The dynamics of cellular automata in shift-invariant topologies

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Abstract. We study the dynamics of cellular automata, and more specifically their transitivity and expansivity, when the set of configurations is endowed with a shift-invariant (pseudo-)distance. We first give an original proof of the non-transitivity of cellular automata when the set of configurations is endowed with the Besicovitch pseudo-distance. We then show that the Besicovitch pseudo-distance induces a distance on the set of shift-invariant measures and on the whole space of measures, and we prove that in these spaces also, cellular automata cannot be expansive nor transitive.

1 Introduction

Cellular automata were introduced by J. von Neumann as a simple formal model for cellular growth and replication. They consist in a discrete lattice of finite-state machines, called *cells* which evolve sequentially and synchronously according to a local rule. This local rule is the same for all cells and determines how a cell will evolve given the states of a finite number of neighboring cells. A snapshot of the states of the cells is called a *configuration*, and a cellular automaton can be seen as a map from the set of configurations to itself. Despite the apparent simplicity of their definition, cellular automata, seen as discrete dynamical systems, can have very complex behaviors, some of which not even being fully understood yet. This behavior is typically studied by endowing the set of configurations with the Cantor distance. For this distance the so-called shift maps, which spacially shift the states of cells according to a fixed vector, can have highly chaotic behaviors.

Other distances can also be defined on the space of configurations for which the shift maps are non-chaotic. An example of such a distance is the Besicovitch distance (in fact, pseudo-distance), introduced by Cattaneo et al. [CFMM97]. It was proven by Blanchard et al. [BCF03] that no cellular automaton can be transitive for this pseudo-distance. Their proof uses Kolmogorov complexity, which is an algorithmic measure of information content. We first provide new simple proof of this fact, also based on Kolmogorov complexity, and we show that our proof can be turned into a purely analytic one, based on Hausdorff dimension.

Suppose now that a measure μ is defined on the set of configurations. A cellular automaton acts on the set of configurations and canonically transforms μ into another measure. Hence, instead of its action on the set of configurations, a cellular automaton can be studied via its action on the set of measures. If μ is shift-invariant, then its image by any cellular automaton is also shift-invariant. Hence, cellular automata also have a natural action on the set of shift-invariant measures. In [Sab07], it is shown that any pseudo-distance on the set of configurations induces a pseudo-distance on the set of shift-invariant measures. Thus, both the Cantor and the Besicovitch distances induce a distance on the set of shift-invariant measures. We show that in this framework also, no cellular automaton is transitive nor expansive on the set of shift-invariant measures endowed with the distance induced by the Besicovitch distance.

The last section of the paper unifies the two proofs of non-transitivity, in the space of configurations and in the space of shift-invariant measures respectively, by embedding these two spaces in the (much) bigger one containing all measures (non-necessarily shift-invariant). Here again, Kolmogorov complexity and effective Hausdorff dimension turn out to be the cornerstone of the proof.

Before moving on to our discussion, we recall the formal definition of the main concepts of the paper, namely transitivity and expansivity. Let (X, d) be a metric space, and $f : X \to X$. The map f is said to be *transitive* if for any $x, y \in X$ and any $\varepsilon > 0$, there exists $x', y' \in X$ and $n \in \mathbb{N}$ such that $d(x, x') < \varepsilon$, $d(y, y') < \varepsilon$ and $f^n(x') = y'$. It is said to be *expansive* if there exists $\varepsilon > 0$ such that for all x, y with $x \neq y$, there exists an $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) > \varepsilon$.

Informally, transitivity is a mixing property, while expansivity is a sign of sensitivity to initial conditions. Hence, both these conditions are often seen as symptomatic of chaotic dynamical systems.

2 Action of cellular automata on $\mathcal{A}^{\mathbb{M}}$

Formally speaking, a *cellular automaton* is a tuple $\langle \mathcal{A}, \mathbb{M}, \mathbb{U}, \delta \rangle$ where $\mathcal{A}$ is a finite alphabet (the *set of states*), $\mathbb{M}$ is a semi-group (the set of indices of cells), $\mathbb{U}$ is a finite subset of $\mathbb{M}$ (the *neighborhood*), and δ is a function from $\mathcal{A}^{\mathbb{U}}$ into $\mathcal{A}$ (*the local rule*). In this setting, the set of configurations is the set $\mathcal{A}^{\mathbb{M}}$. The cellular automaton acts on it via its *global rule*, defined as follows: for all $x \in \mathcal{A}^{\mathbb{M}}$, and all $i \in \mathbb{M}$, the i-th coordinate of $F(x)$ is given by the rule $F(x)_i = \delta((x_{i+k} : k \in \mathbb{U}))$. In the sequel, when this create no confusion, we will make no distinction between a CA and its global rule.

In this paper, the semi-group $\mathbb{M}$ will be of the form $\mathbb{M} = \mathbb{Z}^{d'} \times \mathbb{N}^{d''}$, but most of the results we will present can be generalized to a larger class of semi-groups. Let $\mathbb{M} = \mathbb{Z}^{d'} \times \mathbb{N}^{d''}$. For all $m \in \mathbb{M}$, we denote by $|m|$ the distance of m to the origin point. This allows us to define the radius of the cellular automaton: $r(F) = \max\{|m| : m \in \mathbb{U}\}$ where $\mathbb{U}$ is the neighborhood of F.

Cantor topology One can define a topology on $\mathcal{A}^{\mathbb{M}}$ by endowing $\mathcal{A}$ with the discrete topology, and considering the product topology (or Cantor topology) on $\mathcal{A}^{\mathbb{M}}$. For this topology, $\mathcal{A}^{\mathbb{M}}$ is compact, perfect and totally disconnected. Moreover one can define a metric (which we call the *Cantor distance*) on $\mathcal{A}^{\mathbb{M}}$ which is compatible with the Cantor topology:

$$\forall x, y \in \mathcal{A}^{\mathbb{M}}, \quad d_C(x, y) = 2^{-\min\{|i| : x_i \neq y_i \ i \in \mathbb{M}\}}.$$

Let $\mathbb{U} \subset \mathbb{M}$. For $x \in \mathcal{A}^{\mathbb{M}}$, we denote by $x_{\mathbb{U}} \in \mathcal{A}^{\mathbb{U}}$ the restriction of x to $\mathbb{U}$. For a pattern $w \in \mathcal{A}^{\mathbb{U}}$, one defines the cylinder centered on w by $[w]_{\mathbb{U}} = \{x \in \mathcal{A}^{\mathbb{Z}} : x_{\mathbb{U}} = w\}$.

The action of $\mathbb{M}$ on itself allows to define an action on $\mathcal{A}^{\mathbb{M}}$ by *shift*. For all $m \in \mathbb{M}$ this action is defined by:

$$\begin{aligned} \sigma^m : \mathcal{A}^{\mathbb{M}} &\longrightarrow \mathcal{A}^{\mathbb{M}} \\ (x_i)_{i \in \mathbb{M}} &\longmapsto (x_{i+m})_{i \in \mathbb{M}} \end{aligned}$$

Cellular automata commute with the shift maps: for every cellular automaton $F : \mathcal{A}^{\mathbb{M}} \to \mathcal{A}^{\mathbb{M}}$ and all $m \in \mathbb{M}$, $F \circ \sigma^m = \sigma^m \circ F$. In fact, this a fundamental characteristic of CA. Indeed, Hedlund's theorem [Hed69] states that the cellular automata on $(\mathcal{A}^{\mathbb{M}}, d_C)$ are exactly the continuous functions which commute with the shift maps. It is easy to remark that any cellular automaton F is Lipschitz for the distance d_C. More precisely, for all $x, y \in \mathcal{A}^{\mathbb{M}}$, one has:

$$d_C(F(x), F(y)) \leq 2^{-r(F)} d_C(x, y).$$

It is well-known and easy to see that the action of any shift σ^m on $(\mathcal{A}^{\mathbb{M}}, d_C)$ is transitive. More generally, for all surjective cellular automaton $F : \mathcal{A}^{\mathbb{M}} \to \mathcal{A}^{\mathbb{M}}$ of neighborhood $\mathbb{U}$ one can easily check that the action of $F \circ \sigma^m$ on $(\mathcal{A}^{\mathbb{M}}, d_C)$ is transitive for all $m \in \mathbb{M} \setminus \mathbb{U}$. The reason for this is that the distance d_C is non-homogeneous, hence a simple transport of information is enough to obtain transitivity. This can seem counter-intuitive, and a natural way to overcome this problem is to look at the action of cellular automata on spaces where the distance is shift-invariant or even where the points of the space are themselves shift-invariant. In such spaces, transitivity will not come from transport of information, but rather from *creation* of information.

Besicovitch topology Thus, it seems that a shift-invariant distance on $\mathcal{A}^{\mathbb{M}}$ would be very appropriate to study the dynamics of cellular automata. Following this idea, Cattaneo et al. introduced the *Besicovitch pseudo-distance*:

Definition 1 ([CFMM97]). *The Besicovitch pseudo-distance d_B is defined on $\mathcal{A}^{\mathbb{M}}$ by*

$$d_B(x, y) = \limsup_{n \to +\infty} \frac{\mathrm{Card}\big(\{i \in \mathbb{U}_n : x_i \neq y_i\}\big)}{\mathrm{Card}(\mathbb{U}_n)}.$$

Informally speaking, it measures the asymptotic density of the cells on which x and y differ. It is clearly a pseudo-distance, i.e. it satisfies both the symetry property and triangular inequality. However, $d_B(x, y)$ does not imply $x = y$: if x and y coincide everywhere except on a very sparse set of cells, their Besicovitch pseudo-distance is zero, and yet they are different configurations. Hence, the topology induced on $\mathcal{A}^{\mathbb{M}}$ by d_B is not separated. Notice also that d_B is shift-invariant.

It was proven by Blanchard et al. that CA cannot be expansive with respect to d_B:

Theorem 1 ([BFK97]). *There is no expansive CA on* $(\mathcal{A}^{\mathbb{M}}, d_B)$.

Cattaneo et al. asked whether there exist transitive CA for the Besicovitch pseudo-distance. It remained a recurrent open question (see [BFK97], [Man98], [DFM00]) until it was negatively answered by Blanchard et al. [BCF03]. The original proof of this theorem uses the notion of Kolmogorov complexity, but is quite involved. We present here a simpler proof also based on Kolmogorov complexity, which we will extend later to a much more general framework. We assume that the reader is familiar with Kolmogorov complexity (see [LV97] for an extensive survey, see also [Cal02] for Kolmogorov complexity of strings over a non-binary alphabet).

Theorem 2. *There is no transitive CA on* $(\mathcal{A}^{\mathbb{M}}, d_B)$.

Proof. For all $x \in \mathcal{A}^{\mathbb{M}}$, we set

$$\dim_1(x) = \liminf_{n \to +\infty} \frac{K(x_{\mathbb{U}_n})}{\mathrm{Card}(\mathbb{U}_n)}$$

where K denotes Kolmogorov complexity (what version of Kolmogorov complexity we use does not matter, since all versions coincide up to a logarithmic term). Notice that the quantity $\dim_1(x)$ lies in $[0, \log|\mathcal{A}|]$ (here and in the rest of the paper, log is the logartihm of base 2). The notation $\dim_1$ is justified by a result of Mayordomo [May00] who (elaborating on the work of Staiger and others) showed that this quantity is an effectivization of Hausdorff dimension. We start with two easy lemmas, which we will need again later on:

Lemma 1. *For every* $x \in \mathcal{A}^{\mathbb{M}}$ *and every CA* F, *one has* $\dim_1(F(x)) \leq \dim_1(x)$

Indeed, to compute $F(x)_{\mathbb{U}_n}$, one only needs to know $x_{\mathbb{U}_{n+r(F)}}$, by definition of a CA. Hence $K(F(x)_{\mathbb{U}_n}) \leq K(x_{\mathbb{U}_{n+r(F)}})$. But as $\mathbb{M} = \mathbb{N}^{d'} \times \mathbb{Z}^{d''}$, there are at most $O(n^{d'+d''-1})$ cells in $\mathbb{U}_{n+r(F)} \setminus \mathbb{U}_n$. Hence,

$$K(F(x)_{\mathbb{U}_n}) \leq K(x_{\mathbb{U}_{n+r(F)}}) \leq K(x_{\mathbb{U}_n}) + O(n^{d'+d''-1}).$$

Since the quantity $O(n^{d'+d''-1})$ is a $o(\mathrm{Card}(\mathbb{U}_n))$ (because $\mathrm{Card}(\mathbb{U}_n) = O(n^{d'+d''})$), the lemma is proved.

Lemma 2. *For all $x, y \in \mathcal{A}^{\mathbb{M}}$:*

$$|\dim_1(x) - \dim_1(y)| \leq \hbar(d_B(x,y))$$

with $\hbar(x) = -(1-x)\log(1-x) - x\log(x) + x\log|\mathcal{A}|$ *(notice that* $\hbar(x)$ *is concave, and tends towards* 0 *as* x *tends towards* 0*, which proves that* $\dim_1$ *is uniformly continuous w.r.t* d_B*).*

Let $k = |\mathcal{A}|$. We identify $\mathcal{A}$ with $(\mathbb{Z}/k\mathbb{Z}) = \{\overline{0}...\overline{k-1}\}$, and hence $\mathcal{A}^{\mathbb{M}}$ with $(\mathbb{Z}/k\mathbb{Z})^{\mathbb{M}}$, which is a group (and we denote its addition by $\oplus$). If $d_B(x,y) \leq \varepsilon$ then by definition of d_B, one can write $x = y \oplus z$, where z is a configuration such that for all n, $\frac{\mathrm{Card}\left(\{i \in \mathbb{U}_n : z_i \neq \overline{0}\}\right)}{\mathrm{Card}(\mathbb{U}_n)} \leq \varepsilon + o(1)$. For a given n, setting $N = \mathrm{Card}(\mathbb{U}_n)$, the number of patterns consisting of N cells, with at least $(1-\varepsilon)N$ cells labeled by $\overline{0}$ is bounded by

$$\varepsilon N \binom{N}{\varepsilon N} |\mathcal{A}|^{\varepsilon N}$$

Hence, the Kolmogorov complexity of $z_{\mathbb{U}_n}$ is not greater than the logarithm of this quantity, which, by Stirling's formula, is equal to $\hbar(\varepsilon)N + o(N)$. Since $x_{\mathbb{U}_n}$ can be computed from $y_{\mathbb{U}_n}$ and $z_{\mathbb{U}_n}$, it follows that for all n,

$$K(x_{\mathbb{U}_n}) \leq K(y_{\mathbb{U}_n}) + K(z_{\mathbb{U}_n}) + o(\mathrm{Card}(\mathbb{U}_n)) \leq K(y_{\mathbb{U}_n}) + \hbar(\varepsilon)\mathrm{Card}(\mathbb{U}_n) + o(\mathrm{Card}(\mathbb{U}_n))$$

By definition of $\dim_1$, the lemma follows. We are now ready to prove Theorem 2. Let F be a CA on $\mathcal{A}^{\mathbb{M}}$. Let x be a configuration such that $\dim_1(x) = 0$ and y such that $\dim_1(y) = \log|\mathcal{A}|$ (such sequences exist, see for example [Lut00]). Let $\varepsilon > 0$. If F were transitive, then there would exist $x', y' \in \mathcal{A}^{\mathbb{M}}$ and $n \in \mathbb{N}$ such that $d_B(x,x') \leq \varepsilon$, $d_B(y,y') \leq \varepsilon$ and $F^n(x') = y'$. By Lemma 2, we would then have $\dim_1(x') \leq \hbar(\varepsilon)$, and $\dim_1(y') \geq 1 - \hbar(\varepsilon)$. But also, applying inductively Lemma 1 on x', we would have $\dim_1(F^n(x')) \leq \dim_1(x') \leq \hbar(\varepsilon)$, i.e, $\dim_1(y') \leq \hbar(\varepsilon)$. For ε small enough, this contradicts $\dim_1(y') \geq 1 - \hbar(\varepsilon)$. $\square$

3 Action of cellular automata on $\mathcal{M}_{\sigma}(\mathcal{A}^{\mathbb{M}})$

Measures on $\mathcal{A}^{\mathbb{M}}$ Let $\mathfrak{B}$ be the Borel sigma-algebra of $\mathcal{A}^{\mathbb{M}}$. We denote by $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$ the set of probability measures on $\mathcal{A}^{\mathbb{M}}$ defined on the sigma-algebra $\mathfrak{B}$. Usually $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$ is endowed with weak* topology: a sequence $(\mu_n)_{n\in\mathbb{N}}$ of $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$ converges to $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{M}})$ if and only if for all finite subset $\mathbb{U} \subset \mathbb{M}$ and for all pattern $u \in \mathcal{A}^{\mathbb{U}}$, one has $\lim_{n\to\infty} \mu_n([u]_{\mathbb{U}}) = \mu([u]_{\mathbb{U}})$.

In the weak* topology, the set $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$ is compact and metrizable. One defines a distance compatible with the weak* topology by for all $\mu, \nu \in \mathcal{M}(\mathcal{A}^{\mathbb{M}})$:

$$d_*^{\mathcal{M}}(\mu,\nu) = \sum_{n\in\mathbb{N}} \frac{1}{\mathrm{Card}(\mathbb{U}_n)} \sum_{u \in \mathcal{A}^{\mathbb{U}_n}} \left|\mu([u]_{\mathbb{U}_n}) - \nu([u]_{\mathbb{U}_n})\right|,$$

where $\mathbb{U}_n = \{m \in \mathbb{M} : |m| \leq n\}$.

Let $F : X \to Y$ be a mesurable function between the measurable spaces X and Y and let $\mu \in \mathcal{M}(X)$. It is possible to consider the mesure $F_*\mu$ on Y defined by $F_*\mu(B) = \mu(F^{-1}(B))$ for all measurable set $B \subset Y$. Thus, the $\mathbb{M}$-action σ acts naturally on $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$ by:

$$\sigma_*^m(\mu(B)) = \mu(\sigma^{-m}(B)), \text{ for all } m \in \mathbb{M},\ \mu \in \mathcal{M}(\mathcal{A}^{\mathbb{M}}) \text{ and } B \in \mathfrak{B}.$$

A measure $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{M}})$ is said *σ-invariant* if $\sigma_*^m\mu = \mu$ for all $m \in \mathbb{M}$; denote $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ the set of σ-invariant probability measure.

The distance $d_B^{\mathcal{M}}$ In [Sab07], a general framework to define a distance on $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ is given: let d be a pseudo-distance on $\mathcal{A}^{\mathbb{M}}$, we want to introduce a pseudo-distance on $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ induced by the pseudo-distance d. Let $\mu, \nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$, the intuitive idea is to calculate the mean of $d(x, y)$ when x is chosen according to the probability measure μ and y according to the probability measure ν. If we just take (x, y) according to the probability $\mu \times \nu$, when $\nu = \mu$, one obtains $\int d(x, y)\mathrm{d}(\mu \times \mu)$ which is in general positive. Hence it is important to allow some kind of correlation in the choice of x and y. This is why we introduce the notion of joint measure.

Let μ and ν be two σ-invariant probability measures on $\mathcal{A}^{\mathbb{M}}$. A probability measure λ on $\mathcal{A}^{\mathbb{M}} \times \mathcal{A}^{\mathbb{M}}$ is a *joint measure* according to μ and ν if λ is $\sigma \times \sigma$-invariant and $\pi_*^1\lambda = \mu$ and $\pi_*^2\lambda = \nu$, where π^1 and π^2 are respectively the projections according the first and second coordinate. Denote $\mathcal{J}(\mu, \nu)$ the set of joint measures according μ and ν. Of course, one has $\mathcal{J}(\mu, \nu) \subset \mathcal{M}_{\sigma\times\sigma}(\mathcal{A}^{\mathbb{M}} \times \mathcal{A}^{\mathbb{M}})$. Moreover $\mathcal{J}(\mu, \nu)$ is convex and compact for the weak topology.

Definition 2. *Let d be a pseudo-distance on $\mathcal{A}^{\mathbb{M}}$ such that $(x, y) \mapsto d(x, y)$ is Borel-measurable (this is the case for d_C and d_B). One defines a function $d^{\mathcal{M}}$ from $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}) \times \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ on $\mathbb{R}^+$ by:*

$$d^{\mathcal{M}}(\mu, \nu) = \inf_{\lambda \in \mathcal{J}(\mu,\nu)} \int d(x, y)\mathrm{d}\lambda(x, y) \quad \textit{for all } \mu, \nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}).$$

In [Sab07], we prove that $d_C^{\mathcal{M}}$ is equivalent to $d_*^{\mathcal{M}}$ and that $d_B^{\mathcal{M}}$ defines a distance on $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$, which is not equivalent to $d_*^{\mathcal{M}}$. Moreover we give general properties about this type of measure. In particular we have the following lemma:

Lemma 3. *Let $\mu, \nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ and let $\mathbb{U} \subset \mathbb{M}$ be a finite subset. One has:*

$$d_B^{\mathcal{M}}(\mu, \nu) \geq \frac{1}{\mathrm{Card}(\mathbb{U})} \inf_{\lambda \in \mathcal{J}(\mu,\nu)} \lambda([u]_{\mathbb{U}} \times [v]_{\mathbb{U}} : u, v \in \mathcal{A}^{\mathbb{U}},\ u \neq v).$$

Proof. Let $\mu, \nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ and let $\lambda \in \mathcal{J}(\mu, \nu)$. Let $u, v \in \mathcal{A}^{\mathbb{U}}$, one has:

$$\bigcup_{u,v \in \mathcal{A}^{\mathbb{U}}, u \neq v} [u]_{\mathbb{U}} \times [v]_{\mathbb{U}} \subset \bigcup_{m \in \mathbb{U}} \left(\bigcup_{a,b \in \mathcal{A}, a \neq b} [a]_m \times [b]_m \right).$$

One deduces the following inequality:

$$\begin{aligned}\lambda([u]_{\mathbb{U}} \times [v]_{\mathbb{U}} : u, v \in \mathcal{A}^{\mathbb{U}},\, u \neq v) &\leq \sum_{m \in \mathbb{U}} \lambda([a]_m \times [b]_m : a, b \in \mathcal{A},\, a \neq b) \\ &\underset{(\star)}{=} \mathrm{Card}(\mathbb{U})\, \lambda([a]_0 \times [b]_0 : a, b \in \mathcal{A},\, a \neq b),\end{aligned}$$

where $(\star)$ follows from the $\sigma \times \sigma$-invariance of λ.

This lemma allows in particular to prove that $d_B^{\mathcal{M}}$ is a distance.

Action of a cellular automaton on $\boldsymbol{\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})}$ Let $(\mathcal{A}^{\mathbb{M}}, F)$ be a CA and $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$. Since F commutes with the shift, if $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ then $F_*\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$. Let d be a pseudo-distance on $\mathcal{A}^{\mathbb{M}}$. To study the $\mathbb{N}$-action of F_* on $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d^{\mathcal{M}})$ as a dynamical system, we are going to prove the continuity of the function F_* on $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d^{\mathcal{M}})$.

Proposition 1. *Let d be a pseudo-distance on $\mathcal{A}^{\mathbb{M}}$ and let $F : \mathcal{A}^{\mathbb{M}} \to \mathcal{A}^{\mathbb{M}}$ be a function d-Lipschitz of constant K on $\mathcal{A}^{\mathbb{M}}$. For all $\mu, \nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$, one has:*

$$d^{\mathcal{M}}(F_*\mu, F_*\nu) \leq K d^{\mathcal{M}}(\mu, \nu).$$

In particular F_* is continuous on $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d^{\mathcal{M}})$.

Proof. Let $\lambda \in \mathcal{J}(\mu, \nu)$, one has $(F_* \times F_*)\lambda \in \mathcal{J}(F_*\mu, F_*\nu)$, thus:

$$\int d(x,y)\mathrm{d}(F_* \times F_*)\lambda = \int d(F(x), F(y))\,\mathrm{d}\lambda \leq \int K d(x,y)\,\mathrm{d}\lambda.$$

One deduces that $d(F_*\mu, F_*\nu) \leq K d(\mu, \nu)$.

Since all CA are Lipschitz for d_C and d_B, this proposition holds for all CA. Thus one can study the dynamical system $F_* : \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}) \to \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ according to the distance $d_*^{\mathcal{M}}$ or $d_B^{\mathcal{M}}$.

Non-expansivity of CA on $\boldsymbol{(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})}$ In the space of measures $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$, we have the following counterpart to Theorem 1:

Proposition 2. *Let $(\mathcal{A}^{\mathbb{M}}, F)$ be a CA. F_* does not act expansively on $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$*

Proof. Let $\mu, \nu \in (\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ and $\varepsilon > 0$. Consider $\mu' = (1-\varepsilon)\mu + \varepsilon\nu$. Let $\lambda' \in \mathcal{J}(\mu, \mu)$ and $\lambda'' \in \mathcal{J}(\mu, \nu)$, such that $\lambda = (1-\varepsilon)\lambda' + \varepsilon\lambda'' \in \mathcal{J}(\mu, \mu')$. One then has:

$$(1-\varepsilon)\int d(x,y)\mathrm{d}\lambda' + \varepsilon\int d(x,y)\mathrm{d}\lambda'' = \int d(x,y)\mathrm{d}\lambda \geq d^{\mathcal{M}}(\mu, \mu')$$

Thus,

$$\varepsilon d^{\mathcal{M}}(\mu, \nu) = (1-\varepsilon)d^{\mathcal{M}}(\mu, \mu) + \varepsilon d^{\mathcal{M}}(\mu, \nu) \geq d^{\mathcal{M}}(\mu, (1-\varepsilon)\mu + \varepsilon\nu)$$

Since F_* preserves convex combinations, one has $F_*^n\mu' = (1-\varepsilon)F_*^n\mu + \varepsilon F_*^n\nu$ for all $n \in \mathbb{N}$, so $d^{\mathcal{M}}(F_*^n\mu, F_*^n\mu') \leq \varepsilon d^{\mathcal{M}}(F_*^n\mu, F_*^n\nu)$. Hence, F_* is not enxpansive in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$.

Continuity of the entropy of σ The information contained in a a generic configuration can be expressed by the entropy of the shift. A comparative study of the entropy of the shift and Kolmogorov complexity was carried out by Brudno [Bru82]. As we will see, the entropy of the shift is continuous with respect to the underlying measure.

Definition 3. *Let $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$, the entropy of the shift $\mathbb{M}$-action can be defined as:*

$$h_\mu(\sigma) = \lim_{n\to\infty} \frac{H_\mu(\mathcal{P}_{\mathbb{U}_n})}{\mathrm{Card}(\mathbb{U}_n)},$$

where $\mathcal{P}_{\mathbb{U}_n}$ is the partition of cylinders centered on $\mathbb{U}_n$ and $H_\mu(\mathcal{P}_{\mathbb{U}_n})$ is the entropy of the partition $\mathcal{P}_{\mathbb{U}_n}$ according to the measure μ, defined by:

$$H_\mu(\mathcal{P}_{\mathbb{U}_n}) = -\sum_{u\in\mathcal{A}^{\mathbb{U}_n}} \mu([u]_{\mathbb{U}_n}) \log(\mu([u]_{\mathbb{U}_n})).$$

One recalls that $\mathbb{U}_n = \{m \in \mathbb{M} : |m| \leq n\}$.

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two partitions of $\mathcal{A}^{\mathbb{M}}$. We define the refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$ by

$$\mathcal{P}_1 \vee \mathcal{P}_2 = \{A \cap B : A \in \mathcal{P}_1 \text{ and } B \in \mathcal{P}_2\}.$$

Moreover it is possible to define the conditional entropy of $\mathcal{P}_1$ given $\mathcal{P}_2$:

$$H_\mu(\mathcal{P}_1|\mathcal{P}_2) = -\sum_{B\in\mathcal{P}_2} \mu(B) \sum_{A\in\mathcal{P}_1} \frac{\mu(A\cap B)}{\mu(B)} \log(\mu(A)).$$

Thanks conditional entropy, it is possible to decompose the entropy of a refinement:

$$H_\mu(\mathcal{P}_1 \vee \mathcal{P}_2) = H_\mu(\mathcal{P}_2) + H_\mu(\mathcal{P}_1|\mathcal{P}_2).$$

It is well known that the function $\mu \mapsto h_\mu(\sigma)$ is upper semi-continuous in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_*^{\mathcal{M}})$, see [DGS76] for more detail.

Theorem 3. *The function $\mu \mapsto h_\mu(\sigma)$ is uniformly continuous in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$.*

Proof. Let μ and ν in $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$. By definition of the entropy of σ, one has

$$h_\mu(\sigma) = \lim_{n\to\infty} \frac{H_\mu(\mathcal{P}_{\mathbb{U}_n})}{\mathrm{Card}(\mathbb{U}_n)} \text{ and } h_\nu(\sigma) = \lim_{n\to\infty} \frac{H_\nu(\mathcal{P}_{\mathbb{U}_n})}{\mathrm{Card}(\mathbb{U}_n)}.$$

However, for all $\lambda \in \mathcal{J}(\mu,\nu)$ one has:

$$\begin{aligned}
|H_\mu(\mathcal{P}_{\mathbb{U}_n}) - H_\nu(\mathcal{P}_{\mathbb{U}_n})| &= |H_\lambda(\mathcal{P}_{\mathbb{U}_n} \times \mathcal{A}^{\mathbb{M}}) - H_\lambda(\mathcal{A}^{\mathbb{M}} \times \mathcal{P}_{\mathbb{U}_n})| \\
&= |\big(H_\lambda(\mathcal{P}_{\mathbb{U}_n} \times \mathcal{A}^{\mathbb{M}}) - H_\lambda(\mathcal{P}_{\mathbb{U}_n} \times \mathcal{A}^{\mathbb{M}} \vee \mathcal{A}^{\mathbb{M}} \times \mathcal{P}_{\mathbb{U}_n})\big) \\
&\quad -\big(H_\lambda(\mathcal{A}^{\mathbb{M}} \times \mathcal{P}_{\mathbb{U}_n}) - H_\lambda(\mathcal{P}_{U_n} \times \mathcal{A}^{\mathbb{M}} \vee \mathcal{A}^{\mathbb{M}} \times \mathcal{P}_{\mathbb{U}_n})\big)| \\
&\leq H_\lambda(\mathcal{P}_{\mathbb{U}_n} \times \mathcal{A}^{\mathbb{M}} | \mathcal{A}^{\mathbb{M}} \times \mathcal{P}_{\mathbb{U}_n}) + H_\lambda(\mathcal{A}^{\mathbb{M}} \times \mathcal{P}_{\mathbb{U}_n} | \mathcal{P}_{\mathbb{U}_n} \times \mathcal{A}^{\mathbb{M}}).
\end{aligned}$$

Moreover, one has:

$$\begin{aligned}H_\lambda(\mathcal{P}_{\mathbb{U}_n}\times\mathcal{A}^{\mathbb{M}}|\mathcal{A}^{\mathbb{M}}\times\mathcal{P}_{\mathbb{U}_n}) &\leq \sum_{i\in\mathbb{U}_n} H_\lambda(\mathcal{P}_i\times\mathcal{A}^{\mathbb{M}}|\mathcal{A}^{\mathbb{M}}\times\mathcal{P}_{\mathbb{U}_n})\\ &\leq \mathrm{Card}(\mathbb{U}_n)H_\lambda(\mathcal{P}_0\times\mathcal{A}^{\mathbb{M}}|\mathcal{A}^{\mathbb{M}}\times\mathcal{P}_{\mathbb{U}_n})\\ &\leq \mathrm{Card}(\mathbb{U}_n)H_\lambda(\mathcal{P}_0\times\mathcal{A}^{\mathbb{M}}|\mathcal{A}^{\mathbb{M}}\times\mathcal{P}_0),\end{aligned}$$

where $\mathcal{P}_0=\mathcal{P}_{\mathbb{U}_0}$. Symmetrically one obtains

$$H_\lambda(\mathcal{A}^{\mathbb{M}}\times\mathcal{P}_{\mathbb{U}_n}|\mathcal{P}_{\mathbb{U}_n}\times\mathcal{A}^{\mathbb{M}}) \leq \mathrm{Card}(\mathbb{U}_n)H_\lambda(\mathcal{A}^{\mathbb{M}}\times\mathcal{P}_0|\mathcal{P}_0\times\mathcal{A}^{\mathbb{M}}).$$

Thus, by summation one has:

$$|h_\mu(\sigma)-h_\nu(\sigma)| \leq H_\lambda(\mathcal{P}_0\times\mathcal{A}^{\mathbb{M}}|\mathcal{A}^{\mathbb{M}}\times\mathcal{P}_0)+H_\lambda(\mathcal{A}^{\mathbb{M}}\times\mathcal{P}_0|\mathcal{P}_0\times\mathcal{A}^{\mathbb{M}}).$$

Consider $\alpha=(\cup_{a,b\in\mathcal{A},a\neq b}[a]_0\times[b]_0;\cup_{a\in\mathcal{A}}[a]_0\times[a]_0)$, the partition of $\mathcal{A}^{\mathbb{M}}\times\mathcal{A}^{\mathbb{M}}$ formed of two elements. Set $\delta=\lambda(\cup_{a,b\in\mathcal{A},a\neq b}[a]_0\times[b]_0)$. One has:

$$H_\lambda(\mathcal{P}_0\times\mathcal{A}^{\mathbb{M}}|\mathcal{A}^{\mathbb{M}}\times\mathcal{P}_0)\leq H_\lambda(\alpha)\leq -(\delta\log(\delta)+(1-\delta)\log(1-\delta)).$$

Let $\varepsilon>0$. The function $\delta\to\delta\log(\delta)+(1-\delta)\log(1-\delta)$ tends towards 0 when δ tends towards 0. Thus, there exists $\delta_0>0$ such that $\delta\log(\delta)+(1-\delta)\log(1-\delta)\leq\frac{\varepsilon}{2}$ for all $\delta<\delta_0$. Let $\mu,\nu\in\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ such that $d_B^{\mathcal{M}}(\mu,\nu)<\delta_0$. According to Lemma 3, there exists $\lambda\in\mathcal{J}(\mu,\nu)$ such that

$$\lambda([a]_0\times[b]_0 : a,b\in\mathcal{A}, a\neq b)\leq d_B^{\mathcal{M}}(\mu,\nu)<\delta_0.$$

In this case, one has $H_\lambda(\mathcal{P}_0\times\mathcal{A}^{\mathbb{M}}|\mathcal{A}^{\mathbb{M}}\times\mathcal{P}_0)\leq\frac{\varepsilon}{2}$, and symetically $H_\lambda(\mathcal{A}^{\mathbb{M}}\times\mathcal{P}_0,\mathcal{P}_0\times\mathcal{A}^{\mathbb{M}})\leq\frac{\varepsilon}{2}$. We deduce that for all $\varepsilon>0$, there exists δ_0 such that if $d_B^{\mathcal{M}}(\mu,\nu)\leq\delta_0$ then

$$|h_\mu(\sigma)-h_\nu(\sigma)| \leq H_\lambda(\mathcal{P}_0\times\mathcal{A}^{\mathbb{M}}|\mathcal{A}^{\mathbb{M}}\times\mathcal{P}_0)+H_\lambda(\mathcal{A}^{\mathbb{M}}\times\mathcal{P}_0|\mathcal{P}_0\times\mathcal{A}^{\mathbb{M}})\leq\varepsilon.$$

This proves the uniform continuity of $\mu\to h_\mu(\sigma)$ in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}),d_B^{\mathcal{M}})$.

Application to transitivity

Theorem 4. *Let $(\mathcal{A}^{\mathbb{M}},F)$ be a CA. F_* cannot be transitive in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}),d_B^{\mathcal{M}})$.*

Proof. Let

$$\mathcal{U}=\{\mu\in\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}) : h_\mu(\sigma)<1/3\} \text{ and } \mathcal{V}=\{\mu\in\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}) : h_\mu(\sigma)>2/3\}.$$

By Theorem 3, $\mathcal{U}$ and $\mathcal{V}$ are open sets of $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}),d_B^{\mathcal{M}})$. Since F commutes with σ, it can be view as a factor map from $(\mathcal{A}^{\mathbb{M}},\mu,\sigma)$ to $(\mathcal{A}^{\mathbb{M}},F_*\mu,\sigma)$, so one has $h_\mu(\sigma)\geq h_{F_*\mu}(\sigma)$. Thus $F_*(\mathcal{U})\subset\mathcal{U}$. One deduces that $\mathcal{V}\cap F_*^n(\mathcal{U})=\emptyset$ for all $n\in\mathbb{N}$, thus F_* can not be transitive in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}),d_B^{\mathcal{M}})$.

In $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}),d_*^{\mathcal{M}})$, the function $\mu\to h_\mu(\sigma)$ is just upper semi-continuous, so $\mathcal{V}$ is not open and the previous proof does not hold. In the space $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}),d_*^{\mathcal{M}})$, the existence of transitive CA is open.

4 Action of cellular automata on $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$

In this section, we do not restrict ourselves to the space of shift-invariant measures: we instead consider the whole space $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$. The distance $d_B^{\mathcal{M}}$ defined in the previous section can be extended to arbitrary measures, hence endowing $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$ with a Besicovitch-like topology. On the space $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$, $d_B^{\mathcal{M}}$ is only a pseudo-distance, as for example two measures which are equal up to a shift are at distance 0 from each other. Similarly to $(\mathcal{A}^{\mathbb{M}}, d_B)$, the space $(\mathcal{M}(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$ is not separated.

The space $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$ can clearly be viewed as a subspace of $(\mathcal{M}(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$. Moreover, $(\mathcal{A}^{\mathbb{M}}, d_B)$ can also be viewed as a subspace of $(\mathcal{M}(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$ via the isometric embedding

$$\begin{aligned} \mathcal{A}^{\mathbb{M}} &\longrightarrow \mathcal{M}(\mathcal{A}^{\mathbb{M}}) \\ x &\longmapsto \delta_x \end{aligned}$$

where δ_x is the measure concentrated on x (i.e. $\delta_x(A) = 1$ if $x \in A$, $\delta_x(A) = 0$ otherwise).

The proof of non-expansivity of CA which was proven in the previous section naturally extends to the whole space $(\mathcal{M}(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$. On the other hand, the proofs of non-transitivity we presented respectively for $(\mathcal{A}^{\mathbb{M}}, d_B)$ and $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$ cannot be extended in a completely straightforward way to $(\mathcal{M}(\mathcal{A}^{\mathbb{M}}), d_B)$. It is true however that no CA is transitive in this space. In fact, non-transitivity happens in the larger class of Lipschitz funtions:

Theorem 5. *Let $F : \mathcal{A}^{\mathbb{M}} \to \mathcal{A}^{\mathbb{M}}$ be a function that is Lipschitz w.r.t. the distance d_C. The action of F_* on $(\mathcal{M}(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$ is not transitive.*

Proof. We adapt the proof of Theorem 2. First notice that for a function F that is Lipschitz in d_C with constant 2^r one only needs to know $F\ x_{\mathbb{U}_{n+r}}$ to compute $F(x)_{\mathbb{U}_n}$, hence Lemma 1 remains true if one takes F to be a Lipschitz function w.r.t. d_C and one replaces Kolmogorov complexity K by $K^{(F)}$, i.e. Kolmogorov complexity relativized to oracle F (F being a Lipschitz function, it can be given as an oracle), and $\dim_1$ by $\dim_1^{(F)}$.

For a measure μ, we set

$$\mathbb{E}\dim_1(\mu) = \int \dim_1(x)\, \mathrm{d}\mu(x)$$

We will need the following analogue of Lemma 2 (which can be relativized to any given oracle):

Lemma 4. *There exists a constant c such that for all μ, ν:*

$$d_B^{\mathcal{M}}(\mu,\nu) < c \Rightarrow \left|\mathbb{E}dim_1(\mu) - \mathbb{E}dim_1(\nu)\right| \le \hbar(d_B^{\mathcal{M}}(\mu,\nu))$$

and thus, $\mathbb{E}dim_1$ is uniformly continuous w.r.t. $d_B^{\mathcal{M}}$.

Let c be the constant such that $\hbar$ is increasing on $[0, c]$, and let μ, ν be such that $d_B^{\mathcal{M}}(\mu, \nu) < c$. Let $\varepsilon \in (d_B^{\mathcal{M}}(\mu, \nu), c)$. By definition of $d_B^{\mathcal{M}}$, there exists a measure $\lambda \in \mathcal{J}(\mu, \nu)$ such that

$$\int d_B(x, y) \, \mathrm{d}\lambda \leq \varepsilon$$

Since $\hbar$ is increasing on $[0, c]$ and concave:

$$\int \hbar(d_B(x, y)) \leq \hbar\Big(\int d_B(x, y) \, \mathrm{d}\lambda \Big) \leq \hbar(\varepsilon)$$

which by Lemma 1 implies:

$$\int |\dim_1(x) - \dim_1(y)| \, \mathrm{d}\lambda \leq \hbar(\varepsilon)$$

and thus

$$\big| \mathbb{E}\mathrm{dim}_1(\mu) - \mathbb{E}\mathrm{dim}_1(\nu) \big| \leq \hbar(\varepsilon)$$

which implies the desired result, as ε can be chosen arbitrarily close to $d_B^{\mathcal{M}}(\mu, \nu)$.

We are now ready to prove Theorem 5. Let F be a Lipschitz function w.r.t. d_C. Let δ_0 be the measure concentrated on the configuration where all cells have state 0, and ν be Lebesgue measure. Let μ' be a measure such that $d_B^{\mathcal{M}}(\delta_0, \mu') \leq \varepsilon$ and μ'' be a measure such that $d_B^{\mathcal{M}}(\nu, \mu'') \leq \varepsilon$ with ε small enough. Since $\mathbb{E}\mathrm{dim}_1(\delta_0) = 0$ and $\mathbb{E}\mathrm{dim}_1(\nu) = \log |\mathcal{A}|$, by Lemma 4, one has $\mathbb{E}\mathrm{dim}_1(\mu') \leq \hbar(\varepsilon)$ and $\mathbb{E}\mathrm{dim}_1(\mu'') \geq 1 - \hbar(\varepsilon)$.

By Lemma 1, for all $x \in \mathcal{A}^{\mathbb{M}}$, one has $\dim_1^{(F)}(F(x)) \leq \dim_1^{(F)}(x)$, hence for every $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{M}})$, $\mathbb{E}\mathrm{dim}_1^{(F)}(F_*(\mu)) \leq \mathbb{E}\mathrm{dim}_1^{(F)}(\mu)$. Hence, by the above discussion, if $d_B^{\mathcal{M}}(\delta_0, \mu) \leq \varepsilon$, for all $n \in \mathbb{N}$, $\mathbb{E}\mathrm{dim}_1^{(F)}(F_*^n(\mu)) \leq \hbar(\varepsilon)$, which (still by the above discussion) means that $F_*^n(\mu)$ will never be $d_B^{\mathcal{M}}$-close to Lebesgue measure. This finishes the proof of the theorem.

The non-tranisitivity of CA in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$ (as stated in Theorem 4) immediately follows from the above proof, as δ_0 and Lebesgue measure are shift-invariant measures. On can also modifiy the above proof to get Theorem 2: instead of Lebesgue measure, take ν equal to δ_z for some $z \in \mathcal{A}^{\mathbb{M}}$ such that $\dim_1^{(F)}(z) = \log |\mathcal{A}|$, the rest of the proof remaining the same.

Conclusion

It appears that in the shift-invariant topologies we considered, cellular automata cannot be expansive nor transitive. This is mainly due to the unability of cellular automata to create information. Indeed, for the non-transitivity of CA, the three proofs we gave all have the same scheme. First, we define on a (pseudo-)metric space (E, d) where d is a shift-invariant distance (in this paper, resp. $(\mathcal{A}^{\mathbb{M}}, d_B)$, $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$ and $(\mathcal{M}(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$) a quantity which in some sense measures the amount of information, $\mathcal{I} : E \to \mathbb{R}_+$, (resp. $\dim_1$, $h_\mu(\sigma)$ and $\mathbb{E}\dim_1$), which we prove to be uniformly continuous w.r.t. the distance d. This amount of information is non-increasing under the action of a cellular automaton (or even Lipschitz functions), i.e. $\mathcal{I}(F(x)) \leq \mathcal{I}(x)$ for all $x \in E$. Since in all cases there are elements of the space which contain little information (i.e. $\mathcal{I}(x) = 0$) and some which contain a lot of information (i.e. in our case $\mathcal{I}(x) = \log|\mathcal{A}|$). Hence, the two open sets $\mathcal{U} = \{x \in E : \mathcal{I}(x) < \varepsilon\}$ and $\mathcal{V} = \{x \in E : \mathcal{I}(x) > \log|\mathcal{A}| - \varepsilon\}$ witness, for ε small enough, the non-transitivity of cellular automata.

References

[BCF03] F. Blanchard, J. Cervelle, and E. Formenti. Periodicity and transitivity for cellular automata in Besicovitch topologies. In *Mathematical foundations of computer science 2003*, volume 2747 of *Lecture Notes in Comput. Sci.*, pages 228–238. Springer, Berlin, 2003.

[BFK97] Cellular automata in the Cantor, Besicovitch, and Weyl topological spaces. *Complex Systems*, 11(2):107–123, 1997.

[Bru82] A. A. Brudno. Entropy and the complexity of the trajectories of a dynamic system. *Trudy Moskov. Mat. Obshch.*, 44:124–149, 1982.

[Cal02] C. Calude. Information theory and randomness: an algorithmic perspective, second ed. Spinger Berlin, 2002.

[CFMM97] G. Cattaneo, E. Formenti, L. Margara, and J. Mazoyer. A shift-invariant metric on $S^{\mathbf{Z}}$ inducing a non-trivial topology. In *Mathematical foundations of computer science 1997 (Bratislava)*, volume 1295 of *Lecture Notes in Comput. Sci.*, pages 179–188. Springer, Berlin, 1997.

[DFM00] M. Delorme, E. Formenti, J. Mazoyer. Open problems on cellular automata. *Technical report RR-2000-25, Ecole Normale Supérieure de Lyon*, 2000.

[DGS76] M. Denker, C. Grillenberger, and K. Sigmund. Ergodic theory on compact spaces. Springer-Verlag, Berlin, 1976. Lecture Notes in Mathematics, Vol. 527.

[Egg49] H. G. Eggleston. The fractional dimension of a set defined by decimal properties. *Quarterly Journal of Mathematics*, Oxford Series 20:31–36, 1949.

[Fal85] K. Falconer. The geometry of fractal sets. Cambridge University Press, 1985.

[Hed69] G. A. Hedlund. Endormorphisms and automorphisms of the shift dynamical system. *Math. Systems Theory*, 3:320–375, 1969.

[Lut00] J. Lutz. Dimension in complexity classes. In *Conference on Computational Complexity, CCC'00, IEEE Computer Society, 158–169*, 2000.

[LV97] M. Li, P. Vitanyi. An introduction to Kolmogorov complexity and its applications, second ed.. Spinger Berlin, 1997.

[Man98] G. Manzini. Characterization of sensitive linear automata with respect to the couting distance. In *Mathematical Foundations of Computer Science 1998*, 1998.

[May00] E. Mayordomo. A Kolmogorov complexity characterization of constructive Hausdorff dimension. *Information Processing Letters 84:1-3*, 2000.

[Sab07] M. Sablik. Action of Cellular automata on σ-invariant probability measure. *Prépublication de l'UMPA, www.umpa.ens-lyon.fr/∼msablik*, September 2006.