their measure is one.

Remark 2.7. One has $Eq^{\alpha}_{\mathbb{Z}}(F, \mu) \subset Eq^{\alpha}_{\mathbb{N}}(F, \mu)$.

2.2.2. *Directional almost equicontinuity and almost blocking particle*

As in the topological case, we define the notion of μ-almost blocking wall which allows to see the μ-almost equicontinuity of slope α as a property of the space-time diagrams. In this case, particles are given by μ-almost blocking walls which stop the information μ-almost surely.

Definition 2.4. Assume $\mathbb{K} = \mathbb{N}$ or $\mathbb{Z}$. Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA with neighborhood $\mathbb{U} = [r, s]$ (also F^{-1} if $\mathbb{K} = \mathbb{Z}$) and $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$. Let $\alpha \in \mathbb{R}$, $e \geq \max(\lfloor\alpha\rfloor + 1 + s, -\lfloor\alpha\rfloor + 1 - r)$ and $U = (u_n)_{n\in\mathbb{N}} \in (\mathcal{A}^e)^{\mathbb{N}}$. The sequence U is a $(\mathbb{K}, \mu)$-*almost blocking wall of slope* α *and width* e if:

$$\mu\left(\left\{x \in \mathcal{A}^{\mathbb{Z}} : F^n \circ \sigma^{\lfloor n\alpha\rfloor}(x)_{[0,e-1]} = u_n \; \forall n \in \mathbb{K}\right\}\right) > 0.$$

When U has these properties, denote $W^{\alpha}_{\mathbb{K}}(U, i) = \{x \in \mathcal{A}^{\mathbb{Z}} : F^n \circ \sigma^{\lfloor n\alpha\rfloor}(x)_{[i,i+e-1]} = u_n \; \forall n \in \mathbb{K}\}$ for $i \in \mathbb{Z}$. For all $i \in \mathbb{Z}$, one has $\mu(W^{\alpha}_{\mathbb{K}}(U, i)) > 0$ since μ is σ-invariant and $\sigma^{-i}(W^{\alpha}_{\mathbb{K}}(U, 0)) = W^{\alpha}_{\mathbb{K}}(U, i)$.

For all $i \in \mathbb{Z}$, define:

$$W^{\alpha}_{\mathbb{K}}(U, i)' = \left\{x \in W^{\alpha}_{\mathbb{K}}(U, i) : \lim_{n\to\infty} \frac{\mu(W^{\alpha}_{\mathbb{K}}(U, i + j) \cap B(\sigma^j(x), 2^{-n}))}{\mu(B(\sigma^j(x), 2^{-n}))} = 1 \; \forall j \in \mathbb{Z}\right\}.$$

Since μ is σ-invariant, by Lemma 2.3, one has $\mu(W^{\alpha}_{\mathbb{K}}(U, 0)) = \mu(W^{\alpha}_{\mathbb{K}}(U, 0)')$.

Remark 2.8. If the interior of $W^{\alpha}_{\mathbb{K}}(U, 0)$ is not empty, there exists $u \in \mathcal{A}^*$ and $p \in \mathbb{N}$ such that $[u] \subset W^{\alpha}_{\mathbb{K}}(U, p)$; u is a $\mathbb{K}$-blocking word of $(\mathcal{A}^{\mathbb{Z}}, F)$.

As in the topological case, there exists a characterization of μ-almost equicontinuity by using μ-almost blocking walls.

Proposition 2.4. *Assume $\mathbb{K} = \mathbb{N}$ or $\mathbb{Z}$. Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA, $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ and $\alpha \in \mathbb{R}$. The following properties are equivalent:*

1. *$(\mathcal{A}^{\mathbb{Z}}, F)$ is $(\mathbb{K}, \mu)$-almost equicontinuous of slope α;*
2. *$Eq^{\alpha}_{\mathbb{K}}(F, \mu) \neq \emptyset$;*
3. *for all $e \geq \max(\lfloor\alpha\rfloor + 1 + s, -\lfloor\alpha\rfloor + 1 - r)$ there exists a $(\mathbb{K}, \mu)$-almost blocking wall of slope α and width e;*
4. *there exist $e \geq \max(\lfloor\alpha\rfloor + 1 + s, -\lfloor\alpha\rfloor + 1 - r)$ and a $(\mathbb{K}, \mu)$-almost blocking wall of slope α and width e.*

Proof. It easily follows that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$. We prove that $(4) \Rightarrow (1)$.

Let $U = (u_n)_{n\in\mathbb{N}} \in (\mathcal{A}^e)^{\mathbb{N}}$ be a $(\mathbb{K}, \mu)$-blocking wall of slope $\alpha \in \mathbb{R}$. Let Y be the set of $y \in \mathcal{A}^{\mathbb{Z}}$ whose forward and backward orbits for the shift intersect $W^{\alpha}_{\mathbb{K}}(U, 0)'$ infinitely many times, that is:

$$Y = \bigcap_{n\in\mathbb{N}} \Big(\big(\cup_{i\geq n} W^{\alpha}_{\mathbb{K}}(U, i)' \big) \cap \big(\cup_{i\leq -n} W^{\alpha}_{\mathbb{K}}(U, i)' \big) \Big).$$

By Poincaré's recurrence Theorem $\mu(Y) > 0$. We want to show that $Y \cap \mathrm{supp}(\mu) \subset Eq^{\alpha}_{\mathbb{K}}(F, \mu)$.

Let $y \in Y \cap \mathrm{supp}(\mu)$ and $m \in \mathbb{N}$. Consider $i, j \in \mathbb{N}$ such that $i \leq -m$, $j \geq m - E$ and $y \in W^{\alpha}_{\mathbb{K}}(U, i)' \cap W^{\alpha}_{\mathbb{K}}(U, j)'$. By definition of $W^{\alpha}_{\mathbb{K}}(U, i)'$, there exists $p \geq \max(-i, j)$ such that

$$\mu(W^{\alpha}_{\mathbb{K}}(U, i) \cap B(y, 2^{-p})) \geq \frac{2}{3}\mu(B(y, 2^{-p})) \quad \text{and} \quad \mu(W^{\alpha}_{\mathbb{K}}(U, j) \cap B(y, 2^{-p})) \geq \frac{2}{3}\mu(B(y, 2^{-p})).$$

We deduce that

$$\mu(W^{\alpha}_{\mathbb{K}}(U, i) \cap W^{\alpha}_{\mathbb{K}}(U, j) \cap B(y, 2^{-p})) \geq \frac{1}{3}\mu(B(y, 2^{-p})).$$

Moreover, for all $x \in W^{\alpha}_{\mathbb{K}}(U, i) \cap W^{\alpha}_{\mathbb{K}}(U, j) \cap B(y, 2^{-p})$, it is easy to see that $\sigma^{\lfloor\alpha n\rfloor} \circ F^n(x)_{[i,j+e-1]} = \sigma^{\lfloor\alpha n\rfloor} \circ F^n(y)_{[i,j+e-1]}$ for all $n \in \mathbb{K}$, thus $x \in D^{\alpha}(y, 2^{-m}, \mathbb{K})$. Since $y \in \mathrm{supp}(\mu)$, one has $\mu(D^{\alpha}(y, 2^{-m}, \mathbb{K})) \geq \frac{1}{3}\mu(B(y, 2^{-p})) > 0$. It follows that $y \in Eq^{\alpha}_{\mathbb{K}}(F, \mu)$. □

2.2.3. A directional classification

The next proposition characterizes the μ-almost equicontinuous points of slope α which are not equicontinuous:

Proposition 2.5. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA, $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ and $\alpha \in \mathbb{R}$. A point $x \in Eq^{\alpha}(F, \mu) \setminus Eq^{\alpha}(\mathrm{supp}(\mu), F)$ if and only if $\mu(B(x, \delta)) > \mu(B(x, \delta) \cap D^{\alpha}(x, \varepsilon)) > 0$ for all $\varepsilon > 0$ and $\delta > 0$.*

Proof. Assume there exist $\varepsilon > 0$ and $\delta > 0$ such that $\mu(B(x, \delta)) = \mu(B(x, \delta) \cap D^{\alpha}(x, \varepsilon))$. One has $\mu(B(x, \delta) \setminus D^{\alpha}(x, \varepsilon)) = 0$. However $B(x, \delta) \setminus D^{\alpha}(x, \varepsilon)$ is open. Thus, $B(x, \delta) = D^{\alpha}(x, \varepsilon)$ since $x \in \mathrm{supp}(\mu)$.

The converse follows from the definition. □

As in the topological case, we have a similar classification in the measurable case.

Theorem 2.6. *Assume $\mathbb{K} = \mathbb{N}$ or $\mathbb{Z}$. Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA, $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ and $\alpha \in \mathbb{R}$. One of the following cases hold:*

1. *$(\mathcal{A}^{\mathbb{Z}}, F)$ is $(\mathbb{K}, \mu)$-equicontinuous of slope $\alpha \iff$ there exists u, a $\mathbb{K}$-blocking word of slope α, such that $\mu([u]_0) > 0$;*
2. *$(\mathcal{A}^{\mathbb{Z}}, F)$ is $(\mathbb{K}, \mu)$-almost equicontinuous of slope α but not $(\mathbb{K}, \mu)$-equicontinuous $\iff Eq^{\alpha}_{\mathbb{K}}(F, \mu) \setminus Eq^{\alpha}_{\mathbb{K}}(\mathcal{A}^{\mathbb{Z}}, F) \neq \emptyset \iff$ there exists U, a $(\mathbb{K}, \mu)$-almost blocking wall of slope α, such that the interior of $W^{\alpha}(U, 0)$ is empty;*
3. *$(\mathcal{A}^{\mathbb{Z}}, F)$ is $(\mathbb{K}, \mu)$-almost expansive of slope α.*

Proof. The classification in three classes follows from Proposition 2.4. It is just necessary to prove the first equivalence. One can adapt the proof of [5] to the case of directional dynamics.

Assume that $(\mathcal{A}^{\mathbb{Z}}, F)$ is $(\mathbb{K}, \mu)$-equicontinuous of slope α. Every point of $Eq^{\alpha}_{\mathbb{K}}(\mathcal{A}^{\mathbb{Z}}, F)$ contains a $\mathbb{K}$-blocking word of slope α. Moreover, there are countable many $\mathbb{K}$-blocking words. Since $\mu(Eq^{\alpha}_{\mathbb{K}}(\mathcal{A}^{\mathbb{Z}}, F)) > 0$, one deduces that there exists a $\mathbb{K}$-blocking word u such that $\mu([u]_0) > 0$.

In the other direction, we consider u, a $\mathbb{K}$-blocking word of slope α, such that $\mu([u]_0) > 0$. Define the set:

$$Y = \bigcap_{n\in\mathbb{N}} \Big(\big(\cup_{i\geq n} [u]_i \big) \cap \big(\cup_{i\leq n} [u]_i \big) \Big).$$

By Poincaré's recurrence Theorem $\mu(Y) > 0$. Moreover, by Proposition 2.2, one has $Y \subset Eq^{\alpha}_{\mathbb{K}}(\mathcal{A}^{\mathbb{Z}}, F)$. One deduces that $(\mathcal{A}^{\mathbb{Z}}, F)$ is $(\mathbb{K}, \mu)$-equicontinuous. □

3. Sets of directions with a certain kind of dynamics

When we look to the action induced by a CA in a given direction we obtain similar classifications as P. Kůrka or R. H. Gilman. Thus, this is not revealing of the spatio-temporal behavior of the CA. Our proposition is to consider sets of directions which have a certain kind of dynamical behavior.

3.1. Sets of directions and their relations

Assume $\mathbb{K}=\mathbb{N}$ or $\mathbb{Z}$. Let $(\mathcal{A}^{\mathbb{Z}},F)$ be a CA, Σ be a subshift and $\mu\in\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$.

• Sets with topological equicontinuous properties:

$$\mathbf{A}_{\mathbb{K}}(\Sigma,F)=\{\alpha\in\mathbb{R}:Eq^{\alpha}_{\mathbb{K}}(\Sigma,F)\neq\emptyset\},$$
$$\text{and } \mathbf{A}'_{\mathbb{K}}(\Sigma,F)=\{\alpha\in\mathbb{R}:Eq^{\alpha}_{\mathbb{K}}(\Sigma,F)=\Sigma\}.$$

One has $\mathbf{A}'_{\mathbb{K}}(\Sigma,F)\subset\mathbf{A}_{\mathbb{K}}(\Sigma,F)$.
Moreover, $\mathbf{A}_{\mathbb{Z}}(\Sigma,F)\subset\mathbf{A}_{\mathbb{N}}(\Sigma,F)\cap\mathbf{A}_{\mathbb{N}}(\Sigma,F^{-1})$ and $\mathbf{A}'_{\mathbb{Z}}(\Sigma,F)\subset\mathbf{A}'_{\mathbb{N}}(\Sigma,F)\cap\mathbf{A}'_{\mathbb{N}}(\Sigma,F^{-1})$.

• Sets with topological expansive properties:

$$\mathbf{B}_{\mathbb{K}}(\Sigma,F)=\{\alpha\in\mathbb{R}:(\mathcal{A}^{\mathbb{Z}},F)\text{ is }(\mathbb{K},\Sigma)\text{-expansive of slope }\alpha\},$$
$$\mathbf{B}^r_{\mathbb{K}}(\Sigma,F)=\{\alpha\in\mathbb{R}:(\mathcal{A}^{\mathbb{Z}},F)\text{ is }(\mathbb{K},\Sigma)\text{-right-expansive of slope }\alpha\},$$
$$\text{and }\mathbf{B}^l_{\mathbb{K}}(\Sigma,F)=\{\alpha\in\mathbb{R}:(\mathcal{A}^{\mathbb{Z}},F)\text{ is }(\mathbb{K},\Sigma)\text{-left-expansive of slope }\alpha\}.$$

One has $\mathbf{B}_{\mathbb{K}}(\Sigma,F)=\mathbf{B}^r_{\mathbb{K}}(\Sigma,F)\cap\mathbf{B}^l_{\mathbb{K}}(\Sigma,F)$.
Moreover, $\left(\mathbf{B}^l_{\mathbb{N}}(\Sigma,F)\cap\mathbf{B}^r_{\mathbb{N}}(\Sigma,F^{-1})\right)\cup\left(\mathbf{B}^r_{\mathbb{N}}(\Sigma,F)\cap\mathbf{B}^l_{\mathbb{N}}(\Sigma,F^{-1})\right)\subset\mathbf{B}_{\mathbb{Z}}(\Sigma,F)$.

• Sets with measurable equicontinuous properties:

$$\mathbf{C}_{\mathbb{K}}(F,\mu)=\{\alpha\in\mathbb{R}:(\mathcal{A}^{\mathbb{Z}},F)\text{ is }(\mathbb{K},\mu)\text{-almost equicontinuous}\},$$
$$\text{and }\mathbf{C}'_{\mathbb{K}}(F,\mu)=\{\alpha\in\mathbb{R}:(\mathcal{A}^{\mathbb{Z}},F)\text{is }(\mathbb{K},\mu)\text{-equicontinuous}\}.$$

One has $\mathbf{C}'_{\mathbb{K}}(F,\mu)\subset\mathbf{C}_{\mathbb{K}}(F,\mu)$.
Moreover $\mathbf{C}_{\mathbb{Z}}(F,\mu)\subset\mathbf{C}_{\mathbb{N}}(F,\mu)\cap\mathbf{C}_{\mathbb{N}}(F^{-1},\mu)$ and $\mathbf{C}'_{\mathbb{Z}}(F,\mu)\subset\mathbf{C}'_{\mathbb{N}}(F,\mu)\cap\mathbf{C}'_{\mathbb{N}}(F^{-1},\mu)$.

Remark 3.1. The set of directions which are $(\mathbb{K},\Sigma)$-sensitive is $\mathbb{R}\backslash\mathbf{A}_{\mathbb{K}}(\Sigma,F)$ and the set of directions which are $(\mathbb{K},\mu)$-almost expansive is $\mathbb{R}\setminus\mathbf{C}_{\mathbb{K}}(F,\mu)$. So it is not necessary to study these sets.

The next proposition shows the link between topological and measure-theoretical equicontinuous properties.

Proposition 3.1. *Assume $\mathbb{K}=\mathbb{N}$ or $\mathbb{Z}$. Let $(\mathcal{A}^{\mathbb{Z}},F)$ be a CA and $\mu\in\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$. One has:*

$$\mathbf{C}'_{\mathbb{K}}(F,\mu)\subset\mathbf{A}_{\mathbb{K}}(\mathrm{supp}(\mu),F)\subset\mathbf{C}_{\mathbb{K}}(F,\mu).$$

Proof. For $\alpha\in\mathbb{R}$ one has $Eq^{\alpha}_{\mathbb{K}}(\mathcal{A}^{\mathbb{Z}},F)\cap\mathrm{supp}(\mu)\subset Eq^{\alpha}_{\mathbb{K}}(\mathrm{supp}(\mu),F)$, so

$$\mathbf{C}'_{\mathbb{K}}(F,\mu)\subset\mathbf{A}_{\mathbb{K}}(\mathrm{supp}(\mu),F).$$

Let $\alpha\in\mathbb{R}$, $x\in Eq^{\alpha}_{\mathbb{K}}(\mathrm{supp}(\mu),F)$ and $\varepsilon>0$. There exists $p\in\mathbb{N}$ such that $[x_{[-p,p]}]\cap\mathrm{supp}(\mu)\subset D^{\alpha}_{\mathrm{supp}(\mu)}(x,\varepsilon,\mathbb{K})$. One deduces that $\mu(D^{\alpha}_{\mathrm{supp}(\mu)}(x,\varepsilon,\mathbb{K}))\geq\mu([x_{[-p,p]}])>0$ since $x\in\mathrm{supp}(\mu)$. That is, $x\in Eq^{\alpha}_{\mathbb{K}}(F,\mu)$. Thus $Eq^{\alpha}_{\mathbb{K}}(\mathrm{supp}(\mu),F)\subset Eq^{\alpha}_{\mathbb{K}}(F,\mu)$; and consequently

$$\mathbf{A}_{\mathbb{K}}(\mathrm{supp}(\mu),F)\subset\mathbf{C}_{\mathbb{K}}(F,\mu).\quad\square$$

The next proposition shows the link between expansivity and equicontinuous properties.

Proposition 3.2. *Assume $\mathbb{K}=\mathbb{N}$ or $\mathbb{Z}$. Let $(\mathcal{A}^{\mathbb{Z}},F)$ be a CA, Σ be an infinite subshift and $\mu\in\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ which does not charge atoms (that is $\mu(\{x\})=0$ for all $x\in\mathcal{A}^{\mathbb{Z}}$).*

In the topological case,

$$\left(\mathbf{B}^r_{\mathbb{K}}(\Sigma,F)\cup\mathbf{B}^l_{\mathbb{K}}(\Sigma,F)\right)\cap\mathbf{A}_{\mathbb{K}}(\Sigma,F)=\emptyset.$$

In particular, if $\mathbf{B}_{\mathbb{K}}(\Sigma,F)\neq\emptyset$ then $\mathbf{A}_{\mathbb{K}}(\Sigma,F)=\emptyset$.

In the measure-theoretical case,

$$\left(\mathbf{B}^r_{\mathbb{K}}(\mathrm{supp}(\mu),F)\cup\mathbf{B}^l_{\mathbb{K}}(\mathrm{supp}(\mu),F)\right)\cap\mathbf{C}_{\mathbb{K}}(F,\mu)=\emptyset.$$

In particular, if $\mathbf{B}_{\mathbb{K}}(\mathrm{supp}(\mu),F)\neq\emptyset$ then $\mathbf{C}_{\mathbb{K}}(F,\mu)=\emptyset$.

Proof. Let $(\mathcal{A}^{\mathbb{Z}},F)$ be $(\mathbb{K},\Sigma)$-right expansive of slope α and constant of expansivity ε. One has:

$$D^{\alpha}_{\Sigma}(x,\varepsilon,\mathbb{K})\subset\{y\in\Sigma:y_i=x_i\ \forall i\geq 0\}.$$

Then the interior of $D^{\alpha}_{\Sigma}(x,\varepsilon,\mathbb{K})$ is empty. Thus $Eq^{\alpha}_{\mathbb{K}}(\Sigma,F)=\emptyset$.

Analogously, one proves $\mathbf{B}^l_{\mathbb{K}}(\Sigma,F)\cap\mathbf{A}_{\mathbb{K}}(\Sigma,F)=\emptyset$. In the case $\mathbf{B}_{\mathbb{K}}(\Sigma,F)\neq\emptyset$, one has $\mathbf{B}^r_{\mathbb{K}}(\Sigma,F)\cup\mathbf{B}^l_{\mathbb{K}}(\Sigma,F)=\mathbb{R}$, so $\mathbf{A}_{\mathbb{K}}(\Sigma,F)=\emptyset$.

In the measurable case, $(\mathcal{A}^{\mathbb{Z}},F)$ is $(\mathbb{K},\mathrm{supp}(\mu))$-right expansive with slope α and constant of expansivity ε. Let $x\in\mathrm{supp}(\mu)$. One has:

$$\mu(D^{\alpha}(x,\varepsilon,\mathbb{K}))\leq\mu(\{y\in\mathrm{supp}(\mu):y_i=x_i\ \forall i\geq 0\})=0,$$

since μ does not charge the atom $\{x\}$. One deduces that $x\notin Eq^{\alpha}_{\mathbb{K}}(F,\mu)$. Thus $\mathbf{B}^r_{\mathbb{K}}(\mathrm{supp}(\mu),F)\cap\mathbf{C}_{\mathbb{K}}(F,\mu)=\emptyset$.

Similarly, one deduces $\mathbf{B}^l_{\mathbb{K}}(\mathrm{supp}(\mu),F)\cap\mathbf{C}_{\mathbb{K}}(F,\mu)=\emptyset$. In the case $\mathbf{B}_{\mathbb{K}}(\mathrm{supp}(\mu),F)\neq\emptyset$, one has $\mathbf{B}^r_{\mathbb{K}}(\mathrm{supp}(\mu),F)\cup\mathbf{B}^l_{\mathbb{K}}(\mathrm{supp}(\mu),F)=\mathbb{R}$, so $\mathbf{C}_{\mathbb{K}}(F,\mu)=\emptyset$. $\square$

3.2. Directional dynamics: Factors, inclusions and products

For completeness, we explain how directional dynamics behaves with factor maps, inclusions and products. The proofs are left to the reader.

Suppose $\mathbb{K} = \mathbb{N}$ or $\mathbb{Z}$. Let $(\mathcal{A}^{\mathbb{Z}}, F)$, $(\mathcal{B}^{\mathbb{Z}}, G)$ be Ca and $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$, $\Sigma' \subset \mathcal{B}^{\mathbb{Z}}$ be subshifts. Consider a morphism $\pi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ between the actions (σ, G) and (σ, F) and a measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$.

- If π is surjective and open (send open sets into open sets) and $\pi(\Sigma) = \Sigma'$, then

$$\mathbf{A}'_{\mathbb{K}}(\Sigma, F) \subset \mathbf{A}'_{\mathbb{K}}(\Sigma', G), \qquad \mathbf{A}_{\mathbb{K}}(\Sigma, F) \subset \mathbf{A}_{\mathbb{K}}(\Sigma', G), \quad \text{and} \quad \mathbf{C}_{\mathbb{K}}(F, \mu) \subset \mathbf{C}_{\mathbb{K}}(G, \pi_*\mu).$$

- If π is injective and $\pi(\Sigma) \subseteq \Sigma'$ then:

$$\mathbf{A}'_{\mathbb{K}}(\Sigma', G) \subset \mathbf{A}'_{\mathbb{K}}(\Sigma, F), \qquad \mathbf{A}_{\mathbb{K}}(\Sigma', G) \subset \mathbf{A}_{\mathbb{K}}(\Sigma, F), \qquad \mathbf{C}_{\mathbb{K}}(G, \pi_*\mu) \subset \mathbf{C}_{\mathbb{K}}(F, \mu), \qquad \mathbf{B}_{\mathbb{K}}(\Sigma, F) \subset \mathbf{B}_{\mathbb{K}}(\Sigma', G).$$

- Consider the product CA $((\mathcal{A} \times \mathcal{B})^{\mathbb{Z}}, F \times G)$. One has:

$$\mathbf{A}'_{\mathbb{K}}(\Sigma \times \Sigma', F \times G) = \mathbf{A}'_{\mathbb{K}}(\Sigma, F) \cap \mathbf{A}'_{\mathbb{K}}(\Sigma', G), \qquad \mathbf{A}_{\mathbb{K}}(\Sigma \times \Sigma', F \times G) = \mathbf{A}_{\mathbb{K}}(\Sigma, F) \cap \mathbf{A}_{\mathbb{K}}(\Sigma', G),$$
$$\mathbf{C}_{\mathbb{K}}(F \times G, \mu \times \nu) = \mathbf{C}_{\mathbb{K}}(F, \mu) \cap \mathbf{C}_{\mathbb{K}}(G, \nu), \qquad \mathbf{B}_{\mathbb{K}}(\Sigma \times \Sigma', F \times G) = \mathbf{B}_{\mathbb{K}}(\Sigma, F) \cap \mathbf{B}_{\mathbb{K}}(\Sigma', G).$$

3.3. Some examples

Example 3.1 (*All Directions are Equicontinuous*). Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a nilpotent CA. That is, there exist $y \in \mathcal{A}^{\mathbb{Z}}$ and $n \in \mathbb{N}$ such that $F^n(x) = y$ for all $x \in \mathcal{A}^{\mathbb{Z}}$. Then $\mathbf{A}_{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, F) = \mathbb{R} = \mathbf{A}'_{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, F)$ and $\mathbf{B}^r_{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, F) = \mathbf{B}^l_{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, F) = \emptyset$.

Example 3.2 (*Unique Equicontinuous Integer Direction*). Assume $\mathbb{K} = \mathbb{N}$ or $\mathbb{Z}$. Let $\alpha \in \mathbb{Z}$ and consider the CA $(\mathcal{A}^{\mathbb{Z}}, \sigma^\alpha)$. One proves $\mathbf{A}_{\mathbb{K}}(\mathcal{A}^{\mathbb{Z}}, \sigma^\alpha) = \mathbf{A}'_{\mathbb{K}}(\mathcal{A}^{\mathbb{Z}}, \sigma^\alpha) = \{-\alpha\}$, that is, the CA has a unique equicontinuous direction $-\alpha$. It also holds that $\mathbf{B}^r_{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, \sigma^\alpha) =]-\alpha, +\infty)$, $\mathbf{B}^l_{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, \sigma^\alpha) = (-\infty, -\alpha[$ and $\mathbf{B}_{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, \sigma^\alpha) = \emptyset$, so the CA is right or left $\mathbb{N}$-expansive in all directions different from $-\alpha$. It is not $\mathbb{N}$-expansive but it is $\mathbb{Z}$-expansive and $\mathbf{B}_{\mathbb{Z}}(\mathcal{A}^{\mathbb{Z}}, \sigma^\alpha) = \mathbb{R} \setminus \{-\alpha\}$.

Example 3.3 (*Equicontinuous Points for Directions Contained in a Closed Interval*). Let $\mathcal{A} = \{0, 1\}$ and $[r, s] \subset \mathbb{Z}$. For all $x \in \mathcal{A}^{\mathbb{Z}}$ one define $F(x)_i = x_{r+i} \cdots x_{s+i}$ (the product of the coordinates of $[r, s]$). One remarks that $u = 0$ is a $(\mathbb{N}, \mathcal{A}^{\mathbb{Z}})$-blocking word of slope α, for all $\alpha \in [-s, -r]$. One has $\mathbf{A}_{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, F) = [-s, -r]$. Moreover it is easy to see that $\mathbf{A}'_{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, F) = \emptyset$ and $\mathbf{B}^l_{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, F) = \mathbf{B}^r_{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, F) = \mathbf{B}_{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, F) = \emptyset$.

Example 3.4 (*Open Cone of Expansive Directions*). A CA of neighborhood $\mathbb{U} = [r, s]$ is left-permutative (resp. right-permutative) if for any $u \in \mathcal{A}^{s-r}$ and $b \in \mathcal{A}$ there is a unique $a \in \mathcal{A}$ such that $\overline{F}(au) = b$ (respectively $\overline{F}(ua) = b$). A CA is bipermutative if it is left and right-permutative.

If $(\mathcal{A}^{\mathbb{Z}}, F)$ is left-permutative, then $\mathbf{B}^l_{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, F) = (-\infty, -r[$. If $(\mathcal{A}^{\mathbb{Z}}, F)$ is right-permutative, then $\mathbf{B}^r_{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, F) =]-s, +\infty)$. Thus, if $(\mathcal{A}^{\mathbb{Z}}, F)$ is bipermutative, then $\mathbf{B}_{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, F) =]-s, -r[$.

Example 3.5 (*Just Gliders: μ-almost Equicontinuous Points*). Let $\mathcal{A} = \{-1, 0, 1\}$, $\mathbb{U} = [-1, 1]$ and the local function defined by:

$$\overline{F}(a, b, c) = \begin{cases} 1 & \text{if } a = 1 \text{ and } 2b + c \geq 0, \\ -1 & \text{if } c = -1 \text{ and } a + 2b \leq 0, \\ 0 & \text{in the other cases.} \end{cases}$$

It is possible to interpret space-time diagrams as a background of 0's where there are 1 particles which go to the right and -1 particles which go to the left. Two opposite particles disappear when they collide.

Let $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$. We consider the CA $(\mathcal{A}^{\mathbb{Z}}, F)$. One proves that $\mathbf{A}_{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, F) = \mathbf{A}'_{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, F) = \emptyset$. If $\mu([1]_0) > 0$ then $\mathbf{C}_{\mathbb{N}}(F, \mu) = \{1\}$ and if $\mu([-1]_0) > 0$ then $\mathbf{C}_{\mathbb{N}}(F, \mu) = \{-1\}$.

Example 3.6. Let $\mathcal{A} = \{0, M, R, L, I\}$. Consider $(\mathcal{A}^{\mathbb{Z}}, F)$ be the CA defined by $\overline{F}(abc) = L$ if $c \in \{L, I\}$ and $a, b \in \mathcal{A} \setminus \{R\}$, $\overline{F}(abc) = R$ if $a \in \{D, I\}$ and $b, c \in \mathcal{A} \setminus \{G\}$, $\overline{F}(abc) = I$ if ($a \in \{R, I\}$ and b or $c \in \{L, I\}$) or ($c \in \{L, I\}$ and a or $b \in \{R, I\}$), $\overline{abc} = 0$. In this example there is a "background" of 0 where there are particles M, which act as walls, and there particles L and R which go to the left and to the right respectively, they rebound on the walls and intersect as I. This is a classical example of bijective CA with equicontinous points [16].

We can verify that $\mathbf{B}^l_{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, F) =]-\infty, -1[$ and $\mathbf{B}^r_{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, F) =]1, +\infty[$ so $\mathbf{B}_{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, F) = \emptyset$, but $\mathbf{B}_{\mathbb{Z}}(\mathcal{A}^{\mathbb{Z}}, F) =]-\infty, -1[\cup]1, \infty[$. For the equicontinuity, one has $\mathbf{A}_{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, F) = \mathbf{A}_{\mathbb{Z}}(\mathcal{A}^{\mathbb{Z}}, F) = \{0\}$ and $\mathbf{A}'_{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, F) = \mathbf{A}'_{\mathbb{Z}}(\mathcal{A}^{\mathbb{Z}}, F) = \emptyset$.

Example 3.7 (*All Directions are Sensitive and $\mathbb{N}$-Expansive Direction do not Exist*). The relations given in Section 3.2 allow to say that $\mathbf{A}'_{\mathbb{N}}((\mathcal{A} \times \mathcal{A})^{\mathbb{Z}}, \sigma \times \sigma^{-1}) = \mathbf{A}_{\mathbb{N}}((\mathcal{A} \times \mathcal{A})^{\mathbb{Z}}, \sigma \times \sigma^{-1}) = \mathbf{B}_{\mathbb{N}}((\mathcal{A} \times \mathcal{A})^{\mathbb{Z}}, \sigma \times \sigma^{-1}) = \emptyset$.

3.4. Multiplication and addition

Multiplication by a rational. Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA of neighborhood $\mathbb{U} = [r, s]$. We want to construct a CA where the set of directions which have a certain kind of dynamics is the same as $(\mathcal{A}^{\mathbb{Z}}, F)$ multiplied by a rational $\alpha = \frac{p}{q}$.

Let $\mathcal{A}_\alpha = \mathcal{A}_1 \times \cdots \times \mathcal{A}_q$ where $\mathcal{A}_i = \mathcal{A}$ for all $i \in [1, q]$. One defines the CA $(\mathcal{A}_\alpha^{\mathbb{Z}}, F_\alpha)$ of neighborhood $\mathbb{U}_\alpha = [pr, ps]$ by the local rule:

$$\begin{array}{rccc} \overline{F_\alpha}: & (\mathcal{A}_1 \times \ldots \times \mathcal{A}_q)^{[pr,ps]} & \longrightarrow & \mathcal{A}_1 \times \ldots \times \mathcal{A}_q \\ & (a_r^1, \ldots, a_r^q), \ldots, (a_s^1, \ldots, a_s^q) & \longmapsto & (\overline{F^p}(a_{pr}^q, \ldots, a_{ps}^q), a_0^1, \ldots, a_0^{q-1}). \end{array}$$

It is easy to verify that

$$\mathbf{A}'_{\mathbb{K}}(\mathcal{A}_\alpha^{\mathbb{Z}}, F_\alpha) = \alpha \mathbf{A}'_{\mathbb{K}}(\mathcal{A}^{\mathbb{Z}}, F), \qquad \mathbf{A}_{\mathbb{K}}(\mathcal{A}_\alpha^{\mathbb{Z}}, F_\alpha) = \alpha \mathbf{A}_{\mathbb{K}}(\mathcal{A}^{\mathbb{Z}}, F) \quad \text{and} \quad \mathbf{B}(\mathcal{A}_\alpha^{\mathbb{Z}}, F_\alpha) = \alpha \mathbf{B}(\mathcal{A}^{\mathbb{Z}}, F).$$

Moreover

$$\mathbf{C}_{\mathbb{K}}(F_\alpha, \mu \times \ldots \times \mu) = \alpha \mathbf{C}_{\mathbb{K}}(F, \mu) \quad \text{and} \quad \mathbf{C}'_{\mathbb{K}}(F_\alpha, \mu \times \ldots \times \mu) = \alpha \mathbf{C}'_{\mathbb{K}}(F, \mu).$$

Thus if $F = \sigma^l$ with $l \in \mathbb{N}$, the associated CA $(\mathcal{A}_\alpha^{\mathbb{Z}}, F_\alpha)$ verifies $\mathbf{A}'_{\mathbb{K}}(\mathcal{A}_\alpha^{\mathbb{Z}}, F_\alpha) = \{\alpha\}$.

In the same way, if $(\mathcal{A}^{\mathbb{Z}}, F)$ is the CA defined in the Example 3.3, the associated CA $(\mathcal{A}_\alpha^{\mathbb{Z}}, F_\alpha)$ verifies $\mathbf{A}_{\mathbb{K}}(\mathcal{A}_\alpha^{\mathbb{Z}}, F_\alpha) = [-\alpha s, -\alpha r]$.

To finish, if $(\mathcal{A}^{\mathbb{Z}}, F)$ is bipermutative of neighborhood $[r, s]$ (Example 3.4), the associated CA $(\mathcal{B}^{\mathbb{Z}}, G)$ verifies $\mathbf{B}(\mathcal{A}^{\mathbb{Z}}, G) =]-\alpha s, -\alpha r[$.

Addition of a rational. Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA and $\alpha = \frac{p}{q} \in \mathbb{Q}$. Consider the CA $(\mathcal{A}_\alpha, \widetilde{F})$ where $\widetilde{F} = F \times \ldots \times F$. One verifies that $\mathbf{A}'_{\mathbb{K}}(\mathcal{A}^{\mathbb{Z}}, \widetilde{F} \circ \sigma_\alpha) = \mathbf{A}'_{\mathbb{K}}(\mathcal{A}^{\mathbb{Z}}, F) - \alpha$, $\mathbf{A}_{\mathbb{K}}(\mathcal{A}^{\mathbb{Z}}, \widetilde{F} \circ \sigma_\alpha) = \mathbf{A}_{\mathbb{K}}(\mathcal{A}^{\mathbb{Z}}, F) - \alpha$, $\mathbf{C}_{\mathbb{K}}(\mathcal{A}^{\mathbb{Z}}, \widetilde{F} \circ \sigma_\alpha) = \mathbf{C}_{\mathbb{K}}(\mathcal{A}^{\mathbb{Z}}, F) - \alpha$ and $\mathbf{B}(\mathcal{A}^{\mathbb{Z}}, \widetilde{F} \circ \sigma_\alpha) = \mathbf{B}(\mathcal{A}^{\mathbb{Z}}, F) - \alpha$.

4. Sets of directions with equicontinuous properties

If a CA has a direction verifying an equicontinuous property, then this direction is delimited by blocking walls which indicate the propagation of information. If there exist two directions verifying equicontinuous properties, we are going to see that all the propagation of information disappears and the CA evolves towards a trivial configuration. This phenomenon appears when we characterize $\mathbf{A}_{\mathbb{K}}(\Sigma, F)$, $\mathbf{A}'_{\mathbb{K}}(\Sigma, F)$ and $\mathbf{C}_{\mathbb{K}}(F, \mu)$.

4.1. Directions with equicontinuous points

Definition 4.1. Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA and Σ be a subshift. $(\mathcal{A}^{\mathbb{Z}}, F)$ is *Σ-nilpotent* if the *Σ-limit set* defined by

$$\Lambda_\Sigma(F) = \cap_{n \in \mathbb{N}} \overline{\cup_{m \geq n} F^m(\Sigma)},$$

is finite. By compacity, there exists $n \in \mathbb{N}$ such that $F^n(\Sigma) = \Lambda_F(\Sigma)$.

We observe that in general Σ is not F-invariant.

We are interested in the directions containing equicontinuous points.

Theorem 4.1. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA of neighborhood $\mathbb{U} = [r, s]$ and $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ be a weakly-specified subshift. One of the following cases holds:*

- *$\mathbf{A}_{\mathbb{N}}(\Sigma, F) = \mathbb{R}$, which is equivalent to $(\mathcal{A}^{\mathbb{Z}}, F)$ is Σ-nilpotent;*
- *there exist $\alpha', \alpha'' \in [-s, -r]$, with $\alpha' < \alpha''$, such that $]\alpha', \alpha''[\subset \mathbf{A}_{\mathbb{N}}(\Sigma, F) \subset [\alpha', \alpha''] \subset [-s, -r]$;*
- *there exists $\alpha \in [-s, -r]$ such that $\mathbf{A}_{\mathbb{N}}(\Sigma, F) = \{\alpha\}$;*
- *$\mathbf{A}_{\mathbb{N}}(\Sigma, F) = \emptyset$.*

Proof. CLAIM 1: *$\mathbf{A}_{\mathbb{N}}(\Sigma, F)$ is convex (we just need the transitivity of Σ).*

Proof: Assume there exist $\alpha' < \alpha''$ such that $Eq_{\mathbb{N}}^{\alpha'}(\Sigma, F) \neq \emptyset$ and $Eq_{\mathbb{N}}^{\alpha''}(\Sigma, F) \neq \emptyset$. We can consider two $(\mathbb{N}, \Sigma)$-blocking words u' and u'' of slope α' and α'' respectively. So there exist $e', e'' \geq \max(\lfloor \alpha'' \rfloor + 1 + s, -\lfloor \alpha' \rfloor + 1 - r)$, $p' \in [0, |u'| - e']$ and $p'' \in [0, |u''| - e'']$ such that for all $x', y' \in [u']_0 \cap \Sigma$, for all $x'', y'' \in [u'']_0 \cap \Sigma$ and for all $n \in \mathbb{N}$:

$$\begin{aligned} \sigma^{\lfloor n\alpha' \rfloor} \circ F^n(x')_{[p', p'+e'-1]} &= \sigma^{\lfloor n\alpha' \rfloor} \circ F^n(y')_{[p', p'+e'-1]} \\ \text{and} \quad \sigma^{\lfloor n\alpha'' \rfloor} \circ F^n(x'')_{[p'', p''+e''-1]} &= \sigma^{\lfloor n\alpha'' \rfloor} \circ F^n(y'')_{[p'', p''+e''-1]}. \end{aligned}$$

Since Σ is transitive, there exists $w \in \mathcal{L}_\Sigma$ such that $u = u'wu'' \in \mathcal{L}_\Sigma$. For all $x, y \in [u]_0 \cap \Sigma$ and for all $n \in \mathbb{N}$ one has:

$$F^n(x)_{[p'+\lfloor \alpha' n \rfloor, |u'|+p''+e''-1+\lfloor \alpha'' n \rfloor]} = F^n(y)_{[p'+\lfloor \alpha' n \rfloor, |u'|+p''+e''-1+\lfloor \alpha'' n \rfloor]}.$$

This implies that u is a $(\mathbb{N}, \Sigma)$-blocking word of slope α for all $\alpha \in [\alpha', \alpha'']$. ◇ Claim 1

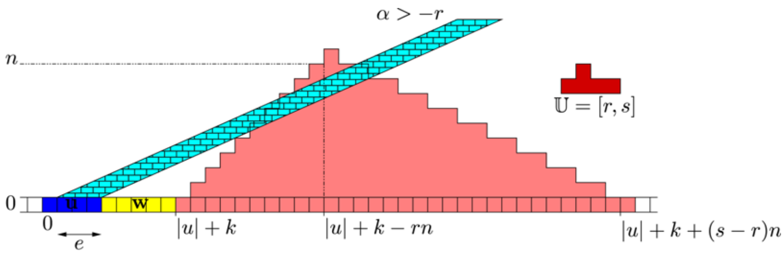

Fig. 2. $\mathbf{A}_{\mathbb{N}}(\Sigma, F) \subset [-s, -r]$.

Claim 2: *If $(\mathcal{A}^{\mathbb{Z}}, F)$ is not Σ-nilpotent then $\mathbf{A}_{\mathbb{N}}(\Sigma, F) \subset [-s, -r]$.*

Proof: Let u be a $(\mathbb{N}, \Sigma)$-blocking word of slope $\alpha > -r$ and width e. There exists $p \in [0, |u| - e]$ such that

$$\forall n \in \mathbb{N}, \forall x, y \in [u]_0 \cap \Sigma, F^n(x)_{[\lfloor n\alpha \rfloor + p, \lfloor n\alpha \rfloor + p + e - 1]} = F^n(y)_{[\lfloor n\alpha \rfloor + p, \lfloor n\alpha \rfloor + p + e - 1]}.$$

Let $z \in \Sigma \cap [u]_0$ be a σ-periodic configuration. The sequence $(F^n(z))_{n \in \mathbb{N}}$ is ultimately periodic of preperiod m and period p. Denote Σ' the subshift generated by $(F^n(z))_{n \in [m, m+p-1]}$, Σ' is finite since $F^n(z)$ is a σ-periodic configuration for all $n \in \mathbb{N}$. Let q be the order of the subshift of finite type Σ'.

Since Σ is a weakly-specified subshift, there exists $N \in \mathbb{N}$ such that for all $u, v \in \mathcal{L}_\Sigma$ there exist $k \leq N$ and $x \in \Sigma$ a σ-periodic point such that $x_{[0, |u|-1]} = u$ and $x_{[k+|u|, k+|u|+|v|-1]} = v$. Let $n \in \mathbb{N}$ such that $|u| + N - rn + q \leq \alpha n + q$ (it is possible since $\alpha > -r$), we want to prove that $F^n(\Sigma) \subset \Sigma'$.

The set $[rn, sn]$ is a neighborhood of $(\mathcal{A}^{\mathbb{Z}}, F^n)$. Let $v \in \mathcal{L}_\Sigma((s - r)n + q)$. There exist $x \in \Sigma$ and $k \leq N$, such that $x_{(-\infty, |u|-1]} = z_{(-\infty, |u|-1]}$ and $x_{[|u|+k, |u|+k+|v|-1]} = v$. Since u is a $(\mathbb{N}, \Sigma)$-blocking word of slope α, the choice of n implies that $F^n(x)_{[|u|+N-rn, |u|+N-rn+q-1]} = F^n(z)_{[|u|+N-rn, |u|+N-rn+q-1]}$ (see Fig. 2). One deduces that the image of the function $\overline{F^n} : \mathcal{L}_\Sigma([rn, sn + q]) \to \mathcal{A}^q$ is contained in $\mathcal{L}_{\Sigma'}(q)$. One deduces that $F^n(\Sigma) \subset \Sigma'$ so $(\mathcal{A}^{\mathbb{Z}}, F)$ is Σ-nilpotent.

The same proof holds for $\alpha < -s$. ◇ Claim 2

Since $\mathbf{A}_{\mathbb{N}}(\Sigma, F)$ is convex, if $\mathbf{A}_{\mathbb{N}}(\Sigma, F) \neq \emptyset$ then it is a segment of $\mathbb{R}$. If $(\mathcal{A}^{\mathbb{Z}}, F)$ is not Σ-nilpotent, the conditions on the boundaries of $\mathbf{A}_{\mathbb{N}}(\Sigma, F)$ follow from Claim 2. □

Remark 4.1. If moreover Σ is specified, the same proof shows that there exists $\mathcal{A}_\infty \subset \mathcal{A}$ such that $\Lambda_F(\Sigma) = \{^\infty a^\infty : a \in \mathcal{A}_\infty\}$.

Example 4.1. Consider $(\{0, 1\}^{\mathbb{Z}}, \sigma)$ on $\Sigma = \{\delta_{^\infty(01)^\infty}, \delta_{^\infty(10)^\infty}\}$. Σ is a transitive subshift of finite type (so it is weakly-specified) which is not mixing. One has $\mathbf{A}_{\mathbb{K}}(\Sigma, F) = \mathbf{A}'_{\mathbb{K}}(\Sigma, F) = \mathbb{R}$ and $\Lambda_\sigma(\Sigma) = \Sigma$ which does not contain σ-uniform configuration.

Example 4.2. Consider $(\{0, 1\}^{\mathcal{A}^{\mathbb{Z}}}, F)$ such that $F(x)_i = x_{i-1} \cdot x_i \cdot x_{i+1}$. Let Σ the subshift such that $\mathcal{L}_\Sigma \cap \{0^m 1^n : n \geq \log_2(m)\} \cap \{1^n 0^m : n \geq \log_2(m)\} = \emptyset$. Σ is a transitive F-invariant subshift and according the relation verifies by Σ, the word 100001 is blocking of slope 2. However $\Lambda_F(\Sigma) = \{x \in \{0, 1\}^{\mathbb{Z}} : 10^n 1 \text{ is not a subword of } x\}$ is infinite.

Corollary 4.2. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a bijective CA and $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ be a mixing sofic subshift. Denote $\mathbb{U} = [r, s]$ a neighborhood of F and $\mathbb{U}' = [r', s']$ a neighborhood of F^{-1}. One of the following cases holds:*

- *there exists $\alpha \in [\max(-s, -s'), \min(-r, -r')]$ such that $\mathbf{A}_{\mathbb{Z}}(\Sigma, F) = \{\alpha\}$;*
- *$\mathbf{A}_{\mathbb{Z}}(\Sigma, F) = \emptyset$.*

Proof. One has $\mathbf{A}_{\mathbb{Z}}(\Sigma, F) \subset \mathbf{A}_{\mathbb{N}}(\Sigma, F) \cap \mathbf{A}_{\mathbb{N}}(\Sigma, F^{-1})$, so we can apply Theorem 4.1 to characterize $\mathbf{A}_{\mathbb{Z}}(\Sigma, F)$. Since F is bijective, if Σ is infinite, it is impossible that $\mathbf{A}_{\mathbb{N}}(\Sigma, F)$ contains two different directions with equicontinuous points. Thus the two first cases of Theorem 4.1 are eliminated. □

4.2. Directions of equicontinuity

Theorem 4.3. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA of neighborhood $\mathbb{U} = [r, s]$ and $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ be a weakly-specifed subshift. One of the following cases holds:*

1. *$\mathbf{A}'_{\mathbb{N}}(\Sigma, F) = \mathbb{R}$, in this case $(\mathcal{A}^{\mathbb{Z}}, F)$ is Σ-nilpotent;*
2. *there exists $\alpha \in [-s, -r] \cap \mathbb{Q}$ such that $\mathbf{A}'_{\mathbb{N}}(\Sigma, F) = \{\alpha\}$, in this case there exist a preperiod $m \geq 0$ and a period $p > 0$ such that $\sigma^{\lfloor (m+p)\alpha \rfloor} \circ F^{m+p} = \sigma^{\lfloor m\alpha \rfloor} \circ F^m$;*
3. *$\mathbf{A}'_{\mathbb{N}}(\Sigma, F) = \emptyset$.*

Proof. Claim 1: *$\mathbf{A}'_{\mathbb{N}}(\Sigma, F)$ contains two distinct directions if and only if $(\mathcal{A}^{\mathbb{Z}}, F)$ is Σ-nilpotent.*

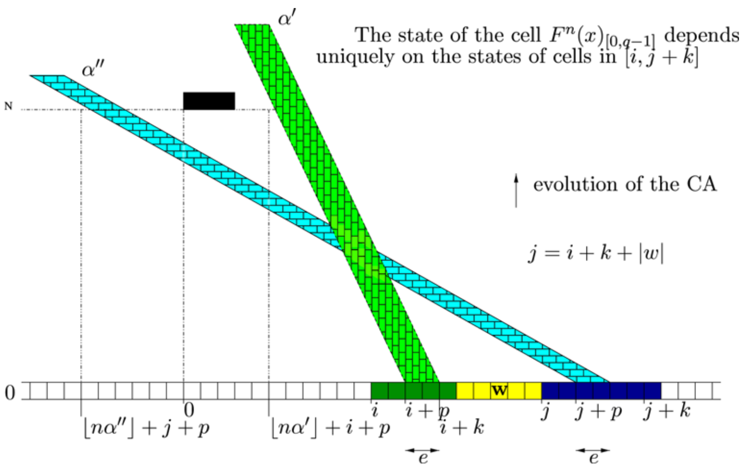

Fig. 3. $\mathbb{N}$-Equicontinuity of slope α' and α'' and nilpotent CA.

Proof: If $(\mathcal{A}^{\mathbb{Z}}, F)$ is Σ-nilpotent, one obtains a σ-periodic configuration after finitely many steps. One deduces that $Eq^{\alpha}_{\mathbb{N}}(\Sigma, F) = \Sigma$ for all $\alpha \in \mathbb{R}$.

Assume that $(\mathcal{A}^{\mathbb{Z}}, F)$ is $(\mathbb{N}, \Sigma)$-equicontinuous of slope α' and α'' with $\alpha' > \alpha''$. By definition, there exist $e \geq \max(\lfloor \alpha' \rfloor + 1 + s, -\lfloor \alpha'' \rfloor + 1 - r)$ and $k, p \in \mathbb{N}$ such that for all $x \in \Sigma$, for all $i \in \mathbb{Z}$, $n \in \mathbb{N}$ and $z \in [x_{[i,i+k]}]_i \cap \Sigma$ one has:

$$\begin{aligned} \sigma^{\lfloor \alpha' n \rfloor} \circ F^n(x)_{[i+p,i+p+e-1]} &= \sigma^{\lfloor \alpha' n \rfloor} \circ F^n(z)_{[i+p,i+p+e-1]} \\ \text{and} \quad \sigma^{\lfloor \alpha'' n \rfloor} \circ F^n(x)_{[i+p,i+p+e-1]} &= \sigma^{\lfloor \alpha'' n \rfloor} \circ F^n(z)_{[i+p,i+p+e-1]}. \end{aligned}$$

Let $z \in \Sigma$ be a σ-periodic configuration. The sequence $(F^n(z))_{n \in \mathbb{N}}$ is ultimately periodic of preperiod m and period p. Denote Σ' the subshift generated by $(F^n(z))_{n \in [m,m+p-1]}$, Σ' is finite since $F^n(z)$ is a σ-periodic configuration for all $n \in \mathbb{N}$. Let q be the order of the subshift of finite type Σ'.

Since Σ is a weakly-specified subshift, there exists $N \in \mathbb{N}$ such that for all $x, y \in \Sigma$, for all $i \in \mathbb{Z}$, there exists $w \in \mathcal{L}_{\Sigma}$, $|w| \leq N$, and $j \in \mathbb{Z}$ such that $x_{(-\infty,i[} w \sigma^j(y)_{[i+|w|,+\infty)} \in \Sigma$.

Let $n \in \mathbb{N}$ and $i \in \mathbb{Z}$ be such that $\lfloor n\alpha'' \rfloor + i + k + p + N \leq 0 \leq q \leq \lfloor n\alpha' \rfloor + i + p$; such integers exist because $\alpha' > \alpha''$. Let $x \in \Sigma$, there exist $w \in \mathcal{A}^*$ such that $|w| \leq N$, $j \in \mathbb{Z}$ and $y = x_{[-\infty,i+k]} w \sigma^j(z)_{[i+k+1+|w|,\infty]}$. Since $y \in [x_{[i,i+k]}]_i$, the states $F^n(y)_{[0,q-1]}$ do not depend of coordinates of y larger than $i + k$ (see Fig. 3). One deduces that $F^n(x)_{[0,q-1]} = F^n(y)_{[0,q-1]}$. Analogously in the other side, one obtains that $F^n(\sigma^j(z))_{[0,q-1]} = F^n(y)_{[0,q-1]}$. Thus $F^n(x)_{[0,q-1]} = F^n(\sigma^j(z))_{[0,q-1]}$. Since x is arbitrary, one deduces that $F^n(\Sigma) \subset \Sigma'$, so $(\mathcal{A}^{\mathbb{Z}}, F)$ is Σ-nilpotent.

Thus, if $\mathbf{A}'_{\mathbb{N}}(\Sigma, F) \neq \emptyset$ and $(\mathcal{A}^{\mathbb{Z}}, F)$ is not Σ-nilpotent, the set $\mathbf{A}'_{\mathbb{N}}(\Sigma, F)$ is reduced to $\{\alpha\}$. By Theorem 4.1, one has $\alpha \in [-s, -r]$. The claim follows.

◇ Claim 1

Claim 2: *If $\mathbf{A}'_{\mathbb{N}}(\Sigma, F) = \{\alpha\}$ then $\alpha \in \mathbb{Q}$ and $(\sigma^{\lfloor \alpha n \rfloor} \circ F^n)_n$ is ultimately periodic.*

Proof: By definition, there exist $k, p \in \mathbb{N}$ such that for all $x \in \Sigma$, for all $n \in \mathbb{N}$ and $z \in [x_{[-p,k]}]_i \cap \Sigma$ one has:

$$\sigma^{\lfloor \alpha n \rfloor} \circ F^n(x)_0 = \sigma^{\lfloor \alpha n \rfloor} \circ F^n(z)_0.$$

Thus the sequence $(\sigma^{\lfloor \alpha n \rfloor} \circ F^n(x)_0)_{n \in \mathbb{N}}$ is uniquely determined by the knowledge of $x_{[-p,k]}$. For all $n \in \mathbb{N}$, put $p_n = \lfloor \alpha n \rfloor$ and consider the function

$$\begin{aligned} f_n : \quad \mathcal{L}_{\Sigma}([-p,k]) &\longrightarrow \quad \mathcal{A} \\ u \quad &\longmapsto \quad F^n(x)_0 \text{ where } x \in [u]_{[p_n-p,p_n+k]} \cap \Sigma. \end{aligned}$$

Since there exist a finite number of functions from $\mathcal{L}_{\Sigma}([-p,k])$ to $\mathcal{A}$, we deduce that there exist $n_1 < n_2$ such that $f_{n_1} = f_{n_2}$. We want to prove that $f_{n_1+1} = f_{n_2+1}$ and $p_{n_1+1} - p_{n_1} = p_{n_2+1} - p_{n_2}$.

For all $x \in \Sigma$,

$$F^{n_1+1}(x)_0 = f_{n_1+1}(x_{[p_{n_1+1}-p,p_{n_1+1}+k]}) \underset{\star}{=} \overline{F}((f_{n_1}(x_{[p_{n_1}-p+u,p_{n_1}+k+u]}))_{u \in \mathbb{U}}),$$

where $\mathbb{U}$ is the neighborhood of $(\mathcal{A}^{\mathbb{Z}}, F)$. Moreover,

$$\begin{aligned} F^{n_2+1}(x)_0 &= \overline{F}((f_{n_2}(x_{[p_{n_2}-p+u,p_{n_2}+k+u]}))_{u \in \mathbb{U}}) \\ &= \overline{F}((f_{n_1}(x_{[p_{n_2}-p+u,p_{n_2}+k+u]}))_{u \in \mathbb{U}}) \\ &\underset{*}{=} f_{n_1+1}(x_{[p_{n_2}+p_{n_1+1}-p_{n_1}-p,p_{n_2}+p_{n_1+1}-p_{n_1}+k]}), \end{aligned}$$

where $*$ follows from $\star$. Since $F^{n_2+1}(x)_0 = f_{n_2+1}(x_{[p_{n_2+1}-p, p_{n_2+1}+k]})$, one deduces that $f_{n_1+1} = f_{n_2+1}$ and $p_{n_1+1} - p_{n_1} = p_{n_2+1} - p_{n_2}$. Thus the sequences $(f_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$ are ultimately periodic. We deduce that $\alpha \in \mathbb{Q}$ and $(\sigma^{\lfloor \alpha n\rfloor} \circ F^n)_n$ is ultimately periodic. $\diamond$ Claim 2

□

Corollary 4.4. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA and Σ be a mixing sofic subshift. If $\mathbf{A}'_{\mathbb{N}}(\Sigma, F) = \{\alpha\}$ then there exixts $m \in \mathbb{N}$ such that $\mathbf{B}^l_{\mathbb{N}}(F^m(\Sigma), F) =]-\infty, \alpha[$ and $\mathbf{B}^r_{\mathbb{N}}(F^m(\Sigma), F) =]\alpha, +\infty[$.*

Proof. By previous theorem, the sequence $(\sigma^{\lfloor \alpha n\rfloor} \circ F^n)_{n\in\mathbb{N}}$ is ultimately periodic of preperiod m and period p. Thus, for all $n \in \mathbb{N}$ the restriction of $\sigma^{\lfloor \alpha(n+p)\rfloor} \circ F^{n+p}$ to $F^m(\Sigma)$ is the identity. The result is plain. □

Remark 4.2. In this case, since $F^m(\Sigma)$ is not reduced to one point (F is not Σ-nilpotent), one deduces that $\mathbf{A}_{\mathbb{N}}(\Sigma, F) = \mathbf{A}'_{\mathbb{N}}(\Sigma, F)$.

Corollary 4.5. *Given a CA $(\mathcal{A}^{\mathbb{Z}}, F)$ and $\alpha \in \mathbb{R}$, it is undecidable to know whether $(\mathcal{A}^{\mathbb{Z}}, F)$ is $\mathbb{N}$-equicontinuous of slope α.*

Proof. By the result of Culik, Pachl and Yu [6], it is undecidable to know if a CA is nilpotent. However, if it is possible to know the $\mathbb{N}$-equicontinuity of slope α, then to decide if a CA is nilpotent, it would be enough to know if $(\mathcal{A}^{\mathbb{Z}}, F)$ and $(\mathcal{A}^{\mathbb{Z}}, \sigma \circ F)$ are $\mathbb{N}$-equicontinuous of slope α. Thus, the $\mathbb{N}$-equicontinuity of slope α is undecidable. □

Remark 4.3. There is another proof of this result in [7].

Corollary 4.6. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA and $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ a mixing sofic subshift. If $(\mathcal{A}^{\mathbb{Z}}, F)$ is bijective and $(\mathbb{N}, \Sigma)$-equicontinuous of slope $\alpha \in \mathbb{Q}$, then $(\mathcal{A}^{\mathbb{Z}}, F^{-1})$ is $(\mathbb{N}, \Sigma)$-equicontinuous of slope $-\alpha$. Thus, one has $\mathbf{A}'_{\mathbb{Z}}(\Sigma, F) = \mathbf{A}'_{\mathbb{N}}(\Sigma, F)$.*

Proof. By previous proposition, there exist $m, p \in \mathbb{N}$ such that $F^{m+p} \circ \sigma^{\lfloor (m+p)\alpha\rfloor} = F^m \circ \sigma^{\lfloor m\alpha\rfloor}$ where $F^p \circ \sigma^{\lfloor p\alpha\rfloor} = \mathrm{Id}$. One deduces that $(F^{-n} \circ \sigma^{-\lfloor n\alpha\rfloor})_{n\in\mathbb{N}}$ is periodic of period p. It follows the $(\mathbb{N}, \Sigma)$-equicontinuity of slope $-\alpha$ of $(\mathcal{A}^{\mathbb{Z}}, F^{-1})$. □

4.3. Directions of μ-almost equicontinuity

In the measurable point of view, the property $Eq^{\alpha}_{\mathbb{K}}(\Sigma, F) = \Sigma$ is translated to $\mu(Eq^{\alpha}_{\mathbb{K}}(F, \mu)) = 1$ which is equivalent to $Eq^{\alpha}_{\mathbb{K}}(F, \mu) \neq \emptyset$ when μ is σ-ergodic. Now we study μ-almost equicontinuous directions.

Definition 4.2. The following notion was introduced in [14]. Let $(\mathcal{A}^{\mathbb{Z}}, F)$ and $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$. The μ-limit set of F is given by

$$u \notin \mathcal{L}(\Lambda_\mu(F)) \iff \lim_{n\to\infty} \mu(F^{-n}([u]_0)) = 0.$$

Thus $(\mathcal{A}^{\mathbb{Z}}, F)$ is said μ-nilpotent if $\Lambda_\mu(F)$ is finite.

Proposition 4.7. *Assume $\mathbb{K} = \mathbb{N}$ or $\mathbb{Z}$. Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA and $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ be a σ-ergodic probability measure. The set $\mathbf{C}_{\mathbb{K}}(F, \mu)$ is convex.*

Proof. Let U' and U'' be two $(\mathbb{K}, \mu)$-almost blocking walls of slope α' and α'' respectively with $\alpha' < \alpha''$ and common width e. By σ-ergodicity of μ, there exists $k \in \mathbb{N}$ such that $\mu(W^{\alpha'}_{\mathbb{K}}(U', 0) \cap W^{\alpha''}_{\mathbb{K}}(U'', k)) > 0$. By Lemma 2.3, there exist $x \in W^{\alpha'}_{\mathbb{K}}(U', 0) \cap W^{\alpha''}_{\mathbb{K}}(U'', k)$ and $r \in \mathbb{N}$ such that $\mu(X) > 0$ where

$$X = W^{\alpha'}_{\mathbb{K}}(U', 0) \cap W^{\alpha''}_{\mathbb{K}}(U'', k) \cap [x_{[-r,k+r]}]_{-r}.$$

Moreover, for all $z \in X$ and for all $n \in \mathbb{N}$, one has $F^n(x)_i = F^n(z)_i$ for all $i \in [\lfloor \alpha' n\rfloor, \lfloor \alpha'' n\rfloor + k + e]$.

Let $\alpha \in [\alpha', \alpha'']$. Put $U = (F^n(x)_{[\lfloor \alpha n\rfloor, \lfloor \alpha n\rfloor + e-1]})_{n\in\mathbb{N}}$. Since U' and U'' are $(\mathbb{K}, \mu)$-almost blocking walls, one deduces that $X \subset W^{\alpha}_{\mathbb{K}}(U, 0)$. Then $\mu(W^{\alpha}_{\mathbb{K}}(U, 0)) > 0$, that is to say U is a μ-almost blocking wall of slope α and width e. □

Theorem 4.8. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA of neighborhood $\mathbb{U} = [r, s]$ and $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ be a σ-strongly mixing probability measure. One of the following cases hold:*

1. *$(\mathcal{A}^{\mathbb{Z}}, F)$ is μ-nilpotent;*
2. *there exists α such that $\mathbf{C}_{\mathbb{N}}(F, \mu) = \{\alpha\}$;*
3. *$\mathbf{C}_{\mathbb{N}}(\Sigma, F) = \emptyset$.*

Proof. We are going to prove that if $\mathbf{C}_{\mathbb{N}}(F, \mu)$ has at least two distinct elements, then there exists $\mathcal{A}_\infty \subset \mathcal{A}$ such that $\Lambda_\mu(F) = \{^\infty a^\infty : a \in \mathcal{A}_\infty\}$.

Let U' and U'' be two $(\mathbb{N}, \mu)$-almost blocking walls of slope α' and α'' respectively, with $\alpha' < \alpha''$. By σ-ergodicity of μ, there exists $k \in \mathbb{N}$ such that $\mu(W^{\alpha'}_{\mathbb{N}}(U', 0) \cap W^{\alpha''}_{\mathbb{N}}(U'', k)) > 0$. By Lemma 2.3 there exist $x \in W^{\alpha'}_{\mathbb{N}}(U', 0) \cap W^{\alpha''}_{\mathbb{N}}(U'', k)$ and $r \in \mathbb{N}$ such that $\mu(X) > 0$ where

$$X = W^{\alpha'}_{\mathbb{N}}(U', 0) \cap W^{\alpha''}_{\mathbb{N}}(U'', k) \cap [x_{[-r,k+r]}]_{-r}.$$

Moreover, for all $z \in X$ and for all $n \in \mathbb{N}$, one has $F^n(x)_i = F^n(z)_i$ for all $i \in [\lfloor \alpha' n\rfloor, \lfloor \alpha'' n\rfloor + k]$.

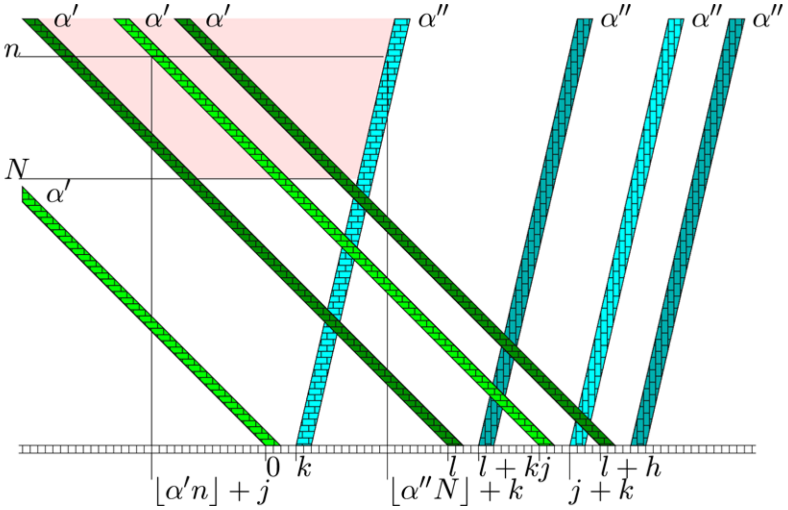

Fig. 4. μ-almost equicontinuity of slope α' and α'' and the set $\Lambda_\mu(F)$.

Assume $\alpha' \leq 0$. Let $\varepsilon > 0$. Since μ is σ strongly mixing, there exists $l \in \mathbb{N}$ such that for all $j \geq l$ we have $\mu(X \cap \sigma^{-j}(X)) > 0$. By σ-ergodicity of μ, there exists $h \in \mathbb{N}$ such that the set

$$Y_l^\varepsilon = \{y \in \mathcal{A}^{\mathbb{Z}} : \exists j \in [l, l+h] \text{ such that } \sigma^j(y) \in X\}$$

verifies $\mu(Y_l^\varepsilon) \geq 1 - \varepsilon$. Consider $N \in \mathbb{N}$ such that $\lfloor \alpha' N \rfloor + l + h \leq \lfloor \alpha'' N \rfloor + k$, this is possible since $\alpha' < \alpha''$; this inequality is verified for all $n \geq N$. Let $n \geq N$ and $y \in Y_l^\varepsilon$. There exists $j \in [l, l+h]$ such that $\sigma^j(y) \in X$. So for all $z \in \sigma^{-j}(X)$, one has $F^n(y)_i = F^n(z)_i$ for all $i \in [\lfloor \alpha' n \rfloor + j, \lfloor \alpha'' n \rfloor + k + j]$ (see Fig. 4). Since $\mu(X \cap \sigma^{-j}(X)) > 0$ there exists $z \in X \cap \sigma^{-j}(X)$. We deduce that

$$F^n(y)_i = F^n(z)_i = F^n(x)_i \quad \text{for all } i \in [\lfloor \alpha' n \rfloor, \lfloor \alpha'' n \rfloor + k] \cap [\lfloor \alpha' n \rfloor + j, \lfloor \alpha'' n \rfloor + k + j] \underset{(*)}{\neq} \emptyset,$$

where $(*)$ follows from the inequality verified by every $n \geq N$. So, there exists $a \in \mathcal{A}$ such that:

$$F_*^n \mu[a]_{\lfloor \alpha' n \rfloor + j} = F_*^n \mu[a] \geq \mu(Y_l^\varepsilon) \geq 1 - \varepsilon.$$

It follows that if u is not a power of an element of $\mathcal{A}$ then $F^n \mu[u]$ converges to 0. This proves the result when $\alpha' \leq 0$. The case $\alpha' \geq 0$ is analogous. □

Corollary 4.9. *Given a CA $(\mathcal{A}^{\mathbb{Z}}, F)$, $\alpha \in \mathbb{R}$ and μ a non-trivial Bernoulli measure, it is undecidable to know if $(\mathcal{A}^{\mathbb{Z}}, F)$ is $(\mathbb{N}, \mu)$-equicontinuous of slope α.*

Proof. By the result of Boyer, Poupet and Theyssier [4], it is undecidable to know if a CA is μ-nilpotent. However, if it is possible to decide the $(\mathbb{N}, \mu)$-equicontinuity of slope α, to know if a CA is nilpotent it would be enough to verify if $(\mathcal{A}^{\mathbb{Z}}, F)$ and $(\mathcal{A}^{\mathbb{Z}}, \sigma \circ F)$ are $(\mathbb{N}, \mu)$-equicontinuous of slope α. Thus, the $(\mathbb{N}, \mu)$-equicontinuity of slope α is undecidable. □

5. Directional expansivity

The directional expansivity was first studied for general $\mathbb{Z}^d$-action in [1]. There are a lot of studies of expansive properties for CA. The most connected to directional dynamics is the work of Nasu [18] or [19]. In this section we establish that expansive directions are contained in an open cone. This shows that CA with an expansive direction have a strong information transfer. Indeed, each direction in the open cone contains all the information of the initial configuration. Of course this transfer of information is bounded by the neighborhood of the CA; the information cannot go faster than the computations of the CA. However, we do not know if the bounds of the cone can be irrational.

5.1. Characterization of $\mathbb{K}$-expansivity

We will need some additional notations.

Notation. Let $\mathbb{V}, \mathbb{V}' \subset \mathbb{Z} \times \mathbb{K}$. We say that $\mathbb{V}$ *codes* $\mathbb{V}'$ relatively to (Σ, F), denoted $\mathbb{V} \vdash_{\Sigma,F}^{\mathbb{K}} \mathbb{V}'$, if for all $x, y \in \Sigma$,

$$(F^n(x)_m)_{(m,n)\in\mathbb{V}} = (F^n(y)_m)_{(m,n)\in\mathbb{V}} \Longrightarrow (F^n(x)_m)_{(m,n)\in\mathbb{V}'} = (F^n(y)_m)_{(m,n)\in\mathbb{V}'}.$$

That is, for all $x \in \Sigma$ the knowledge of $(F^n(x)_m)_{(m,n)\in\mathbb{V}}$ allows us to know $(F^n(x)_m)_{(m,n)\in\mathbb{V}'}$ with the local rule $\overline{F}$ (and the local rule $\overline{F^{-1}}$ if $\mathbb{K} = \mathbb{Z}$). If there is no ambiguity, we write $\vdash$ instead of $\vdash_{\Sigma,F}^{\mathbb{K}}$. If $\mathbb{V} \subset \mathbb{V}'$ one has $\mathbb{V}' \vdash \mathbb{V}$.

Let $\alpha \in \mathbb{R}$, $r', r'' \in \mathbb{Z}$ and $N \in \mathbb{N}$. Denote

$$\mathbb{V}_{\mathbb{K}}^{\alpha}([r', r''], N) = \{(\lfloor m + n\alpha \rfloor, n) \in \mathbb{Z} \times \mathbb{K} : n \in [-N, N] \cap \mathbb{K}, r' \leq m \leq r''\}.$$

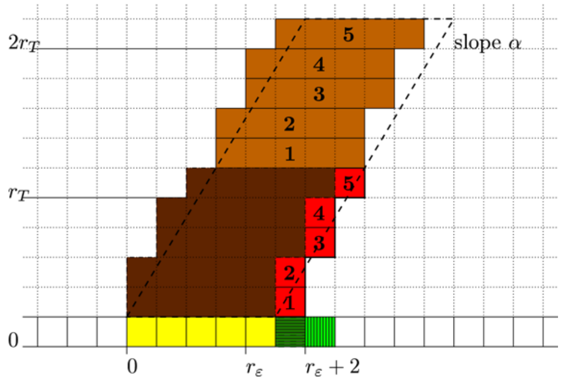

Fig. 5. $\mathbb{V}^{\alpha}_{\mathbb{N}}([0,r_\varepsilon],2r_T)\vdash\{0\}\times[0,r_\varepsilon+2]$.

First of all, let us establish a lemma which allows to characterize the $\mathbb{K}$-right-expansivity and $\mathbb{K}$-left-expansivity.

Lemma 5.1. *Assume $\mathbb{K}=\mathbb{N}$ or $\mathbb{Z}$. Let $(\mathcal{A}^{\mathbb{Z}},F)$ be a CA, $\Sigma\subset\mathcal{A}^{\mathbb{Z}}$ be a subshift and $\alpha\in\mathbb{R}$. The following properties are equivalent:*

1. *$(\mathcal{A}^{\mathbb{Z}},F)$ is $(\mathbb{K},\Sigma)$-right-expansive (resp. $(\mathbb{K},\Sigma)$-left-expansive) of slope α with expansivity constant $\varepsilon=2^{-r_\varepsilon}$;*
2. *there exists $r_\varepsilon\in\mathbb{N}$ such that for all $R>r_\varepsilon$ there is $r_T\in\mathbb{N}$ which verifies $\mathbb{V}^{\alpha}_{\mathbb{K}}([0,r_\varepsilon],r_T)\vdash\mathbb{V}^{\alpha}_{\mathbb{K}}([0,R],0)$ (resp. $\mathbb{V}^{\alpha}_{\mathbb{K}}([-r_\varepsilon,0],r_T)\vdash\mathbb{V}^{\alpha}_{\mathbb{K}}([-R,0],0)$);*
3. *there exists $r_\varepsilon,r_T\in\mathbb{N}$ such that $\mathbb{V}^{\alpha}_{\mathbb{K}}([0,r_\varepsilon],r_T)\vdash\mathbb{V}^{\alpha}_{\mathbb{K}}([0,r_\varepsilon+1],0)$ (resp. $\mathbb{V}^{\alpha}_{\mathbb{K}}([-r_\varepsilon,0],r_T)\vdash\mathbb{V}^{\alpha}_{\mathbb{K}}([-r_\varepsilon-1,0],0)$).*

Proof. The definition of the $(\mathbb{K},\Sigma)$-right-expansivity is translated by the coding formalism: $(\mathcal{A}^{\mathbb{Z}},F)$ is $(\mathbb{K},\Sigma)$-right-expansive of slope α if and only if there exists $r_\varepsilon\in\mathbb{N}$ such that $\mathbb{V}^{\alpha}_{\mathbb{K}}([0,r_\varepsilon],+\infty)\vdash\mathbb{V}^{\alpha}_{\mathbb{K}}([0,+\infty),0)$.

(1) $\Rightarrow$ (2) By expansivity of slope α, there exists r_ε such that $\mathbb{V}^{\alpha}_{\mathbb{K}}([0,r_\varepsilon],+\infty)\vdash\mathbb{V}^{\alpha}_{\mathbb{K}}([0,+\infty),0)$. Let $R>r_\varepsilon$, then $\mathbb{V}^{\alpha}_{\mathbb{K}}([0,R],0)$ is a finite subset of $\mathbb{V}^{\alpha}_{\mathbb{K}}([0,+\infty),0)$. One deduces that there exists a finite subset of $\mathbb{V}^{\alpha}_{\mathbb{K}}([0,r_\varepsilon],+\infty)$ which allows us to code $\mathbb{V}^{\alpha}_{\mathbb{K}}([0,R],0)$ relatively to (Σ,F). So, there exists $r_T\in\mathbb{N}$ such that $\mathbb{V}^{\alpha}_{\mathbb{K}}([0,r_\varepsilon],r_T)\vdash\mathbb{V}^{\alpha}_{\mathbb{K}}([0,R],0)$.

(2) $\Rightarrow$ (3) This implication is direct.

(3) $\Rightarrow$ (1) Assume there exists $r_\varepsilon,r_T\in\mathbb{N}$ such that $\mathbb{V}^{\alpha}_{\mathbb{K}}([0,r_\varepsilon],r_T)\vdash\mathbb{V}^{\alpha}_{\mathbb{K}}([0,r_\varepsilon+1],0)$. So, one has $\mathbb{V}^{\alpha}_{\mathbb{K}}([0,r_\varepsilon],r_T+1)\vdash\mathbb{V}^{\alpha}_{\mathbb{K}}([0,r_\varepsilon+1],1)$. By r_T iterations of the same process, one obtains $\mathbb{V}^{\alpha}_{\mathbb{K}}([0,r_\varepsilon],2r_T)\vdash\mathbb{V}^{\alpha}_{\mathbb{K}}([0,r_\varepsilon+1],r_T)$ (see Fig. 5). By σ-invariance, one has $\mathbb{V}^{\alpha}_{\mathbb{K}}([0,r_\varepsilon+1],r_T)\vdash\mathbb{V}^{\alpha}_{\mathbb{K}}([0,r_\varepsilon+2],0)$, so $\mathbb{V}^{\alpha}_{\mathbb{K}}([0,r_\varepsilon],2r_T)\vdash\mathbb{V}^{\alpha}_{\mathbb{K}}([0,r_\varepsilon+2],0)$. By recurrence one deduces that for all $n\in\mathbb{N}$ it holds that $\mathbb{V}^{\alpha}_{\mathbb{K}}([0,r_\varepsilon],nr_T)\vdash\mathbb{V}^{\alpha}_{\mathbb{K}}([0,r_\varepsilon+n],0)$. The $(\mathbb{K},\Sigma)$-right-expansivity of slope α of $(\mathcal{A}^{\mathbb{Z}},F)$ follows. □

Remark 5.1. The integers r_ε and r_T depend on α; in case of ambiguity we can denote them by r^{α}_{ε} and r^{α}_{T}. In fact, r_T corresponds to the radius of the transverse CA (see [2]).

5.2. Directional $\mathbb{N}$-expansivity

The previous lemma allows to describe the sets $\mathbf{B}^{r}_{\mathbb{N}}(\Sigma,F)$, $\mathbf{B}^{l}_{\mathbb{N}}(\Sigma,F)$ and $\mathbf{B}_{\mathbb{N}}(\Sigma,F)$. For more convenience, we denote $\mathbb{V}^{\alpha}$ instead of $\mathbb{V}^{\alpha}_{\mathbb{N}}$.

Theorem 5.2. *Let $(\mathcal{A}^{\mathbb{Z}},F)$ be a CA of neighborhood $\mathbb{U}=[r,s]$ and $\Sigma\subset\mathcal{A}^{\mathbb{Z}}$ an infinite transitive subshift.*

- *If $\mathbf{B}^{r}_{\mathbb{N}}(\Sigma,F)\neq\emptyset$ then there exists $\alpha'\geq-s$ such that $\mathbf{B}^{r}_{\mathbb{N}}(\Sigma,F)=]\alpha',+\infty)\subset]-s,+\infty)$.*
- *If $\mathbf{B}^{l}_{\mathbb{N}}(\Sigma,F)\neq\emptyset$ then there exists $\alpha''\leq-r$ such that $\mathbf{B}^{l}_{\mathbb{N}}(\Sigma,F)=(-\infty,\alpha''[\subset(-\infty,-r[$.*
- *If $\mathbf{B}_{\mathbb{N}}(\Sigma,F)\neq\emptyset$ then there exist $\alpha',\alpha''\in\mathbb{R}$ with $-s\leq\alpha'\leq\alpha''\leq-r$ such that $\mathbf{B}_{\mathbb{N}}(\Sigma,F)=]\alpha',\alpha''[\subset]-s,-r[$.*

Proof. Claim 1: *If $\alpha'\in\mathbf{B}^{r}_{\mathbb{N}}(\Sigma,F)$ then $[\alpha',+\infty)\subset\mathbf{B}^{r}_{\mathbb{N}}(\Sigma,F)$.*

Proof: By Lemma 5.1, there exists $r^{\alpha'}_{\varepsilon},r^{\alpha'}_{T}\in\mathbb{N}$ such that $\mathbb{V}^{\alpha'}_{\mathbb{K}}([0,r^{\alpha'}_{\varepsilon}],r_T)\vdash\mathbb{V}^{\alpha'}_{\mathbb{K}}([0,r^{\alpha'}_{\varepsilon}+1],0)$.

Let $\alpha\in[\alpha',+\infty)$. Put $r^{\alpha}_{\varepsilon}=\lfloor(\alpha-\alpha')r^{\alpha'}_{T}\rfloor+r^{\alpha'}_{\varepsilon}$. One has

$$\mathbb{V}^{\alpha}_{\mathbb{K}}([0,r^{\alpha}_{\varepsilon}],r_T)\vdash\mathbb{V}^{\alpha}_{K}([0,r^{\alpha}_{\varepsilon}],0)\cup\mathbb{V}^{\alpha'}_{\mathbb{K}}([r^{\alpha}_{\varepsilon}-r^{\alpha'}_{\varepsilon},r^{\alpha}_{\varepsilon}],r^{\alpha'}_{T})\vdash\mathbb{V}^{\alpha}_{K}([0,r^{\alpha}_{\varepsilon}+1],0).$$

One deduces that $\alpha\in\mathbf{B}^{r}_{\mathbb{N}}(\Sigma,F)$. ◇ Claim 1

Claim 2: *$\mathbf{B}^{r}_{\mathbb{N}}(\Sigma,F)$ is open.*

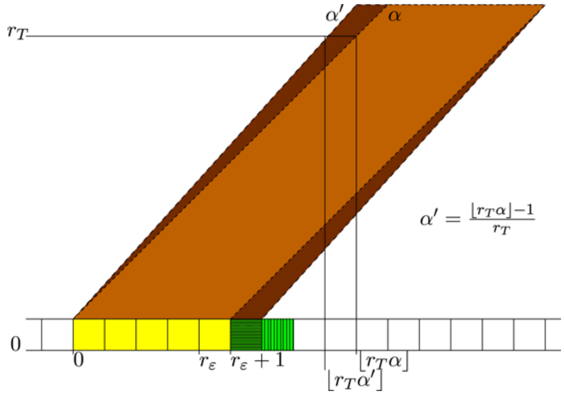

Fig. 6. $\mathbf{B}^r_{\mathbb{N}}(\Sigma, F)$ is open.

Proof: Let $\alpha \in \mathbf{B}^r_{\mathbb{N}}(\Sigma, F)$. By Lemma 5.1, there exist $r_\varepsilon, r_T \in \mathbb{N}$ such that $\mathbb{V}^\alpha([0, r_\varepsilon], r_T) \vdash \mathbb{V}^\alpha([0, r_\varepsilon + 2], 0)$. One defines

$$\alpha' = \frac{\lfloor \alpha\, r_T\rfloor - 1}{r_T}.$$

Thus, $\mathbb{V}^{\alpha'}([0, r_\varepsilon + 1], r_T) \supset \mathbb{V}^\alpha([0, r_\varepsilon], r_T)$ and $\mathbb{V}^{\alpha'}([0, r_\varepsilon + 1], r_T) \vdash \mathbb{V}^\alpha([0, r_\varepsilon + 2], 0)$ (see Fig. 6). One deduces that $\alpha' \in \mathbf{B}^r_{\mathbb{N}}(\Sigma, F)$. According to the previous step, one has $\alpha \in]\alpha', +\infty) \subset \mathbf{B}^r_{\mathbb{N}}(\Sigma, F)$. ◇ `Claim 2`

CLAIM 3: $\mathbf{B}^r_{\mathbb{N}}(\Sigma, F) \subset] - s, +\infty)$.

Proof: If $(\mathcal{A}^{\mathbb{Z}}, F)$ is $(\mathbb{N}, \Sigma)$-right-expansive of slope α, there exists $r_\varepsilon \in \mathbb{N}$ such that if $x_{[0,+\infty)} \neq y_{[0,+\infty)}$ then $F^n(x)_{[-r_\varepsilon+\lfloor\alpha n\rfloor, r_\varepsilon+\lfloor\alpha n\rfloor]} \neq F^n(y)_{[-r_\varepsilon+\lfloor\alpha n\rfloor, r_\varepsilon+\lfloor\alpha n\rfloor]}$ for some $n \in \mathbb{N}$. Since the neighborhood of $(\mathcal{A}^{\mathbb{Z}}, F^n)$ is $[nr, ns]$, the differences between $x_{[0,+\infty)}$ and $y_{[0,+\infty)}$ coming from $(F^i(x)_{[-k+\lfloor\alpha n\rfloor, k+\lfloor\alpha n\rfloor]})_{i\in[0,n]} \neq (F^i(y)_{[-k+\lfloor\alpha n\rfloor, k+\lfloor\alpha n\rfloor]})_{i\in[0,n]}$ appear in the interval $[0, k + \lfloor\alpha n\rfloor + sn]$. If Σ is infinite, then to have $(\mathbb{N}, \Sigma)$-right-expansivity of slope α the right extremity of the interval must tend to $+\infty$. This needs $\alpha \in] - s, +\infty[$. ◇ `Claim 3`

From previous steps, one deduces that if $\mathbf{B}^r_{\mathbb{N}}(\Sigma, F) \neq \emptyset$, there exists $\alpha' \geq -s$ such that $\mathbf{B}^r_{\mathbb{N}}(\Sigma, F) =]\alpha', +\infty)$. Symmetrically, if $\mathbf{B}^l_{\mathbb{N}}(\Sigma, F) \neq \emptyset$, then there exists $\alpha'' \leq -r$ such that $\mathbf{B}^l_{\mathbb{N}}(\Sigma, F) = (-\infty, \alpha''[$. The result on $\mathbf{B}_{\mathbb{N}}(\Sigma, F)$ follows from the intersection of previous sets. □

Remark 5.2. Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA and let $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ be a subshift. If $\mathbf{B}^r_{\mathbb{N}}(\Sigma, F) \neq \emptyset$ (resp. $\mathbf{B}^l_{\mathbb{N}}(\Sigma, F) \neq \emptyset$) then for all $x, y \in \Sigma$ such that $x_{(-\infty,k]} = y_{(-\infty,k]}$ (resp. $x_{[k,+\infty)} = y_{[k,+\infty)}$) for some $k \in \mathbb{Z}$ then $F^n(x) \neq F^n_T(y)$ (if not, it is not possible to distinguish $\sigma^i(x)$ and $\sigma^i(y)$ for all $i \in \mathbb{Z}$). By definition $(\mathcal{A}^{\mathbb{Z}}, F)$ is said right-closing (resp. left-closing) on Σ.

Therefore, if Σ is an F-invariant transitive subshift of finite type such that $\mathbf{B}^r_{\mathbb{N}}(\Sigma, F) \cup \mathbf{B}^l_{\mathbb{N}}(\Sigma, F) \neq \emptyset$ then $F : \Sigma \to \Sigma$ is surjective since it is right or left-closing (see [17]).

5.3. Directional $\mathbb{Z}$-expansivity

Here we are going to consider bijective CA to be able to speak about directional $\mathbb{Z}$-expansivity. As for the $\mathbb{Z}$-expansive CA, the study of the directional $\mathbb{Z}$-expansivity is not easy. Already, the set $\mathbf{B}_{\mathbb{Z}}(\mathcal{A}^{\mathbb{Z}}, F)$ is not necessarily convex (consider the identity map where $\mathbf{B}_{\mathbb{Z}}(\mathcal{A}^{\mathbb{Z}}, \mathrm{Id}) = \mathbb{R} \setminus \{0\}$; or Example 3.6). However, Claim 2 of Theorem 5.2 can be adapted to show that $\mathbf{B}_{\mathbb{Z}}(\Sigma, F)$ is open. This proof does not hold for $\mathbf{B}^r_{\mathbb{Z}}(\Sigma, F)$ and $\mathbf{B}^l_{\mathbb{Z}}(\Sigma, F)$; those last ones are not necessarily open.

Proposition 5.3. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a bijective CA and $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ be a subshift. The set of $(\mathbb{Z}, \Sigma)$-expansive directions, $\mathbf{B}_{\mathbb{Z}}(\Sigma, F)$, is open and there exist $\alpha' < \alpha''$ such that $(-\infty, \alpha'[\cup]\alpha'', +\infty) \subset \mathbf{B}_{\mathbb{Z}}(\Sigma, F)$.*

Proof. Let $\alpha \in \mathbf{B}_{\mathbb{Z}}(\Sigma, F)$. By Lemma 5.1, which characterizes the $(\mathbb{Z}, \Sigma)$-expansivity, there exists $r_\varepsilon, r_T \in \mathbb{N}$ such that $\mathbb{V}^\alpha_{\mathbb{Z}}([-r_\varepsilon, r_\varepsilon], r_T) \vdash \mathbb{V}^\alpha_{\mathbb{Z}}([-r_\varepsilon - 2, r_\varepsilon + 2], 0)$. One defines

$$\alpha_1 = \frac{\lfloor \alpha\, r_T\rfloor - 1}{r_T} \quad \text{and} \quad \alpha_2 = \frac{\lfloor \alpha\, r_T\rfloor + 1}{r_T}.$$

Let $\alpha''' \in [\alpha_1, \alpha_2]$. One has $\mathbb{V}^{\alpha'''}_{\mathbb{Z}}([-r_\varepsilon - 1, r_\varepsilon + 1], r_T) \supset \mathbb{V}^\alpha_{\mathbb{Z}}([-r_\varepsilon, r_\varepsilon], r_T)$, so $\mathbb{V}^{\alpha'''}_{\mathbb{Z}}([-r_\varepsilon - 1, r_\varepsilon + 1], r_T) \vdash \mathbb{V}^\alpha_{\mathbb{Z}}([-r_\varepsilon - 2, r_\varepsilon + 2], 0)$. One deduces that $\alpha''' \in \mathbf{B}_{\mathbb{Z}}(\Sigma, F)$. Thus $[\alpha_1, \alpha_2]$ is a neighborhood of α included in $\mathbf{B}_{\mathbb{Z}}(\Sigma, F)$. Thus $\mathbf{B}_{\mathbb{Z}}(\Sigma, F)$ is open.

A bijective CA is open, so it is right and left-closing. Since it is right-closing, there exists $m'_F \in \mathbb{N}$ such that if $x_{[-m'_F, 0[} = y_{[-m'_F, 0[}$ and $F(x)_{[-m'_F, m'_F]} = F(y)_{[-m'_F, m'_F]}$ then $x_0 = y_0$. By Lemma 5.1, one deduces that $m'_F \in \mathbf{B}^r_{\mathbb{N}}(\Sigma, F)$. In the same way there exist $m''_F, m'_{F^{-1}}, m''_{F^{-1}} \in \mathbb{N}$ such that $m''_F \in \mathbf{B}^l_{\mathbb{N}}(\Sigma, F)$, $m'_{F^{-1}} \in \mathbf{B}^r_{\mathbb{N}}(\Sigma, F^{-1})$ and $m''_{F^{-1}} \in \mathbf{B}^l_{\mathbb{N}}(\Sigma, F^{-1})$. One deduces that $(-\infty, \min(m''_F, -m'_{F^{-1}})[\cup]\max(m'_F, -m''_{F^{-1}}), +\infty) \subset \mathbf{B}_{\mathbb{Z}}(\Sigma, F)$. □

Modifying Example 3.6 thanks to Sections 3.2 and 3.4, it is possible to construct a CA $(\mathcal{A}^{\mathbb{Z}}, F)$ such that

$$\mathbf{B}_{\mathbb{Z}}(\mathcal{A}^{\mathbb{Z}}, F) = \bigcup_{k\in[1,n]}]\alpha_k^l, \alpha_k^r[,$$

where $-\infty = \alpha_1^l < \alpha_1^r < \alpha_2^l \leq \alpha_2^r < \cdots < \alpha_{n-1}^l \leq \alpha_{n-1}^r < \alpha_n^l < \alpha_n^r = +\infty$ with α_k^l and α_k^r in $\mathbb{Q}$ for all $k \in [1, n]$. However, we do not know examples where $\mathbf{B}_{\mathbb{Z}}(\Sigma, F)$ has another form.

6. Conclusion: A directional classification according to dynamical properties

We have studied CA as $\mathbb{Z} \times \mathbb{N}$-action (or $\mathbb{Z} \times \mathbb{Z}$-action when we want to consider the past for a bijective CA) in order to emphasize the spatio-temporal structures produced by this type of dynamics. We study the sensibility to initial conditions from two points of view. The first one privileges points chosen in a subshift Σ and the second one privileges points which are chosen according to a σ-invariant probability measure μ. We also study sets of directions which have extreme and opposite dynamical behaviors, namely, equicontinuous and expansive properties:

- If $\mathbf{A}'_{\mathbb{N}}(\Sigma, F) \neq \emptyset$ or $\mathbf{A}_{\mathbb{N}}(\Sigma, F) \neq \emptyset$ or $\mathbf{C}_{\mathbb{N}}(F, \mu) \neq \emptyset$, there is a direction delimited by blocking walls which indicates the propagation of information. If there are two directions, all propagation is killed and the CA evolves towards a trivial configuration.
- If $\mathbf{B}_{\mathbb{N}}(\Sigma, F) \neq \emptyset$, there exists an open cone where from each direction it is possible to recover from it the information contained in the initial configuration.

According to the topological point of view, the different results of this paper can be summarized to obtain the following classification.

Theorem 6.1. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA of neighborhood $\mathbb{U} = [r, s]$. Let $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ be a weakly-specified subshift. One of the following cases hold:*

C1. *$\mathbf{A}'_{\mathbb{N}}(\Sigma, F) = \mathbf{A}_{\mathbb{N}}(\Sigma, F) = \mathbb{R}$. In this case $(\mathcal{A}^{\mathbb{Z}}, F)$ is Σ-nilpotent, moreover $\mathbf{B}^r_{\mathbb{N}}(\Sigma, F) = \mathbf{B}^l_{\mathbb{N}}(\Sigma, F) = \emptyset$.*

C2. *There exists $\alpha \in [-s, -r] \cap \mathbb{Q}$ such that $\mathbf{A}'_{\mathbb{N}}(\Sigma, F) = \mathbf{A}_{\mathbb{N}}(\Sigma, F) = \{\alpha\}$. In this case there exist $m, p \in \mathbb{N}$ such that the sequence $(F^n \circ \sigma^{\lfloor \alpha n \rfloor})_{n\in\mathbb{N}}$ is ultimately periodic of preperiod m and period p. Moreover, $\mathbf{B}^l_{\mathbb{N}}(F^m(\Sigma), F) =]-\infty, \alpha[$ and $\mathbf{B}^r(F^m(\Sigma), F) =]\alpha, +\infty[$.*

C3. *There exists $\alpha', \alpha'' \in [-s, -r]$, $\alpha' \leq \alpha''$ such that $]\alpha', \alpha''[\subset \mathbf{A}_{\mathbb{N}}(\Sigma, F) \subset [\alpha'\alpha'']$. In this case $\mathbf{A}'_{\mathbb{N}}(\Sigma, F) = \mathbf{B}^r_{\mathbb{N}}(\Sigma, F) = \mathbf{B}^l_{\mathbb{N}}(\Sigma, F) = \emptyset$.*

C4. *There exists $\alpha \in [-s, -r]$ such that $\mathbf{A}_{\mathbb{N}}(\Sigma, F) = \{\alpha\}$ and $\mathbf{A}'_{\mathbb{N}}(\Sigma, F) = \emptyset$. In this case $\mathbf{B}^r_{\mathbb{N}}(\Sigma, F)$ and $\mathbf{B}^l_{\mathbb{N}}(\Sigma, F)$ can be empty or not, but $\mathbf{B}_{\mathbb{N}}(\Sigma, F) = \emptyset$.*

C5. *There exist $\alpha', \alpha'' \in [-s, -r]$ such that $\mathbf{B}_{\mathbb{N}}(\Sigma, F) =]\alpha', \alpha''[$. In this case $\mathbf{A}_{\mathbb{N}}(\Sigma, F) = \mathbf{A}'_{\mathbb{K}}(\Sigma, F) = \emptyset$.*

C6. *All directions are sensitive and $\mathbb{N}$-expansive directions do not exist. In this case $\mathbf{A}_{\mathbb{N}}(\Sigma, F) = \mathbf{A}'_{\mathbb{N}}(\Sigma, F) = \mathbf{B}_{\mathbb{N}}(\Sigma, F) = \emptyset$ but $\mathbf{B}^r_{\mathbb{N}}(\Sigma, F)$ and $\mathbf{B}^l_{\mathbb{N}}(\Sigma, F)$ can be empty or not.*

There exist examples in each class of the preceding classification:

Class	**C1.**	**C2.**	**C3.**	**C4.**	**C5.**	**C6.**
Example	3.1	3.2	3.3	3.6	3.4	3.7

There are still a lot of open questions. Here we address some of them:

- Can the bounds of $\mathbf{A}_{\mathbb{N}}(\Sigma, F)$, $\mathbf{C}_{\mathbb{N}}(F, \mu)$ and $\mathbf{B}_{\mathbb{N}}(\Sigma, F)$ be irrational ?
- Is $\mathbf{A}_{\mathbb{N}}(\Sigma, F)$ closed ?
- Can $\mathbf{B}_{\mathbb{Z}}(\Sigma, F)$ be different than a union of open segments with rational bounds ?
- Is it true that:

$$\mathbf{A}_{\mathbb{N}}(\Sigma, F) \stackrel{?}{=} \mathbf{A}_{\mathbb{Z}}(\Sigma, F) \quad \text{and} \quad \mathbf{C}_{\mathbb{N}}(\Sigma, F) \stackrel{?}{=} \mathbf{C}_{\mathbb{Z}}(\Sigma, F).$$

- It is possible to define dynamics according to a curve defined by $h : \mathbb{K} \to \mathbb{Z}$. One only has to define the tube $D^h_{\Sigma}(x, \varepsilon, \mathbb{K}) = \{y \in \Sigma : d_C(F^n \circ \sigma^{h(n)}(x), F^n \circ \sigma^{h(n)}(y)) < \varepsilon, \forall n \in \mathbb{K}\}$ and replace $D^{\alpha}_{\Sigma}(x, \varepsilon, \mathbb{K})$ by $D^h_{\Sigma}(x, \varepsilon, \mathbb{K})$ in the definitions of Subsection 1.3. It is easy to do a similar theory but we have no example where the slope is not linear.

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