Measure rigidity for algebraic bipermutative cellular automata

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Abstract

Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a bipermutative algebraic cellular automaton. We present conditions which force a probability measure which is invariant for the $\mathbb{N}\times\mathbb{Z}$-action of F and the shift map σ to be the Haar measure on Σ, a closed shift-invariant subgroup of the Abelian compact group $\mathcal{A}^{\mathbb{Z}}$. This generalizes simultaneously results of B. Host, A. Maass and S. Martínez [HMM03] and M. Pivato [Piv05]. This result is applied to give conditions which also force an (F,σ)-invariant probability measure to be the uniform Bernoulli measure when F is a particular invertible affine expansive cellular automaton on $\mathcal{A}^{\mathbb{N}}$.

1 Introduction

Let $F:\mathcal{A}^{\mathbb{M}}\to\mathcal{A}^{\mathbb{M}}$ with $\mathbb{M}=\mathbb{N}$ or $\mathbb{Z}$ be a one-dimensional cellular automaton (CA). The study of invariant measures under the action of F has been addressed from different points of view in the last two decades. As ergodic theory is the study of invariant measures, it is thus natural to characterize them. In addition, since F commutes with the shift map σ, it is important to describe invariant probability measures for the semi-group action generated by F and σ. We remark that it easy to prove the existence of such measures by considering a cluster point of the Cesàro mean under iteration of F of a σ-invariant measure. This problem is related to Furstenberg's conjecture [Fur67] that the Lebesgue measure on the torus is the unique invariant measure under multiplication by two relatively prime integers. In the algebraic setting, the study of invariant measures under a group action on a zero-dimensional group like Ledrappier's example [Led78], has been extensively considered in [Sch95] and [Ein05].

The uniform Bernoulli measure has an important role in the study of (F,σ)-invariant measures. G.A. Hedlund has shown in [Hed69] that a CA is surjective iff the uniform Bernoulli measure on $\mathcal{A}^{\mathbb{M}}$ is (F,σ)-invariant. Later, D. Lind [Lin84] shows for the radius 1 $\bmod 2$ automaton that starting from any Bernoulli measure the Cesàro mean of the iterates by the CA converges to the uniform measure. This result is generalized for a large class of algebraic CA and a large class of measures with tools from stochastic processes in [MM98] and [FMMN00], and with harmonic analysis tools in [PY02] and [PY04].

However, the uniform Bernoulli measure is not the only (F,σ)-invariant measure, indeed every uniform measure supported on a (F,σ)-periodic orbit is (F,σ)-invariant. We want to obtain additional conditions which allow us to characterize the uniform Bernoulli measure. We limit the study to CA which have algebraic and strong combinatorial properties: the algebraic bipermutative CA. Let $(\mathcal{A}^{\mathbb{Z}},F)$ be a bipermutative algebraic CA; we examine the conditions that force an (F,σ)-invariant measure μ to be the Haar measure of $\mathcal{A}^{\mathbb{Z}}$, denoted by $\lambda_{\mathcal{A}^{\mathbb{Z}}}$. When $\mathcal{A}^{\mathbb{Z}}$ is an infinite product of the finite group $\mathcal{A}$, the Haar measure is the uniform Bernoulli measure. B. Host, A. Maass and S. Martínez take this direction in [HMM03] and characterize (F,σ)-invariant measure of affine bipermutative CA of radius 1 when the alphabet is $\mathbb{Z}/p\mathbb{Z}$ with p prime. They show two theorems with different assumptions on the measure μ. M. Pivato gives in [Piv05] an extension of the first one considering a larger class of algebraic CA but with extra conditions on the measure and the kernel of F. The main result in the present paper provides a generalization of the second theorem of [HMM03] which also generalizes Pivato's result.

To introduce more precisely previous work and this article, we need to provide definitions and introduce some classes of CA. Let $\mathcal{A}$ be a finite set and $\mathbb{M} = \mathbb{N}$ or $\mathbb{Z}$. We consider $\mathcal{A}^{\mathbb{M}}$, the configuration space of $\mathbb{M}$-indexed sequences in $\mathcal{A}$. If $\mathcal{A}$ is endowed with the discrete topology, $\mathcal{A}^{\mathbb{M}}$ is compact and totally disconnected in the product topology. The *shift* map $\sigma : \mathcal{A}^{\mathbb{M}} \to \mathcal{A}^{\mathbb{M}}$ is defined by $\sigma(x)_i = x_{i+1}$ for $x = (x_m)_{m\in\mathbb{M}} \in \mathcal{A}^{\mathbb{M}}$ and $i \in \mathbb{M}$. Denote by $\mathcal{A}^*$ the set of all finite sequences or words $w = w_0...w_{n-1}$ with letters in $\mathcal{A}$; by $|w|$ we mean the length of $w \in \mathcal{A}^*$. Given $w \in \mathcal{A}^*$ and $i \in \mathbb{M}$, the *cylinder set* starting at coordinate i with the word w is $[w]_i = \{x \in \mathcal{A}^{\mathbb{M}} : x_{i,i+|w|-1} = w\}$, the cylinder set starting at 0 is simply denoted by $[w]$.

A *cellular automaton* (CA) is a pair $(\mathcal{A}^{\mathbb{M}}, F)$ where $\mathcal{A}^{\mathbb{M}}$ is called the *configuration space*, and $F : \mathcal{A}^{\mathbb{M}} \to \mathcal{A}^{\mathbb{M}}$ is a continuous function which commutes with the shift. We can therefore consider (F, σ) as a $\mathbb{N} \times \mathbb{M}$-action. By Hedlund's theorem [Hed69], it is equivalent to give a local function which acts uniformly and synchronously on the configuration space, that is to say, there is a finite segment $\mathbb{U} \subset \mathbb{M}$ (named *neighborhood*) and a *local rule* $\overline{F} : \mathcal{A}^{\mathbb{U}} \to \mathcal{A}$, such that $F(x)_m = \overline{F}((x_{m+u})_{u\in\mathbb{U}})$ for all $x \in \mathcal{A}^{\mathbb{M}}$ and $m \in \mathbb{M}$. The *radius* of F is $r(F) = \max\{|u| : u \in \mathbb{U}\}$; when $\mathbb{U}$ is as small as possible, it is called the *smallest neighborhood.* If the smallest neighborhood is reduced to one point we say that F is *trivial.*

Let $\mathfrak{B}$ be the Borel sigma-algebra of $\mathcal{A}^{\mathbb{M}}$, we denote by $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$ the set of probability measures on $\mathcal{A}^{\mathbb{M}}$ defined on the sigma-algebra $\mathfrak{B}$. As usual, $\sigma\mu$ (respectively $F\mu$) denotes the measure given by $\sigma\mu(B) = \mu(\sigma^{-1}(B))$ (respectively $F\mu(B) = \mu(F^{-1}(B))$) for B a Borel set; this allows us to consider the (F, σ)-action on $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$. We say that $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{M}})$ is *σ-invariant* (respectively *F-invariant*) iff $\sigma\mu = \mu$ (respectively $F\mu = \mu$); obviously μ is *(F, σ)-invariant* iff μ is σ-invariant and F-invariant. We denote $\mathcal{I}_\mu(\sigma) = \{B \in \mathfrak{B} : \mu(\sigma^{-1}(B)\triangle B) = 0\}$ the algebra of σ-invariant sets mod μ. If $\mathcal{A}^{\mathbb{M}}$ has a group structure and Σ is a closed σ-invariant subgroup of $\mathcal{A}^{\mathbb{M}}$, the *Haar measure* on Σ, denoted λ_Σ, is the unique measure in $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$ with $\mathrm{supp}(\mu) \subset \Sigma$ which is invariant by the action of Σ. We can characterize λ_Σ using *characters* in $\widehat{\mathcal{A}^{\mathbb{M}}}$, which are continuous morphisms from $\mathcal{A}^{\mathbb{M}}$ to $\mathbb{C}$: indeed, $\mu = \lambda_\Sigma$ iff $\mathrm{supp}(\mu) \subset \Sigma$ and $\mu(\chi) = 0$ for all $\chi \in \widehat{\mathcal{A}^{\mathbb{M}}}$ such that $\chi(\Sigma) \neq \{1\}$, see [Gui68] for more detail. If $\mathcal{A}$ is a finite group and $\mathcal{A}^{\mathbb{M}}$ is a product group, the Haar measure of $\mathcal{A}^{\mathbb{M}}$ corresponds to the *uniform Bernoulli measure* defined on a cylinder set $[u]_i$ by:

$$\lambda_{\mathcal{A}^{\mathbb{M}}}([u]_i) = \frac{1}{|\mathcal{A}|^{|u|}}.$$

Let $(\mathcal{A}^{\mathbb{M}}, F)$ be a CA of smallest neighborhood $\mathbb{U} = [r, s] = \{r, ..., s\}$. F is *left-permutative* iff for any $u \in \mathcal{A}^{s-r}$ and $b \in \mathcal{A}$, there is a unique $a \in \mathcal{A}$ such that $\overline{F}(au) = b$; F is *right-permutative* iff for any $u \in \mathcal{A}^{s-r}$ and $b \in \mathcal{A}$ there is a unique $a \in \mathcal{A}$ such that $\overline{F}(ua) = b$. F is *bipermutative* iff it is both left and right permutative.

If $\mathcal{A}^{\mathbb{M}}$ has a topological group structure and if $\sigma : \mathcal{A}^{\mathbb{M}} \to \mathcal{A}^{\mathbb{M}}$ is a continuous group endomorphism, $\mathcal{A}^{\mathbb{M}}$ is called a *group shift.* By Hedlund's Theorem [Hed69], the σ-commuting multiplication operator is given by a local rule $\overline{*} : \mathcal{A}^{[r,s]} \times \mathcal{A}^{[r,s]} \to \mathcal{A}$. We refer to [Kit87] for more details. If $\mathcal{A}^{\mathbb{M}}$ is an Abelian group shift and $F : \mathcal{A}^{\mathbb{M}} \to \mathcal{A}^{\mathbb{M}}$ is a group endomorphisms which commutes with σ, then the CA $(\mathcal{A}^{\mathbb{M}}, F)$ is called *algebraic.* If $\mathcal{A}$ has an Abelian group structure, $\mathcal{A}^{\mathbb{M}}$ is a compact Abelian group. We say that $(\mathcal{A}^{\mathbb{M}}, F)$ is a *linear* CA if F is a group endomorphism or equivalently if $\overline{F}$ is a morphism from $\mathcal{A}^{\mathbb{U}}$ to $\mathcal{A}$. In this case F can be written:

$$F = \sum_{u\in\mathbb{U}} f_u \circ \sigma^u$$

where for all $u \in \mathbb{U}$, f_u is an endomorphism of $\mathcal{A}$ which is extended coordinate by coordinate to $\mathcal{A}^{\mathbb{M}}$. We can write F as a polynomial of σ, $F = P_F(\sigma)$, where $P_F \in \mathrm{Hom}(\mathcal{A})[X, X^{-1}]$. If $\mathcal{A} = \mathbb{Z}/n\mathbb{Z}$, then an endomorphism of $\mathcal{A}$ is the multiplication by an element of $\mathbb{Z}/n\mathbb{Z}$. We say that $(\mathcal{A}^{\mathbb{M}}, F)$ is an *affine* CA if there exists $(\mathcal{A}^{\mathbb{M}}, G)$ a linear CA and a constant $c \in \mathcal{A}^{\mathbb{M}}$ such that $F = G + c$. The constant must be σ-invariant.

A linear CA $(\mathcal{A}^{\mathbb{M}}, F)$ where $F = \sum_{u\in[r,s]} f_u \circ \sigma^u$ is left (right) permutative of smallest neighborhood $[r, s]$ if f_r (f_s) is a group automorphism. An affine CA $(\mathcal{A}^{\mathbb{M}}, F + c)$, where $(\mathcal{A}^{\mathbb{M}}, F)$ is linear and $c \in \mathcal{A}^{\mathbb{M}}$, is bipermutative if $(\mathcal{A}^{\mathbb{M}}, F)$ is bipermutative. So if $\mathcal{A} = \mathbb{Z}/p\mathbb{Z}$ where p is prime, then any nontrivial affine CA is bipermutative. However, if p is composite, then F is left (right) permutative iff the leftmost (rightmost) coefficient of $\overline{F}$ is relatively prime to p.

Now we can recall the first theorem of [HMM03]:

Theorem 1.1 ([HMM03]). *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be an affine bipermutative CA of smallest neighborhood $\mathbb{U} = [0, 1]$ with $\mathcal{A} = \mathbb{Z}/p\mathbb{Z}$, where p is prime, and let μ be an (F, σ)-invariant probability measure. Assume that:*

1. *μ is ergodic for σ;*
2. *the measure entropy of F is positive ($h_\mu(F) > 0$).*

Then $\mu = \lambda_{\mathcal{A}^{\mathbb{Z}}}$.

The second theorem of [HMM03] relaxes the σ-ergodicity into (F, σ)-ergodicity provided the measure satisfies a technical condition on the sigma-algebra of invariant sets for powers of σ:

Theorem 1.2 ([HMM03]). *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be an affine bipermutative CA of smallest neighborhood $\mathbb{U} = [0, 1]$ with $\mathcal{A} = \mathbb{Z}/p\mathbb{Z}$ where p is prime, and let μ be an (F, σ)-invariant probability measure. Assume that:*

1. *μ is ergodic for the $\mathbb{N} \times \mathbb{Z}$-action (F, σ);*
2. *$\mathcal{I}_\mu(\sigma) = \mathcal{I}_\mu(\sigma^{p(p-1)}) \mod \mu$;*
3. *$h_\mu(F) > 0$.*

Then $\mu = \lambda_{\mathcal{A}^{\mathbb{Z}}}$.

M. Pivato gives in [Piv05] a result similar to Theorem 1.1, which applies to a larger class of algebraic CA but with extra conditions on the measure and $\mathrm{Ker}(F)$:

Theorem 1.3 ([Piv05]). *Let $\mathcal{A}^{\mathbb{Z}}$ be any Abelian group shift, let $(\mathcal{A}^{\mathbb{Z}}, F)$ be an algebraic bipermutative CA of smallest neighborhood $\mathbb{U} = [0, 1]$ and let μ be an (F, σ)-invariant probability measure. Assume that:*

1. *μ is totally ergodic for σ;*
2. *$h_\mu(F) > 0$;*
3. *$\mathrm{Ker}(F)$ contains no nontrivial σ-invariant subgroups.*

Then $\mu = \lambda_{\mathcal{A}^{\mathbb{Z}}}$.

It is possible to extend Theorem 1.3 to a nontrivial algebraic bipermutative CA without restriction on the neighborhood. In Section 2 of this paper we give entropy formulas for bipermutative CA without restrictions on the neighborhood. These formulas are the first step to adapt the proof of Theorem 1.2 in Section 3 in order to obtain our main result:

Theorem 3.3. *Let $\mathcal{A}^{\mathbb{Z}}$ be any Abelian group shift, let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a nontrivial algebraic bipermutative CA, let Σ be a closed (F, σ)-invariant subgroup of $\mathcal{A}^{\mathbb{Z}}$, let $k \in \mathbb{N}$ such that every prime factor of $|\mathcal{A}|$ divides k and let μ be an (F, σ)-invariant probability measure on $\mathcal{A}^{\mathbb{Z}}$ with $\mathrm{supp}(\mu) \subset \Sigma$. Assume that:*

1. *μ is ergodic for the $\mathbb{N} \times \mathbb{Z}$-action (F, σ);*
2. *$\mathcal{I}_\mu(\sigma) = \mathcal{I}_\mu(\sigma^{kp_1})$ with p_1 the smallest common period of all elements of $\mathrm{Ker}(F)$;*
3. *$h_\mu(F) > 0$;*
4. *every σ-invariant infinite subgroup of $D^{\Sigma}_{\infty}(F) = \cup_{n \in \mathbb{N}} \mathrm{Ker}(F^n) \cap \Sigma$ is dense in Σ.*

Then $\mu = \lambda_\Sigma$.

Theorem 3.3 is a common generalization of Theorem 1.2 and Theorem 1.3 when $\mathcal{A}$ is a cyclic group and $\mathcal{A}^{\mathbb{Z}}$ is the product group. To obtain a generalization of Theorem 1.3 for any Abelian group $\mathcal{A}^{\mathbb{Z}}$, we must take a weaker assumption for D^{Σ}_{∞}, however we need a further restriction for the probability measure:

Theorem 3.4. *Let $\mathcal{A}^{\mathbb{Z}}$ be any Abelian group shift, let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a nontrivial algebraic bipermutative CA, let Σ be a closed (F, σ)-invariant subgroup of $\mathcal{A}^{\mathbb{Z}}$, let $k \in \mathbb{N}$ such that every prime factor of $|\mathcal{A}|$ divides k and let μ be an (F, σ)-invariant probability measure on $\mathcal{A}^{\mathbb{Z}}$ with $\mathrm{supp}(\mu) \subset \Sigma$. Assume that:*

1. μ is ergodic for σ;

2. $\mathcal{I}_\mu(\sigma) = \mathcal{I}_\mu(\sigma^{kp_1})$ with p_1 the smallest common period of all elements of $\mathrm{Ker}(F)$*;*

3. $h_\mu(F) > 0$;

4. every (F,σ)-invariant infinite subgroup of $D_\infty^\Sigma(F) = \cup_{n\in\mathbb{N}}\mathrm{Ker}(F^n) \cap \Sigma$ is dense in Σ.

Then $\mu = \lambda_\Sigma$.

To do this some technical work is required on each of the assumptions. Presently we do not know how to obtain a common generalization of Theorems 3.3 and 3.4.

In Section 4 we show how to replace and relax some assumptions of Theorems 3.3 and 3.4, in particular how one obtains Theorems 1.2 and 1.3 as consequences. First we replace the assumption of positive entropy of F by the positive entropy of $F^n \circ \sigma^m$ for some $(n,m) \in \mathbb{N}\times\mathbb{Z}$. Then we give a necessary and sufficient condition for D_∞^Σ to contain no nontrivial (F,σ)-invariant infinite subgroups. This condition is implied by the assumption that $\mathrm{Ker}(F)$ contains no nontrivial σ-invariant subgroups.

In Section 5 we restrict the study to linear CA and obtain rigidity results which cannot be deduced from Theorem 1.2 and 1.3. For example, in Subsection 5.1, we can see that Theorem 3.3 works for $F = P_F(\sigma)$ any nontrivial linear CA on $(\mathbb{Z}/p\mathbb{Z})^\mathbb{Z}$ with p prime. In this case Theorem 1.2 works only for CA of radius 1 and Pivato's result works only if P_F is irreducible on $\mathbb{Z}/p\mathbb{Z}$. In Section 6 we give an application of this work. We stray from the algebraic bipermutative CA case and show measure rigidity for some affine one-sided invertible expansive CA (not necessary bipermutative) with the help of previous results.

2 Entropy formulas for bipermutative CA

Let $(\mathcal{A}^\mathbb{Z}, F)$ be a CA, $\mathfrak{B}$ be the Borel sigma-algebra of $\mathcal{A}^\mathbb{Z}$ and $\mu \in \mathcal{M}(\mathcal{A}^\mathbb{Z})$. We put $\mathfrak{B}_n = F^{-n}(\mathfrak{B})$ for $n \in \mathbb{N}$. For $\mathcal{P}$ a finite partition of $\mathcal{A}^\mathbb{Z}$ and for $\mathfrak{B}'$ a sub sigma-algebra of $\mathfrak{B}$ we denote $H_\mu(\mathcal{P}) = -\sum_{A\in\mathcal{P}} \mu(A)\log(\mu(A))$ the entropy of $\mathcal{P}$ and $H_\mu(\mathcal{P}|\mathfrak{B}') = -\sum_{A\in\mathcal{P}}\int_A \log(\mathbb{E}_\mu(1_A|\mathfrak{B}'))d\mu$ the conditional entropy of $\mathcal{P}$ given $\mathfrak{B}'$. Furthermore $h_\mu(F)$ denotes the entropy of the measure-preserving dynamical system $(\mathcal{A}^\mathbb{M}, \mathfrak{B}, \mu, F)$. We refer to [Pet89] or [Wal82] for the definition and main properties.

We define the cylinder partitions $\mathcal{P} = \{[a] : a \in \mathcal{A}\}$ and $\mathcal{P}_{[r,s]} = \{[u]_r : u \in \mathcal{A}^{s-r}\}$. The following lemma is a more general version of the entropy formula in Lemma 4.3. of [HMM03] (where this Lemma is proved for CA with radius 1):

Lemma 2.1. *Let $(\mathcal{A}^\mathbb{Z}, F)$ be a bipermutative CA of smallest neighborhood $\mathbb{U} = [r,s]$ with $r \le 0 \le s$ and let μ be an F-invariant probability measure on $\mathcal{A}^\mathbb{Z}$. Then $h_\mu(F) = H_\mu(\mathcal{P}_{[0,s-r-1]}|\mathfrak{B}_1)$.*

Proof. We have $h_\mu(F) = \lim_{l\to\infty} h_\mu(F, \mathcal{P}_{[-l,l]})$ with:

$$h_\mu(F,\mathcal{P}_{[-l,l]}) = \lim_{T\to\infty} H_\mu(\mathcal{P}_{[-l,l]}|\bigvee_{n=1}^{T} F^{-n}(\mathcal{P}_{[-l,l]})) = H_\mu(\mathcal{P}_{[-l,l]}|\bigvee_{n=1}^{\infty} F^{-n}(\mathcal{P}_{[-l,l]})).$$

Let $l \ge s-r$. By bipermutativity of F, for $T \ge 1$, it is equivalent to know $(F^n(x)_{[-l,l]})_{n\in[1,T]}$ and to know $F(x)_{[Tr-l,Ts+l]}$. This means that $\bigvee_{n=1}^T F^{-n}(\mathcal{P}_{[-l,l]}) = F^{-1}(\mathcal{P}_{[Tr-l,Ts+l]})$. By taking the limit as $l\to\infty$, we deduce (with the convention $\infty.0 = 0$):

$$\bigvee_{n=1}^{\infty} F^{-n}(\mathcal{P}_{[-l,l]}) = F^{-1}(\mathcal{P}_{[\infty.r-l,\infty.s+l]}).$$

So we have:

$$h_\mu(F,\mathcal{P}_{[-l,l]}) = H_\mu(\mathcal{P}_{[-l,l]}|F^{-1}\mathcal{P}_{[\infty.r-l,\infty.s+l]}).$$

Similarly, by bipermutativity of F, the knowledge of $F(x)_{[\infty.r-l,\infty.s+l]}$ and $x_{[0,s-r-1]}$ allows us to know $x_{[-l,l]}$ and vice versa. We deduce:

$$\mathcal{P}_{[0,s-r-1]} \vee F^{-1}(\mathcal{P}_{[\infty.r-l,\infty.s+l]}) = \mathcal{P}_{[-l,l]} \vee F^{-1}(\mathcal{P}_{[\infty.r-l,\infty.s+l]}).$$

Therefore,

$$h_\mu(F, \mathcal{P}_{[-l,l]}) = H_\mu(\mathcal{P}_{[0,s-r+1]} | F^{-1}(\mathcal{P}_{[\infty.r-l,\infty.s+l]})).$$

If $r < 0 < s$, then $\mathcal{P}_{[\infty.r-l,\infty.s+l]} = \mathfrak{B}_1$. Otherwise, by taking the limit as $l \to \infty$ and using the martingale convergence theorem, we obtain $h_\mu(F) = H_\mu(\mathcal{P}_{[0,s-r-1]} | \mathfrak{B}_1)$. □

When μ is an (F, σ)-invariant probability measure, it is possible to express the entropy of a right-permutative CA according to the entropy of σ.

Proposition 2.2. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a right-permutative CA of neighborhood $\mathbb{U} = [0, s]$, where s is the smallest possible value and let μ be an (F, σ)-invariant probability measure. Then $h_\mu(F) = s\, h_\mu(\sigma)$.*

Proof. Let $N \in \mathbb{N}$ and $l \geq s$. By right-permutativity, since $\mathbb{U} = [0, s]$, for all $x \in \mathcal{A}^{\mathbb{Z}}$ it is equivalent to know $(F^n(x)_{[-l,l]})_{n\in[0,N]}$ and $x_{[-l,l+Ns]}$; this means that:

$$\bigvee_{n=0}^{N} F^{-n}(\mathcal{P}_{[-l,l]}) = \mathcal{P}_{[-l,l+Ns]}.$$

So for $l \geq s$ we have:

$$\begin{aligned}
h_\mu(F, \mathcal{P}_{[-l,l]}) &= \lim_{N\to\infty} \frac{1}{N} H_\mu(\bigvee_{n=0}^{N} F^{-n}(\mathcal{P}_{[-l,l]})) \\
&= \lim_{N\to\infty} \frac{1}{N} H_\mu(\mathcal{P}_{[-l,l+Ns]}) \\
&= \lim_{N\to\infty} -\frac{1}{N} \sum_{u\in\mathcal{A}^{Ns+2l}} \mu([u]) \log(\mu[u]) \\
&= \lim_{N\to\infty} -\frac{Ns+2l}{N} \frac{1}{Ns+2l} \sum_{u\in\mathcal{A}^{Ns+2l}} \mu([u]) \log(\mu[u]) \\
&= s\, h_\mu(\sigma).
\end{aligned}$$

We deduce that $h_\mu(F) = \lim_{l\to\infty} h_\mu(F, \mathcal{P}_{[-l,l]}) = s\, h_\mu(\sigma)$. □

Remark 2.1. We have a similar formula for a left-permutative CA of neighborhood $\mathbb{U} = [r, 0]$. Moreover, it is easy to see that this proof is true for a right-permutative CA on $\mathcal{A}^{\mathbb{N}}$.

Corollary 2.3. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a bipermutative CA of smallest neighborhood $\mathbb{U} = [r, s]$, and let μ be an (F, σ)-invariant probability measure on $\mathcal{A}^{\mathbb{Z}}$. We have:*

$$h_\mu(F) = \begin{cases} s\, h_\mu(\sigma) & \text{if } s \geq r \geq 0, \\ (s - r)\, h_\mu(\sigma) & \text{if } s \geq 0 \geq r, \\ -r\, h_\mu(\sigma) & \text{if } 0 \geq s \geq r. \end{cases}$$

Proof. Cases where $s \geq r \geq 0$ or $0 \geq s \geq r$ can be directly deduced from Proposition 2.2.

When $s \geq 0 \geq r$, the CA $(\mathcal{A}^{\mathbb{Z}}, \sigma^{-r} \circ F)$ is bipermutative of smallest neighborhood $[0, s-r]$. Since σ is bijective, we deduce that $\mathfrak{B}$ is σ-invariant. Thus, $F^{-1}(\mathfrak{B}) = (\sigma^{-r} \circ F)^{-1}(\mathfrak{B})$. Since μ is (F, σ)-invariant, by Lemma 2.1, one has:

$$h_\mu(F) = H_\mu(\mathcal{P}_{[0,s-r-1]} | F^{-1}(\mathfrak{B})) = H_\mu(\mathcal{P}_{[0,s-r-1]} | (\sigma^{-r} \circ F)^{-1}(\mathfrak{B})) = h_\mu(\sigma^{-r} \circ F).$$

The result follows from Proposition 2.2. □

Remark 2.2. It is not necessary to use Lemma 2.1. Corollary 2.3 can be proved by a similar method of Proposition 2.2.

A bipermutative CA $(\mathcal{A}^{\mathbb{Z}}, F)$ of smallest neighborhood $\mathbb{U}$ is topologically conjugate to $((\mathcal{A}^t)^{\mathbb{N}}, \sigma)$ where $t = \max(\mathbb{U} \cup \{0\}) - \min(\mathbb{U} \cup \{0\})$, via the conjugacy $\varphi : x \in \mathcal{A}^{\mathbb{Z}} \to (F(x)_{[0,t]})_{n\in\mathbb{N}}$. So the uniform Bernoulli measure is a maximal entropy measure. Thus from Corollary 2.3 we deduce an expression of $h_{\text{top}}(F)$. This implies a result of [War00] which compute the topological entropy for linear CA on $(\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}}$ with p prime by algebraic methods. Moreover this formula gives Lyapunov exponents for permutative CA according to the definition of [She92] or [Tis00].

3 Proof of main theorems

Now we consider $(\mathcal{A}^{\mathbb{Z}}, F)$ a bipermutative algebraic CA of smallest neighborhood $\mathbb{U} = [r, s]$. For $y \in \mathcal{A}^{\mathbb{Z}}$ call T_y the translation $x \mapsto x + y$ on $\mathcal{A}^{\mathbb{Z}}$. For every $n \in \mathbb{N}$, we write $D_n(F) = \mathrm{Ker}(F^n)$; if there is no ambiguity we just denote it by D_n. Clearly D_n is a subgroup of D_{n+1}. Denote $\partial D_{n+1} = D_{n+1} \setminus D_n$ for all $n \in \mathbb{N}$. By bipermutativity we have $|D_n| = |D_1|^n = |\mathcal{A}|^{(s-r)n}$ where $|.|$ denotes the cardinality of the set. We can consider the subgroup $D_\infty(F) = \cup_{n\in\mathbb{N}} D_n(F)$ of $\mathcal{A}^{\mathbb{Z}}$, we denote it by D_∞ if there is no ambiguity; it is dense in $\mathcal{A}^{\mathbb{Z}}$ since F is bipermutative. Every D_n is finite and σ-invariant so every $x \in D_n$ is σ-periodic. Let p_n be the smallest common period of all elements of D_n. Then p_n divides $|D_n|!$.

Let $\mathfrak{B}$ be the Borel sigma-algebra of $\mathcal{A}^{\mathbb{Z}}$ and let μ be a probability measure on $\mathcal{A}^{\mathbb{Z}}$. Put $\mathfrak{B}_n = F^{-n}(\mathfrak{B})$ for every $n \in \mathbb{N}$, it is the sigma-algebra generated by all cosets of D_n. For every $n \in \mathbb{N}$ and μ-almost every $x \in \mathcal{A}^{\mathbb{Z}}$, the conditional measure $\mu_{n,x}$ is defined for every measurable set $U \subset \mathcal{A}^{\mathbb{Z}}$ by $\mu_{n,x}(U) = \mathbb{E}_\mu(\mathbf{1}_U|\mathfrak{B}_n)(x)$. Its main properties are:

(A) For μ-almost every $x \in \mathcal{A}^{\mathbb{Z}}$, $\mu_{n,x}$ is a probability measure on $\mathcal{A}^{\mathbb{Z}}$ and $\mathrm{supp}(\mu_{n,x}) \subset F^{-n}(\{F^n(x)\}) = x + D_n$.

(B) For all measurable sets $U \subset \mathcal{A}^{\mathbb{Z}}$, the function $x \to \mu_{n,x}(U)$ is $\mathfrak{B}_n$-measurable and $\mu_{n,x} = \mu_{n,y}$ for every $y \in F^{-n}(\{F^n(x)\}) = x + D_n$.

(C) Let $G : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a measurable map and let U be a measurable set. For μ-almost every $x \in \mathcal{A}^{\mathbb{Z}}$ one has $\mathbb{E}_\mu(\mathbf{1}_{G^{-1}(U)}|G^{-1}(\mathfrak{B}))(x) = \mathbb{E}_\mu(\mathbf{1}_U|\mathfrak{B})(G(x))$. So $\sigma^m\mu_{n,x} = \mu_{n,\sigma^m(x)}$ and $F\mu_{n+1,x} = \mu_{n,F(x)}$ for μ-almost every $x \in \mathcal{A}^{\mathbb{Z}}$ and every $n \in \mathbb{N}$.

(D) Since $\mathfrak{B}_n$ is T_d-invariant for $d \in D_n$, by (C) one has $\mu_{n,x} = \mu_{n,x+d}$.

For all $n \in \mathbb{N}$ define $\zeta_{n,x} = T_{-x}\mu_{n,x}$; it is a probability measure concentrated on D_n. The previous four properties of conditional measures can be transposed to $\zeta_{n,x}$:

Lemma 3.1. *Fix $n \in \mathbb{N}$. For μ-almost all $x \in \mathcal{A}^{\mathbb{Z}}$, the following are true:*

(a) $\zeta_{n,x+d} = T_{-d}\zeta_{n,x}$ for every $d \in D_n$.

(b) $\sigma^m\zeta_{n,x} = \zeta_{n,\sigma^m(x)}$ for every $m \in \mathbb{Z}$ and $F\zeta_{n+1,x} = \zeta_{n,F(x)}$.

(c) For every $m \in p_n\mathbb{Z}$, we have $\sigma^m\zeta_{n,x} = \zeta_{n,x}$. Hence $x \to \zeta_{n,x}$ is σ^m-invariant.

Proof. (a) is by Property (D). (b) is by Property (C). And (c) is because $\mathrm{supp}(\zeta_{n,x}) \subset D_n$. □

For $n > 0$ and $d \in D_n$ we define:

$$E_{n,d} = \{x \in \mathcal{A}^{\mathbb{Z}} : \zeta_{n,x}(\{d\}) > 0\} \text{ and } E_n = \bigcup_{d\in\partial D_n} E_{n,d}.$$

Then $E_{n,d}$ is σ^{p_n}-invariant by Lemma 3.1(c), and E_n is σ-invariant, because ∂D_n is σ-invariant. We write $\eta(x) = \zeta_{1,x}(\{0\}) = \mu_{1,x}(\{x\})$. The function η is σ-invariant and $E_1 = \{x \in \mathcal{A}^{\mathbb{Z}} : \eta(x) < 1\}$. Therefore one has:

$$\eta(F^{n-1}(x)) = \mu_{1,F^{n-1}(x)}(\{F^{n-1}(x)\}) = \mu_{1,F^{n-1}(x)}(F^{n-1}(x + D_{n-1})) \underset{(*)}{=} \mu_{n,x}(x + D_{n-1}) = \zeta_{n,x}(D_{n-1}),$$

where $(*)$ is by property (C). Thus $E_n = \{x \in \mathcal{A}^{\mathbb{Z}} : \zeta_{n,x}(D_{n-1}) < 1\} = F^{-n+1}(E_1)$.

Let $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ be a closed (F, σ)-invariant subgroup of $\mathcal{A}^{\mathbb{Z}}$. We denote $D_n^\Sigma = D_n \cap \Sigma$ and $\partial D_{n+1}^\Sigma = D_{n+1}^\Sigma \setminus D_n^\Sigma$ for all $n \in \mathbb{N}$ and $D_\infty^\Sigma = D_\infty \cap \Sigma$.

Remark 3.1. For μ an (F, σ)-invariant probability measure such that $\mathrm{supp}(\mu) \subset \Sigma$, we remark that for every $n \in \mathbb{N}$ and μ-almost every $x \in \mathcal{A}^{\mathbb{Z}}$, $\mathrm{supp}(\mu_{n,x}) \subset x + D_n^\Sigma \subset \Sigma$ and $\mathrm{supp}(\zeta_{n,x}) \subset D_n^\Sigma$. So for all $n \in \mathbb{N}$ and $d \in \partial D_n$, if $d \notin \Sigma$ one has $\mu(E_{n,d}) = 0$.

Lemma 3.2. *Let μ be a σ-invariant measure on $\mathcal{A}^{\mathbb{Z}}$. If there exist $k \in \mathbb{N}$ such that $\mathcal{I}_\mu(\sigma) = \mathcal{I}_\mu(\sigma^k)$ then for all $n \geq 1$ one has $\mathcal{I}_\mu(\sigma) = \mathcal{I}_\mu(\sigma^{k^n})$.*

Proof. Applying the ergodic decomposition theorem to $(\mathcal{A}^{\mathbb{Z}}, \mathfrak{B}, \mu, \sigma)$, to prove $\mathcal{I}_\mu(\sigma) = \mathcal{I}_\mu(\sigma^{k^n})$ it is equivalent to prove that almost every σ-ergodic component δ of μ is ergodic for σ^{k^n}. The proof is done by induction.

The base case $\mathcal{I}_\mu(\sigma) = \mathcal{I}_\mu(\sigma^{k^1})$ is true by hypothesis. Let $n \geq 2$ and assume that this property holds for $n-1$ and does not hold for n. That is to say we consider a σ-ergodic component δ of μ (by induction it is also $\sigma^{k^{n-1}}$-ergodic) which is not σ^{k^n}-ergodic. There exist $\lambda \in \mathbb{C}$ such that $\lambda^{k^n} = 1$ and $\lambda^{k^{n-1}} \neq 1$ and a non constant function $h : \mathcal{A}^{\mathbb{Z}} \to \mathbb{C}$ such that $h(\sigma(x)) = \lambda h(x)$ for δ-almost every $x \in \mathcal{A}^{\mathbb{Z}}$. We deduce that $h^k(\sigma(x)) = \lambda^k h^k(x)$ and $h^k(\sigma^{k^{n-1}}(x)) = \lambda^{k^n} h^k(x) = h^k(x)$ for δ-almost every $x \in \mathcal{A}^{\mathbb{Z}}$. By $\sigma^{k^{n-1}}$-ergodicity of δ, h^k is constant δ-almost everywhere, so $\lambda^k = 1$ which is a contradiction. $\square$

Remark 3.2. If k divides k' then $\mathcal{I}_\mu(\sigma) \subset \mathcal{I}_\mu(\sigma^k) \subset \mathcal{I}_\mu(\sigma^{k'})$. So if $\mathcal{I}_\mu(\sigma) = \mathcal{I}_\mu(\sigma^{k'})$ one also has $\mathcal{I}_\mu(\sigma) = \mathcal{I}_\mu(\sigma^k)$.

We recall the main theorem:

Theorem 3.3. *Let $\mathcal{A}^{\mathbb{Z}}$ be any Abelian group shift, let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a nontrivial algebraic bipermutative CA, let Σ be a closed (F,σ)-invariant subgroup of $\mathcal{A}^{\mathbb{Z}}$, let $k \in \mathbb{N}$ such that every prime factor of $|\mathcal{A}|$ divides k and let μ be an (F,σ)-invariant probability measure on $\mathcal{A}^{\mathbb{Z}}$ with* $\operatorname{supp}(\mu) \subset \Sigma$. *Assume that:*

1. *μ is ergodic for the $\mathbb{N} \times \mathbb{Z}$-action (F,σ);*
2. *$\mathcal{I}_\mu(\sigma) = \mathcal{I}_\mu(\sigma^{kp_1})$ with p_1 the smallest common period of all elements of* $\operatorname{Ker}(F)$;
3. *$h_\mu(F) > 0$;*
4. *every σ-invariant infinite subgroup of $D^{\Sigma}_{\infty}(F) = \cup_{n\in\mathbb{N}} \operatorname{Ker}(F^n) \cap \Sigma$ is dense in Σ.*

Then $\mu = \lambda_\Sigma$.

Proof. For all $n \in \mathbb{Z}$, F is bipermutative iff $\sigma^n \circ F$ is bipermutative. Since F is nontrivial, by Corollary 2.3, we deduce that $h_\mu(\sigma^n \circ F) > 0$ for all $n \in \mathbb{Z}$. Moreover μ is σ-invariant. So we can assume that the smallest neighborhood of F is $[0, r]$ with $r \in \mathbb{N}$.

Claim 1: *For all $n \in \mathbb{N}$, $\mathcal{I}_\mu(\sigma) = \mathcal{I}_\mu(\sigma^{kp_n})$, where p_n is the smallest common σ-period of D_n.*

Proof: Let $n \in \mathbb{N}$. Every $x \in D_n$ is a σ-periodic point of σ-period p_n, so by bipermutativity, every $y \in F^{-1}(\{x\})$ is σ-periodic. Since $\sigma^{p_n}(y) \in F^{-1}(\{x\})$, one has that p_n divides the σ-period of y. We deduce that p_n divides p_{n+1}. Moreover there exists $d \in D_1$ such that $\sigma^{p_n}(y) = y + d$, so $\sigma^{|D_1|p_n}(y) = y + |D_1|d = y$. We deduce that p_{n+1} divides $|\mathcal{A}|^r p_n$, because $|D_1| = |\mathcal{A}|^r$. By induction p_n divides $|\mathcal{A}|^{r(n-1)} p_1$. If m is large enough, then $|\mathcal{A}|^{r(n-1)}$ divides k^m, hence p_n divides $|\mathcal{A}^{r(n-1)} p_1|$ which divides $(kp_1)^m$. Thus, $\mathcal{I}_\mu(\sigma^{kp_n}) = \mathcal{I}_\mu(\sigma)$ by Remark 3.2, because $\mathcal{I}_\mu(\sigma^{(kp_n)^m}) = \mathcal{I}_\mu(\sigma)$ by Lemma 3.2 and hypothesis (2) of Theorem 3.3. $\diamond$ `Claim 1`

Claim 2: *For $n \in \mathbb{N}$ and $d \in D_n$, the measure $T_d(\mathbf{1}_{E_{n,d}}\mu)$ is absolutely continuous with respect to μ.*

Proof: Let $A \in \mathfrak{B}$ be such that $\mu(A) = 0$. Since $\mu(A) = \int_{\mathcal{A}^{\mathbb{Z}}} \mu_{n,x}(A) d\mu(x)$, we deduce that $\mu_{n,x}(A) = 0$ for μ-almost every $x \in \mathcal{A}^{\mathbb{Z}}$. In particular, $0 = \mu_{n,x}(A) \geq \mu_{n,x}(\{x + d\}) = \zeta_{n,x}(\{d\})$, for μ-almost every $x \in T_{-d}(A)$ because $x + d \in A$. Thus $x \notin E_{n,d}$ so $\mu(T_{-d}(A) \cap E_{n,d}) = 0$. This implies that $T_d(\mathbf{1}_{E_{n,d}}\mu)(A) = 0$, so $T_d(\mathbf{1}_{E_{n,d}}\mu)$ is absolutely continuous with respect to μ. $\diamond$ `Claim 2`

To prove the theorem, we consider $\chi \in \widehat{\mathcal{A}^{\mathbb{Z}}}$ with $\mu(\chi) \neq 0$ and we show that $\chi(x) = 1$ for all $x \in \Sigma$. We consider $\Gamma = \{d \in D^{\Sigma}_{\infty} : \chi(d) = \chi(\sigma^m(d)), \forall m \in \mathbb{Z}\}$, a σ-invariant subgroup of D^{Σ}_{∞}. We want to show that Γ is infinite and hence, dense in Σ by hypothesis (4). From this we will deduce that χ must be constant.

Claim 3: *There exists $N \subset \mathcal{A}^{\mathbb{Z}}$ with $\mu(N) = 1$ and $F(N) = N$ (up to a set of measure zero), satisfying the following property: For any $n \in \mathbb{N}$ and $d \in \partial D^{\Sigma}_n$, if there exists $x \in E_{n,d} \cap N$ with $\zeta_{n,x}(\chi) \neq 0$, then $d \in \Gamma$.*

Proof: For $n \in \mathbb{N}$, the function $x \to \zeta_{n,x}$ is σ^{kp_n}-invariant by Lemma 3.1(c). Since $\mathcal{I}_\mu(\sigma) = \mathcal{I}_\mu(\sigma^{kp_n})$ by Claim 1, we deduce that $\zeta_{n,x}$ is σ-invariant. So for μ-almost every $x \in \mathcal{A}^{\mathbb{Z}}$ and for any $m \in \mathbb{Z}$, we have $\sigma^m \zeta_{n,x} = \zeta_{n,x}$ (†). Since $T_d(\mathbf{1}_{E_{n,d}}\mu)$ is absolutely continuous with respect to μ by Claim 2, we have $\sigma^m \zeta_{n,x+d} = \zeta_{n,x+d}$ (‡) too, for μ-almost every $x \in E_{n,d}$, for every $d \in D_n$ and for every $m \in \mathbb{Z}$. We can compute:

$$T_{-\sigma^m d}\zeta_{n,x} \underset{(\dagger)}{=} T_{-\sigma^m d}\sigma^m \zeta_{n,x} = \sigma^m T_{-d}\zeta_{n,x} \underset{(*)}{=} \sigma^m \zeta_{n,x+d} \underset{(\ddagger)}{=} \zeta_{n,x+d} \underset{(*)}{=} T_{-d}\zeta_{n,x},$$

where (†) and (‡) are as above, and ($*$) is by Lemma 3.1(a). So $T_{\sigma^m d - d}\zeta_{n,x} = \zeta_{n,x}$ and by integration $(1 - \chi(\sigma^m d - d))\zeta_{n,x}(\chi) = 0$ for μ-almost every $x \in E_{n,d}$. Thus, there exists $N \subset \mathcal{A}^{\mathbb{Z}}$ with $\mu(N) = 1$, such that for all $d \in D_n$ and $x \in E_{n,d} \cap N$, if $\zeta_{n,x}(\chi) \neq 0$, then $\chi(\sigma^m(d))\chi(d)^{-1} = \chi(\sigma^m(d) - d) = 1$. Hence $\chi(\sigma^m(d)) = \chi(d)$ for all $m \in \mathbb{Z}$, which means $d \in \Gamma$. Moreover the set N is F-invariant up to a set of measure 0, because μ is F-invariant, thus $\mu(F(N)) = F\mu(F(N)) = \mu(F^{-1}(F(N))) \geq \mu(N) = 1$.
◇ Claim 3

Claim 4: *There exists $n_0 \in \mathbb{N}$ such that, if we define $B = \{x \in N : \mathbb{E}_\mu(\chi|\mathfrak{B}_n)(x) \neq 0, \forall n \geq n_0\}$, then $\mu(B) > 0$. Moreover for all $n \geq n_0$, and any $d \in \partial D_n^\Sigma$, if $E_{n,d} \cap B \neq \emptyset$, then $d \in \Gamma$.*

Proof: One has $\lim_{n\to\infty} \mathbb{E}_\mu(\chi|\mathfrak{B}_n) = \mathbb{E}_\mu(\chi| \cap_{m>1} \mathfrak{B}_m)$ by the Martingale Convergence Theorem, and this function is not identically 0 because its integral is equal to $\mu(\chi) \neq 0$. Thus we can choose n_0 such that $B = \{x \in N : \mathbb{E}_\mu(\chi|\mathfrak{B}_n)(x) \neq 0, \forall n \geq n_0\}$ satisfies $\mu(B) > 0$. Moreover, we have:

$$\mathbb{E}_\mu(\chi|\mathfrak{B}_n)(x) = \int_{\mathcal{A}^{\mathbb{Z}}} \chi d\mu_{n,x} = \chi(x)\zeta_{n,x}(\chi).$$

By Claim 3, for any $n \geq n_0$ and any $d \in \partial D_n^\Sigma$, if there is $x \in E_{n,d} \cap B$ then $d \in \Gamma$. ◇ Claim 4

Claim 5: $\mu(E_1) > 0$.

Proof: Let $A \in \mathcal{P}_{[0,r-1]}$. Let $x \in A$ and $d \in D_1$ such that $x + d \in A$. One has $x_{[0,r-1]} = (x+d)_{[0,r-1]}$ and $F(x) = F(x+d)$. By bipermutativity, one deduces that $x = x + d$, that is to say $d = 0$. Therefore, for any $x \in A$ and for any $d \in \partial D_1$, we have $x + d \notin A$. Thus, $A \cap F^{-1}(\{F(x)\}) = A \cap (x + D_1) = \{x\}$. Thus, (A) implies that $\mathbb{E}_\mu(\mathbf{1}_A|\mathfrak{B}_1)(x) = \mu_{1,x}(A) = \mu_{1,x}(\{x\}) = \eta(x)$. By Lemma 2.1,

$$\begin{aligned}
h_\mu(F) &= H_\mu(\mathcal{P}_{[0,r-1]}|\mathfrak{B}_1) \\
&= -\sum_{A \in \mathcal{P}_{[0,r-1]}} \int_A \log\left(\mathbb{E}_\mu(\mathbf{1}_A|\mathfrak{B}_1)\right) d\mu \\
&= \int_{\mathcal{A}^{\mathbb{Z}}} -\log(\eta(x)) d\mu(x) \\
&\underset{(*)}{\leq} \int_{E_1} -\log(\eta(x)) d\mu(x),
\end{aligned}$$

where ($*$) is because $E_1 = \{x \in \mathcal{A}^{\mathbb{Z}} : \eta(x) < 1\}$. But $h_\mu(F) > 0$ by hypothesis (3). This proves Claim 5.
◇ Claim 5

Claim 6: *Γ is infinite.*

Proof: For μ-almost every $x \in \mathcal{A}^{\mathbb{Z}}$ one has:

$$\frac{1}{n}\sum_{j=1}^{n+1} \mathbf{1}_{E_j}(x) \underset{(1)}{=} \frac{1}{n}\sum_{j=1}^{n+1} \mathbf{1}_{F^{-j+1}(E_1)}(x) = \frac{1}{n}\sum_{j=0}^{n} \mathbf{1}_{E_1}(F^j(x)) \underset{(2)}{=} \frac{1}{n^2}\sum_{j,k=0}^{n} \mathbf{1}_{E_1}(\sigma^k F^j(x)) \underset{(3)}{\longrightarrow} \mu(E_1) \underset{(4)}{>} 0.$$

Here, (1) is because $E_j = F^{-j+1}(E_1)$ for all $j \in \mathbb{N}$, (2) is because E_1 is σ-invariant, (3) is the Ergodic Theorem and hypothesis 1, and (4) is by Claim 5.

It follows that for μ-almost every $x \in \mathcal{A}^{\mathbb{Z}}$, there are infinitely many values of $n > 0$ such that $x \in E_n$. Thus $\mu(\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} E_n) = 1$. Since $\mu(B) > 0$, we deduce that $\mu(\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} E_n \cap B) > 0$. For all $n \in \mathbb{N}$, if $d \notin \mathrm{supp}(\mu) \subset \Sigma$, then Remark 3.1 implies that $\mu(E_{n,d}) = 0$. We can conclude that $\{d \in D_\infty^\Sigma : \exists n \in \mathbb{N} \text{ such that } d \in \partial D_n \text{ and } E_{n,d} \cap B \neq \emptyset\}$ is infinite and by Claim 4, it is a subset of Γ. Therefore Γ is infinite. ◇ Claim 6

If we consider $\Gamma' = (\mathrm{Id}_{\mathcal{A}^{\mathbb{Z}}} - \sigma)\Gamma$, we have an infinite σ-invariant subgroup of D_∞^Σ because $\mathrm{Ker}(\mathrm{Id}_{\mathcal{A}^{\mathbb{Z}}} - \sigma)$ is finite. Hypothesis (4) then implies that Γ' is dense in Σ, but by construction, $\chi(\Gamma') = \{1\}$, so by continuity of χ, $\chi(x) = 1$ for all $x \in \Sigma$. Contrapositively, we must have $\mu(\chi) = 0$ for all $\chi \in \widehat{\mathcal{A}^{\mathbb{Z}}}$ such that $\chi(\Sigma) \neq \{1\}$. Since $\mathrm{supp}(\mu) \subset \Sigma$, we conclude that $\mu = \lambda_\Sigma$. □

Remark 3.3. The proof of this theorem works if $(\mathcal{A}^{\mathbb{N}}, F)$ is a right-permutative algebraic CA where all $x \in D_1 = \mathrm{Ker}(F)$ are σ-periodic, but this last assumption is possible only if F is also left-permutative, therefore it is a false generalization.

Remark 3.4. Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a nontrivial algebraic bipermutative CA and let Σ be a closed (F,σ)-invariant subgroup of $\mathcal{A}^{\mathbb{Z}}$ which verifies hypothesis (4) of Theorem 3.3. Let $c \in \Sigma$ be a σ-invariant configuration. We define the CA $G = F + c$. Let μ be a (G,σ)-invariant probability measure on $\mathcal{A}^{\mathbb{Z}}$. If μ verifies the assumptions of Theorem 3.3 for the $\mathbb{N} \times \mathbb{Z}$-action induced by (G,σ), then $\mu = \lambda_\Sigma$.

Assumption (4) becomes more natural when it is replaced by "every (F,σ)-invariant infinite subgroup of $D_\infty^\Sigma(F) = \cup_{n\in\mathbb{N}} \mathrm{Ker}(F^n) \cap \Sigma$ is dense in Σ". It is not clear that this condition is implied by the assumptions of Theorem 3.3. However if we consider a σ-ergodic measure we can prove:

Theorem 3.4. *Let $\mathcal{A}^{\mathbb{Z}}$ be any Abelian group shift, let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a nontrivial algebraic bipermutative CA, let Σ be a closed (F,σ)-invariant subgroup of $\mathcal{A}^{\mathbb{Z}}$, let $k \in \mathbb{N}$ such that every prime factor of $|\mathcal{A}|$ divides k and let μ be an (F,σ)-invariant probability measure on $\mathcal{A}^{\mathbb{Z}}$ with* $\mathrm{supp}(\mu) \subset \Sigma$. *Assume that:*

1. *μ is ergodic for σ;*
2. *$\mathcal{I}_\mu(\sigma) = \mathcal{I}_\mu(\sigma^{kp_1})$ with p_1 the smallest common period of all elements of* $\mathrm{Ker}(F)$*;*
3. *$h_\mu(F) > 0$;*
4. *every (F,σ)-invariant infinite subgroup of $D_\infty^\Sigma(F) = \cup_{n\in\mathbb{N}} \mathrm{Ker}(F^n) \cap \Sigma$ is dense in Σ.*

Then $\mu = \lambda_\Sigma$.

Proof. A measure σ-ergodic is (F,σ)-ergodic so results from Claim 1 to Claim 6 hold.

Claim 7: *Let $B' = \cup_{j\in\mathbb{Z}} \sigma^j(\{x \in N : \mathbb{E}_\mu(\chi|\mathfrak{B}_n)(x) \neq 0, \forall n \in \mathbb{N}\})$. Then $\mu(B') = 1$.*

Proof: By Claim 4, $\mu(B) > 0$ where $B = \{x \in N : \mathbb{E}_\mu(\chi|\mathfrak{B}_n)(x) \neq 0, \forall n \geq n_0\}$. Thus there exists $k \in [0,3]$ such that $B_{n_0} = \{x \in N : \Re(\mathbf{i}^k \mathbb{E}_\mu(\chi|\mathfrak{B}_{n_0})) > 0, \forall n \geq n_0\}$ verifies $\mu(B_{n_0}) > 0$, where $\mathbf{i}^2 = -1$. Since $B_{n_0} \in \mathfrak{B}_{n_0} \subset \mathfrak{B}_{n_0-1}$, one has:

$$\int_{B_{n_0}} \Re(\mathbf{i}^k \mathbb{E}_\mu(\chi|\mathfrak{B}_{n_0-1}))(x) d\mu = \int_{B_{n_0}} \Re(\mathbf{i}^k \mathbb{E}_\mu(\chi|\mathfrak{B}_{n_0})(x)) d\mu > 0$$

So $B_{n_0-1} = \{x \in B_{n_0} : \Re(\mathbf{i}^k \mathbb{E}_\mu(\chi|\mathfrak{B}_{n_0-1})(x)) > 0\} = \{x \in N : \Re(\mathbf{i}^k \mathbb{E}_\mu(\chi|\mathfrak{B}_n)(x)) > 0, \forall n \geq n_0 - 1\}$ verify $\mu(B_{n_0-1}) > 0$. By induction $\mu(B_0) > 0$, so $\mu(B') > 0$. Since B' is σ-invariant, $\mu(B') = 1$ by σ-ergodicity from hypothesis (1). ◇ `Claim 7`

Claim 8: *Let $n \in \mathbb{N}$ and let $d \in \partial D_n^\Sigma$. If $E_{n,d} \cap B'$ is nonempty then $d \in \Gamma = \{d' \in D_\infty^\Sigma : \chi(d') = \chi(\sigma^m(d')), \forall m \in \mathbb{Z}\}$*

Proof: Let $d \in \partial D_n^\Sigma$ and let $x \in E_{n,d} \cap B'$. There exists $j \in \mathbb{Z}$ such that:

$$0 \neq \mathbb{E}_\mu(\chi|\mathfrak{B}_n)(\sigma^j(x)) = \int_{\mathcal{A}^{\mathbb{Z}}} \chi d\mu_{n,\sigma^j(x)} = \chi(\sigma^j(x))\zeta_{n,\sigma^j(x)}(\chi) \underset{(*)}{=} \chi(\sigma^j(x))\zeta_{n,x}(\chi).$$

Here $(*)$ is because $x \to \zeta_{n,x}$ is σ^{kp_n}-invariant by Lemma 3.1(c) and $\mathcal{I}_\mu(\sigma) = \mathcal{I}_\mu(\sigma^{kp_n})$ by Claim 1, so $x \to \zeta_{n,x}$ is σ-invariant. One deduces that $\zeta_{n,x}(\chi) \neq 0$. But $x \in E_{n,d} \cap N$, so $d \in \Gamma$ by Claim 3. ◇ `Claim 8`

Claim 9: *Let $n \geq 1$ and let $d \in \partial D_n^\Sigma$. For μ-almost all $x \in E_{n,d} \cap B'$ one has $F(x) \in E_{n-1,F(d)} \cap B'$.*

Proof: Let $d \in \partial D_n^\Sigma$ and $x \in E_{n,d} \cap B'$. One has:

$$\zeta_{n-1,F(x)}(\{F(d)\}) \underset{(1)}{=} \zeta_{n,x}(F^{-1}(\{F(d)\})) \geq \zeta_{n,x}(\{d\}) \underset{(2)}{>} 0.$$

Here (1) is by Lemma 3.1(b) and (2) is because $x \in E_{n,d}$. We deduce that $F(x) \in E_{n-1,F(d)}$. Since $\mu(B') = 1$ by Claim 7 and μ is F-invariant, one has $\mu(\cap_{n\in\mathbb{N}} F^{-n}(B')) = 1$ so $F(x) \in E_{n-1,F(d)} \cap B'$ for μ-almost all $x \in E_{n,d} \cap B'$. $\diamondsuit$ `Claim 9`

Claim 10: $\cap_{n\in\mathbb{N}} F^{-n}\Gamma$ *is infinite.*

Proof: Let $n \geq 0$. The set $E_n = F^{-n+1}(E_1)$ is σ-invariant since E_1 is σ-invariant and F commutes with σ. Moreover $\mu(E_n) = \mu(E_1) > 0$ by Claim 5. By σ-ergodicity (hypothesis (1)), $\mu(E_n) = 1$ so $\mu(E_n \cap B') = 1$ by Claim 7. For all $n \geq 1$, there exists $d_n \in \partial D_n^\Sigma$ such that $\mu(E_{n,d_n} \cap B') > 0$, and thus, by Claim 9, $\mu(E_{n-k,F^k(d_n)} \cap B') > 0$ for all $k \in [0, n]$. That is to say $F^k(d_n) \in \Gamma$ for $k \in [0, n]$ by Claim 8. One deduces that $\cap_{n\in\mathbb{N}} F^{-n}\Gamma$ is infinite since it contains d_n for all $n \in \mathbb{N}$. $\diamondsuit$ `Claim 10`

If we consider $\Gamma'' = (\mathrm{Id}_{\mathcal{A}^\mathbb{Z}} - \sigma)(\cap_{n\in\mathbb{N}} F^{-n}\Gamma)$, we have an infinite (F,σ)-invariant subgroup of D_∞^Σ because $\mathrm{Ker}(\mathrm{Id}_{\mathcal{A}^\mathbb{Z}} - \sigma)$ is finite. We deduce that Γ'' is dense in Σ by condition (4), but $\chi(\Gamma'') = \{1\}$ by construction, so by continuity of χ, $\chi(x) = 1$ for all $x \in \Sigma$. Contrapositively, we must have $\mu(\chi) = 0$ for all $\chi \in \widehat{\mathcal{A}^\mathbb{Z}}$ such that $\chi(\Sigma) \neq \{1\}$. Since $\mathrm{supp}(\mu) \subset \Sigma$, we conclude that $\mu = \lambda_\Sigma$. $\square$

Corollary 3.5. *Let $\mathcal{A}^\mathbb{Z}$ be any Abelian group shift, let $(\mathcal{A}^\mathbb{Z}, F)$ be an algebraic bipermutative CA. Let Σ be a closed (F,σ)-invariant subgroups of $\mathcal{A}^\mathbb{Z}$ such that there exists $\pi : \mathcal{A}^\mathbb{Z} \to \Sigma$ a surjective continuous morphism which commutes with F and σ ((Σ, σ, F) is a dynamical and algebraic factor of $(\mathcal{A}^\mathbb{Z}, \sigma, F)$). Let $k \in \mathbb{N}$ be such that every prime factor of $|\mathcal{A}|$ divides k. Let μ be an (F,σ)-invariant probability measure on $\mathcal{A}^\mathbb{Z}$. Assume that:*

1. *μ is ergodic for the $\mathbb{N} \times \mathbb{Z}$-action (F, σ);*
2. *$\mathcal{I}_\mu(\sigma) = \mathcal{I}_\mu(\sigma^{kp_1})$ with p_1 the smallest common period of all elements of $\mathrm{Ker}(F)$;*
3. *$h_{\pi\mu}(F) > 0$;*
4. *every σ-invariant infinite subgroup of $D_\infty^\Sigma = \cup_{n\in\mathbb{N}} \mathrm{Ker}(F^n) \cap \Sigma$ is dense in Σ.*

Then $\pi\mu = \lambda_\Sigma$.

4 A discussion about the assumptions

Comparing the assumptions of Theorems 3.3 and 3.4 with those of Theorems 1.1, 1.2 and 1.3 is not completely obvious. Already Theorems 3.3 and 3.4 consider bipermutative algebraic CA without restriction on the neighborhood. In this section we discuss about the assumptions of these theorems and show that Theorems 3.3 and 3.4 generalize Theorems 1.2 and 1.3 but the ergodic assumptions cannot be compared with these of Theorem 1.1.

4.1 Class of CA considered

Theorems 3.3 and 3.4 consider algebraic bipermutative CA without restriction on the neighborhood. The bipermutativity is principally used to prove the entropy formula of Lemma 2.1. We can hope such formula for expansive CA. Subsection 4.3 gives a result in this direction. The next proposition shows that it is equivalent to consider algebraic CA or the restriction of a linear CA.

Proposition 4.1. *Let $\mathcal{A}^\mathbb{Z}$ be any Abelian group shift and let $(\mathcal{A}^\mathbb{Z}, F)$ be an algebraic CA. There exist $(\mathcal{B}^\mathbb{Z}, G)$ a linear CA and Γ a $\sigma_{\mathcal{B}^\mathbb{Z}}$-invariant subgroup of $\mathcal{B}^\mathbb{Z}$ such that $(\mathcal{A}^\mathbb{Z}, \sigma, F)$ is isomorphic to $(\Gamma, \sigma_{\mathcal{B}^\mathbb{Z}}, G)$ in both dynamical and algebraical sense.*

Proof. Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be an algebraic CA. By B.P. Kitchens [Kit87, Proposition 2], there exists $\mathcal{B}'$ a finite Abelian group, Γ a Markov subgroup of $\mathcal{B}'^{\mathbb{Z}}$ and φ a continuous group isomorphism such that $\varphi \circ \sigma = \sigma_{\mathcal{B}'^{\mathbb{Z}}} \circ \varphi$. Define $G' = \varphi \circ F \circ \varphi^{-1}$. One has the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A}^{\mathbb{Z}} & \xrightarrow{\sigma, F} & \mathcal{A}^{\mathbb{Z}} \\ \big\downarrow \varphi & & \big\downarrow \varphi \\ \Gamma & \xrightarrow{\sigma_{\mathcal{B}'^{\mathbb{Z}}}, G'} & \Gamma \end{array} \qquad \begin{array}{c} \varphi \circ \sigma = \sigma_{\mathcal{B}'^{\mathbb{Z}}} \circ \varphi \\ \varphi \circ F = G' \circ \varphi \end{array}$$

G' is continuous and commutes with $\sigma_{\mathcal{B}'^{\mathbb{Z}}}$, so it is a CA on Γ'. We want to extend G' to obtain a linear CA. By G.A. Hedlund [Hed69], there exist a neighborhood $\mathbb{U}$, H a subgroup of $\mathcal{B}'^{\mathbb{U}}$ and a local function $\overline{G'} : H \to \mathcal{B}'$ which define G'. Moreover, by linearity, $\overline{G'}$ is a group morphism. If we could extend $\overline{G'}$ to a morphism from $\mathcal{B}^{\mathbb{U}}$ to $\mathcal{B}$ (where $\mathcal{B}'$ was a subgroup of $\mathcal{B}$), we would obtain the local rule of a linear CA.

There exist $d, k \in \mathbb{N}$ such that $\mathcal{B}'$ can be viewed as a subgroup of $(\mathbb{Z}/d\mathbb{Z})^k$. If $\mathcal{B} = (\mathbb{Z}/d\mathbb{Z})^k$, then H can be viewed as a subgroup of $\mathcal{B}^{\mathbb{U}}$. By the Fundamental Theorem of Finitely Generated Abelian Group [Lan02, Theorem 7.8], there exist $e_1, ..., e_{k|\mathbb{U}|}$ a basis of $\mathcal{B}^{\mathbb{U}}$ and $a_1, ..., a_{k|\mathbb{U}|} \in \mathbb{N}$ such that $\mathcal{B}^{\mathbb{U}} = \bigoplus_i \langle e_i \rangle$ and $H = \bigoplus_i \langle a_i e_i \rangle$. For all $i \in [1, k|\mathbb{U}|]$, there exist $f_i \in \mathcal{B}$ such that $\overline{G'}(a_i e_i) = a_i f_i$ because the order of $\overline{G'}(a_i e_i)$ is at most $\frac{d}{a_i}$. We define the morphism $\overline{G} : \mathcal{B}^{\mathbb{U}} \to \mathcal{B}$ by $\overline{G}(e_i) = f_i$ for all $i \in [1, k|\mathbb{U}|]$. $\overline{G}$ defines a linear CA on $\mathcal{B}^{\mathbb{Z}}$ denoted G whose the restriction is (Γ, G'). □

Remark 4.1. The study of algebraic CA can be restricted to the study of the restriction of linear CA to Markov subgroups.

Since we consider σ-invariant measures, we can assume that the neighborhood of the CA is $\mathbb{U} = [0, r]$. Moreover it is easy to show the next Proposition and consider CA of neighborhood $\mathbb{U} = [0, 1]$.

Proposition 4.2. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA of neighborhood $\mathbb{U} = [0, r]$. There is a CA $((\mathcal{A}^r)^{\mathbb{Z}}, G)$ of neighborhood $\mathbb{U} = [0, 1]$ so that the topological system $(\mathcal{A}^{\mathbb{Z}}, F)$ is isomorphic to the system $((\mathcal{A}^r)^{\mathbb{Z}}, G)$ via the conjugacy*

$$\phi_r : (x_i)_{i \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}} \to ((x_{[ri, ri+r-1]})_{i \in \mathbb{Z}}) \in (\mathcal{A}^r)^{\mathbb{Z}}.$$

Furthermore one has:

$$\begin{aligned} (\mathcal{A}^{\mathbb{Z}}, F) \text{ is bipermutative} &\iff ((\mathcal{A}^r)^{\mathbb{Z}}, G) \text{ is bipermutative;} \\ (\mathcal{A}^{\mathbb{Z}}, F) \text{ is algebraic} &\iff ((\mathcal{A}^r)^{\mathbb{Z}}, G) \text{ is algebraic;} \\ (\mathcal{A}^{\mathbb{Z}}, F) \text{ is linear} &\iff ((\mathcal{A}^r)^{\mathbb{Z}}, G) \text{ is linear.} \end{aligned}$$

If $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ is σ-totally ergodic then $\phi_r \mu \in \mathcal{M}((\mathcal{A}^r)^{\mathbb{Z}})$ is $\sigma_{(\mathcal{A}^r)^{\mathbb{Z}}}$-totally ergodic. Moreover, by conjugacy, $h_\mu(F) > 0$ is equivalent to $h_{\phi_r \mu}(G) > 0$. So, as suggested in [Piv05], Theorem 1.3 holds for algebraic bipermutative CA without any restriction on the neighborhood.

Corollary 4.3. *Let $\mathcal{A}^{\mathbb{Z}}$ be any Abelian group shift, let $(\mathcal{A}^{\mathbb{Z}}, F)$ be an algebraic bipermutative CA (without restriction on the neighborhood) and let μ be an (F, σ)-invariant probability measure. Assume that:*

1. *μ is totally ergodic for σ;*
2. *$h_\mu(F) > 0$;*
3. $\mathrm{Ker}(F)$ *contains no nontrivial σ-invariant subgroups.*

Then $\mu = \lambda_{\mathcal{A}^{\mathbb{Z}}}$.

Remark 4.2. The correspondence holds only if μ is supposed to be σ-totally ergodic. Indeed if μ is σ-ergodic, $\phi_r \mu$ is not necessarily $\sigma_{(\mathcal{A}^r)^{\mathbb{Z}}}$-ergodic

4.2 Ergodicity of action

Assumption (1) of Theorem 3.3 characterizes the ergodicity of the action (F,σ) on the measure space $(\mathcal{A}^{\mathbb{Z}},\mathfrak{B},\mu)$. Since we want to characterize (F,σ)-invariant measures, it is natural to assume that μ is (F,σ)-ergodic because every (F,σ)-invariant measure can be decomposed into (F,σ)-ergodic components. The next relations are easy to check for an (F,σ)-invariant probability measure μ:

$$\mu \text{ is } (F,\sigma)\text{-totally ergodic} \Rightarrow \mu \text{ is } \sigma\text{-totally ergodic} \Rightarrow \mu \text{ is } \sigma\text{-ergodic} \Rightarrow \mu \text{ is } (F,\sigma)\text{-ergodic};$$

$$\mu \text{ is } \sigma\text{-totally ergodic} \Rightarrow \mu \text{ is } (F,\sigma)\text{-ergodic and } \mathcal{I}_\mu(\sigma)=\mathcal{I}_\mu(\sigma^k) \text{ for every } k\geq 1.$$

Thus, hypothesis (1) of Theorem 1.3 implies hypothesis (1) and (2) of Theorem 3.4 which imply hypothesis (1) and (2) of Theorem 3.3. However, we remark that the ergodicity assumption (1) of Theorem 1.1 cannot be compared with hypothesis of Theorem 3.3. Indeed, there are probability measures which are (F,σ)-ergodic with $\mathcal{I}_\mu(\sigma)=\mathcal{I}_\mu(\sigma^k)$ for some $k\geq 1$ which are not σ-ergodic. Conversely there exist probability measures which are σ-ergodic with $\mathcal{I}_\mu(\sigma)\neq\mathcal{I}_\mu(\sigma^k)$ for some $k\geq 1$.

Secondly, if $\mathcal{A}=\mathbb{Z}/p\mathbb{Z}$ and $F=a\,\mathrm{Id}+b\,\sigma$ on $\mathcal{A}^{\mathbb{Z}}$ then $p-1$ is a multiple of the common period of every element of $\mathrm{Ker}(F)$. So the spectrum assumption (2) of Theorem 1.2 implies hypothesis (2) of Theorems 3.3 and 3.4. For Theorem 1.3 the total ergodicity of μ under σ is required. This property does not seem to be very far from hypothesis (2) of Theorems 3.3 and 3.4. But condition (2) of Theorems 3.3 and 3.4 (concerning the σ-invariant set) shows the importance of the algebraic characteristic of the system. The property of (F,σ)-total ergodicity of μ is more restrictive. With such an assumption Einsiedler [Ein05] proves rigidity results for a class of algebraic actions that are not necessarily CA. To finish, the next example shows that assumption (2) of Theorems 3.3 and 3.4 is necessary to obtain the characterization of the uniform Bernoulli measure.

Example 4.3. Let $\mathcal{A}=\mathbb{Z}/2\mathbb{Z}$ and $F=\mathrm{Id}+\sigma$ on $\mathcal{A}^{\mathbb{Z}}$. We consider the subgroup $X_1=\{x\in\mathcal{A}^{\mathbb{Z}}: x_{2n}=x_{2n+1},\forall n\in\mathbb{Z}\}$, it is neither σ-invariant nor F-invariant. Let $X_2=\sigma(X_1)=\{x\in\mathcal{A}^{\mathbb{Z}}: x_{2n}=x_{2n-1},\forall n\in\mathbb{Z}\}$, $X_3=F(X_1)=\{x\in\mathcal{A}^{\mathbb{Z}}: x_{2n}=0,\forall n\in\mathbb{Z}\}$ and $X_4=F(X_2)=\{x\in\mathcal{A}^{\mathbb{Z}}: x_{2n+1}=0,\forall n\in\mathbb{Z}\}$. The set $X=X_1\cup X_2\cup X_3\cup X_4$ is (F,σ)-invariant. Let ν be the Haar measure on X_1. We consider $\mu=\frac{1}{4}(\nu+\sigma\nu+F\nu+F\sigma\nu)$. It is easy to verify that μ is an (F,σ)-ergodic measure such that $h_\mu(\sigma)>0$. However $X_i\in\mathcal{I}_\mu(\sigma^2)\smallsetminus\mathcal{I}_\mu(\sigma)$ for all $i\in[1,4]$, hence hypothesis (2) is false, so we cannot apply Theorem 3.3 and μ it is not the uniform Bernoulli measure. S. Silberger propose similar constructions in [Sil05].

4.3 Positive entropy

Corollary 2.3 shows that for a nontrivial bipermutative CA $(\mathcal{A}^{\mathbb{Z}},F)$, the assumption of positive entropy of F can be replaced by the positive entropy of $F^n\circ\sigma^m$ for some $(n,m)\in\mathbb{N}\times\mathbb{Z}$. So the positive entropy hypothesis (3) of Theorems 3.3 and 3.4 can be replaced by the positive entropy of the action (F,σ) in some given direction. We can find this type of assumption in [Ein05].

We can also expect a similar formula for an expansive CA F but in this case we have the inequality: $h_\mu(F)>0$ iff $h_\mu(\sigma)>0$. To begin we show an inequality for a general CA.

Proposition 4.4. *Let $(\mathcal{A}^{\mathbb{Z}},F)$ be a CA of neighborhood $\mathbb{U}=[r,s]\ni 0$ (not necessarily the smallest possible one) and μ be an (F,σ)-invariant probability measure on $\mathcal{A}^{\mathbb{Z}}$. Then $h_\mu(F)\leq(r-s)\,h_\mu(\sigma)$.*

Proof. By definition, for $N\in\mathbb{N}$, $l\in\mathbb{N}$ and $x\in\mathcal{A}^{\mathbb{Z}}$, the knowledge of $x_{[rN-l,sN+l]}$ determines $(F^n(x)_{[-l,l]})_{n\in[0,N]}$. This means that $\mathcal{P}_{[rN-l,sN+l]}$ is a refinement of $\bigvee_{n=0}^{N}F^{-n}(\mathcal{P}_{[-l,l]})$. So for $l\geq\max(s,-r)$ we have:

$$\begin{aligned}
h_\mu(F,\mathcal{P}_{[-l,l]}) &= \lim_{N\to\infty}\frac{1}{N}H_\mu\Big(\bigvee_{n=0}^{N}F^{-n}(\mathcal{P}_{[-l,l]})\Big)\\
&\leq \lim_{N\to\infty}\frac{1}{N}H_\mu(\mathcal{P}_{[rN-l,Ns+l]})\\
&= \lim_{N\to\infty} -\frac{N(s-r)+2l}{N}\,\frac{1}{N(s-r)+2l}\sum_{u\in\mathcal{A}^{N(s-r)+2l}}\mu([u])\log(\mu[u])\\
&= (s-r)\,h_\mu(\sigma).
\end{aligned}$$

We deduce that $h_\mu(F)=\lim_{l\to\infty}h_\mu(F,\mathcal{P}_{[-l,l]})\leq(s-r)\,h_\mu(\sigma)$. □

Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a positively expansive CA. There exists r_e, the *constant of expansivity*, such as for all $x, y \in \mathcal{A}^{\mathbb{Z}}$ if $x \neq y$ there exists $n \in \mathbb{N}$ which verifies $F^n(x)_{[-r_e,r_e]} \neq F^n(y)_{[-r_e,r_e]}$. Then $(\mathcal{A}^{\mathbb{Z}}, F)$ is topologically conjugate to the one-sided subshift (S_F, σ), where $S_F \subset \mathcal{B}^{\mathbb{N}}$, with $\mathcal{B} = \mathcal{A}^{2r_e+1}$, and where $S_F = \{(F^i(x)_{[-r_e,r_e]})_{i\in\mathbb{N}} : x \in \mathcal{A}^{\mathbb{Z}}\}$, via the conjugacy $\phi_F : x \in \mathcal{A}^{\mathbb{Z}} \to (F^i(x)_{[-r_e,r_e]})_{i\in\mathbb{N}} \in S_F$. Define $F_T : S_F \to S_F$ by $F_T \circ \phi_F(x) = \phi_F \circ \sigma^{r_e}(x)$ for every $x \in \mathcal{A}^{\mathbb{Z}}$. (S_F, F_T) is an invertible one-sided CA. Define the *radius of expansivity* $r_T = \max\{r(F_T), r(F_T^{-1})\}$.

Proposition 4.5. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a positively expansive CA and μ an (F,σ)-invariant probability measure, then $h_\mu(F) \geq \frac{1}{r_T} h_\mu(\sigma)$.*

Proof. By definition of r_T, for $N \in \mathbb{N}$, $l \geq r_e$ and $x \in \mathcal{A}^{\mathbb{Z}}$, the knowledge of $(F^n(x)_{[-l,l]})_{n\in[0,r_T N]}$ implies the knowledge of $x_{[-N-l,N+l]}$. This means that $\bigvee_{n=0}^{r_T N} F^{-n}(\mathcal{P}_{[-l,l]})$ is a refinement of $\mathcal{P}_{[-N-l,N+l]}$. A computation similar to that in the previous proof shows that $r_T\, h_\mu(F) \geq h_\mu(\sigma)$. □

This result can be viewed as a rigidity result. Indeed for an expansive CA $(\mathcal{A}^{\mathbb{Z}}, F)$, the measure entropy of F and σ are linked for an (F,σ)-invariant measure. This is a first step in the research of Lyapunov exponents for expansive CA [Tis00].

4.4 (F,σ)-invariant subgroups of D_∞

Now let us discuss assumption (4) of Theorems 3.3 and 3.4 which is an algebraic condition on the CA. We can remark that Theorems 1.1 and 1.2 have no such assumption because they concern a particular class of CA which verifies this assumption: $F = a\,\mathrm{Id} + b\,\sigma$ on $(\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}}$ with p prime. By Proposition 4.2 it is easy to modify the proof of Theorem 1.3 to consider nontrivial algebraic bipermutative CA without restriction on the neighborhood (Corollary 4.3). But it is necessary to compare the assumption "Ker(F) contains no nontrivial σ-invariant subgroups" with "every σ-invariant infinite subgroup of D_∞ is dense in $\mathcal{A}^{\mathbb{Z}}$". We show that the second property is more general and give in Subsection 5.1 a general class of examples where it is the case.

If $H \subset \mathcal{A}^{\mathbb{Z}}$, denote by $\langle H\rangle$ the subgroup generated by H, $\langle H\rangle_\sigma$ the smallest σ-invariant subgroup which contains H and $\langle H\rangle_{F,\sigma}$ the smallest (F,σ)-invariant subgroup which contains H. Let Σ be a closed (F,σ)-invariant subgroup. If $H \subset \Sigma$, then we remark that $\langle H\rangle$, $\langle H\rangle_\sigma$ and $\langle H\rangle_{F,\sigma}$ are subgroups of Σ.

Proposition 4.6. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be an algebraic CA and let Σ be a closed (F,σ)-invariant subgroup of $\mathcal{A}^{\mathbb{Z}}$. The following propositions are equivalent:*

1. *D_∞^Σ contains no nontrivial (F,σ)-invariant infinite subgroups.*
2. *There exist $m \in \mathbb{N}$ and $n_0 \geq 0$ such that $D_{n_0}^\Sigma \subset \langle d\rangle_{F,\sigma}$ for all $d \in \partial D_{n_0+m}^\Sigma$.*
3. *There exists $m \in \mathbb{N}$ such that $D_{n_0}^\Sigma \subset \langle d\rangle_{F,\sigma}$ for all $n_0 \in \mathbb{N}^*$ and $d \in \partial D_{n_0+m}^\Sigma$.*
4. *There exists $m \in \mathbb{N}$ such that $D_1^\Sigma \subset \langle d\rangle_{F,\sigma}$ for all $d \in \partial D_{m+1}^\Sigma$.*

Proof. (2) $\Rightarrow$ (1) Let Γ be an (F,σ)-invariant infinite subgroup of D_∞^Σ. We prove by induction that $D_n^\Sigma \subset \Gamma$ for all $n \geq n_0$. Since Γ is infinite and D_n^Σ is finite for all $n \in \mathbb{N}$, we deduce that there exists $n' \geq 0$ such that there exists $d \in \Gamma \cap \partial D_{n'+n_0+m}^\Sigma$. By F-invariance of Γ we have $F^{n'}(d) \in \Gamma \cap \partial D_{n_0+m}^\Sigma$, thus $D_{n_0}^\Sigma \subset \langle F^{n'}(d)\rangle_{F,\sigma} \subset \Gamma$.

Let $n \geq n_0$ and assume that $D_n^\Sigma \subset \Gamma$. We want to show that $D_{n+1}^\Sigma \subset \Gamma$. As before, since Γ is infinite and F-invariant we can find $d \in \Gamma \cap \partial D_{n+1+m}^\Sigma$. From $F^{n+1-n_0}(d) \in \Gamma \cap \partial D_{n_0+m}^\Sigma$, we deduce $D_{n_0}^\Sigma \subset \langle F^{n+1-n_0}(d)\rangle_{F,\sigma}$. Let $d' \in D_{n+1}^\Sigma$. Then $F^{n+1-n_0}(d') \in D_{n_0}^\Sigma \subset \langle F^{n+1-n_0}(d)\rangle_{F,\sigma}$ and consequently there exists a finite subset $\mathbb{V} \subset \mathbb{Z} \times \mathbb{N}$ such that $F^{n+1-n_0}(d') = \sum_{(u,m')\in\mathbb{V}} c_{u,m'}\sigma^u \circ F^{m'+n+1-n_0}(d)$ where $c_{u,m'} \in \mathbb{Z}$. We deduce that $d' - \sum_{(u,m')\in\mathbb{V}} c_{u,m'}\sigma^u \circ F^{m'}(d) \in D_{n+1-n_0}^\Sigma \subset D_n^\Sigma \subset \Gamma$. But $d \in \Gamma$, so $\sigma^n \circ F^{m'} \in \Gamma$ for all $(n, m') \in \mathbb{V}$. Thus, $d' \in \Gamma$. This holds for any $d' \in D_{n+1}^\Sigma$. Thus, $D_{n+1}^\Sigma \subset \Gamma$. By induction, $D_k^\Sigma \subset \Gamma$ for all $k \in \mathbb{N}$. Finally, $D_\infty^\Sigma = \cup_{n\in\mathbb{N}} D_n^\Sigma \subset \Gamma$.

(1) $\Rightarrow$ (4) By contradiction, we assume that for all $m \in \mathbb{N}$ there exists $d \in \partial D_{m+1}^\Sigma$ such that $\langle d\rangle_{F,\sigma} \cap D_1^\Sigma \neq D_1^\Sigma$. Since D_1^Σ is a finite group there exists a strict subgroup H of D_1^Σ such that $\Delta = \{d \in D_\infty^\Sigma | \langle d\rangle_{F,\sigma} \cap D_1^\Sigma \subset H\}$ is infinite. Observe that $F(\Delta) \subset \Delta$. For all $d' \in \Delta$ we denote $\Delta_{d'} = \{d \in \Delta | d' \in \langle d\rangle_{F,\sigma}\}$. Let $(n_i)_{i\in\mathbb{N}}$ be an increasing sequence such that $\Delta \cap \partial D_{n_i}^\Sigma \neq \emptyset$. If

$d \in \Delta \cap \partial D^{\Sigma}_{n_{i+1}}$, we have $d' = F^{n_{i+1}-n_i}(d) \in \langle d \rangle_{F,\sigma}$, so that $d \in \Delta_{d'}$, and also $d' \in \Delta \cap \partial D^{\Sigma}_{n_i}$. So we can construct by induction an infinite sequence $(d_i)_{i\in\mathbb{N}}$ of D^{Σ}_{∞} such that $d_i \in \Delta \cap \partial D^{\Sigma}_{n_i}$ and $d_{i+1} \in \Delta_{d_i}$ for all $i \in \mathbb{N}$. Thus $\Gamma = \bigcup_{i\in\mathbb{N}} \langle d_i \rangle_{F,\sigma}$ is an infinite (F,σ)-invariant subgroup of D^{Σ}_{∞} such that $\Gamma \cap D^{\Sigma}_1 \subset H$, which contradicts (1).

(4) $\Rightarrow$ (3) Let $m \in \mathbb{N}$ such that $D^{\Sigma}_1 \subset \langle d \rangle_{F,\sigma}$ for all $d \in \partial D^{\Sigma}_{m+1}$. We prove by induction that for all $n \geq 1$ and $d \in \partial D^{\Sigma}_{n+m}$ one has $D^{\Sigma}_n \subset \langle d \rangle_{F,\sigma}$. For $n = 1$ it is the assumption. Assume that the property is true for $n \in \mathbb{N}^*$. Let $d \in \partial D^{\Sigma}_{n+1+m}$, since $F^n(d) \in \partial D^{\Sigma}_{m+1}$, one has $D^{\Sigma}_1 \subset \langle F^n(d) \rangle_{F,\sigma}$. If $d' \in D^{\Sigma}_{n+1}$, then $F^n(d') \in D^{\Sigma}_1$ and we deduce the existence of $\mathbb{V} \subset \mathbb{Z}\times\mathbb{N}$ such that $F^n(d') = \sum_{(u,m')\in\mathbb{V}} c_{u,m'} \sigma^u \circ F^{m'+n}(d)$ where $c_{u,m'} \in \mathbb{Z}$. From $d' - \sum_{(u,m')\in\mathbb{V}} c_{u,m'} \sigma^u \circ F^{m'}(d) \in D^{\Sigma}_n$ and from the fact that $D^{\Sigma}_n \subset \langle F(d) \rangle_{F,\sigma}$ because $F(d) \in \partial D^{\Sigma}_{n+m}$, we deduce that $d' - \sum_{(u,m')\in\mathbb{V}} c_{u,m'} \sigma^u \circ F^{m'}(d) \in \langle F(d) \rangle_{F,\sigma} \subset \langle d \rangle_{F,\sigma}$. Thus, $d' \in \langle d \rangle_{F,\sigma}$. One deduces that $D^{\Sigma}_{n+1} \subset \langle d \rangle_{F,\sigma}$.

(3) $\Rightarrow$ (2) is trivial. □

Corollary 4.7. *If $D^{\Sigma}_1 = \mathrm{Ker}(F) \cap \Sigma$ contains no nontrivial σ-invariant subgroups then D^{Σ}_{∞} contains no nontrivial (F,σ)-invariant infinite subgroups.*

Proof. If $D^{\Sigma}_1 = \mathrm{Ker}(F) \cap \Sigma$ contains no nontrivial σ-invariant subgroups, for all $d \in \partial D^{\Sigma}_1$, the subgroup $\langle d \rangle_{F,\sigma}$ must be equal to D^{Σ}_1. By Proposition 4.6, one deduce that D^{Σ}_{∞} contains no nontrivial (F,σ)-invariant infinite subgroups. □

For a linear CA $(\mathcal{A}^{\mathbb{Z}}, F)$ where $\mathcal{A} = \mathbb{Z}/n\mathbb{Z}$, the σ-invariant subgroups coincide with the (F,σ)-invariant subgroups. From Corollary 4.7 we get directly that Theorem 3.3 is stronger than Theorem 1.3 in this case. Moreover, if we consider the case of the Theorem 1.2, that is to say that $\mathcal{A} = \mathbb{Z}/p\mathbb{Z}$ with p prime and $F = a\,\mathrm{Id} + b\,\sigma$ with $a \neq 0$ and $b \neq 0$, then $\mathrm{Ker}(F) \simeq \mathbb{Z}/p\mathbb{Z}$ does not contain nontrivial σ-invariant subgroups. So Theorem 3.3 generalizes also Theorem 1.2.

When $\mathcal{A}$ is not cyclic, the σ-invariant subgroups does not necessarily coincide with the (F,σ)-invariant subgroups. In this case we do not know if Theorem 3.3 implies Theorem 1.3. However Corollary 4.7 implies that Theorem 3.4 is stronger than Theorem 1.3 for every algebraic bipermutative CA.

5 Extensions to some linear CA

5.1 The case $\mathcal{A} = \mathbb{Z}/p\mathbb{Z}$

Theorem 1.2 concerns linear CA on $(\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}}$ of smallest neighborhood $\mathbb{U} = [0,1]$. We will show that this implies the fourth assumption of Theorem 3.3. In fact we can show that the fourth assumption is directly implied when we consider a nontrivial linear CA on $(\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}}$. This allows us to prove the following result.

Proposition 5.1. *Let $\mathcal{A} = \mathbb{Z}/p\mathbb{Z}$, let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a nontrivial linear CA with p prime and let μ be an (F,σ)-invariant probability measure on $\mathcal{A}^{\mathbb{Z}}$. Assume that:*

1. *μ is ergodic for the $\mathbb{N}\times\mathbb{Z}$-action (F,σ);*
2. *$\mathcal{I}_{\mu}(\sigma) = \mathcal{I}_{\mu}(\sigma^{p\,p_1})$ with $k \in \mathbb{N}^*$ and p_1 the smallest common period of all elements of* $\mathrm{Ker}(F)$*;*
3. *$h_{\mu}(F) > 0$.*

Then: (a) $\mu = \lambda_{\mathcal{A}^{\mathbb{Z}}}$.

(b) Moreover p_1 divides $\prod_{i=0}^{r-1}(p^r - p^i)$ where $r = \max\{\mathbb{U}, 0\} - \min\{\mathbb{U}, 0\}$ and $\mathbb{U}$ is the smallest neighborhood of F.

Proof. Proof of (a): By (F,σ)-invariance of μ, we can compose F with σ and assume that the smallest neighborhood of F is $[0,r]$ with $r \in \mathbb{N}\setminus\{0\}$. So $F = \sum_{u\in[0,r]} f_u \circ \sigma^u = P_F(\sigma)$ where P_F is a polynomial with coefficients in $\mathbb{Z}/p\mathbb{Z}$ with $f_0 \neq 0$ and $f_r \neq 0$. We remark that F is bipermutative.

Case 1: First we assume that P_F is irreducible on $\mathbb{Z}/p\mathbb{Z}$. We can view $D_1(F)$ as a $\mathbb{Z}/p\mathbb{Z}$ vector space and consider the isomorphism $\sigma_1 : D_1(F) \to D_1(F)$, the restriction of σ at the subgroup $D_1(F)$. By bipermutativity of F, $D_1 \simeq (\mathbb{Z}/p\mathbb{Z})^r$. Moreover $P_F(\sigma_1) = 0$; since P_F is irreducible and its degree is equal to the dimension of D_1, we deduce that P_F is the characteristic polynomial of σ_1. Since P_F is irreducible, $D_1(F)$ is σ_1-simple, so $D_1(F)$ contains no nontrivial σ-invariant subgroups, see [AB93, §VI.8] for more detail. By Corollary 4.7, $D_\infty(F)$ also contains no nontrivial (F,σ)-invariant infinite subgroup, so hypothesis (4) of Theorem 3.3 is verified.

Case 2: Now we assume that $P_F = P^\alpha$ where P is irreducible on $\mathbb{Z}/p\mathbb{Z}$ and $\alpha \in \mathbb{N}$. We have $D_n(P_F(\sigma)) = \mathrm{Ker}(P^{\alpha n}(\sigma)) = D_{\alpha n}(P(\sigma))$ for all $n \in \mathbb{N}$. So $D_\infty(P_F(\sigma)) = D_\infty(P(\sigma))$. Now we are in the previous case and the fourth condition of Theorem 3.3 is verified.

Case 3: In the general case $P_F = P_1^{\alpha_1}...P_l^{\alpha_l}$ where P_i is irreducible and $\alpha_i \in \mathbb{N}$ for all $i \in [1,l]$. Let Γ be an (F,σ)-invariant infinite subgroup of $D_\infty(P_F(\sigma))$. By the kernel decomposition Lemma [AB93, §VI.4], we have $D_n(P_F(\sigma)) = D_n(P_1^{\alpha_1}(\sigma)) \oplus ... \oplus D_n(P_l^{\alpha_l}(\sigma))$ for every $n \in \mathbb{N}$. Moreover $D_n(P_F(\sigma)) \cap \Gamma$ is a σ-invariant subspace of $D_n(P_F(\sigma))$ considered as a $\mathbb{Z}/p\mathbb{Z}$-vector space and $D_n(P_F(\sigma)) \cap \Gamma = (D_n(P_1^{\alpha_1}(\sigma)) \cap \Gamma) \oplus ... \oplus (D_n(P_l^{\alpha_l}(\sigma)) \cap \Gamma)$. We deduce that

$$D_\infty(P_F(\sigma)) \cap \Gamma = \bigoplus_{i\in[1,l]} (D_\infty(P_i^{\alpha_1}(\sigma)) \cap \Gamma) \underset{(*)}{=} \bigoplus_{i\in[1,l]} (D_\infty(P_i(\sigma)) \cap \Gamma),$$

where $(*)$ follows as in Case 2. There exists $i \in [1,l]$ such that $\Gamma \cap D_\infty(P_i(\sigma))$ is an infinite subgroup. By Case 1, one has $\Gamma \cap D_\infty(P_i(\sigma)) = D_\infty(P_i(\sigma))$, so $D_\infty(P_i(\sigma)) \subset \Gamma$. We deduce that Γ is dense, because $D_\infty(P_i(\sigma))$ is dense, because $P_i(\sigma)$ is bipermutative. Thus the fourth condition of Theorem 3.3 is verified; part (a) of the proposition follows.

Proof of (b): If $x \in \mathrm{Ker}(F)$, then the coordinates of x verify $x_{n+r} = -f_r^{-1} \sum_{i=0}^{r-1} f_i x_{n+i}$ for all $n \in \mathbb{Z}$. This recurrence relation can be expressed with a matrix. For all $n \in \mathbb{Z}$ one has $X_{n+1} = AX_n$ where

$$X_n = \begin{pmatrix} x_{n+r-1} \\ \vdots \\ x_n \end{pmatrix} \text{ and } A = \begin{bmatrix} -f_{r-1}f_r^{-1} & \cdots & \cdots & \cdots & -f_0 f_r^{-1} \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

A is invertible because $f_0 \neq 0 \neq f_r$, and for all $n \in \mathbb{Z}$ one has $X_n = A^n X_0$. Thus the period of X_n divides the period of A, which divides the cardinality of the set of invertible matrices on $\mathbb{Z}/p\mathbb{Z}$ of size r, that is to say the number of bases of $(\mathbb{Z}/p\mathbb{Z})^r$, which is $\prod_{i=0}^{r-1}(p^r - p^i)$. □

Remark 5.1. Proposition 5.1 still holds if $((\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}}, F)$ is an affine CA.

Remark 5.2. Proposition 5.1 extends to the case when $\mathcal{A}$ is a finite field and $F = \sum_{u\in\mathbb{U}} f_u \sigma^u$ is a linear CA where each coefficient f_u is the multiplication by an element of the field.

Let $((\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}}, F)$ be a nontrivial linear CA where $P_F(\sigma) = \sum_{u\in[0,r]} f_u \circ \sigma^u$ is a polynomial with coefficients in $\mathbb{Z}/p\mathbb{Z}$ with $f_0 \neq 0$ and $f_r \neq 0$. In this case Theorem 1.3, generalized to nontrivial algebraic bipermutative CA without restriction on the neighborhood, holds only if $\mathrm{Ker}(F)$ contains no nontrivial σ-invariant subgroups, which is equivalent to the irreducibility of P_F. Proposition 5.1 holds for every linear CA on $(\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}}$.

5.2 The case $\mathcal{A} = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$

Now we consider $\mathcal{A} = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$ with p and q distinct primes and $(\mathcal{A}^{\mathbb{Z}}, F)$ a linear bipermutative CA. In this case D_∞ contains infinite σ-invariant subgroups which are not dense in $\mathcal{A}^{\mathbb{Z}}$. For example $D_\infty^{\Gamma_1}$ and $D_\infty^{\Gamma_2}$ where $\Gamma_1 = (\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}} \times \{0_{(\mathbb{Z}/q\mathbb{Z})^{\mathbb{Z}}}\}$ and $\Gamma_2 = \{0_{(\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}}}\} \times (\mathbb{Z}/q\mathbb{Z})^{\mathbb{Z}}$. The measures λ_{Γ_1} and λ_{Γ_2} are (F,σ)-totally ergodic with positive entropy for σ. If μ is an (F,σ)-invariant measure which verifies conditions of the Theorem 3.3, we cannot conclude that $\mu = \lambda_{\mathcal{A}^{\mathbb{Z}}}$. But if we consider the natural factor $\pi_1 : \mathcal{A}^{\mathbb{Z}} \to \Gamma_1$ and $\pi_2 : \mathcal{A}^{\mathbb{Z}} \to \Gamma_2$, then by Corollary 3.5, one has $\pi_1\mu = \lambda_{\Gamma_1}$ or $\pi_2\mu = \lambda_{\Gamma_2}$. A natural conjecture is this: if every cellular automaton factor of F has positive entropy, then $\mu = \lambda_{\mathcal{A}^{\mathbb{Z}}}$. The problem is to rebuild the measure starting from $\pi_1\mu$ and $\pi_2\mu$.

5.3 The case $\mathcal{A} = \mathbb{Z}/p^k\mathbb{Z}$

In this case we do not know under what extra conditions an (F, σ)-invariant measure is the Haar measure. Moreover some linear CA are not bipermutative. The next lemma shows how to remove this condition when you consider a power of the CA.

Lemma 5.2. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a linear CA with $\mathcal{A} = \mathbb{Z}/p^k\mathbb{Z}$, where p is prime, $k \geq 1$ and $F = \sum_{i\in[r,s]} f_i\,\sigma^i$, with $f_i \in \mathbb{Z}/p^k\mathbb{Z}$. Let $\widehat{\mathbb{U}} = \{i \in [r,s] : f_i$ coprime with $p\}$, $\hat{r} = \min \widehat{\mathbb{U}}$ and $\hat{s} = \max \widehat{\mathbb{U}}$. Assume $\widehat{\mathbb{U}}$ is not empty and $\hat{r} < \hat{s}$.*

Then $F^{p^{k-1}}$ is bipermutative of smallest neighborhood $\mathbb{U}' = [p^{k-1}\hat{r}, p^{k-1}\hat{s}]$.

Proof. We can write $F = P_F(\sigma)$ with $P_F \in \mathbb{Z}/p^k\mathbb{Z}[X, X^{-1}]$. We decompose $P_F = P_1 + pP_2$ where $P_1 = \sum_{i\in\widehat{\mathbb{U}}} f_i\, X^i$. By Fermat's little theorem and induction on $j \geq 1$, we can easily prove that:

$$(P_1 + pP_2)^{p^j} = (P_1)^{p^j} \mod p^{j+1}.$$

So we have $P_F^{p^{k-1}} = P_1^{p^{k-1}} = \sum_{i\in[p^{k-1}\hat{r},p^{k-1}\hat{s}]} g_i X^i$ where $g_i \in \mathbb{Z}/p^k\mathbb{Z}$. Moreover $g_{p^{k-1}\hat{r}} = f_{\hat{r}}^{p^{k-1}}$ and $g_{p^{k-1}\hat{s}} = f_{\hat{s}}^{p^{k-1}}$ are relatively prime to p. We deduce that $F^{p^{k-1}} = P_F^{p^{k-1}}(\sigma)$ is bipermutative of smallest neighborhood $\mathbb{U}' = [p^{k-1}\hat{r}, p^{k-1}\hat{s}]$. ☐

Now we can deduce from Corollary 2.3 an entropy formula for general linear CA on $(\mathbb{Z}/p^k\mathbb{Z})^{\mathbb{Z}}$.

Corollary 5.3. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a linear CA with $\mathcal{A} = \mathbb{Z}/p^k\mathbb{Z}$, where p is prime, $k \geq 1$, and $F = \sum_{i\in[s,r]} f_i\,\sigma^i$ with $f_i \in \mathbb{Z}/p^k\mathbb{Z}$. Let $\hat{r} < \hat{s}$ be as in Lemma 5.2. Let μ be an (F, σ)-invariant probability measure on $\mathcal{A}^{\mathbb{Z}}$. Then $h_\mu(F) = (\max(\hat{r}, 0) - \min(\hat{s}, 0))h_\mu(\sigma)$.*

Corollary 5.4. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a linear CA with $\mathcal{A} = \mathbb{Z}/p^k\mathbb{Z}$, where p is prime, $k \geq 1$, and $F = \sum_{i\in[s,r]} f_i\,\sigma^i$ with $f_i \in \mathbb{Z}/p^k\mathbb{Z}$. Assume that for at least two $i \in [s,r]$, f_i is relatively prime with p. Let Σ be a closed (F, σ)-invariant subgroup of $\mathcal{A}^{\mathbb{Z}}$ and let μ be an (F, σ)-invariant probability measure on $\mathcal{A}^{\mathbb{Z}}$ with $\mathrm{supp}(\mu) \subset \Sigma$. Assume that:*

1. *μ is ergodic for the $\mathbb{N} \times \mathbb{Z}$-action induced by (F, σ);*
2. *$\mathcal{I}_\mu(\sigma) = \mathcal{I}_\mu(\sigma^{pp_1})$ with p_1 the smallest common period of all elements of $\mathrm{Ker}(F)$;*
3. *$h_\mu(\sigma) > 0$;*
4. *every σ-invariant infinite subgroup of $D_\infty^\Sigma(F) = \cup_{n\in\mathbb{N}}\mathrm{Ker}(F^n) \cap \Sigma$ is dense in Σ.*

Then $\mu = \lambda_\Sigma$.

Example 5.3. Let $\mathcal{A} = \mathbb{Z}/4\mathbb{Z}$, we consider the CA $(\mathcal{A}^{\mathbb{Z}}, F)$ defined by $F = \mathrm{Id} + \sigma + 2\sigma^2$. Then $\Sigma = \{0, 2\}^{\mathbb{Z}}$ satisfies the conditions of Corollary 5.4. In this case the only (F, σ)-invariant probability measure of positive entropy known are $\lambda_{\mathcal{A}^{\mathbb{Z}}}$ and λ_Σ.

6 Measure rigidity for some affine one-sided expansive CA

An invertible onesided CA $(\mathcal{A}^{\mathbb{N}}, F)$ is called expansive if there exists a constant $r_e \in \mathbb{N}$ such that for all $x, y \in \mathcal{A}^{\mathbb{N}}$, if $x \neq y$ there exists $n \in \mathbb{Z}$ which verifies $F^n(x)_{[0,r_e]} \neq F^n(y)_{[0,r_e]}$. Expansive CA are different from positively expansive CA because we look also the past of the orbit. M. Boyle and A. Maass introduced in [BM00] a class of onesided invertible expansive CA which have remarkable combinatorial properties. Further properties were obtained in [DMS03]. We study this class of examples from the point of view of measure rigidity. This class of CA is not bipermutative so we cannot apply directly Theorem 3.3. However, in some case, it is possible to associate a "dual" CA which correspond to the assumptions of Theorem 3.3. This is a first step to study measure rigidity for expansive CA.

We are going to recall some properties obtained in [BM00]. Let $F : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ be a CA such that $r(F) = 1$. Associate to F the equivalence relation over $\mathcal{A}$: $a\mathcal{R}_F b$ iff $\overline{F}(\cdot\, a) = \overline{F}(\cdot\, b)$ as a function from $\mathcal{A}$ to $\mathcal{A}$; and we write $\mathcal{P}_{\mathcal{R}_F}$ the partition induced by $\mathcal{R}_F$ and $C_{\mathcal{R}_F}(a)$ the class associated to a. Define also $\pi_F : \mathcal{A} \to \mathcal{A}$ by $\pi_F(a) = \overline{F}(aa)$ for any $a \in \mathcal{A}$.

Proposition 6.1 ([BM00])**.** *A onesided CA $F : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ with $r(F) = 1$ is invertible with $r(F^{-1}) = 1$ iff the following conditions hold:*

1. *π_F is a permutation,*
2. *F is left permutative,*
3. *$\forall a \in A$, $\mathrm{Succ}_F(a) := \mathrm{Im}(\overline{F}(a\,\cdot)) \subset \pi_F(C_{\mathcal{R}_F}(a))$.*

If F is an expansive invertible CA with $r(F) = r(F^{-1}) = 1$, then $(\mathcal{A}^{\mathbb{N}}, F)$ is topologically conjugate to the bilateral subshift (S_F, σ) where $S_F = \{(F^i(x)_0)_{i\in\mathbb{Z}} : x \in \mathcal{A}^{\mathbb{N}}\}$ via the conjugacy $\phi_F : x \in \mathcal{A}^{\mathbb{N}} \to (F^i(x)_0)_{i\in\mathbb{N}} \in S_F$. Define $F_T : S_F \to S_F$ by $F_T(\phi_F(x)) = \phi_F(\sigma(x))$ for every $x \in \mathcal{A}^{\mathbb{N}}$. If F is expansive then (S_F, F_T) is a CA (defined on S_F instead of a fullshift). Invertible expansive CA with $r(F) = r(F^{-1}) = r(F_T) = 1$ can be characterized as follows:

Proposition 6.2 ([BM00])**.** *A onesided invertible CA $F : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ with $r(F) = r(F^{-1}) = 1$ is expansive with $r(F_T) = 1$ iff the following conditions are verified:*

1. *$|C \cap \pi_F(C')| \leq 1$ for any C, $C' \in \mathcal{P}_{\mathcal{R}_F}$,*
2. *$\forall a \in \mathcal{A}$, $\mathrm{Succ}_F(a) := \mathrm{Im}(\overline{F}(a\,\cdot)) = \pi_F(C_{\mathcal{R}_F}(a))$.*

Such a CA is said to be in Class (A). The alphabet $\mathcal{A}$ of a CA in Class (A) has cardinality n^2 for some $n \in \mathbb{N}$.

Write $\mathcal{B} = \mathcal{P}_{\mathcal{R}_F}$. In [BM00], the authors show that (S_F, σ) is conjugate to the full shift $(\mathcal{B}^{\mathbb{Z}}, \sigma)$ by $\varphi : S_F \to \mathcal{B}^{\mathbb{Z}}$ such that $\varphi((a_i)_{i\in\mathbb{Z}}) = (C_{\mathcal{R}_F}(a_i))_{i\in\mathbb{Z}}$. The CA (S_F, F_T) determines by φ a CA $(\mathcal{B}^{\mathbb{Z}}, \widetilde{F_T})$ on $\mathcal{B}^{\mathbb{Z}}$ and (S_F, F_T) is conjugate to $(\mathcal{B}^{\mathbb{Z}}, \widetilde{F_T})$. To sum up we have:

$$\begin{array}{ccccc} (\mathcal{A}^{\mathbb{N}}, \sigma) & \equiv & (S_F, F_T) & \equiv & (\mathcal{B}^{\mathbb{Z}}, \widetilde{F_T}), \\ (\mathcal{A}^{\mathbb{N}}, F) & \equiv & (S_F, \sigma) & \equiv & (\mathcal{B}^{\mathbb{Z}}, \sigma), \end{array}$$

(where $\equiv$ means topologically conjugate).

Proposition 6.3. *If F is in Class (A) then $\widetilde{F_T}$ is bipermutative.*

Proof. Let $(\mathcal{A}^{\mathbb{N}}, F)$ be a CA in the class (A) and let $\alpha, \alpha', \beta, \gamma, \delta \in \mathcal{B}$ such that $\overline{\widetilde{F_T}}(\alpha, \beta, \gamma) = \overline{\widetilde{F_T}}(\alpha', \beta, \gamma) = \delta$. Suppose $\beta = \varphi(b)$, for some $b \in S_F$. Then $b \in \pi_F(\gamma)$ by condition (2) of Proposition 6.2, so $b \in \beta \cap \pi_F(\gamma)$, which is a singleton set by condition (1). Hence β and γ uniquely determine b. Likewise, if $\alpha = \varphi(a)$ and $\alpha' = \varphi(a')$ for some $a, a' \in S_F$, then we must have $a, a' \in \overline{F}(b, \delta)$. But $\overline{F}(b, .) : \mathcal{A} \to \mathcal{A}$ is constant on δ by definition of the partition $\mathcal{P}_{\mathcal{R}_F}$, so $a = a'$ so $\alpha = \alpha'$. We deduce that the function $\overline{\widetilde{F_T}}(\cdot, \beta, \gamma) : \mathcal{B} \to \mathcal{B}$ is injective. So it is bijective because $\mathcal{B}$ is finite. Thus, $(\mathcal{B}, \widetilde{F_T})$ is left-permutative.

In the same way we can prove that $(\mathcal{B}, \widetilde{F_T})$ is right-permutative by applying Propositions 6.1 and 6.2 to F^{-1} instead. The result follows. $\square$

A natural question after this proposition is to characterize the CA F in class (A) such that $\widetilde{F_T}$ is algebraic to apply previous theorems. We have only the next sufficient condition:

Proposition 6.4. *Let $(\mathcal{A}^{\mathbb{N}}, F)$ be a linear CA, $F = f_0\mathrm{Id} + f_1\sigma$ where f_0 and f_1 are endomorphisms of $\mathcal{A}$ extended coordinate by coordinate to $\mathcal{A}^{\mathbb{N}}$.*

(a) F is invertible with $r(F^{-1}) = 1$ iff f_0 is an automorphism and $f_1 \circ f_0^{-1} \circ f_1 = 0$.

(b) F is in Class (A) iff f_0 is an automorphism, $\mathrm{Im} f_1 = f_0(\mathrm{Ker} f_1)$ and $\mathrm{Im} f_1 \cap \mathrm{Ker} f_1 = \{0\}$.

(c) When $(\mathcal{A}^{\mathbb{Z}}, F)$ is in Class (A), the CA $(\mathcal{P}_{\mathcal{R}_F}^{\mathbb{Z}}, \widetilde{F_T})$ is linear.

Proof. First we remark that $b \in C_{\mathcal{R}_F}(b')$ iff $f_0(a) + f_1(b) = f_0(a) + f_1(b')$ for all $a \in \mathcal{A}$; this is equivalent to $b \in b' + \mathrm{Ker} f_1$. So $C_{\mathcal{R}_F}(b) = b + \mathrm{Ker} f_1$ for all $b \in \mathcal{A}$. Thus, $\mathcal{P}_{\mathcal{R}_F} \cong \mathcal{A}/\mathrm{Ker} f_1$. Moreover $\mathrm{Succ}_F(a) = \mathrm{Im}(\overline{F}(a\,\cdot)) = f_0(a) + \mathrm{Im} f_1$ for all $a \in \mathcal{A}$, and $\pi_F = f_0 + f_1$.

Proof of (a): Assuming f_0 is an automorphism and $f_1 \circ f_0^{-1} \circ f_1 = 0$, it is possible to express F^{-1} as: $F^{-1} = f_0^{-1}\mathrm{Id} - f_0^{-1} \circ f_1 \circ f_0^{-1}\sigma$. Conversely, if F is invertible with $r(F^{-1}) = 1$, by Proposition 6.1, f_0 is an automorphism because F is left-permutative and $f_1 \circ f_0^{-1} \circ f_1 = 0$ because for some $a \in \mathcal{A}$ one has:

$$f_0(a) + \mathrm{Im}(f_1) = \mathrm{Succ}_F(a) \subset \pi_F(C_{\mathcal{R}_F}(a)) = f_0(a) + f_1(a) + f_0(\mathrm{Ker} f_1),$$

that is to say $\mathrm{Im} f_1 \subset f_0(\mathrm{Ker} f_1)$.

Proof of (b): As in the proof of (a), one has $\mathrm{Succ}_F(a) = \pi_F(C_{\mathcal{R}_F}(a))$ for any $a \in \mathcal{A}$ iff $\mathrm{Im} f_1 = f_0(\mathrm{Ker} f_1)$. Moreover, if $|C \cap \pi_F(C')| \leq 1$ for any $C, C' \in \mathcal{P}_{\mathcal{R}_F}$, then $0 + \mathrm{Ker} f_1 \cap \pi_F(0 + \mathrm{Ker} f_1) = \mathrm{Ker} f_1 \cap f_0(\mathrm{Ker} f_1) = \mathrm{Ker} f_1 \cap \mathrm{Im} f_1 = \{0\}$. Conversely, for any $b, b' \in \mathcal{A}$ one has $C_{\mathcal{R}_F}(b) \cap \pi_F(C_{\mathcal{R}_F}(b)) = b + \mathrm{Ker} f_1 \cap \pi_F(b') + \mathrm{Im} f_1$, so if $\mathrm{Ker} f_1 \cap \mathrm{Im} f_1 = \{0\}$ then $C_{\mathcal{R}_F}(b) \cap \pi_F(C_{\mathcal{R}_F}(b))$ contains at most one element. Characterization of linear CA in Class (A) follows from Proposition 6.2.

Proof of (c): Let $(\mathcal{A}^{\mathbb{N}}, F)$ be a CA in the class (A). We will show that $(\mathcal{P}_{\mathcal{R}_F}^{\mathbb{Z}}, \widetilde{F_T})$ is linear. Since $\mathcal{A}$ is finite Abelian and $\mathrm{Im} f_1 \cap \mathrm{Ker} f_1 = \{0\}$ by (b), one has $\mathrm{Im} f_1 \oplus \mathrm{Ker} f_1 = \mathcal{A}$. Moreover $\mathrm{Im} f_1$ and $\mathrm{Ker} f_1$ are isomorphic to the same group, denoted $\mathcal{B}$, because f_0 is an automorphism and $\mathrm{Im} f_1 = f_0(\mathrm{Ker} f_1)$ by (b). An element $a \in \mathcal{A}$ is written $\binom{x}{y}$ where $x \in \mathrm{Im} f_1 \simeq \mathcal{B}$ and $y \in \mathrm{Ker} f_1 \simeq \mathcal{B}$. One has $\mathcal{P}_{\mathcal{R}_F} \simeq \mathcal{A}/\mathrm{Ker} f_1 \simeq \mathrm{Im} f_1 \simeq \mathcal{B}$. We want to show that $(\mathcal{B}^{\mathbb{Z}}, \widetilde{F_T})$ is linear. We can write f_0 and f_1 as 2×2-matrices with coefficients in $\mathrm{Hom}(\mathcal{B})$:

$$f_0 = \begin{bmatrix} f_{0,11} & f_{0,12} \\ f_{0,21} & f_{0,22} \end{bmatrix} \text{ and } f_1 = \begin{bmatrix} f_{1,11} & f_{1,12} \\ f_{1,21} & f_{1,22} \end{bmatrix}.$$

Since $\mathrm{Im} f_1 = f_0(\mathrm{Ker} f_1)$ one has $f_{0,22} = 0$ and since f_0 is an automorphism we deduce that $f_{0,12}$ and $f_{0,21}$ are automorphisms of $\mathcal{B}$. Since the second coordinate corresponds to the kernel of f_1, one has $f_{12}^1 = f_{22}^1 = 0$ and since $\mathrm{Im} f_1 \cap \mathrm{Ker} f_1 = \{0\}$ one has $f_{1,21} = 0$. Moreover $f_{1,11}$ is an automorphism of $\mathcal{B}$ since it is the restriction of f_1 at $\mathrm{Im} f_1$. So we have:

$$f_0 = \begin{bmatrix} f_{0,11} & f_{0,12} \\ f_{0,21} & 0 \end{bmatrix}, f_0^{-1} = \begin{bmatrix} 0 & f_{0,21}^{-1} \\ f_{0,12}^{-1} & -f_{0,12}^{-1} \circ f_{0,11} \circ f_{0,21}^{-1} \end{bmatrix} \text{ and } f_1 = \begin{bmatrix} f_{1,11} & 0 \\ 0 & 0 \end{bmatrix}.$$

These formulas are illustrated by the next diagram which represents the action of $\overline{F}$ and $\overline{F^{-1}}$ on a neighborhood:

$$\begin{array}{ccc}
 & \vdots & \\
 & \begin{pmatrix} f_{0,11}(x_0) + f_{0,12}(y_0) + f_{1,11}(x_1) \\ f_{0,21}(x_0) \end{pmatrix} & \\
 & & \\
 & \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} & \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \cdots \\
 & & \\
 & \begin{pmatrix} f_{0,21}^{-1}(y_0) \\ f_{0,12}^{-1}(x_0) - f_{0,12}^{-1} \circ f_{0,11} \circ f_{0,21}^{-1}(y_0) + f_{0,12}^{-1} \circ f_{1,11} \circ f_{0,21}^{-1}(y_1) \end{pmatrix} & \\
 & \vdots &
\end{array}$$

We deduce that $\widetilde{F_T} = f_{1,11}^{-1} \circ \sigma - f_{1,11}^{-1} \circ f_{0,11} \circ \mathrm{Id} - f_{1,11}^{-1} \circ f_{0,12} \circ f_{0,21} \circ \sigma^{-1}$, so $(\mathcal{B}^{\mathbb{Z}}, \widetilde{F_T})$ is linear. □

With Proposition 5.1 and the conjugacy relations it is possible to characterize the uniform Bernoulli measure of some linear CA in Class (A):

Proposition 6.5. *Let $(\mathcal{A}^{\mathbb{N}}, F)$ be an affine invertible CA in Class (A) with $|\mathcal{A}| = p^2$ with p prime. Let μ be an (F, σ)-invariant probability measure on $\mathcal{A}^{\mathbb{N}}$. Assume that:*

1. *μ is ergodic for the $\mathbb{Z} \times \mathbb{N}$-action (F, σ);*
2. *$\mathcal{I}_\mu(F) = \mathcal{I}_\mu(F^{p(p-1)(p^2-1)})$;*
3. *$h_\mu(\sigma) > 0$.*

Then $\mu = \lambda_{\mathcal{A}^{\mathbb{N}}}$.

Proof. By Proposition 6.4, $\widetilde{F_T}$ is a linear bipermutative CA of neighborhood $[-1, 1]$ on $\mathcal{B}^{\mathbb{Z}}$, where $\mathcal{B} = \mathbb{Z}/p\mathbb{Z}$. There exist $\phi : \mathcal{A}^{\mathbb{N}} \to \mathcal{B}^{\mathbb{Z}}$ such that $(\mathcal{A}^{\mathbb{N}}, F, \sigma)$ and $(\mathcal{B}^{\mathbb{Z}}, \sigma, \widetilde{F_T})$ are conjugate via ϕ, so:

1. $\phi\mu$ is ergodic for the $\mathbb{N} \times \mathbb{Z}$-action $(\widetilde{F_T}, \sigma)$;

2. $\mathcal{I}_{\phi\mu}(\sigma) = \mathcal{I}_{\phi\mu}(\sigma^{p(p-1)(p^2-1)}) \underset{(*)}{=} \mathcal{I}_{\phi\mu}(\sigma^{p(p-1)})$, where $(*)$ is by Remark 3.2;

3. $h_{\phi\mu}(\widetilde{F_T}) > 0$.

By Proposition 5.1(a) we deduce that $\phi\mu = \lambda_{\mathcal{B}^{\mathbb{Z}}}$ so $\mu = \lambda_{\mathcal{A}^{\mathbb{N}}}$. $\square$

The next example shows two CA of class (A) with $|\mathcal{A}| = 2^2$.

Example 6.1. Let $\mathcal{A} = \mathbb{Z}/2Z \times \mathbb{Z}/2\mathbb{Z}$, we define two CA $(\mathcal{A}^{\mathbb{N}}, F_1)$ and $(\mathcal{A}^{\mathbb{N}}, F_2)$ in Class (A) by:

$$\overline{F_1}\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

and

$$\overline{F_2}\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

The first coordinate corresponds to the class of $\mathcal{P}_{\mathcal{R}_{F_i}}$ and the second coordinate corresponds to the class of $\mathcal{P}_{\mathcal{R}_{F_i^{-1}}}$. For $i \in \{1, 2\}$, let μ_i be such that:

1. μ_i is (F_i, σ)-ergodic and $\mathcal{I}_{\mu_i}(F) = \mathcal{I}_{\mu_i}(F^6)$
2. $\exists (n, m) \in \mathbb{N} \times \mathbb{Z}$ such that $h_{\mu_i}(\sigma^n \circ F_i^m) > 0$

All the hypothesis of Proposition 6.5 are satisfied, we can conclude that $\mu_i = \lambda_{\mathcal{A}^{\mathbb{N}}}$ for all $i \in \{1, 2\}$. To see where Theorem 1.3 does not hold when we assume μ σ-totally ergodic, we are going to exhibit $\mathrm{Ker}(\widetilde{F_i^T})$ for $i \in \{1, 2\}$.

For F_1 one has:

$$\overline{\widetilde{F_1^T}}(\alpha, \beta, \gamma) = \alpha + \beta + \gamma.$$

This formula is illustrated by the next diagram which represents the action of $\overline{F_1}$ and $\overline{F_1^{-1}}$ on a neighborhood:

$$\begin{array}{ccc} & \vdots & \\ & \begin{pmatrix} x_0 + x_1 + y_0 \\ x_0 \end{pmatrix} & \\ & & \\ & \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \qquad\qquad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} & \cdots \\ & & \\ & \begin{pmatrix} y_0 \\ y_0 + y_1 + x_0 \end{pmatrix} & \\ & \vdots & \end{array}$$

So we have:

$$D_1(\widetilde{F_1^T}) = \mathrm{Ker}(\widetilde{F_1^T}) = \{^\infty 000^\infty, ^\infty 011^\infty, ^\infty 110^\infty, ^\infty 101^\infty\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

$\mathrm{Ker}(F_1^T)$ contains no nontrivial σ-invariant subgroups. Then $\mu_1 = \lambda_{\mathcal{A}^{\mathbb{N}}}$ by Theorem 3.3 and Corollary 4.7. In this case, if μ was σ-totally ergodic, then we could have also applied Theorem 1.3 to conclude that $\mu = \lambda_{\mathcal{A}^{\mathbb{N}}}$.

For F_2 one has:

$$\overline{\widetilde{F_2^T}}(\alpha, \beta, \gamma) = \alpha + \gamma.$$

This formula is illustrated by the next diagram which represents the action of $\overline{F_2}$ and $\overline{F_2^{-1}}$ on a neighborhood:

$$\begin{array}{ccc} \vdots & & \\ \begin{pmatrix} x_1+y_0 \\ x_0 \end{pmatrix} & & \\ \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} & \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} & \cdots \\ \begin{pmatrix} y_0 \\ y_1+x_0 \end{pmatrix} & & \\ \vdots & & \end{array}$$

One obtains:

$$D_1(\widetilde{F_2^T}) = \mathrm{Ker}(\widetilde{F_2^T}) = \{^\infty 00^\infty, ^\infty 11^\infty, ^\infty 01^\infty, ^\infty 10^\infty\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

$$D_2(\widetilde{F_2^T}) = \langle D_1 \cup \{^\infty 0001^\infty, ^\infty 0111^\infty, ^\infty 0011^\infty\}\rangle_\sigma.$$

We remark that $\forall d \in \partial D_2$ one has $D_1 \subset \langle d\rangle_\sigma$, so $\mu_2 = \lambda_{\mathcal{A}^{\mathbb{N}}}$ by Theorem 3.3 and Proposition 4.6. We can also remark that in this case $\{^\infty 00^\infty, ^\infty 11^\infty\}$ is a nontrivial σ-invariant subgroup of $\mathrm{Ker}(F)$ so Theorem 1.3 would not apply, even if we assumed that μ was σ-totally ergodic.

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