ACTION OF CELLULAR AUTOMATA ON SHIFT-INVARIANT MEASURE

MATHIEU SABLIK

Abstract. In this article we introduce a general process to construct σ-invariant pseudo-distance. An other σ-invariant object is the set of σ-invariant probability measures. We give a general framework for studying the action of cellular automata on this set and establish some properties of the dynamics of the action of cellular automata on this space.

Introduction

A cellular automaton is a complex system defined by a local rule which acts synchronously and uniformly on the configuration space. These simple models have a wide variety of different dynamical behaviors. Let $\mathcal{A}$ be a finite alphabet and $\mathbb{M} = \mathbb{Z}^{d'} \times \mathbb{N}^{d''}$. A cellular automaton can be defined as a continuous function on the full-shift $\mathcal{A}^{\mathbb{M}}$ endowed with the product topology (also called the Cantor topology) which commutes with the shift σ. Generally a cellular automaton is studied as a $\mathbb{N}$-action on $\mathcal{A}^{\mathbb{M}}$ endowed with the distance of Cantor without worrying about the $\mathbb{M}$-action σ. There is a lot of studies of the dynamical properties of the $\mathbb{N}$-action generated by a cellular automaton which classify them according to certain behaviors.

These classifications are based on the product topology. So, the distance considered privileges the central coordinates whereas there may be no reason to give more importance to coordinates around the origin. Thus, simple cellular automata such as the powers of the shift are sensitive to the initial conditions. This does not correspond to the intuitive idea which computer scientists or physicists have when they observe the extremely regular space-time diagrams of these cellular automata. The principal shortcoming of these classifications is to consider only the action of the cellular automaton without considering the shift $\mathbb{M}$-action. Indeed, space-time diagrams of a cellular automata $(\mathcal{A}^{\mathbb{M}}, F)$ are not so different from that of $(\mathcal{A}^{\mathbb{M}}, \sigma^m \circ F)$ for $m \in \mathbb{M}$. However, if F is not nilpotent, $\sigma^m \circ F$ is sensitive for m taked quite far from the origin. The reason is that Cantor topology is non-homogeneous, thus a simple transport of information is enough to obtain sensitivity.

One point of view can be to address the $\mathbb{M}\times\mathbb{N}$-action (σ, F) in order to put emphasize the spatiotemporal structure. Since the $\mathbb{M}$-action σ is $\mathbb{M}$-expansive, the $\mathbb{M} \times \mathbb{N}$-action (σ, F) is also $\mathbb{M} \times \mathbb{N}$-expansive and the dynamic is so strong: it contains the dynamic of σ. Thus we must to study the dynamic of restrictions of this action at sub-semi-group of $\mathbb{M} \times \mathbb{N}$. In [Sab06] we give general definitions to talk about directional dynamics even in irrational directions; the purpose is to study the sets of direction which have a certain kind of dynamics.

An other point of view is to kill the $\mathbb{M}$-action of σ and consider the $\mathbb{N}$-action of F on σ-invariant object in order to make disappear the notion of signal. In this direction, G. Cattaneo, E. Formenti, L. Margara et J. Mazoyer [CFMM97] introduce another topology defined by the Besicovitch pseudo distance which measures the density of the differences between two configurations in order to give the same importance at all cells. For this distance, the shift is an isometry. However, with this topology we lose the compactness of the space which is the traditional framework of topological dynamics. There exists other σ-invariant distances as the Weyl pseudo-distance [BFK97]

In this article, we begin by giving a general framework to define σ-invariant pseudo-distances using sub-measures on $\mathbb{M}$. This type of distance measure the quantity of defect between two configurations according to the sub-measure. We give some properties of the pseudo-distances obtained according the properties of the sub-measures. This allows to find again properties of Cantor, Besicovitch and Weyl pseudo-distances.

Date: Janvier 2007.

2010 *Mathematics Subject Classification.* Primary 37B10, Secondary 37B50, 03D10.

Key words and phrases. Symbolic Dynamics, Multi-dimensional shifts of finite type, Subaction, Projective subaction, Effectively closed subshifts,Turing machines, Substitutive subshifts.

An other natural σ-invariant object is the set of σ-invariant probability measures $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$. Indeed, a cellular automaton acts on this space by:

$$F_*\mu(B) = \mu(F^{-1}(B)) \text{ for all Borel sets } B \subset \mathcal{A}^{\mathbb{M}}.$$

For the weak* topology, this set is compact and metrizable; moreover the shift has the same behavior as the identity. So this space seems more adapted than Besicovitch space. We can consider the $\mathbb{N}$-action of F_* on $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$. In this space the shift has the same behavior as the identity and a sensitive cellular automaton is not only capable of *transporting* information but it is also able to *create*new information.

P. Kurka shows in [Kůr03] that it has an unique attractor and particulary studies the support of the measure in this attractor. In this article we consider properties of sensibility to initial condition. This approach can be interesting when we use cellular automata to simulate. Indeed, generally we choose an arbitrary configuration according a distribution, this approach describes how evolute the mistake when we chose the initial configuration with a distribution near from the expected distribution.

The different pseudo-distances defined on $\mathcal{A}^{\mathbb{M}}$ induce pseudo-distances on $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ by way of joinings. We begin by doing a general study of the action of a cellular automaton on these spaces. Then we study more precisely two of them, the distance induced by the distance of Cantor and the distance induced by the Besicovitch pseudo-distance. For the Cantor distance, we obtain a distance compatible with the weak* topology denoted d_*. For the Besicovitch pseudo-distance we obtain a distance, denoted $d_B^{\mathcal{M}}$, similar to the distance of Kantorivich, see [Ver04] for a historical approach. First we recover some of the remarkable properties of these metrics even if it is well known by specialists of egodic theory in view to give a self contain article. After we establish some properties of the induced map F_* on the space of σ-invariant measures with these different metrics. The more interesting dynamical properties obtained in these spaces are:

- There is not expansive cellular automata in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_*)$ and $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$. But we do not know results if we restrict the action to the space to σ-ergodic probability measures.
- There is not transitive cellular automaton in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$. This result can be related to the similar result in $\mathcal{A}^{\mathbb{M}}$ endowed with the Besicovitch topology which is proved thanks to Kolmogorov complexity [BCF03]. In the space $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_*)$, we do not know if there is a such result.
- If the cellular automata has equicontinuous points of slope α for the Cantor metric then $Eq_{d_*}(F_*) \neq \emptyset$ and $Eq_{d_B^{\mathcal{M}}}(F_*) \neq \emptyset$ where $Eq_{d_*}(F_*)$ and $Eq_{d_B^{\mathcal{M}}}(F_*)$ are respectively the sets of equicontinuous points in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_*)$ and in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$. However we do not know if these two sets coincide.
- If $(\mathcal{A}^{\mathbb{M}}, F)$ is a linear cellular automata, then $Eq_{d_*}(F_*) = \emptyset$ but we do not know if F_* is d_*-sensitive.

1. Action of a cellular automaton on $\mathcal{A}^{\mathbb{M}}$ and $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$ (Backgrounds)

1.1. **Space of configurations.** Let $\mathbb{M} = \mathbb{Z}^{d'} \times \mathbb{N}^{d''}$. For all $m \in \mathbb{M}$, denote $|m|$ the distance of m to the origin point. Let $\mathcal{A}$ be a finite set. We consider $\mathcal{A}^{\mathbb{M}}$, the *configuration space* of $\mathbb{M}$-indexed sequences in $\mathcal{A}$. If $\mathcal{A}$ is endowed with the discrete topology, $\mathcal{A}^{\mathbb{M}}$ is compact, perfect and totally disconnected in the product topology. Moreover one can define a metric on $\mathcal{A}^{\mathbb{M}}$ compatible with this topology:

$$\forall x, y \in \mathcal{A}^{\mathbb{M}}, \quad d_C(x,y) = 2^{-\min\{|i| : x_i \neq y_i \ \ i \in \mathbb{M}\}}.$$

Let $\mathbb{U} \subset \mathbb{M}$. For $x \in \mathcal{A}^{\mathbb{M}}$, denote $x_{\mathbb{U}} \in \mathcal{A}^{\mathbb{U}}$ the restriction of x to $\mathbb{U}$. For a pattern $w \in \mathcal{A}^{\mathbb{U}}$, one defines the cylinder centered on w by $[w]_{\mathbb{U}} = \{x \in \mathcal{A}^{\mathbb{Z}} : x_{\mathbb{U}} = w\}$.

The action of $\mathbb{M}$ on itself allows to define an action on $\mathcal{A}^{\mathbb{M}}$ by *shift*. For all $m \in \mathbb{M}$ this action is defined by:

$$\begin{array}{rccc} \sigma^m : & \mathcal{A}^{\mathbb{M}} & \longrightarrow & \mathcal{A}^{\mathbb{M}} \\ & (x_i)_{i \in \mathbb{M}} & \longmapsto & (x_{i+m})_{i \in \mathbb{M}}. \end{array}$$

If $m \in \mathbb{M}$ is invertible in $(\mathbb{M}, +)$, then the map σ^m is an homeomorphism. When $\mathbb{M} = \mathbb{N}$ or $\mathbb{Z}$, we just denote σ instead of σ^1.

1.2. **Measures on $\mathcal{A}^{\mathbb{M}}$.** Let $\mathfrak{B}$ be the Borel sigma-algebra of $\mathcal{A}^{\mathbb{M}}$. Denote by $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$ the set of probability measures on $\mathcal{A}^{\mathbb{M}}$ defined on the sigma-algebra $\mathfrak{B}$. Usually $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$ is endowed with weak* topology: a sequence $(\mu_n)_{n \in \mathbb{N}}$ of $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$ converge to $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{M}})$ if and only if for all finite subset $\mathbb{U} \subset \mathbb{M}$ and for all pattern $u \in \mathcal{A}^{\mathbb{U}}$, one has $\lim_{n\to\infty} \mu_n([u]_{\mathbb{U}}) = \mu([u]_{\mathbb{U}})$. In the weak* topology, the set $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$ is compact and metrizable. A metric is defined by:

$$d_*(\mu,\nu) = \sum_{n\in\mathbb{N}} \frac{1}{\mathrm{Card}(A^{\mathbb{U}_n})} \sum_{u\in\mathcal{A}^{\mathbb{U}_n}} |\mu([u]_{\mathbb{U}_n}) - \nu([u]_{\mathbb{U}_n})|,$$

where $\mathbb{U}_n = \{m \in \mathbb{M} : |m| \leq n\}$.

The $\mathbb{M}$-action σ acts naturally on $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$ by:

$$\sigma_*^m(\mu(B)) = \mu(\sigma^{-m}(B)), \text{ for all } m \in \mathbb{M}, \mu \in \mathcal{M}(\mathcal{A}^{\mathbb{M}}) \text{ and } B \in \mathfrak{B}.$$

A probability measure $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{M}})$ is said *σ-invariant* if $\sigma_*^m\mu = \mu$ for all $m \in \mathbb{M}$. Denote $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ the set of σ-invariant probability measure. A probability measure $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{M}})$ is *σ-ergodic* if for all σ-invariant subset $B \in \mathfrak{B}$ ($\sigma^{-m}(B) = B$ μ-almost everywhere for all $m \in \mathbb{M}$) are trivial (i.e. $\mu(B) = 0$ or 1). The set of σ-ergodic probability measure is denoted by $\mathcal{M}_\sigma^{\mathrm{erg}}(\mathcal{A}^{\mathbb{Z}})$. By Birkoff's theorem, one has:

$$\frac{1}{\mathrm{Card}(\mathbb{U}_n)} \sum_{m\in\mathbb{U}_n} \mathbf{1}_{[u]_{\mathbb{U}}} \xrightarrow[n\to\infty]{\mu-pp} \mu([u]_{\mathbb{U}}) \qquad \text{for all pattern } u \in \mathcal{A}^{\mathbb{U}}.$$

$\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ is a compact convex subset of $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$ and $\mathcal{M}_\sigma^{\mathrm{erg}}(\mathcal{A}^{\mathbb{M}})$ is the set of extremal points of $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$, see [DGS76] for more details. Now we are going to give some examples of measure.

Example 1.1. Let $x \in \mathcal{A}^{\mathbb{M}}$. Define the *Dirac measure in x* by $\delta_x(A) = 1$ if $x \in A$ and 0 if not, where $A \in \mathfrak{B}$. The set of Dirac's measure is dense in $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$ for the weak* topology.

Moreover, one has $d'_C(x,y) = d^{\mathcal{M}}(\delta_x,\delta_y)$ where d'_C is a distance equivalent to d_C defined by:

$$d'_C(x,y) = \sum_{n\in\mathbb{N}} \frac{1}{2^n\mathrm{Card}(\mathcal{A}^{\mathbb{U}_n})} \sum_{u\in\mathcal{A}^{\mathbb{U}_n}} |\mathbf{1}_{[u]_{\mathbb{U}_n}}(x) - \mathbf{1}_{[u]_{\mathbb{U}_n}}(y)| \quad \text{ for all } x, y \in \mathcal{A}^{\mathbb{M}}.$$

One remarks that if the configuration is not σ-uniform, the Dirac measure associated is not σ-invariant. However, if we take a σ-periodic configuration x of periodic pattern $x_{\mathbb{P}}$ where $\mathbb{P} \subset \mathbb{M}$ is a finite subset, one constructs a σ-ergodic measure by taking the mean of the Dirac's measures of the σ-orbit:

$$\widetilde{\delta_x} = \frac{1}{\mathrm{Card}(\mathbb{P})} \sum_{m\in\mathbb{P}} \delta_{\sigma^m(x)}.$$

Example 1.2. For all $a \in \mathcal{A}$, put $p_a \in [0,1]$ a real such that $\sum_{a\in\mathcal{A}} p_a = 1$. Define *the Bernoulli measure according to the probability vector* $(p_a)_{a\in\mathcal{A}}$ by:

$$\lambda_{(p_a)_{a\in\mathcal{A}}}([u]_{\mathbb{U}}) = \prod_{m\in\mathbb{U}} p_{u_m} \quad \text{ for all } u \in \mathcal{L}_{\mathcal{A}^{\mathbb{M}}}([u]_{\mathbb{U}}).$$

If all p_a are equal to $\frac{1}{\mathrm{Card}(\mathcal{A})}$, one obtains the *uniform Bernoulli measure* which is just denoted by $\lambda_{\mathcal{A}^{\mathbb{M}}}$.

1.3. **Action of a cellular automaton.**

1.3.1. *Definition of cellular automaton.* A *cellular automaton* (CA) is a pair $(\mathcal{A}^{\mathbb{M}}, F)$ where $F : \mathcal{A}^{\mathbb{M}} \to \mathcal{A}^{\mathbb{M}}$ is defined by $F(x)_m = \overline{F}((x_{m+u})_{u\in\mathbb{U}})$ for all $x \in \mathcal{A}^{\mathbb{M}}$ and $m \in \mathbb{M}$ where $\mathbb{U} \subset \mathbb{M}$ is a finite set named *neighborhood* and $\overline{F} : \mathcal{A}^{\mathbb{U}} \to \mathcal{A}$ is a *local rule*. By Hedlund's theorem [Hed69], it is equivalent to say that it is a continuous function which commutes with the shift ($\sigma^m \circ F = F \circ \sigma^m$ for all $m \in \mathbb{M}$). If the smallest neighborhood is reduced to one point we say that F is *trivial*.

Remark 1.3. It is easy to remark that F is lipschitz for the distance d_C. More precisely, for all $x, y \in \mathcal{A}^{\mathbb{M}}$, one has:

$$d_C(F(x), F(y)) \leq 2^{-r(F)} d_C(x,y).$$

1.3.2. *General definitions about dynamical systems.* Let (X, d) be a metric space and $F : X \to X$ be a continuous function. There is a lot of definitions to precise the sensitivity to initial conditions of the dynamical system generated by the $\mathbb{N}$-action of F on X. We recall here some of them:

• $x \in X$ is an *equicontinuous point* if for all $\varepsilon > 0$, there exists $\delta > 0$, such that for all $y \in X$, if $d(x, y) < \delta$ then $d(F^n(x), F^n(y)) < \varepsilon$ for all $n \in \mathbb{N}$. Denote $Eq_d(F)$ the set of equicontinuous points. If $x \notin Eq_d(F)$, it is a sensitive point.

• (X, F) is *equicontinuous* if for all $\varepsilon > 0$, there exists $\delta > 0$, such that for all $x, y \in X$, if $d(x, y) < \delta$ then $d(F^n(x), F^n(y)) < \varepsilon$ for all $n \in \mathbb{N}$.

• (X, F) is *sensitive* if there exists $\varepsilon > 0$ such that for all $\delta > 0$ and $x \in X$, there exists $y \in X$ and $n \in \mathbb{N}$ such that $d(x, y) < \delta$ and $d(F^n(x), F^n(y)) > \varepsilon$

• (X, F) is $\mathbb{N}$-*expansive* if there exists $\varepsilon > 0$ such that for all $x \neq y$ there exists $n \in \mathbb{N}$ such that $d(F^n(x), F^n(y)) > \varepsilon$.

In an intuitive sense, sensitivity and expansivity denote a certain complexity of the dynamical system whereas equicontinuity denotes a strong regularity.

Proposition 1.1. *Let $(X, d, \mathbb{M}, T)$ be a dynamical system, it is easy to show the next property:*

- *If (X, T) is sensitive then $Eq^{\mathbb{M}}(X, T) = \emptyset$. But generally, the reciprocal is not true.*
- *If X is perfect, then the $\mathbb{M}$-expansivity of (X, T) imply the $\mathbb{M}$-sensitivity of (X, T).*
- *If X is compact, (X, T) is $\mathbb{M}$-equicontinuous if and only if $Eq^{\mathbb{M}}(X, T) = X$.*

1.3.3. *Action of a cellular automaton on $\mathcal{A}^{\mathbb{M}}$.* Let $(\mathcal{A}^{\mathbb{M}}, F)$ be a CA. Since $F : \mathcal{A}^{\mathbb{M}} \to \mathcal{A}^{\mathbb{M}}$ is continuous and commutes with σ, it is possible to consider the $\mathbb{M} \times \mathbb{N}$-action (σ, F) on $\mathcal{A}^{\mathbb{M}}$. Since the $\mathbb{M}$-action σ is expansive, the $\mathbb{M} \times \mathbb{N}$-action (σ, F) is also $\mathbb{M} \times \mathbb{N}$-expansive and the dynamic is so strong: it contains the dynamic of σ. Thus we must to study the dynamic of restrictions of this action at sub-semi-group of $\mathbb{M} \times \mathbb{N}$. In [Sab06] we give general definitions to talk about directional dynamics. We recall here the definition of equicontinuity of slope α, equicontinuous point of slope α and blocking wall of slope α which are used in this article; we remark that in this case $\mathbb{M} = \mathbb{Z}$.

Definition. Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA and let $\alpha \in \mathbb{R}$.

For all $x \in \mathcal{A}^{\mathbb{Z}}$ and $\varepsilon > 0$ one defines the ball centered on x of ray ε and the tube of slope α centered on the point x of width e by:

$$\begin{aligned} B_{d_C}(x, \varepsilon) &= \{y \in \mathcal{A}^{\mathbb{Z}} : d_C(x, y) < \varepsilon\}, \\ E^{\alpha}_{d_C}(x, \varepsilon) &= \{y \in \mathcal{A}^{\mathbb{Z}} : d_C(\sigma^{\lfloor n\alpha \rfloor} \circ F^n(x), \sigma^{\lfloor n\alpha \rfloor} \circ F^n(y)) < \varepsilon, \forall n \in \mathbb{N}\}. \end{aligned}$$

It is possible to define dynamics of slope α:

- The set $Eq^{\alpha}_{d_C}(F)$ of d_C-*equicontinuous point of slope* α is defined by
$$x \in Eq^{\alpha}(F) \Longleftrightarrow \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } B_{d_C}(x, \varepsilon) \subset E^{\alpha}_{d_C}(x, \varepsilon).$$
- $(\mathcal{A}^{\mathbb{Z}}, F)$ is d_C-*equicontinuous of slope* α if
$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in \mathcal{A}^{\mathbb{Z}},\ B_{d_C}(x, \varepsilon) \subset E^{\alpha}_{d_C}(x, \varepsilon).$$
By compacity, this is equivalent to $Eq^{\alpha}_{d_C}(F) = \mathcal{A}^{\mathbb{Z}}$.

To traduce equicontinuous properties in space-time diagrams, we need the notion of blocking word of slope α. The wall generated by the blocking word can be interpreted as a particle which has the direction α and which kills every information that can come from the right or from the left.

Definition. Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA, let $\mathbb{U} = [r, s]$ be a neighborhood of F, let $\alpha \in \mathbb{R}$, let $e \in \mathbb{N}$ with $e \geq \max(\lfloor \alpha \rfloor + 1 + s, -\lfloor \alpha \rfloor + 1 - r)$ and let $u \in \mathcal{L}_{\Sigma}$ with $|u| \geq e$. The word u is a *blocking word of slope α and width e* if there exists $p \in [0, |u| - e]$ such that:

$$\forall x, y \in [u]_0 \cap \Sigma, \forall n \in \mathbb{N}, \sigma^{\lfloor n\alpha \rfloor} \circ F^n(x)_{[p,p+e]} = \sigma^{\lfloor n\alpha \rfloor} \circ F^n(y)_{[p,p+e]}.$$

The evolution of a cell $i \in \mathbb{Z}$ depends on the cells $[i + r, i + s]$. Thus, due to the inequality fulfilled by e, it is easy to see that if u is a blocking word of slope α and width e, for all $j \in \mathbb{Z}$, for all $x, y \in [u]_j \cap \Sigma$ such that $x_{]-\infty,j]} = y_{]-\infty,j]}$, for $n \in \mathbb{N}$ one has $F^n(x)_i = F^n(y)_i$ for all $i \leq \lfloor \alpha n \rfloor + p + e + j$. Similarly for all

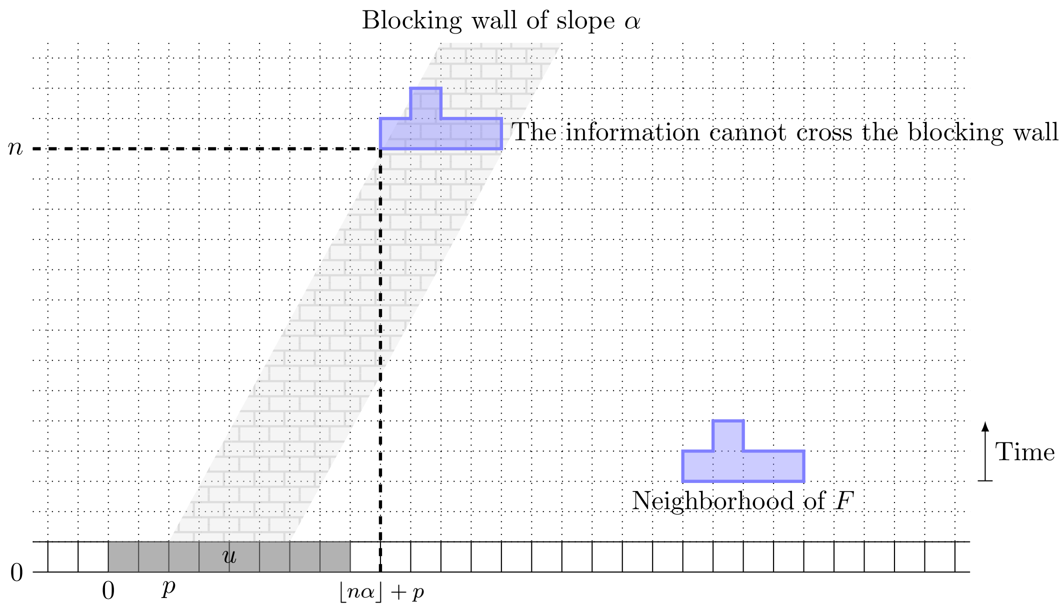

FIGURE 1. u is a blocking word of slope α.

$x, y \in [u]_j \cap \Sigma$ such that $x_{[j,\infty[} = y_{[j,\infty[}$, one has $F^n(x)_i = F^n(y)_i$ for all $i \geq \lfloor \alpha n \rfloor + p$. Concretely, in other words, no information can cross the wall of slope α and width ε generated by the blocking word.

Adapting prove of [Kůr97], it is shown in [Sab06] that $Eq^{\alpha}_{d_C}(F) \neq \emptyset$ if and only if there exists a blocking word of slope α.

1.3.4. *Action of a cellular automaton on $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$.* Let $(\mathcal{A}^{\mathbb{M}}, F)$ be a CA and $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{M}})$. The cellular automaton acts on $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$ by

$$F_*\mu(B) = \mu(F^{-1}(B)) \text{ for all } B \in \mathfrak{B}.$$

The map $\mu \mapsto F_*\mu$ is called the *extension* of F. It is continuous and preserves convex combinations. Thus, $F_* : \mathcal{M}(\mathcal{A}^{\mathbb{M}}) \to \mathcal{M}(\mathcal{A}^{\mathbb{M}})$ defines a dynamical system. However, for all $x \in \mathcal{A}^{\mathbb{M}}$ one has $F_*\delta_x = \delta_{F(x)}$. Thus the map $x \mapsto \delta_x$ allows to consider $(\mathcal{A}^{\mathbb{M}}, (F_*, \sigma))$ as a sub-system of $(\mathcal{M}(\mathcal{A}^{\mathbb{M}}), (F_*, \sigma_*))$, so the dynamics of $F_* : \mathcal{M}(\mathcal{A}^{\mathbb{M}}) \to \mathcal{M}(\mathcal{A}^{\mathbb{M}})$ contains the dynamics of $F : \mathcal{A}^{\mathbb{M}} \to \mathcal{A}^{\mathbb{M}}$. Moreover, the weak$*$ topology on $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$ privileges the origin. Thus it is preferable to restrict the initial space.

Since F commutes with the shift, if $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ then $F_*\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$. In the weak* topology, $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ is closed, so compact. Thus one can study the dynamical system $F_* : \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}) \to \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$.

In the same way, if $\mu \in \mathcal{M}^{\text{erg}}_\sigma(\mathcal{A}^{\mathbb{M}})$ then $F_*\mu \in \mathcal{M}^{\text{erg}}_\sigma(\mathcal{A}^{\mathbb{M}})$. However $\mathcal{M}^{\text{erg}}_\sigma(\mathcal{A}^{\mathbb{M}})$ is not necessary closed for the weak* topology.

In section 3, we endowed $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ with different distances to study the action F_*.

2. Different σ-invariant pseudo-distances on $\mathcal{A}^{\mathbb{M}}$

In this section one describes a generic method to obtain pseudo-distances on $\mathcal{A}^{\mathbb{M}}$ from sub-measures on the lattice $\mathbb{M}$. By this method we obtain classical pseudo-distance like the distance of Cantor, but also the pseudo-distances of Besicovitch and Weyl introduced in [CFMM97] and [BFK97].

2.1. **Sub-measures and pseudo-distances associated.**

Definition. A *sub-measure* on the set $\mathbb{M}$ is a function $\varphi : \mathcal{P}(\mathbb{M}) \to [0, \infty]$ such that:

- $\varphi(\emptyset) = 0$,
- $\varphi(\mathbb{U}) < \infty$ if $\mathbb{U}$ is finite,
- $\varphi(\mathbb{U}') \leq \varphi(\mathbb{U}' \cup \mathbb{U}'') \leq \varphi(\mathbb{U}') + \varphi(\mathbb{U}'')$ for all subset $\mathbb{U}'$ and $\mathbb{U}''$ of $\mathbb{M}$.

A sub-measure φ is *bounded* if there exists M such that $\varphi(\mathbb{U}) < M$ for all $\mathbb{U} \subset \mathbb{M}$. Let φ and ψ be two sub-measures, one says $\varphi \leq \psi$ if $\varphi(\mathbb{U}) \leq \psi(\mathbb{U})$ for all $\mathbb{U} \subset \mathbb{M}$.

Recall $\mathbb{U}_n = \{m \in \mathbb{M} : |m| \leq n\}$. A sub-measure is *finitely approximated* if there exists a subsequence of sub-measures $(\varphi_n)_{n\in\mathbb{N}}$, where φ_n is defined on $\mathbb{U}_n$, such that $\varphi(\mathbb{U}) = \lim_{n\to\infty} \varphi_n(\mathbb{U})$ for all $\mathbb{U} \subset \mathbb{M}$.

For a sub-measure φ, one defines the sub-measure φ_∞ by $\varphi_\infty(\mathbb{U}) = \lim_{n\to\infty} \varphi(\mathbb{U} \setminus \mathbb{U}_n)$ for all $\mathbb{U} \subset \mathbb{M}$.

Notation. Let φ be a sub-measure, one defines $\mathrm{Null}(\varphi) = \{\mathbb{U} \subset \mathbb{M} : \varphi(\mathbb{U}) = 0\}$. This set is an ideal of the ring $(\mathcal{P}(\mathbb{M}), \triangle, \cap)$ where $\triangle$ is the symmetric difference.

Let $x, y \in \mathcal{A}^{\mathbb{M}}$, denote $\Delta(x, y) = \{m \in \mathbb{M} : x_m \neq y_m\}$ the set of *defaults* between two configurations. It is easy to see that $\Delta(x, z) \subset \Delta(x, y) \cup \Delta(y, z)$ for all $x, y, z \in \mathcal{A}^{\mathbb{M}}$. One defines $x \simeq_\varphi y$ if $\Delta(x, y) \in \mathrm{Null}(\varphi)$. Since $\mathrm{Null}(\varphi)$ is an ideal, $\simeq_\varphi$ defines an equivalence relation on $\mathcal{A}^{\mathbb{M}}$.

Proposition 2.1. *Let φ be a sub-measure on $\mathbb{M}$. Put $d_\varphi(x, y) = \varphi(\Delta(x, y))$, for all $x, y \in \mathcal{A}^{\mathbb{M}}$. The function d_φ is a pseudo-distance on $\mathcal{A}^{\mathbb{M}}$. By quotient, $(X_\varphi, \widetilde{d_\varphi})$ is a metric space where $X_\varphi = \mathcal{A}^{\mathbb{M}}/\simeq_\varphi$.*

Proof. For all $x, y, z \in \mathcal{A}^{\mathbb{M}}$ one has $\Delta(x, z) \subset \Delta(x, y) \cup \Delta(y, z)$, the triangular inequality of d_φ follows. Moreover, $\Delta(x, x) = \emptyset$ and $\Delta(x, y) = \Delta(y, x)$, so one obtains the reflexivity and the symmetry of d_φ. It is possible to quotient $\mathcal{A}^{\mathbb{M}}$ by the equivalence relation $\simeq_\varphi$, the pseudo-distance d_φ becomes a distance $\widetilde{d_\varphi}$ on X_φ. $\square$

Example 2.1. Let

$$\varphi_C(\mathbb{U}) = 2^{-\min\{|m|:m\in\mathbb{U}\}},$$

for all $\mathbb{U} \subset \mathbb{M}$. The distance associated correspond at the distance of Cantor on $\mathcal{A}^{\mathbb{M}}$.

Example 2.2. Let

$$\varphi_B(\mathbb{U}) = \limsup_{n\to\infty} \frac{\mathrm{Card}(\mathbb{U} \cap \mathbb{U}_n)}{\mathrm{Card}(\mathbb{U}_n)},$$

for all $\mathbb{U} \subset \mathbb{M}$. The pseudo-distance associated is the pseudo-distance of Besicovitch denoted d_B which is introduced in [CFMM97]. This distance is called Hamming distance in ergodic theory, see [Gla03].

One remarks that φ_B is finitely approximated by the family of sub-measures defined by:

$$\varphi_n(\mathbb{U}) = \sup_{k\geq n} \frac{\mathrm{Card}(\mathbb{U} \cap \mathbb{U}_k)}{\mathrm{Card}(\mathbb{U}_k)} \text{ for all } \mathbb{U} \subset \mathbb{M} \text{ and all } n \in \mathbb{N}.$$

Example 2.3. Let

$$\varphi_W(\mathbb{U}) = \limsup_{n\to\infty} \sup_{k\in\mathbb{M}} \frac{\mathrm{Card}(\mathbb{U} \cap (k + \mathbb{U}_n))}{\mathrm{Card}(\mathbb{U}_n)},$$

for all $\mathbb{U} \subset \mathbb{M}$. The pseudo-distance associated is the pseudo-distance of Weyl denoted d_W [BFK97].

Example 2.4. Let φ a sub-measure. It is possible to define a sub-measure invariant under the action of $\mathbb{M}$; for all $\mathbb{U} \subset \mathbb{M}$ put:

$$\varphi'(\mathbb{U}) = \limsup_{n\to\infty} \frac{1}{\mathrm{Card}(\mathbb{U}_n)} \sum_{i\in\mathbb{U}_n} \varphi((\mathbb{U} - i) \cap \mathbb{M}).$$

If $\varphi(\mathbb{U}) = 1$ if and only if $0 \in \mathbb{U}$, one obtains $\varphi' = \varphi_B$.

Example 2.5. Let $f : \mathbb{M} \to \mathbb{R}^+$. For all $\mathbb{U} \subset \mathbb{M}$, denote $\varphi_f(\mathbb{U}) = \sum_{m\in\mathbb{U}} f(m)$. It is a sub-measure of $\mathbb{M}$.

Following the construction of φ_B, it is possible to construct a family of sub-measures. For all $\mathbb{U} \subset \mathbb{M}$ put

$$\widehat{\varphi_f}(\mathbb{U}) = \limsup_{n\to\infty} \frac{\varphi_f(\mathbb{U} \cap \mathbb{U}_n)}{\varphi_f(\mathbb{U}_n)}.$$

Following the construction of φ_W, it is possible to construct a family of sub-measures. For all $\mathbb{U} \subset \mathbb{M}$ put

$$\widetilde{\varphi_f}(\mathbb{U}) = \limsup_{n\to\infty} \sup_{k\in\mathbb{M}} \frac{\varphi_f(\mathbb{U} \cap (\mathbb{U}_n + k))}{\varphi_f(\mathbb{U}_n)}.$$

Put $\partial\mathbb{U}_n = \mathbb{U}_n \setminus \mathbb{U}_{n-1}$. If f verifies $\lim_{n\to\infty} \varphi_f(\mathbb{U}_n) = +\infty$ and $\lim_{n\to\infty} \frac{\varphi_f(\partial\mathbb{U}_n)}{\varphi_f(\mathbb{U}_n)} = 0$ then $\widehat{\varphi_f}$ and $\widetilde{\varphi_f}$ are sub-measure invariant under the action of $\mathbb{M}$.

If f is the constant function equal to 1, we obtain $\widetilde{\varphi_f} = \varphi_B$ and $\widetilde{\varphi_f} = \varphi_W$. If $f(m) = \frac{1}{n^{d'+d''}}$ (we recall that $\mathbb{M} = \mathbb{Z}^{d'} \times \mathbb{Z}^{d''}$), $\widehat{\varphi_f}$ is a sub-measure associated to the logarithmic density.

Example 2.6. Let $\mathbb{V}$ be a finite subset of $\mathbb{M}$. Let φ be a sub-measure on $\mathbb{M}$. One defines the sub-measure $\varphi^{\mathbb{V}}$ by $\varphi^{\mathbb{V}}(\mathbb{U}) = \varphi(\mathbb{U} + \mathbb{V})$. For example, for φ_B, one obtaines

$$d_{\varphi_B^{\mathbb{V}}}(x, y) = \limsup_{n\to\infty} \frac{\mathrm{Card}(\{i \in \mathbb{U}_n : x_{i+\mathbb{V}} = y_{i+\mathbb{V}}\})}{\mathrm{Card}(\mathbb{U}_n)}.$$

2.2. **Properties of the space** $(X_\varphi, \widetilde{d_\varphi})$**.** Now we are going to explore the properties of $(X_\varphi, \widetilde{d_\varphi})$ according to the properties of φ.

Proposition 2.2. *Let φ and ψ be two sub-measures on $\mathcal{P}(\mathbb{M})$.*

1. *If $\varphi \leq \psi$, then $d_\varphi(x, y) \leq d_\psi(x, y)$ for all $x, y \in \mathcal{A}^{\mathbb{M}}$.*
2. *If $\varphi(\mathbb{U}) = \varphi(\mathbb{U} + m)$ for all $\mathbb{U} \subset \mathbb{M}$ and $m \in \mathbb{M}$ (one says that φ is invariant under the action of $\mathbb{M}$) then d_φ is σ-invariant.*
3. $\mathrm{Null}(\varphi) = \{\emptyset\}$ *if and only if d_φ is a distance on $\mathcal{A}^{\mathbb{M}}$.*

Example 2.7. One has $\varphi_B \leq \varphi_W$, thus $d_B(x, y) \leq d_W(x, y)$ for all $x, y \in \mathcal{A}^{\mathbb{M}}$.
It is easy to verify that $\mathrm{Null}(\varphi_C) = \emptyset$.

Proposition 2.3. *Let φ be a bounded sub-measure. The function d_φ is a distance which defines the Cantor topology if and only if φ charges every atom and verifies* $\lim_{n\to\infty} \varphi(\mathbb{M} \setminus \mathbb{U}_n) = 0$.

Proof. If φ allows to define a distance compatible with the Cantor topology then necessary every atom is charged since one has $\mathrm{Null}(\varphi) = \emptyset$. Moreover, since the Cantor topology is compact, there exists a constant K such that $d_\varphi(x, y) \leq K\, d_{\varphi_C}(x, y)$ for all $x, y \in \mathcal{A}^{\mathbb{M}}$. One deduces that $\lim_{n\to\infty} \varphi(\mathbb{M} \setminus \mathbb{U}_n) = 0$.

Reciprocally, let φ be a bounded sub-measure which charges every atoms such that $\lim_{n\to\infty} \varphi(\mathbb{M}\setminus\mathbb{U}_n) = 0$. Since φ charges every atoms, d_φ defines a distance on $\mathcal{A}^{\mathbb{M}}$. Consider $(x^n)_{n\in\mathbb{N}}$ a sequence of $\mathcal{A}^{\mathbb{M}}$ which converges to $x \in \mathcal{A}^{\mathbb{M}}$ for the Cantor topology. Since φ is bounded, for all $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that $\varphi(\mathbb{M} \setminus \mathbb{U}_m) < \varepsilon$ for all $m \geq M$. By definition of the Cantor topology, there exists $N \in \mathbb{N}$ such that $x^n_{\mathbb{U}_m} = x_{\mathbb{U}_m}$ for all $n \geq N$. Thus $d_\varphi(x^n, x) < \varepsilon$ for all $n \geq N$, that is to say $(x^n)_{n\in\mathbb{N}}$ converges to x for the topology defined by d_φ. In the same way, it is possible to prove that a sequence which converges for the topology defined by d_φ, converges also for the Cantor topology. One deduces the equivalence of the topologies. □

Example 2.8. Let $f : \mathbb{N} \to \mathbb{R}^+$ such that $\varphi_f(\mathbb{M}) < +\infty$ and $f(m) \neq 0$ for all $m \in \mathbb{M}$. The distance d_{φ_f} defines the Cantor topology.

Remark 2.9. Since it is impossible to define an equidistributed sub-measure on a countable set, one deduces that does not exist σ-invariant distance compatible with the Cantor topology.

Proposition 2.4. *Let φ be a sub-measure such that for all $\varepsilon > 0$, there exists $\mathbb{U} \subset \mathbb{M}$ which verifies $0 < \varphi(\mathbb{U}) < \varepsilon$; then $(X_\varphi, \widetilde{d_\varphi})$ is perfect.*

Proof. By hypothesis, for all $n \in \mathbb{N}$, there exists $\mathbb{V}_n \subset \mathbb{M}$ such that $0 < \varphi(\mathbb{V}_n) < \frac{1}{n}$. Let $x \in \mathcal{A}^{\mathbb{M}}$. For all $n \in \mathbb{N}$, define $x^n \in \mathcal{A}^{\mathbb{M}}$ by $x^n_m \neq x_m$ if $m \in \mathbb{V}_n$ and $x^n_m = x_m$ in other cases. Thus, one has $d_\varphi(x, x^n) < \frac{1}{n}$ and $\Delta(x, x^n) \notin \mathrm{Null}(\varphi)$. One deduces that the image of x in X_φ is a point of accumulation. □

Example 2.10. Let $f : \mathbb{N} \to \mathbb{R}^+$. If f verifies $\lim_{n\to\infty} \varphi_f(\mathbb{U}_n) = +\infty$ and $\lim_{n\to\infty} \frac{\varphi_f(\partial\mathbb{U}_n)}{\varphi_f(\mathbb{U}_n)} = 0$ then $\widehat{\varphi_f}$ and $\widetilde{\varphi_f}$ verify the hypothesis of the proposition. In particular $(X_B, \widetilde{d_B})$ and $(X_W, \widetilde{d_W})$ are perfects.

Proposition 2.5. *Let φ be a finitely approximated sub-measure such that $\varphi = \varphi_\infty$. Then $(X_{\varphi_\infty}, \widetilde{d_\varphi})$ is complete.*

Proof. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of sub-measures of $\mathbb{M}$ which approximates φ, one recalls that for every $n \in \mathbb{N}$, φ_n is defined on $\mathbb{U}_n$. Let $(x^n)_{n\in\mathbb{N}}$ be a Cauchy's sequence of elements of $\mathcal{A}^{\mathbb{M}}$ for the pseudo-distance d_φ. One considers the sub-sequence $(n_i)_{i\in\mathbb{N}}$ such that $d_\varphi(x^{n_i}, x^{n_j}) \leq 2^{-i-2}$ for all $j \geq i$. Since φ is finitely approximated, there exists an increasing sequence $(l_i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$ and for all $l \geq l_i$, one has $|d_{\varphi_l}(x^{n_i}, x^{n_{i+1}}) - d_\varphi(x^{n_i}, x^{n_{i+1}})| \leq 2^{-i-2}$ for all $j \geq i$. Thus, for all $i \in \mathbb{N}$ and $l \geq l_i$, one has:

$$d_{\varphi_l}(x^{n_i}, x^{n_{i+1}}) \leq 2^{-i-1}.$$

One considers $x \in \mathcal{A}^{\mathbb{M}}$ such that $x_m = x_m^{n_i}$ if $m \in \mathbb{U}_{l_i} \setminus \mathbb{U}_{l_{i-1}}$. We want to show that the sub-sequence $(x^{n_i})_{i\in\mathbb{N}}$ converges toward x for the pseudo-distance d_φ. Let $\varepsilon > 0$. Since $\varphi = \varphi_\infty$, there exist $j', j_0 \in \mathbb{N}$ such that $d_\varphi(x, x^j) \leq \varphi(\Delta(x, x^j) \setminus \mathbb{U}_{j'}) + \frac{\varepsilon}{2}$ for all $j \geq j_0$. Moreover, since φ is finitely approximated, for all $j \geq j_0$, there exists k such that

$$\varphi(\Delta(x,x^j) \setminus \mathbb{U}_{j'}) \leq \varphi_{l_k}(\Delta(x,x^j) \setminus \mathbb{U}_{l_{j-1}}) + \frac{\varepsilon}{2} \leq \sum_{i=j}^{k-1} \varphi_{l_k}(\Delta(x^i, x^{i+1})) + \frac{\varepsilon}{2} \leq \sum_{i=j}^{k-1} \frac{1}{2^{i+1}} + \frac{\varepsilon}{2} \leq \frac{1}{2^j} + \frac{\varepsilon}{2}.$$

One deduces that $d_\varphi(x, x^j) \leq \frac{1}{2^j} + \varepsilon$, thus $(x^i)_{i\in\mathbb{N}}$ converges toward x. □

Example 2.11. Let $f : \mathbb{N} \to \mathbb{R}^+$. If f verifies $\lim_{n\to\infty} \varphi_f(\mathbb{U}_n) = +\infty$ and $\lim_{n\to\infty} \frac{\varphi_f(\partial \mathbb{U}_n)}{\varphi_f(\mathbb{U}_n)} = 0$ then $\widehat{\varphi_f}$ verifies the hypothesis of the proposition. In particular $(X_B, \widetilde{d_B})$ is complete. However φ_W is not finitely approximated, in fact $(X_W, \widetilde{d_W})$ is not complete (see [BFK97]).

2.3. Action of a cellular automaton on $(X_\varphi, \widetilde{d_\varphi})$.

Proposition 2.6. *Let φ be a sub-measure on $\mathbb{M}$ which is invariant under the action of $\mathbb{M}$. Let $(\mathcal{A}^{\mathbb{M}}, F)$ be a CA of neighborhood $\mathbb{U}$. Then for all $x, y \in \mathcal{A}^{\mathbb{M}}$, one has $d_\varphi(F(x), F(y)) \leq \mathrm{Card}(\mathbb{U})\, d_\varphi(x, y)$. In particular F defines a continuous function on X_φ denoted F_φ.*

Proof. Let $x, y \in \mathcal{A}^{\mathbb{M}}$. One has:

$$\begin{array}{rcl} \Delta(F(x), F(y)) & = & \{m \in \mathbb{M} : F(x)_m \neq F(y)_m\} \\ & \subset & \bigcup_{u\in\mathbb{U}} \{m \in \mathbb{M} : x_{m+u} \neq y_{m+u}\} \\ & \subset & \Delta(x,y) - \mathbb{U}. \end{array}$$

By invariance of φ under the action of $\mathbb{M}$, one deduces that $d_\varphi(F(x), F(y)) \leq \mathrm{Card}(\mathbb{U})\, d_\varphi(x, y)$. □

Proposition 2.7. *Assume that $\mathbb{M} = \mathbb{N}$ or $\mathbb{Z}$. Let φ be a sub-measure on $\mathbb{M}$ invariant under the action of $\mathbb{M}$. Let $(\mathcal{A}^{\mathbb{M}}, F)$ be a CA with d_C-equicontinuous points of slope α. Then (X_φ, F_φ) has $\widetilde{d_\varphi}$-equicontinuous points.*

Proof. Let $u \in \mathcal{A}^n$ be a blocking word of slope α and width e. Let $K \in \mathbb{N}$ and $x \in \mathcal{A}^{\mathbb{M}}$ such that the initial letter of any occurrence of u is spaced at the maximum of $K \in \mathbb{N}$.

One considers a configuration x' different of x only for one coordinate. This difference is localized between two occurrences of u spaced at maximum of $2K$. Thus, $F^n(x)$ and $F^n(x')$ has at maximum $2K$ differences for all $n \in \mathbb{N}$. More generally, for all $y \in \mathcal{A}^{\mathbb{M}}$ and $n \in \mathbb{N}$, one has:

$$\Delta(F^n(x), F^n(y)) \subset \Delta(x,y) + [-K, K].$$

Since φ is invariant under the action of $\mathbb{M}$, one deduce that for all $n \in \mathbb{N}$ one has:

$$d_\varphi(F^n(x), F^n(y)) \leq 2K d_\varphi(x, y).$$

Thus the point x is a d_φ-equicontinuous point. □

3. Action of a cellular automaton on $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$

A natural σ-invariant object on which cellular automata can act, is the set of σ-invariant probability measure $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$. The natural topology on $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ is the weak* topology. However the different pseudo-distances on $\mathcal{A}^{\mathbb{M}}$ introduced in the previous section allow to construct pseudo-distances on $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$. In this section, first we exhibit a general process to define these pseudo-distances. Then we gives some general properties of the action of CA in these spaces.

3.1. **Different pseudo-distances on $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$.** Let d be a pseudo-distance on $\mathcal{A}^{\mathbb{M}}$, we want to introduce a pseudo-distance on $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ induced by the pseudo-distance d. Let $\mu,\nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$, the intuitive idea is to calculate the mean of $d(x,y)$ when x is chosen with the probability μ and y with the probability ν. However it is necessary to precise the correlation to choose the pair (x,y). That's why we introduce the notion of joint measure. Of course it is necessary to minimize the influence of the correlation to obtain a pseudo-distance.

Let μ and ν be two σ-invariant probability measures on $\mathcal{A}^{\mathbb{M}}$. A probability measure λ on $\mathcal{A}^{\mathbb{M}} \times \mathcal{A}^{\mathbb{M}}$ is a *joint measure* according μ and ν if λ is $\sigma\times\sigma$-invariant and $\pi^1_*\lambda = \mu$ and $\pi^2_*\lambda = \nu$, where π^1 and π^2 are respectively the projections according the first and second coordinate. Denote $\mathcal{J}(\mu,\nu)$ the set of joint measures according μ and ν.

Of course, one has $\mathcal{J}(\mu,\nu) \subset \mathcal{M}_{\sigma\times\sigma}(\mathcal{A}^{\mathbb{M}} \times \mathcal{A}^{\mathbb{M}})$. Moreover $\mathcal{J}(\mu,\nu)$ is convex and compact for the weak topology. If μ and ν are σ-ergodic, then the extremal points of $\mathcal{J}(\mu,\nu)$ are ergodic. We refer to [Gla03] for more details.

Definition. Let d be a pseudo-distance on $\mathcal{A}^{\mathbb{M}}$ such that $(x,y)\mapsto d(x,y)$ is Borel-measurable (In this article we assume implicitly this property). For example d_φ is Borel-measurable when φ is finitely approximated (like φ_C or φ_B). One defines a function d from $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}) \times \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ on $\mathbb{R}^+$ by:

$$d^{\mathcal{M}}(\mu,\nu) = \inf_{\lambda\in\mathcal{J}(\mu,\nu)} \int d(x,y)\mathrm{d}\lambda(x,y) \qquad \text{for all } \mu,\nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}).$$

Proposition 3.1. *Let d be a pseudo-distance on $\mathcal{A}^{\mathbb{M}}$. The function $d^{\mathcal{M}} : \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}) \times \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}) \to \mathbb{R}^+$ is a pseudo-distance on $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$.*

Proof. Let $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$. One defines $\lambda \in \mathcal{M}_{\sigma\times\sigma}(\mathcal{A}^{\mathbb{M}}\times\mathcal{A}^{\mathbb{M}})$ such that $\lambda(A\times B) = \mu(A\cap B)$ for all $A, B \in \mathfrak{B}$. One obtains $\int d(x,y)\mathrm{d}\lambda = 0$ and $\lambda \in \mathcal{J}(\mu,\mu)$, thus $d^{\mathcal{M}}(\mu,\mu) = 0$.

Now we are going to verify the triangular inequality for $d^{\mathcal{M}}$. Let μ, ν and η in $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ and consider $\lambda_{\mu,\nu} \in \mathcal{J}(\mu,\nu)$ and $\lambda_{\nu,\eta} \in \mathcal{J}(\nu,\eta)$. The disintegration of these measures according ν can be written as:

$$\lambda_{\mu,\nu} = \int \lambda_\mu(y)\mathrm{d}\nu(y) \text{ et } \lambda_{\nu,\eta} = \int \lambda_\eta(y)\mathrm{d}\nu(y).$$

Then put $\lambda = \int \lambda_\mu(y)\times\lambda_\eta(y)\mathrm{d}\nu(y) \in \mathcal{J}(\mu,\eta)$, one obtains:

$$\begin{aligned}
\int d(x,z)d\mathrm{d}\lambda(x,z) &= \iint d(x,z)(\lambda_\mu(y)\times\lambda_\eta(y))(x,z)\mathrm{d}\nu(y) \\
&\leq \iint (d(x,y)+d(y,z))\mathrm{d}(\lambda_\mu(y)\times\lambda_\eta(y))(x,z)\mathrm{d}\nu(y) \\
&= \iint d(x,y)\mathrm{d}\lambda_\mu(y)(x)\mathrm{d}\nu(y) + \iint d(y,z)\mathrm{d}\lambda_\eta(y)(z)\mathrm{d}\nu(y) \\
&= d^{\mathcal{M}}(\mu,\nu) + d^{\mathcal{M}}(\nu,\eta).
\end{aligned}$$

By taking the inferior bound, one deduces that $d^{\mathcal{M}}(\mu,\eta) \leq d^{\mathcal{M}}(\mu,\nu) + d^{\mathcal{M}}(\nu,\eta)$. ☐

Open problem 3.1. It is difficult to obtain more general properties of $d^{\mathcal{M}}$. For example under which conditions the pseudo-distance $d^{\mathcal{M}}$ is a distance, or is complete?

Remark 3.1. In this article, we just consider pseudo-distance generated by sub-measure since this is a general process to construct pseudo-distance on $\mathcal{A}^{\mathbb{M}}$.

For some σ-invariant probability measure, there exist special points of $\mathcal{A}^{\mathbb{M}}$ which represent the measure. That is to say the frequency of apparition of a pattern correspond to the measure of the cylinder centered on this pattern. This allows to give a symbolic interpretation of the pseudo-distance $d_\varphi^{\mathcal{M}}$.

Definition. A point $x \in \mathcal{A}^{\mathbb{M}}$ is *generic* if for all $\mathbb{U}\subset\mathbb{M}$ finite and for every pattern $u \in \mathcal{A}^{\mathbb{U}}$ the sequence $(f(u,x,n))_{n\in\mathbb{N}}$ converges where

$$f(u,x,n) = \frac{1}{\mathrm{Card}(\mathbb{U}_n)} \sum_{m\in\mathbb{U}_n} \mathbf{1}_{[u]_{\mathbb{U}}}(\sigma^m(x)),$$

is the *frequency of apparition of the pattern u in x at the order n*. The limit of this sequence is denoted $f(u,x)$, this is the *frequency of apparition of the pattern u in x*. Denote $\mathcal{G}$ the set of genreric points.

Let $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$. Denote $\mathcal{G}(\mu)$ the set of *generic points of* μ, this is the set of points $x \in \mathcal{G}$ such that for every pattern, the frequency of this pattern in x is equal to the measure of the cylinder centered on this pattern. When μ is σ-ergodic, the Birkhoff's Theorem says that $\mu(\mathcal{G}(\mu)) = 1$.

Let φ be a sub-measure of $\mathbb{M}$ and let $\mu,\nu \in \mathcal{M}_\sigma^{\text{erg}}(\mathcal{A}^{\mathbb{M}})$. Since $\mathcal{G}(\mu)$ is σ-invariant, the function $y \in \mathcal{G}(\nu) \mapsto d_\varphi(\mathcal{G}(\mu), y) = \inf\{d_\varphi(x,y) : x \in \mathcal{G}(\mu)\}$ is σ-invariant. By σ-ergodicity, this function is constant ν-almost everywhere. We denote $d_\varphi(\mathcal{G}(\mu),\mathcal{G}(\nu))$ this value.

Proposition 3.2. *Let φ be a sub-measure on $\mathbb{M}$. Let μ and ν in $\mathcal{M}_\sigma^{erg}(\mathcal{A}^{\mathbb{M}})$. One has $d_\varphi^{\mathcal{M}}(\mu,\nu) \geq d_\varphi(\mathcal{G}(\mu),\mathcal{G}(\nu))$.*

Moreover if φ is invariant under the action of $\mathbb{M}$, then $d_\varphi^{\mathcal{M}}(\mu,\nu) = d_\varphi(\mathcal{G}(\mu),\mathcal{G}(\nu))$.

Proof. Let $\lambda \in \mathcal{J}(\mu,\nu)$, one has $\lambda(\mathcal{G}(\mu)\times\mathcal{G}(\nu)) = 1$. Moreover, for all $(x,y) \in \mathcal{G}(\mu)\times\mathcal{G}(\nu)$, one has $d_\varphi(x,y) \geq d_\varphi(\mathcal{G}(\mu),y)$. We integrate according λ to obtain $\int d_\varphi(x,y)d\lambda \geq d_\varphi(\mathcal{G}(\mu),\mathcal{G}(\nu))$; thus $d_\varphi^{\mathcal{M}}(\mu,\nu) \geq d_\varphi(\mathcal{G}(\mu),\mathcal{G}(\nu))$.

Assume that φ is invariant under the action of $\mathbb{M}$; we want to prove the inequality in the other sense. Since μ and ν are σ-ergodic, the extremal points of $\mathcal{J}(\mu,\nu)$ are ergodics. Thus, the inferior bounds of the Definition 3.1 can be take in $\mathcal{J}(\mu,\nu)\cap\mathcal{M}_{\sigma\times\sigma}^{\text{erg}}(\mathcal{A}^{\mathbb{M}}\times\mathcal{A}^{\mathbb{M}}) = \mathcal{J}^{\text{erg}}(\mu,\nu)$. Let $\lambda \in \mathcal{J}^{\text{erg}}(\mu,\nu)$, Since φ is invariant under the action of $\mathbb{M}$, the function $(x,y) \mapsto d_\varphi(x,y)$ is $\sigma\times\sigma$-invariant, so constant λ-almost everywhere. We integrate and take the inferior bound to obtain the equality expected. $\square$

Remark 3.2. The second part of the Proposition is generally false when φ is not invariant under the action of $\mathbb{M}$. By example for φ_C, the set of generic points for a Bernoulli measure is dense for the distance d_C, but two different Bernoulli measure have a positive distance for $d_C^{\mathcal{M}}$.

This Proposition give a geometric interpretation for the distance $d_\varphi^{\mathcal{M}}$. Two ergodic measures are near if we can go from a generic point to another with modifications on "few" cells. The notion of "few" is given by the sub-measure φ. By example, in the case of φ_C, this signify to do modifications far from the origin; in the case of φ_B, this signify to do modifications on sub-sets of $\mathbb{M}$ which have a weak density.

3.2. **Action of a CA on $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_\varphi^{\mathcal{M}})$.** Now we consider the action F_* on $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$. In the first time we characterize the first propeties of this dynamical system for a general pseudo-measure on $\mathcal{A}^{\mathbb{M}}$.

3.2.1. *Continuity of the action.* To study the $\mathbb{N}$-action of F_* on $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_\varphi^{\mathcal{M}})$ as a dynamical system, we are going to prove the continuity of the function F_* in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_\varphi^{\mathcal{M}})$.

Proposition 3.3. *Let φ be a sub-measure and let $F : \mathcal{A}^{\mathbb{M}} \to \mathcal{A}^{\mathbb{M}}$ be a function d_φ-lipschitz of constant K on $\mathcal{A}^{\mathbb{M}}$. For all $\mu,\nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$, one has:*

$$d_\varphi^{\mathcal{M}}(F_*\mu, F_*\nu) \leq K d_\varphi^{\mathcal{M}}(\mu,\nu).$$

In particular F_* is continuous on $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_\varphi^{\mathcal{M}})$.

Proof. Let $\lambda \in \mathcal{J}(\mu,\nu)$, one has $(F_*\times F_*)\lambda \in \mathcal{J}(F_*\mu, F_*\nu)$, thus:

$$\int d_\varphi^{\mathcal{M}}(x,y)\mathrm{d}(F_*\times F_*)\lambda = \int d_\varphi^{\mathcal{M}}(F(x),F(y))d\lambda \leq \int K d_\varphi^{\mathcal{M}}(x,y)d\lambda.$$

One deduces that $d_\varphi^{\mathcal{M}}(F_*\mu, F_*\nu) \leq K d_\varphi^{\mathcal{M}}(\mu,\nu)$. $\square$

This proposition holds for all CA when we consider the pseudo-distance induced by the Cantor's distance d_C. Indeed, according the Remark **??**, one has $d_C(F(x),F(y)) \leq 2^{-r(F)} d_C(x,y)$ for all $x,y \in \mathcal{A}^{\mathbb{M}}$. By Proposition 2.6, the previous Proposition holds also when we consider pseudo-distance on $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ induced by pseudo-distance σ-invariant on $\mathcal{A}^{\mathbb{M}}$; in particular for the pseudo-distance induced by sub-measures invariant under the action of $\mathbb{M}$, like the pseudo-distances of Besicovitch or Weyl.

3.2.2. *Expansivity of F_*.* The function F_* is affine in $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$, we are going to use this property to show that there not exist CA which acts expansively on $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ for all pseudo-distances iduced.

Theorem 3.4. *Let φ be a bounded sub-measure. There not exist CA $(\mathcal{A}^{\mathbb{M}}, F)$ such that F_* is expansive in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_\varphi^{\mathcal{M}})$.*

Proof. Let $\mu, \nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ and $\varepsilon > 0$. Consider $\mu' = (1-\varepsilon)\mu + \varepsilon\nu$. Let $\lambda' \in \mathcal{J}(\mu,\mu)$ and $\lambda'' \in \mathcal{J}(\mu,\nu)$, so one has $\lambda = (1-\varepsilon)\lambda' + \varepsilon\lambda'' \in \mathcal{J}(\mu,\mu')$. Thus,

$$(1-\varepsilon)\int d_\varphi(x,y)\mathrm{d}\lambda' + \varepsilon\int d_\varphi(x,y)\mathrm{d}\lambda'' = \int d_\varphi(x,y)\mathrm{d}\lambda \geq d_\varphi^{\mathcal{M}}(\mu,\mu').$$

One deduces that:

$$\varepsilon\, d_\varphi^{\mathcal{M}}(\mu,\nu) = (1-\varepsilon)d_\varphi^{\mathcal{M}}(\mu,\mu) + \varepsilon\, d_\varphi^{\mathcal{M}}(\mu,\nu) \geq d_\varphi^{\mathcal{M}}(\mu,(1-\varepsilon)\mu+\varepsilon\nu).$$

Since F_* preserves convex combinations, one has $F_*^n\mu' = (1-\varepsilon)F_*^n\mu + \varepsilon F_*^n\nu$ for all $n\in\mathbb{N}$, so $d_\varphi^{\mathcal{M}}(F_*^n\mu, F_*^n\mu') \leq \varepsilon\, d_\varphi^{\mathcal{M}}(F_*^n\mu, F_*^n\nu)$. If φ is bounded, one deduces that F_* can not be expansive in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}), d_\varphi^{\mathcal{M}})$. □

3.2.3. *Equicontinuity.* Now we consider that the semi-group is $\mathbb{M} = \mathbb{N}$ or $\mathbb{Z}$ in view to use properties of blocking words introduces in [Sab06].

Proposition 3.5. *Assume that $\mathbb{M} = \mathbb{N}$ or $\mathbb{Z}$. Let $(\mathcal{A}^{\mathbb{M}}, F)$ be an equicontinuous CA of slope α. Then F_* is equicontinous in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_\varphi^{\mathcal{M}})$.*

Proof. According to [Sab06], the equicontinuity of slope α implies that the sequence of functions $(F^n \circ \sigma^{\lfloor n\alpha\rfloor})_{n\in\mathbb{N}}$ is ultimately periodic. By σ-invariance the sequence $(F_*^n)_{n\in\mathbb{N}}$ is ultimately periodic. One deduces that F_* is equicontinuous in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_\varphi^{\mathcal{M}})$. □

3.2.4. *Equicontinuous points.*

Proposition 3.6. *Assume that $\mathbb{M} = \mathbb{N}$ or $\mathbb{Z}$. Let φ be a sub-measure invariant under the action of $\mathbb{M}$ and let $(\mathcal{A}^{\mathbb{M}}, F)$ be a CA with equicontinuous points of slope α. Then $Eq_{d_\varphi^{\mathcal{M}}}(F_*) \neq \emptyset$.*

Proof. Let u be a blocking word of slope α. Consider a σ-periodic configuration $z \in \mathcal{A}^{\mathbb{M}}$ of period $K \geq |u|$ with an occurrence of u. Let $\mu_z = \frac{1}{K}\sum_{i=0}^{K-1}\delta_{\sigma^i(z)}$ the σ-invariant measure supported by the σ-orbit of z. Let $\nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ and $\lambda \in \mathcal{J}(\mu_z,\nu)$. Consider the desintegration of ν according to μ_z, for all $i \in [0, K-1]$, there exists $\nu_{\sigma^i(z)} \in \mathcal{M}(\mathcal{A}^{\mathbb{M}})$ such that

$$\lambda = \frac{1}{K}\sum_{i=0}^{K-1}\delta_{\sigma^i(z)} \times \nu_{\sigma^i(z)}.$$

For all $n \in \mathbb{N}$ one has:

$$(F_* \times F_*)^n\lambda = \frac{1}{K}\sum_{i=0}^{K-1}\delta_{\sigma^i(F^n(z))} \times F_*^n\nu_{\sigma^i(z)} \in \mathcal{J}(F_*^n\mu_z, F_*^n\nu).$$

One deduces that:

$$\begin{aligned}\int d_\varphi(x,y)d(F_*^n\times F_*^n)\lambda(x,y) &= \frac{1}{K}\sum_{i=0}^{K-1}\int d_\varphi(\sigma^i(F^n(z)),y)dF_*^n\nu_{\sigma^i(z)}(y)\\ &= \frac{1}{K}\sum_{i=0}^{K-1}\int d_\varphi(F^n(\sigma^i(z)),F^n(y))d\nu_{\sigma^i(z)}(y).\end{aligned}$$

By equicontinuity of slope α, by proposition 2.7, one has $d_\varphi(F^n(\sigma^i(z)),F^n(y)) \leq 2Kd_\varphi(\sigma^i(z),y)$. Thus, after to integrate and to take the inferior bound, one obtains:

$$d_\varphi^{\mathcal{M}}(F_*^n(\mu_z),F_*^n(\nu)) \leq 2Kd_\varphi^{\mathcal{M}}(\mu_z,\nu).$$

Result that $\mu_z \in Eq_{d_\varphi^{\mathcal{M}}}(F_*)$. □

Open problem 3.2. This proposition just give elements of $Eq_{d_\varphi^{\mathcal{M}}}(F_*)$ supported by σ-periodic orbit when φ is $\mathbb{M}$-invariant. In the next section we give more points for the distance $d_C^{\mathcal{M}}$ and $d_B^{\mathcal{M}}$.

The next example shows that there exists cellular automaton with sensitive points in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_\varphi^{\mathcal{M}})$

Example 3.3. Consider the CA defined on $\mathcal{A} = \mathbb{Z}/2\mathbb{Z}$ by $F(x)_i = x_{i-1} \cdot x_i \cdot x_{i+1}$ for all $x \in \mathcal{A}^{\mathbb{Z}}$ and $i \in \mathbb{Z}$. It is easy to see that for all σ-ergodic probability measure μ which verifies $\mu([0]) > 0$, the sequence $(F_*^n\mu)_{n\in\mathbb{N}}$ converges toward $\delta_{^\infty 0^\infty}$ in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}), d_\varphi^{\mathcal{M}})$. So, one has:

$$Eq_{d_\varphi^{\mathcal{M}}}(F_*) \cap \mathcal{M}_\sigma^{\mathrm{erg}}(\mathcal{A}^{\mathbb{Z}}) = \mathcal{M}_\sigma^{\mathrm{erg}}(\mathcal{A}^{\mathbb{Z}}) \setminus \{\delta_{^\infty 1^\infty}\}.$$

4. Two useful distances on $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$: d_* and $d_B^{\mathcal{M}}$

Usually we use two specific distances, d_* and $d_B^{\mathcal{M}}$ in order to differentiate σ-invariant probability measures. The first one, d_*, correspond to the weak* topology. The distance d_* says that two measures are "near" if the measures of sufficiently large cylinder are not too distant. The second distance $d_B^{\mathcal{M}}$ says that two measures are "near" if it is possible to pass from a generic point to another with change only for a little density of cells. This distance takes consideration in the frequency of apparition of a pattern.

The weak* topology is easy to define, is separable and compact. It gives a good context to study dynamical systems. However, the set $\mathcal{M}_\sigma^{\mathrm{erg}}(\mathcal{A}^{\mathbb{M}})$ is not closed and the function $\mu \mapsto h_\mu(\sigma)$ is upper semi-continuous. Even if the distance $d_B^{\mathcal{M}}$ is more complicated because it is defined by joinings, this distance has some properties like the continuity of shift-entropy and the closer of ergodic measure. This allows to show quickly some remarkable properties as there is not transitive cellular automaton in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$ (there is a similar theorem in the space $(X_B, \widetilde{d_B})$ proved thanks to the Kolmogorov complexity [BCF03]). In this section, first we recall some basic properties of these spaces. The properties of continuity of $\mu \to h_\mu(\sigma)$ in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$ and $\mathcal{M}_\sigma^{\mathrm{erg}}$ is closed in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$ (subsection 4.1.3 and 4.1.4) are well known by the ergodic community, but we put here to do a self contain paper. Secondly we study some dynamical properties of the action of a cellular automata on $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_*)$ and $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$.

4.1. **Some special properties.**

4.1.1. *The distances $d_C^{\mathcal{M}}$ and d_*.* In fact, the distance $d_C^{\mathcal{M}}$ correspond to the weak* topology usually used on $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$.

Proposition 4.1. *The distance $d_C^{\mathcal{M}}$ define the weak* topology on $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$.*

Proof. One defines a metric on $\mathcal{A}^{\mathbb{M}}$ by:

$$d'_C(x,y) = \sum_{n\in\mathbb{N}} \frac{1}{2^n \mathrm{Card}(\mathcal{A}^{\mathbb{U}_n})} \sum_{u\in\mathcal{A}^{\mathbb{U}_n}} |\mathbf{1}_{[u]_{\mathbb{U}_n}}(x) - \mathbf{1}_{[u]_{\mathbb{U}_n}}(y)| \quad \text{for all } x,y \in \mathcal{A}^{\mathbb{M}}.$$

We recall that $\mathbb{U}_n = \{m \in \mathbb{M} : |m| \leq n\}$. It is easy to see that d_C and d'_C are equivalent.

Let $\mu,\nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$. For all $\lambda \in \mathcal{J}(\mu,\nu)$, one has:

$$\int d'_C(x,y)d\lambda = \sum_{n\in\mathbb{N}} \frac{1}{2^n \mathrm{Card}(\mathbb{U}_n)} \sum_{u\in\mathcal{A}^{\mathbb{U}_n}} |\mu([u]_{\mathbb{U}_n}) - \nu([u]_{\mathbb{U}_n})|,$$

So the distance $d_C^{\mathcal{M}}$ defines the weak* topology. □

Remark 4.1. Let $x,y \in \mathcal{A}^{\mathbb{M}}$ be σ-periodic configurations. One has $d_*(\widetilde{\delta_x}, \widetilde{\delta_y}) = \min_{m\in\mathbb{M}} d'_C(x, \sigma^m(y))$.

4.1.2. *The distance $d_B^{\mathcal{M}}$.* First we establish a inferior bound of the pseudo-distance $d_B^{\mathcal{M}}$. This allows to associate $d_B^{\mathcal{M}}$ to the Kantorovich metrics, we refer to [Gla03] for more details or [Ver04] for an historical approach.

Lemma 4.2. *Let $\mu,\nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ and let $\mathbb{U} \subset \mathbb{M}$ be a finite subset. One has:*

$$\begin{aligned} d_B^{\mathcal{M}}(\mu,\nu) &= \inf_{\lambda\in\mathcal{J}(\mu,\nu)} \lambda([a]_{\mathbb{U}} \times [b]_{\mathbb{U}} : a,b \in \mathcal{A},\, a \neq b) \\ &\geq \frac{1}{\mathrm{Card}(\mathbb{U})} \inf_{\lambda\in\mathcal{J}(\mu,\nu)} \lambda([u]_{\mathbb{U}} \times [v]_{\mathbb{U}} : u,v \in \mathcal{A}^{\mathbb{U}},\, u \neq v). \end{aligned}$$

Proof. Let $\mu,\nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ and let $\lambda \in \mathcal{J}(\mu,\nu)$.

Assume that λ is $\sigma\times\sigma$-ergodic, by Birkhoff's Theorem, for λ-almost all $(x,y)\in\mathcal{A}^{\mathbb{M}}\times\mathcal{A}^{\mathbb{M}}$, one has:

$$\lim_{n\to\infty}\frac{1}{\mathrm{Card}(\mathbb{U}_n)}\sum_{m\in\mathbb{U}_n}\mathbf{1}_0(\sigma^m(x),\sigma^m(y)) \underset{n\to\infty}{\longrightarrow} \lambda([a]_0\times[b]_0 : a,b\in\mathcal{A},\ a\neq b).$$

One deduces that for λ-almost every $x,y\in\mathcal{A}^{\mathbb{M}}$, one has $d_B(x,y)=\lambda([a]_0\times[b]_0 : a,b\in\mathcal{A},\ a\neq b)$, so

$$\int d_B(x,y)\mathrm{d}\lambda = \lambda([a]_0\times[b]_0 : a,b\in\mathcal{A},\ a\neq b).$$

If λ is not $\sigma\times\sigma$-ergodic, the Theorem of ergodic decomposition (see [DGS76] or [Gla03]) allows to write $\lambda=\int_0^1\lambda^t\mathrm{d}t$ where almost every λ^t are $\sigma\times\sigma$-ergodic. One deduces that:

$$\int_0^1\int d_B(x,y)\mathrm{d}\lambda^t\mathrm{d}t=\int_0^1\lambda^t([a]_0\times[b]_0 : a,b\in\mathcal{A},\ a\neq b)\mathrm{d}t=\lambda([a]_0\times[b]_0 : a,b\in\mathcal{A},\ a\neq b).$$

Thus $d_B^{\mathcal{M}}(\mu,\nu)=\inf_{\lambda\in\mathcal{J}(\mu,\nu)}\lambda([a]_0\times[b]_0 : a,b\in\mathcal{A},\ a\neq b)$.

We are going to prove the last inequality of the Lemma. Let $u,v\in\mathcal{A}^{\mathbb{U}}$, one has:

$$\bigcup_{u,v\in\mathcal{A}^{\mathbb{U}},u\neq v}[u]_{\mathbb{U}}\times[v]_{\mathbb{U}}\subset\bigcup_{m\in\mathbb{U}}\left(\bigcup_{a,b\in\mathcal{A},a\neq b}[a]_m\times[b]_m\right).$$

One deduces the following inequality:

$$\begin{aligned}\lambda([u]_{\mathbb{U}}\times[v]_{\mathbb{U}} : u,v\in\mathcal{A}^{\mathbb{U}},\ u\neq v) &\leq \sum_{m\in\mathbb{U}}\lambda([a]_m\times[b]_m : a,b\in\mathcal{A},\ a\neq b)\\ &\underset{(\star)}{=} \mathrm{Card}(\mathbb{U})\,\lambda([a]_0\times[b]_0 : a,b\in\mathcal{A},\ a\neq b),\end{aligned}$$

where $(\star)$ follows from the $\sigma\times\sigma$-invariance of λ. □

Now it is possible to show that $d_B^{\mathcal{M}}$ is a distance.

Proposition 4.3. *The pseudo-distance $d_B^{\mathcal{M}}$ is a distance. That is to say that for all $\mu,\nu\in\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$, one has $d_B^{\mathcal{M}}(\mu,\nu)=0$ if and only if $\mu=\nu$.*

Proof. Let $\mu,\nu\in\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ such that $d_B^{\mathcal{M}}(\mu,\nu)=0$, it is sufficient to show that $\mu=\nu$. Let $\mathbb{U}\subset\mathbb{M}$ be a finite set and let $w\in\mathcal{A}^{\mathbb{U}}$. Since $\mathcal{J}(\mu,\nu)$ is compact and $\lambda\to\lambda([u]_{\mathbb{U}}\times[v]_{\mathbb{U}} : u,v\in\mathcal{A}^{\mathbb{U}},\ u\neq v)$ is continue, according to lemma 4.2, one deduce that there exists $\lambda\in\mathcal{J}(\mu,\nu)$ such that $\lambda([u]_{\mathbb{U}}\times[v]_{\mathbb{U}} : u,v\in\mathcal{A}^{\mathbb{U}},\ u\neq v)=0$. Thus, one has:

$$|\mu([w]_{\mathbb{U}})-\nu([w]_{\mathbb{U}})|=|\lambda([w]_{\mathbb{U}}\times\mathcal{A}^{\mathbb{M}})-\lambda(\mathcal{A}^{\mathbb{M}}\times[w]_{\mathbb{U}})|\leq\lambda([u]_{\mathbb{U}}\times[v]_{\mathbb{U}} : u,v\in\mathcal{A}^{\mathbb{U}},\ u\neq v)=0.$$

So, μ and ν coincide on every cylinder. That is to say $\mu=\nu$. □

Example 4.2. We want to calculate the distance between two Bernoulli measure. Let $\mathcal{A}=\{0,1\}$, let $p,q\in[0,1]$ such that $p<\frac{1}{2}$ and $p-1>q>p$. Consider λ_p and λ_q two Bernoulli on $\mathcal{A}^{\mathbb{M}}$ such that $\lambda_p([1])=p$ and $\lambda_q([1])=q$. An element $x\in\mathcal{G}(\lambda_p)$ contains a density p of cells 1 whereas an element $y\in\mathcal{G}(\lambda_q)$ contains a density q of cells 1. So $d_B(x,y)\geq q-p$, it follows that $d_B^{\mathcal{M}}(\lambda_p,\lambda_q)\geq q-p$.

Let $r\in[0,1]$ such that $q=p(1-r)+(1-p)r$, considers the Bernoulli measure λ_r defined by $\lambda_r([1])=r$. Consider the function

$$\begin{array}{rlcll} T: & \mathcal{A}^{\mathbb{M}}\times\mathcal{A}^{\mathbb{M}} & \longrightarrow & \mathcal{A}^{\mathbb{M}} & \\ & (x,y) & \longmapsto & x+y & \mod 2.\end{array}$$

Let $(x,y)\in\mathcal{A}^{\mathbb{M}}\times\mathcal{A}^{\mathbb{M}}$ and $m\in\mathbb{M}$. We have $T(x,y)_m=1$ if and only if $x_m=1$ and $y_m=0$ or $x_m=0$ and $y_m=1$. Since every cells are independent for a Bernoulli measure, we deduce that $T_*(\lambda_p\times\lambda_r)=\lambda_q$. Let $x\in\mathcal{G}(\lambda_p)$ and $z\in\mathcal{G}(\lambda_r)$, then put $y=x+z$ which is λ_q-almost certainly in $\mathcal{G}(\lambda_q)$, moreover $d_B(x,y)=q-p$. One deduces that

$$d_B^{\mathcal{M}}(\lambda_p,\lambda_q)=q-p.$$

4.1.3. *Continuity of the entropy of σ.* The information contained in a a generic configuration can be expressed by the entropy of the shift. A. A. Brudno compare the entropy of the shift with the Kolmogorov's complexity [Bru82]. Naturally, this tools can be used to study the dynamic of F_*. We are going interested here to the continuous property of the function $\mu \mapsto h_\mu(\sigma)$.

Definition. Let $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$, the entropy of the shift $\mathbb{M}$-action can be defined as:

$$h_\mu(\sigma) = \lim_{n\to\infty} \frac{H_\mu(\mathcal{P}_{\mathbb{U}_n})}{\mathrm{Card}(\mathbb{U}_n)},$$

where $\mathcal{P}_{\mathbb{U}_n}$ is the partition of cylinders centered on $\mathbb{U}_n$ and $H_\mu(\mathcal{P}_{\mathbb{U}_n})$ is the entropy of the partition $\mathcal{P}_{\mathbb{U}_n}$ according to the measure μ, defined by:

$$H(\mathcal{P}_{\mathbb{U}_n}) = -\sum_{u\in\mathcal{A}^{\mathbb{U}_n}} \mu([u]_{\mathbb{U}_n}) \log(\mu([u]_{\mathbb{U}_n})).$$

One recalls that $\mathbb{U}_n = \{m \in \mathbb{M} : |m| \leq n\}$.

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two partitions of $\mathcal{A}^{\mathbb{M}}$. We define the refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$ by

$$\mathcal{P}_1 \vee \mathcal{P}_2 = \{A \cap B : A \in \mathcal{P}_1 \text{ and } B \in \mathcal{P}_2\}.$$

Moreover it is possible to define the conditional entropy of $\mathcal{P}_1$ given $\mathcal{P}_2$:

$$H_\mu(\mathcal{P}_1|\mathcal{P}_2) = -\sum_{B\in\mathcal{P}_2} \mu(B) \sum_{A\in\mathcal{P}_1} \frac{\mu(A\cap B)}{\mu(B)} \log(\mu(A)).$$

It is well known that the function $\mu \mapsto h_\mu(\sigma)$ is upper semi-continuous in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_*)$ [DGS76]. We are going to see that this function is continuous in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$.

Theorem 4.4. *The function $\mu \mapsto h_\mu(\sigma)$ is uniformly continuous in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$.*

Proof. Let μ and ν in $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$. By definition of the entropy of σ, one has

$$h_\mu(\sigma) = \lim_{n\to\infty} \frac{H_\mu(\mathcal{P}_{\mathbb{U}_n})}{\mathrm{Card}(\mathbb{U}_n)} \text{ and } h_\nu(\sigma) = \lim_{n\to\infty} \frac{H_\nu(\mathcal{P}_{\mathbb{U}_n})}{\mathrm{Card}(\mathbb{U}_n)}.$$

However, for all $\lambda \in \mathcal{J}(\mu,\nu)$ one has:

$$\begin{array}{rcl}
|H_\mu(\mathcal{P}_{\mathbb{U}_n}) - H_\nu(\mathcal{P}_{\mathbb{U}_n})| & = & |H_\lambda(\mathcal{P}_{\mathbb{U}_n} \times \mathcal{A}^{\mathbb{M}}) - H_\lambda(\mathcal{A}^{\mathbb{M}} \times \mathcal{P}_{\mathbb{U}_n})| \\
& = & |\big(H_\lambda(\mathcal{P}_{\mathbb{U}_n} \times \mathcal{A}^{\mathbb{M}}) - H_\lambda(\mathcal{P}_{\mathbb{U}_n} \times \mathcal{A}^{\mathbb{M}} \vee \mathcal{A}^{\mathbb{M}} \times \mathcal{P}_{\mathbb{U}_n})\big) \\
& & -\big(H_\lambda(\mathcal{A}^{\mathbb{M}} \times \mathcal{P}_{\mathbb{U}_n}) - H_\lambda(\mathcal{P}_{U_n} \times \mathcal{A}^{\mathbb{M}} \vee \mathcal{A}^{\mathbb{M}} \times \mathcal{P}_{\mathbb{U}_n})\big)| \\
& \leq & H_\lambda(\mathcal{P}_{\mathbb{U}_n} \times \mathcal{A}^{\mathbb{M}} | \mathcal{A}^{\mathbb{M}} \times \mathcal{P}_{\mathbb{U}_n}) + H_\lambda(\mathcal{A}^{\mathbb{M}} \times \mathcal{P}_{\mathbb{U}_n} | \mathcal{P}_{\mathbb{U}_n} \times \mathcal{A}^{\mathbb{M}}).
\end{array}$$

Moreover, one has:

$$\begin{array}{rcl}
H_\lambda(\mathcal{P}_{\mathbb{U}_n} \times \mathcal{A}^{\mathbb{M}} | \mathcal{A}^{\mathbb{M}} \times \mathcal{P}_{\mathbb{U}_n}) & \leq & \sum_{i\in\mathbb{U}_n} H_\lambda(\mathcal{P}_i \times \mathcal{A}^{\mathbb{M}} | \mathcal{A}^{\mathbb{M}} \times \mathcal{P}_{\mathbb{U}_n}) \\
& \leq & \mathrm{Card}(\mathbb{U}_n) H_\lambda(\mathcal{P}_0 \times \mathcal{A}^{\mathbb{M}} | \mathcal{A}^{\mathbb{M}} \times \mathcal{P}_{\mathbb{U}_n}) \\
& \leq & \mathrm{Card}(\mathbb{U}_n) H_\lambda(\mathcal{P}_0 \times \mathcal{A}^{\mathbb{M}} | \mathcal{A}^{\mathbb{M}} \times \mathcal{P}_0),
\end{array}$$

where $\mathcal{P}_0 = \mathcal{P}_{\mathbb{U}_0}$. Symmetrically one obtains

$$H_\lambda(\mathcal{A}^{\mathbb{M}} \times \mathcal{P}_{\mathbb{U}_n} | \mathcal{P}_{\mathbb{U}_n} \times \mathcal{A}^{\mathbb{M}}) \leq \mathrm{Card}(\mathbb{U}_n) H_\lambda(\mathcal{A}^{\mathbb{M}} \times \mathcal{P}_0 | \mathcal{P}_0 \times \mathcal{A}^{\mathbb{M}}).$$

Thus, by summation one has:

$$|h_\mu(\sigma) - h_\nu(\sigma)| \leq H_\lambda(\mathcal{P}_0 \times \mathcal{A}^{\mathbb{M}} | \mathcal{A}^{\mathbb{M}} \times \mathcal{P}_0) + H_\lambda(\mathcal{A}^{\mathbb{M}} \times \mathcal{P}_0 | \mathcal{P}_0 \times \mathcal{A}^{\mathbb{M}}).$$

Consider $\alpha = (\cup_{a,b\in\mathcal{A},a\neq b}[a]_0 \times [b]_0; \cup_{a\in\mathcal{A}}[a]_0 \times [a]_0)$, the partition of $\mathcal{A}^{\mathbb{M}} \times \mathcal{A}^{\mathbb{M}}$ formed of two elements. Set $\delta = \lambda(\cup_{a,b\in\mathcal{A},a\neq b}[a]_0 \times [b]_0)$. One has:

$$H_\lambda(\mathcal{P}_0 \times \mathcal{A}^{\mathbb{M}} | \mathcal{A}^{\mathbb{M}} \times \mathcal{P}_0) \leq H_\lambda(\alpha) \leq -(\delta \log(\delta) + (1-\delta)\log(1-\delta)).$$

Let $\varepsilon > 0$. The function $\delta \to \delta \log(\delta) + (1-\delta)\log(1-\delta)$ tends towards 0 when δ tends towards 0. Thus, there exists $\delta_0 > 0$ such that $\delta \log(\delta) + (1-\delta)\log(1-\delta) \leq \frac{\varepsilon}{2}$ for all $\delta < \delta_0$. Let $\mu, \nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ such that $d_B^{\mathcal{M}}(\mu,\nu) < \delta_0$. According to Lemma 4.2, there exists $\lambda \in \mathcal{J}(\mu,\nu)$ such that

$$\lambda([a]_0 \times [b]_0 : a, b \in \mathcal{A}, a \neq b) \leq d_B^{\mathcal{M}}(\mu,\nu) < \delta_0.$$

In this case, one has $H_\lambda(\mathcal{P}_0 \times \mathcal{A}^{\mathbb{M}} | \mathcal{A}^{\mathbb{M}} \times \mathcal{P}_0) \leq \frac{\varepsilon}{2}$, and symetically $H_\lambda(\mathcal{A}^{\mathbb{M}} \times \mathcal{P}_0, \mathcal{P}_0 \times \mathcal{A}^{\mathbb{M}}) \leq \frac{\varepsilon}{2}$. We deduce that for all $\varepsilon > 0$, there exists δ_0 such that if $d_B^{\mathcal{M}}(\mu,\nu) \leq \delta_0$ then

$$|h_\mu(\sigma) - h_\nu(\sigma)| \leq H_\lambda(\mathcal{P}_0 \times \mathcal{A}^{\mathbb{M}} | \mathcal{A}^{\mathbb{M}} \times \mathcal{P}_0) + H_\lambda(\mathcal{A}^{\mathbb{M}} \times \mathcal{P}_0 | \mathcal{P}_0 \times \mathcal{A}^{\mathbb{M}}) \leq \varepsilon.$$

This proves the uniform continuity of $\mu \to h_\mu(\sigma)$ in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$. □

Remark 4.3. In fact, the continuity of $\mu \to h_\mu(\sigma)$ it is already known for the distance $\overline{d}$, see [Gla03].

4.1.4. *Limit of ergodic measure in* $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$. The image of an ergodic measure by the action of a CA is also ergodic. It could be natural to consider the restriction to the set $\mathcal{M}_\sigma^{\mathrm{erg}}(\mathcal{A}^{\mathbb{M}})$. However $\mathcal{M}_\sigma^{\mathrm{erg}}(\mathcal{A}^{\mathbb{M}})$ is dense in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_*)$, so it is not pertinent to study asymptotic proprieties for the dynamical system $(\mathcal{M}_\sigma^{\mathrm{erg}}(\mathcal{A}^{\mathbb{M}}), d_*, F_*)$. However, we are going to see that the set $\mathcal{M}_\sigma^{\mathrm{erg}}(\mathcal{A}^{\mathbb{M}})$ is closed in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$.

Proposition 4.5. *The set* $\mathcal{M}_\sigma^{erg}(\mathcal{A}^{\mathbb{M}})$ *is closed in* $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$.

Proof. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of σ-ergodic measures which converges for the distance $d_B^{\mathcal{M}}$ toward the measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$. Let $\varepsilon > 0$. According to Lemma 4.2, since $(\mu_n)_{n\in\mathbb{N}}$ converges for $d_B^{\mathcal{M}}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, there exists $\lambda_n \in \mathcal{J}(\mu, \mu_n)$ which verifies

$$\lambda_n([a]_0 \times [b]_0 : a, b \in \mathcal{A},\ a \neq b) \leq \varepsilon^2.$$

One chooses $n \geq N$. The Theorem of ergodic decomposition allows to write $\lambda_n = \int_0^1 \lambda_n^t \mathrm{d}t$ where $\lambda_n^t \in \mathcal{M}_{\sigma\times\sigma}^{\mathrm{erg}}(\mathcal{A}^{\mathbb{M}} \times \mathcal{A}^{\mathbb{M}})$. Let $\mathbb{U} \subset \mathbb{M}$ be a finite subset and let $u \in \mathcal{A}^{\mathbb{U}}$ be a pattern. Since μ_n is σ-ergodic, for μ_n-almost all $x \in \mathcal{A}^{\mathbb{M}}$, one has:

$$\frac{1}{\mathrm{Card}(\mathbb{U}_k)} \sum_{m\in\mathbb{U}_k} \mathbf{1}_{[u]_{\mathbb{U}}}(\sigma^m(x)) \underset{k\to\infty}{\longrightarrow} \mu_n([u]_{\mathbb{U}}).$$

Moreover, since λ_n^t is $\sigma \times \sigma$-ergodic, for λ_n^t-almost all (x,y) one has:

$$\frac{1}{\mathrm{Card}(\mathbb{U}_k \times \mathbb{U}_k)} \sum_{(m',m'')\in\mathbb{U}_k\times\mathbb{U}_k} \mathbf{1}_{\mathcal{A}^{\mathbb{M}}\times[u]_{\mathbb{U}}}(\sigma^{m'}(x), \sigma^{m''}(y)) \underset{k\to\infty}{\longrightarrow} \pi_*^2 \lambda_n^t(\mathcal{A}^{\mathbb{M}} \times [u]_{\mathbb{U}}).$$

However, for all $k \in \mathbb{N}$ one has

$$\frac{1}{\mathrm{Card}(\mathbb{U}_k)} \sum_{m\in\mathbb{U}_k} \mathbf{1}_{[u]_{\mathbb{U}}}(\sigma^m(x)) = \frac{1}{\mathrm{Card}(\mathbb{U}_k \times \mathbb{U}_k)} \sum_{(m',m'')\in\mathbb{U}_k\times\mathbb{U}_k} \mathbf{1}_{\mathcal{A}^{\mathbb{M}}\times[u]_{\mathbb{U}}}(\sigma^{m'}(x), \sigma^{m''}(x)),$$

thus, by taking the limit, $\pi_*^2\lambda_n^t(\mathcal{A}^{\mathbb{M}} \times [u]_{\mathbb{U}}) = \mu([u]_{\mathbb{U}})$. Since the choice of the pattern is arbitrary, one deduce that $\pi_*^2\lambda_n^t = \mu_n$. Put $\mu^t = \pi_*^1\lambda_n^t$, this give the ergodic decomposition of μ, that is to say $\mu = \int_0^1 \mu^t \mathrm{d}t$; this decomposition is independent of the choose of n by unicity of the ergodic decomposition.

Let $\mathbb{U} \subset \mathbb{M}$ be a finite subset and let $u \in \mathcal{A}^{\mathbb{U}}$ be a pattern. One has:

$$\begin{aligned}
|\mu([u]_{\mathbb{U}}) - \mu^t([u]_{\mathbb{U}})| &= |\mu([u]_{\mathbb{U}}) - \mu_n([u]_{\mathbb{U}}) + \mu_n([u]_{\mathbb{U}}) - \mu^t([u]_{\mathbb{U}})| \\
&\leq |\lambda_n([u]_{\mathbb{U}} \times \mathcal{A}^{\mathbb{M}}) - \lambda_n(\mathcal{A}^{\mathbb{M}} \times [u]_{\mathbb{U}})| + |\lambda_n^t([u]_{\mathbb{U}} \times \mathcal{A}^{\mathbb{M}}) - \lambda_n^t(\mathcal{A}^{\mathbb{M}} \times [u]_{\mathbb{U}})| \\
&\underset{(*)}{\leq} \mathrm{Card}(\mathbb{U}) \left(\lambda_n([a]_0 \times [b]_0 : a, b \in \mathcal{A},\ a \neq b) + \lambda_n^t([a]_0 \times [b]_0 : a, b \in \mathcal{A},\ a \neq b)\right)
\end{aligned}$$

where $(*)$ follows from Lemma 4.2. Since $\lambda_n([a]_0 \times [b]_0 : a, b \in \mathcal{A},\ a \neq b) \leq \varepsilon^2$, there exists a set $A_\varepsilon \subset [0,1]$ of measure superior at $1-\varepsilon$ such that $\lambda_n^t([a]_0 \times [b]_0 : a, b \in \mathcal{A},\ a \neq b) \leq \varepsilon$ for all $t \in A_\varepsilon$. Thus, for all $t \in A_\varepsilon$ one has:

$$|\mu([u]_{\mathbb{U}}) - \mu^t([u]_{\mathbb{U}})| \leq \mathrm{Card}(\mathbb{U})(\varepsilon^2 + \varepsilon).$$

The choice of ε is arbitrary, one deduce that $\mu([u]_{\mathbb{U}}) = \mu^t([u]_{\mathbb{U}})$ for almost all $t \in [0,1]$. Since the pattern u is chosen arbitrary, one has $\mu = \mu^t$ for almost all $t \in [0,1]$. Thus μ is σ-ergodic. □

4.1.5. *Links between the spaces* $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_*)$*,* $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$ *and* $(\mathcal{G}, d_B)$*.* As we have defined two distances (d_* and $d_B^{\mathcal{M}}$) on $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ it is interesting to study how they relate to each other. The following proposition shows that $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$ can be embedded into $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_*)$.

Proposition 4.6. *The function* $\mathrm{Id} : (\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}}) \to (\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_*)$ *is uniformly continuous.*

Proof. Let $\mu, \nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$. Let $n \in \mathbb{N}$ and $u \in \mathcal{A}^{\mathbb{U}_n}$, according to Lemma 4.2, there exists $\lambda \in \mathcal{J}(\mu, \nu)$ such that

$$d_B^{\mathcal{M}}(\mu, \nu) \geq \frac{1}{\mathrm{Card}(\mathbb{U}_n)} \lambda([u]_{\mathbb{U}_n} \times [v]_{\mathbb{U}_n} : u, v \in \mathcal{A}^{\mathbb{U}_n},\, u \neq v).$$

Thus, one has:

$$\begin{aligned} |\mu([u]_{\mathbb{U}_n}) - \nu([u]_{\mathbb{U}_n})| &= |\lambda([u]_{\mathbb{U}_n} \times \mathcal{A}^{\mathbb{M}}) - \lambda(\mathcal{A}^{\mathbb{M}} \times [u]_{\mathbb{U}_n})| \\ &\leq \left| \sum_{v \in \mathcal{A}^{\mathbb{U}_n} \setminus \{u\}} \lambda([u]_{\mathbb{U}_n} \times [v]_{\mathbb{U}_n}) - \sum_{v \in \mathcal{A}^{\mathbb{U}_n} \setminus \{u\}} \lambda([v]_{\mathbb{U}_n} \times [u]_{\mathbb{U}_n}) \right| \\ &\leq \mathrm{Card}(\mathbb{U}_n) d_B^{\mathcal{M}}(\mu, \nu). \end{aligned}$$

Let $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0+1}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2}$. If the measures $\mu, \nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ verifies $d_B^{\mathcal{M}}(\mu, \nu) < \frac{\varepsilon}{2\mathrm{Card}(\mathbb{U}_{n_0})}$ then:

$$d_*(\mu, \nu) \leq \sum_{n=0}^{n_0} \frac{1}{2^n \mathrm{Card}(\mathbb{U}_n)} \sum_{u \in \mathcal{A}^{\mathbb{U}_n}} \left|\mu([u]_{\mathbb{U}_n}) - \nu([u]_{\mathbb{U}_n})\right| + \sum_{n=n_0+1}^{\infty} \frac{1}{2^n} \leq \varepsilon.$$

One deduces that $\mathrm{Id} : (\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}}) \to (\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_*)$ is uniformly continuous. □

The reverse function $\mathrm{Id} : (\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_*) \to (\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$ is not continuous, since otherwise the two distances d_* and $d_B^{\mathcal{M}}$ would be equivalent (and we have said earlier that it is not the case).

For all σ-invariant probability measure, there exist special points of $\mathcal{A}^{\mathbb{M}}$ which represent the measure. That is to say the frequency of occurence of a pattern corresponds to the measure of the cylinder centered on this pattern:

Definition. A point $x \in \mathcal{A}^{\mathbb{M}}$ is *generic* if for all $\mathbb{U} \subset \mathbb{M}$ finite and for every pattern $u \in \mathcal{A}^{\mathbb{U}}$ the sequence $(f(u, x, n))_{n \in \mathbb{N}}$ converges where

$$f(u, x, n) = \frac{1}{\mathrm{Card}(\mathbb{U}_n)} \sum_{m \in \mathbb{U}_n} \mathbf{1}_{[u]_{\mathbb{U}}}(\sigma^m(x)),$$

is the *frequency of the pattern* u *in* x *at the order* n. The limit of this sequence is denoted $f(u, x)$, this is the *frequency of the pattern* u *in* x. Denote $\mathcal{G}$ the set of genreric points.

Let $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$. Denote $\mathcal{G}(\mu)$ the set of *generic points of* μ, this is the set of points $x \in \mathcal{G}$ such that for every pattern, the frequency of this pattern in x is equal to the measure of the cylinder centered on this pattern.

It is possible to establish a correspondence between the space of generic points and the set of σ-invariant probability measures.

Proposition 4.7. *The projection of* $\mathcal{G}$ *in* $(X_B, \widetilde{d_B})$ *is closed.*

Proof. Let $(x^n)_{n \in \mathbb{N}}$ a sequence of elements of $\mathcal{A}^{\mathbb{M}}$ which converges toward $x \in \mathcal{A}^{\mathbb{M}}$ for the pseudo-distance d_B. Let $\mathbb{U} \subset \mathbb{M}$ be a finite subset, let $u \in \mathcal{A}^{\mathbb{U}}$ be a pattern and let $k \in \mathbb{N}$. One has:

$$|f(u, x^n, k) - f(u, x, k)| \leq \frac{\mathrm{Card}(i \in \mathbb{U}_k : x_i^n \neq x_i)}{\mathrm{Card}(\mathbb{U}_k)}.$$

Taking the limit, one deduces that $(f(u, x, k))_{k \in \mathbb{N}}$ converges so $x \in \mathcal{G}$. □

Proposition 4.8. *Let* $\mu, \nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$*,* $x \in \mathcal{G}(\mu)$ *and* $y \in \mathcal{G}(\nu)$*. One has* $d_B(x, y) \geq d_B^{\mathcal{M}}(\mu, \nu)$*.*

Proof. Let $x \in \mathcal{G}(\mu)$ and $y \in \mathcal{G}(\nu)$. Let $\mathbb{U} \subset \mathbb{M}$ and $\mathbb{V} \subset \mathbb{M}$ be two finite subsets and let $u \in \mathcal{A}^{\mathbb{U}}$ and $v \in \mathcal{A}^{\mathbb{V}}$ be two patterns. Since x and y are generic points, one can define:

$$\lambda([u]_{\mathbb{U}_n} \times [v]_{\mathbb{U}_n}) = \lim_{n\to\infty} \frac{1}{\mathrm{Card}(\mathbb{U}_n)} \sum_{m\in\mathbb{U}_n} \mathbf{1}_{[u]_{\mathbb{U}_n}\times[v]_{\mathbb{U}_n}}(\sigma^m(x), \sigma^m(y)).$$

One has $\lambda \in \mathcal{J}(\mu,\nu)$. Moreover, by construction $\int d_B(x',y')d\lambda(x',y') = d_B(x,y)$; the result follows. $\square$

Consider the function $\phi : \mathcal{G} \to \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ which associates a generic point $x \in \mathcal{G}$ to the measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ defined by $\mu([u]_{\mathbb{U}}) = f(u,x)$ for all patterns $u \in \mathcal{A}^{\mathbb{U}}$. According to Proposition 4.8, the function ϕ is continuous. Moreover, it is easy to see that the image of a generic point by F is also a generic point. The following commutative diagram sums up these properties and establishes a correspondence between the different spaces.

$$\begin{tikzcd}
(\mathcal{G}, d_B) \arrow[r, "F"] \arrow[d, "\phi"] & (\mathcal{G}, d_B) \arrow[d, "\phi"] \\
(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}}) \arrow[r, "F_*"] \arrow[d, "\mathrm{Id}"] & (\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}}) \arrow[d, "\mathrm{Id}"] \\
(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_*) \arrow[r, "F_*"] & (\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_*)
\end{tikzcd}$$

4.2. **Results about the dynamic of F_*.** The specifics of the metrics d_* and $d_B^{\mathcal{M}}$ allow to obtain more results on the dynamic of F_*. According to the results more general established in the previous section, it is already known that :

- There not exist CA such that the action is expansive in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_*)$ and $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$ (Theorem 3.4).
- If a CA $(\mathcal{A}^{\mathbb{Z}}, F)$ is equicontinuous of slope $\alpha \in \mathbb{Q}$ then F_* is equicontinuous in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}), d_*)$ and $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}), d_B^{\mathcal{M}})$ (Proposition 3.5).
- If a CA $(\mathcal{A}^{\mathbb{Z}}, F)$ has equicontinuous points of slope α then F_* has equicontinuous points in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}), d_B^{\mathcal{M}})$ (Proposition 3.6).
- There exists CA $(\mathcal{A}^{\mathbb{M}}, F)$ such that some measures are not in $Eq_{d_*}(F_*)$ or in $Eq_{d_B^{\mathcal{M}}}(F_*)$ (Example 3.3).

Open problem 4.1. The set $\mathcal{M}_\sigma^{\mathrm{erg}}(\mathcal{A}^{\mathbb{M}})$ is closed in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$, so we can study the restriction of F_* to this space. We know very few properties of this space, in particular we do not know if there is not expansive cellular automaton. Indeed the prove of Proposition 3.4 use the linearity in the space which is not valid in $\mathcal{M}_\sigma^{\mathrm{erg}}(\mathcal{A}^{\mathbb{M}})$.

4.2.1. *Transitivity.* Theorem 4.4 allows to use the notion of entropy to show that the action of a CA can not be transitive in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$. The utilization of entropy recalls the utilization of the complexity of Kolmogorov in [BCF03] to show that there not exist transitive CA for Besicovitch topology. It is interesting to notice the common point between the two proofs: for both of them, the non-transitivity, which as we said is intuitively an unability to create information, is proven using a measure of the quantity of information, respectively $h_\mu(\sigma)$ and complexity of Kolmogorov. The link between metric entropy and Kolmogorov's complexity is explored in [Bru82].

Corollary 4.9. *There not exist CA $(\mathcal{A}^{\mathbb{M}}, F)$ such that F_* is transitive in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$.*

Proof. Let

$$\mathcal{U} = \{\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}) : h_\mu(\sigma) < \frac{1}{3}\} \text{ and } \mathcal{V} = \{\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}) : h_\mu(\sigma) > \frac{2}{3}\}.$$

By Theorem 4.4, $\mathcal{U}$ and $\mathcal{V}$ are open sets of $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$. Since F commutes with σ, it can be view as a factor map from $(\mathcal{A}^{\mathbb{M}}, \mu, \sigma)$ to $(F(\mathcal{A}^{\mathbb{M}}), F_*, \sigma)$, so one has $h_\mu(\sigma) \geq h_{F_*\mu}(\sigma)$, thus $F_*(\mathcal{U}) \subset \mathcal{U}$. One deduces that $\mathcal{V} \cap F_*^n(\mathcal{U}) = \emptyset$ for all $n \in \mathbb{N}$, thus F_* can not be transitive in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$. $\square$

Remark 4.4. It is interesting to notice the common point between the proofs of Theorem [BCF03] and Corollary 4.9: for both of them, the non-transitivity, which as we said is intuitively an unability to create information, is proven using a measure of the quantity of information, respectively $\dim_1$ and $h_\mu(\sigma)$.

In $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_*)$, the function $\mu \to h_\mu(\sigma)$ is just upper semi-continuous, so $\mathcal{V}$ is not open and the previous proof does not hold. In the space $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_*)$, the existence of transitive CA is open.

4.2.2. *Equicontinuity.* Proposition 3.6 exhibits just measures supported by σ-periodic points as $d_\varphi^{\mathcal{M}}$-equicontinuous points when φ is $\mathbb{M}$-invariant. In fact, it is possible to give a more larger class of measures which are equicontinuous points in the space $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_*)$ and $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}}), d_B^{\mathcal{M}})$. These measures are stable to every perturbations.

Theorem 4.10. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA and let $B \in \mathcal{A}^*$ be a blocking word of slope $\alpha \in \mathbb{R}$. Then every σ-ergodic probability measure $\mu \in \mathcal{M}_\sigma^{erg}(\mathcal{A}^{\mathbb{Z}})$ which verifies $\mu([B]) > 0$ is an equicontinuous point of F_* : $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}), d_*) \to (\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}), d_*)$.*

Proof. Let $\varepsilon > 0$, let μ be a σ-ergodic probability measure which charges B and let ν be a σ-invariant measure. For $n \in \mathbb{N}$, one defines $X_{i,n}^k$, the set of points $x \in \mathcal{A}^{\mathbb{Z}}$ such that there is an occurrence of B in $[-k - \lfloor n\alpha \rfloor, -\lfloor n\alpha \rfloor]$ and another in $[i - 1 - \lfloor n\alpha \rfloor, k + i - 1 - \lfloor n\alpha \rfloor]$.

Let i_0 such that $\sum_{i=i_0+1}^{\infty} \frac{1}{|\mathcal{A}|^i} \leq \varepsilon$ and let $n \in \mathbb{N}$. Since B is charged by μ, by σ-ergodicity, there exists $k \in \mathbb{N}$ such that $\mu(X_{i,n}^k) \geq 1 - \varepsilon$ for all $i \leq i_0$. Moreover $X_{i,n}^k$ can be written as an union of cylinders centered on $[-k - \lfloor n\alpha \rfloor, k + i - 1 - \lfloor n\alpha \rfloor]$ of words of $\mathcal{A}^{i+2k}$. By σ-invariance, one deduces that $|\mu(X_{i,n}^k) - \nu(X_{i,n}^k)| \leq |\mathcal{A}|^{i+2k} d(\mu, \nu)$, so:

$$\nu(X_{i,n}^k) \geq 1 - \varepsilon - |\mathcal{A}|^{i+2k} d(\mu, \nu).$$

Let $i \leq i_0$ and let $u \in \mathcal{A}^i$. Put $X_{u,n}^k = F^{-n} \circ \sigma^{-\lfloor \alpha n \rfloor}([u]_{[0,i-1]}) \cap X_{i,n}^k$. Taking the lower bounds of $\mu(X_{i,n}^k)$ and $\nu(X_{i,n}^k)$, for $i \leq i_0$ one deduces:

$$\begin{aligned} |F_*^n \mu([u]_{[0,i-1]}) - F_*^n \nu([u]_{[0,i-1]})| &\leq |F_*^n \mu([u]_{[0,i-1]}) - \mu(X_{u,n}^k)| + |\mu(X_{u,n}^k) - \nu(X_{u,n}^k)| \\ &\quad + |F_*^n \nu([u]_{[0,i-1]}) - \nu(X_{u,n}^k)| \\ &\leq \varepsilon + |\mu(X_{u,n}^k) - \nu(X_{u,n}^k)| + \varepsilon + |\mathcal{A}|^{|u|+2k} d(\mu, \nu). \end{aligned}$$

A summation gives for all $i \leq i_0$ the following inequality:

$$\sum_{u \in \mathcal{A}^i} |F_*^n \mu([u]_{[0,i-1]}) - F_*^n \nu([u]_{[0,i-1]})| \leq 2\varepsilon |\mathcal{A}|^i + |\mathcal{A}|^{2i+2k} d(\mu, \nu) + \sum_{u \in \mathcal{A}^i} |\mu(X_{u,n}^k) - \nu(X_{u,n}^k)|.$$

Let $Y_{u,n}^k$ the set of words $v \in \mathcal{A}^{|u|+2k}$ such that there exists $y \in F^{-n} \circ \sigma^{-\lfloor \alpha n \rfloor}[u]_{[0,i-1]} \cap [v]_{[-k-\lfloor n\alpha \rfloor, k+|u|-\lfloor n\alpha \rfloor]} \cap X_{|u|,n}^k$. Since B is a blocking word of slope α, for all $v \in Y_{u,n}^k$, for all $x \in [v]_{[-k-\lfloor n\alpha \rfloor, k+|u|-\lfloor n\alpha \rfloor]}$, one has $F^n(x) \circ \sigma^{\lfloor \alpha n \rfloor}(x)_{[0,|u|-1]} = u$. One deduces that:

$$X_{u,n}^k = F^{-n} \circ \sigma^{-\lfloor \alpha n \rfloor}([u]_{[0,i-1]}) \cap X_{|u|,n}^k = \bigcup_{Y_{u,n}^k} [v]_{[-k-\lfloor n\alpha \rfloor, k+|u|-\lfloor n\alpha \rfloor]}.$$

Thus, it is possible to decompose the sets $X_{u,n}^k$ to obtain a sum of measure of cylinder centered on $[-k - \lfloor n\alpha \rfloor, k + |u| - \lfloor n\alpha \rfloor]$ with words in $\mathcal{A}^{i+2k}$:

$$\begin{aligned} \sum_{u \in \mathcal{A}^i} |\mu(X_{u,n}^k) - \nu(X_{u,n}^k)| &= \sum_{u \in A^i} \left| \sum_{v \in Y_{u,n}^k} \mu([v]_{[-k-\lfloor n\alpha \rfloor, k+|u|-\lfloor n\alpha \rfloor]}) - \nu([v]_{[-k-\lfloor n\alpha \rfloor, k+|u|-\lfloor n\alpha \rfloor]}) \right| \\ &\leq \sum_{v \in \mathcal{A}^{i+2k}} |\mu([v]_{[0,i-1]}) - \nu([v]_{[0,i-1]})|. \end{aligned}$$

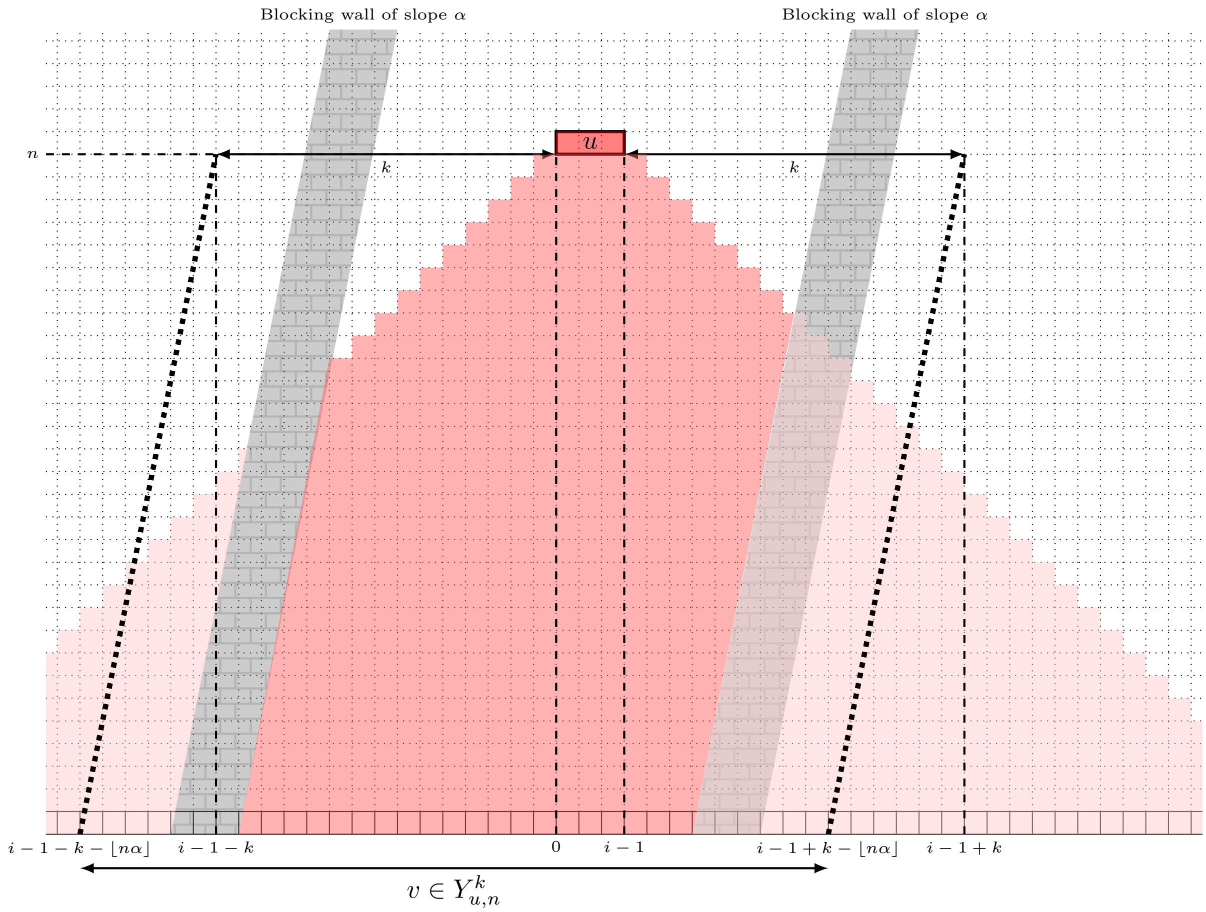

FIGURE 2. Blocking word of slope α and d_*-equicontinuity.

By summation of previous inequalities, it follows that:

$$\begin{aligned} d_*(F_*^n\mu, F_*^n\nu) &= \sum_{i\leq i_0} \frac{1}{|\mathcal{A}|^{2i}} \sum_{u\in\mathcal{A}^i} |F_*^n\mu([u]_{[0,i-1]}) - F_*^n\nu([u]_{[0,i-1]})| \\ &\quad + \sum_{i>i_0} \frac{1}{|\mathcal{A}|^{2i}} \sum_{u\in\mathcal{A}^i} |F_*^n\mu([u]_{[0,i-1]}) - F_*^n\nu([u]_{[0,i-1]})| \\ &\leq \frac{2\varepsilon}{|\mathcal{A}|-1} + |\mathcal{A}|^{2k}(1+|\mathcal{A}|^{i_0})d(\mu,\nu) + \varepsilon. \end{aligned}$$

This shows that the orbits $(F_*^n\mu)_{n\in\mathbb{N}}$ and $(F_*^n\nu)_{n\in\mathbb{N}}$ stay close to each other when μ and ν are close enough. □

Theorem 4.11. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA and let $B \in \mathcal{A}^*$ be a blocking word of slope $\alpha \in \mathbb{R}$. Then every σ-ergodic probability measure $\mu \in \mathcal{M}_\sigma^{erg}(\mathcal{A}^{\mathbb{Z}})$ which verifies $\mu([B]) > 0$ is an equicontinuous point of $F_* : (\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}), d_B^{\mathcal{M}}) \to (\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}), d_B^{\mathcal{M}})$.*

Proof. Let B be a blocking word of slope α and let $\mu \in \mathcal{M}_\sigma^{\mathrm{erg}}(\mathcal{A}^{\mathbb{Z}})$ such that $\mu([B]) > 0$. Let $\varepsilon > 0$. For all $n \in \mathbb{N}$, put $X_k^n \subset \mathcal{A}^{\mathbb{Z}}$ such that for all $x \in X_k^n$ there is an occurrence of B in $[-k-\lfloor\alpha n\rfloor, -\lfloor\alpha n\rfloor]$ and another in $[-\lfloor\alpha n\rfloor, k-\lfloor\alpha n\rfloor]$. By σ-ergodicity of μ, there exists $k \in \mathbb{N}$ which verifies $\mu(X_k^n) = 1-\varepsilon$ for all $n \in \mathbb{N}$.

Let $\nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ and let $\varepsilon > 0$. Let $\lambda \in \mathcal{J}(\mu, \nu)$ and consider the disintegration of λ according to μ; that is to say, for all $x \in \mathcal{A}^{\mathbb{Z}}$, there exists $\nu_x \in \mathcal{M}(\mathcal{A}^{\mathbb{M}})$ such that

$$\lambda = \int \delta_x \times \nu_x \mathrm{d}\mu(x).$$

Thus one has:

$$\begin{aligned} d_B^{\mathcal{M}}(F_*^n\mu, F_*^n\nu) &\leq \lambda((x,y) : F^n(x)_0 \neq F^n(y)_0) \\ &\leq \varepsilon * + \iint_{X_k^n \times \mathcal{A}^{\mathbb{Z}}} \mathbf{1}_{F^n(x)_0 \neq F^n(y)_0} \mathrm{d}\nu_x(y)\mathrm{d}\mu(x) \\ &\leq \varepsilon + \iint_{X_k^n \times \mathcal{A}^{\mathbb{Z}}} \mathbf{1}_{x_{[-k,k]} \neq y_{[-k,k]0}} \mathrm{d}\nu_x(y)\mathrm{d}\mu(x) \\ &\leq \varepsilon + (2k+1)\lambda((x,y) : x_0 \neq y_0). \end{aligned}$$

One deduces that $d_B^{\mathcal{M}}(F_*^n\mu, F_*^n\nu) \leq \varepsilon + (2k+1)d_B^{\mathcal{M}}(\mu,\nu)$, so $\mu \in Eq_{d_B^{\mathcal{M}}}(F_*)$. □

4.2.3. *The case of linear CA.* The uniform Bernoulli measure has an important role in the study of σ-invariant measures. G.A. Hedlund has shown in [Hed69] that a CA is surjective if and only if the uniform Bernoulli measure on $\mathcal{A}^{\mathbb{M}}$ is (F,σ)-invariant. Later, D. Lind [Lin84] shows for the radius 1 mod 2 automaton that starting from any Bernoulli measure the Cesàro mean of the iterates by the CA converges to the uniform measure. This result is generalized for a large class of algebraic CA and a large class of measures with tools from stochastic processes in [MM98] and [FMMN00], and with harmonic analysis tools in [PY02] and [PY04]. We use this result to show the d_*-sensitivity of linear CA.

Definition. Let $\mathcal{A}$ be an Abelian finite group, $(\mathcal{A}^{\mathbb{Z}}, F)$ is a linear CA if it F is a group endomorphism on the product group $\mathcal{A}^{\mathbb{M}}$. A linear CA is not trivial if it is not a product of shift.

In [BK99] they prove in particular that for algebraic cellular automaton $(\mathcal{A}^{\mathbb{Z}}, F)$ the set P of (σ, F)-periodic points is dense in $(\mathcal{A}^{\mathbb{Z}}, d_C)$. The next lemma shows a similar reasult for the measure.

Lemma 4.12. *Let $\mathcal{A} = \mathbb{Z}/p\mathbb{Z}$ with p is prime. Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a non-trivial algebraic cellular automaton of neighborhood $\mathbb{U} = [r,s]$. Let P be the set of (σ,F)-periodic points. Then the set $\{\widetilde{\delta_x} : x \in P\}$ is dense in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}), d_C^{\mathcal{M}})$. According to example 1.1, we recall that:*

$$\widetilde{\delta_x} = \frac{1}{T}\sum_{i=0}^{T-1} \delta_{\sigma^i(x)} \qquad \textit{where } T \textit{ is the } \sigma\textit{-period of } x.$$

Proof. Let $M = \mathrm{Card}(\ker(F))$. A ponit x has M preimages under F. Fix any prime $q > M$, then F cannot map a point of least σ-period q to a point of lower period.

Let Per_q be the set of σ-periodic points with σ-period equal to q. This set consists of $p^q - p$ points of least σ-period q and p σ-fixed points. The restriction of F to the subgroup Per_q is an homomorphism which maps the fixed points to the fixed points and the points of least period q to the points of least period q. Moreover $F^2(Per_q) = F(Per_q)$, so one deduces that the points of $F(Per_q)$ are also F-periodic.

Let $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ and let $\varepsilon > 0$. Let $N \in \mathbb{N}$ such that $\sum_{n>N} 1/2^n < \varepsilon/2$. There exist q prime and x a σ-periodic point of σ-period $q - s + r$ such that $|f(u,x) - \mu([u]_{[0,|u|-1]})| < \varepsilon/2$ for all $u \in \mathcal{A}^n$ with $n \leq N$. We recall that $f(u,x)$ is the frequency of the pattern u in x and this limit is well defined since x is periodic. So one has $d_*(\mu, \delta_x) < \varepsilon$.

Since F is bipermutative, there exist $y \in Per_q$ such that $F(y)_{[0,q-s+r-1]} = x_{[0,q-s+r-1]}$. Let $x' = F(y)$, x' is a (σ, F) periodic point thus $\widetilde{\delta_{x'}}$ is F_*-periodic. Moreover, if q is chose sufficiently large, then $d_*(\widetilde{\delta_x}, \widetilde{\delta_{x'}}) < \varepsilon$. One deduces that $\{\widetilde{\delta_x} : x \in P\}$ is dense in $(\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}), d_C^{\mathcal{M}})$. □

Theorem 4.13. *Let $(\mathcal{A}^{\mathbb{M}}, F)$ be a non trivial linear CA. Then $Eq_{d_*}(F_*) = \emptyset$.*

Proof. Since $(\mathcal{A}^{\mathbb{M}}, F)$ is a non trivial linear CA, there exists p prime and a surjective endomorphism $\pi : \mathcal{A} \to \mathbb{Z}/p\mathbb{Z}$ such that the factor CA $\pi \circ F$ (where π is extended coordinate to coordinate) is a linear CA on $\mathbb{Z}/p\mathbb{Z}$ which is non trivial. Since π_* is open, it is sufficient to prove the sensitivity for this factor of F_*. Thus we assume that we are in this case.

Let $((\mathbb{Z}/p\mathbb{Z})^{\mathbb{M}}, F)$ be a nontrivial linear CA with p prime. In [PY02], they show that there is a weak* dense set in $\mathcal{M}(\mathcal{A}^{\mathbb{M}})$, the harmonic measures set denoted $\mathcal{H}$, such that every measure $\mu \in \mathcal{H}$ verifies

$$\lim_{n\in\mathbb{J}\to\infty} d_*(F_*^n\mu, \lambda_{\mathcal{A}^{\mathbb{M}}}) = 0,$$

where $\lambda_{\mathcal{A}^{\mathbb{M}}}$ is the uniform Bernoulli measure and $\mathbb{J}$ is a set a subset of $\mathbb{N}$ of upper density 1.

Let P be the set of (σ, F)-periodic points. According to Lemma 4.12, the set $\{\widetilde{\delta_x} : x \in P\}$ is weak* dense in $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$.

Let $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{M}})$ such that $\mu \neq \lambda_{\mathcal{A}^{\mathbb{M}}}$ and let $\varepsilon < \frac{1}{2}d_*(\lambda_{\mathcal{A}^{\mathbb{M}}}, \mu)$. For all $\delta < \varepsilon$. There exists $\mu' \in \mathcal{H}$ and $x \in P$ such that $d_*(\mu, \mu') < \delta$ and $d_*(\mu, \widetilde{\delta_x}) < \delta$. Thus one has

$$\lim_{n\in\mathbb{J}\to\infty} d_*(F_*^n\mu', \lambda_{\mathcal{A}^{\mathbb{M}}}) = 0,$$

where $\mathbb{J}$ is a subset of $\mathbb{N}$ of density 1. Moreover, if T is the F-period of x, one has $F_*^{Tn}\widetilde{\delta_x} = \widetilde{\delta_x}$ for all $n \in \mathbb{N}$. Since $\mathcal{J}$ is upper density 1, there exists $n \in \mathbb{N}$ such that $Tn \in \mathcal{J}$ and $d_*(F_*^{Tn}\mu', \lambda_{\mathcal{A}^{\mathbb{M}}}) < \varepsilon$. Thus one has $d_*(F_*^{Tn}\widetilde{\delta_x}, F_*^{Tn}\mu') > \varepsilon$. One deduces that $\mu \notin Eq_{d_*}(F_*)$.

If $\mu = \lambda_{\mathcal{A}^{\mathbb{M}}}$ there exists μ' such that the sequence $(F_*^n\mu')_{n\in\mathcal{A}^{\mathbb{M}}}$ has two adherence points and one of them is $\mu = \lambda_{\mathcal{A}^{\mathbb{M}}}$ [PY02]. Thus (μ, μ') is a Li-York pair ($\mu = \lambda_{\mathcal{A}^{\mathbb{M}}}$ is F_*-invariant), so $\mu \notin Eq_{d_*}(F_*)$.

Thus $Eq_{d_*}(F_*) = \emptyset$, but this method do not allow to obtain an uniform sensitive constant. There is a problem around of $\lambda_{\mathcal{A}^{\mathbb{M}}}$. □

References

[BCF03] François Blanchard, Julien Cervelle, and Enrico Formenti. Periodicity and transitivity for cellular automata in Besicovitch topologies. In *Mathematical foundations of computer science 2003*, volume 2747 of *Lecture Notes in Comput. Sci.*, pages 228–238. Springer, Berlin, 2003.

[BFK97] François Blanchard, Enrico Formenti, and Petr Kůrka. Cellular automata in the Cantor, Besicovitch, and Weyl topological spaces. *Complex Systems*, 11(2):107–123, 1997.

[BK99] Mike Boyle and Bruce Kitchens. Periodic points for onto cellular automata. *Indag. Math. (N.S.)*, 10(4):483–493, 1999.

[Bru82] A. A. Brudno. Entropy and the complexity of the trajectories of a dynamic system. *Trudy Moskov. Mat. Obshch.*, 44:124–149, 1982.

[CFMM97] Gianpiero Cattaneo, Enrico Formenti, Luciano Margara, and Jacques Mazoyer. A shift-invariant metric on $S^{\mathbf{Z}}$ inducing a non-trivial topology. In *Mathematical foundations of computer science 1997 (Bratislava)*, volume 1295 of *Lecture Notes in Comput. Sci.*, pages 179–188. Springer, Berlin, 1997.

[DGS76] Manfred Denker, Christian Grillenberger, and Karl Sigmund. *Ergodic theory on compact spaces*. Springer-Verlag, Berlin, 1976. Lecture Notes in Mathematics, Vol. 527.

[FMMN00] Pablo A. Ferrari, Alejandro Maass, Servet Martínez, and Peter Ney. Cesàro mean distribution of group automata starting from measures with summable decay. *Ergodic Theory Dynam. Systems*, 20(6):1657–1670, 2000.

[Gla03] Eli Glasner. *Ergodic theory via joinings*, volume 101 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.

[Hed69] Gustav A. Hedlund. Endormorphisms and automorphisms of the shift dynamical system. *Math. Systems Theory*, 3:320–375, 1969.

[Kůr97] Petr Kůrka. Languages, equicontinuity and attractors in cellular automata. *Ergodic Theory Dynam. Systems*, 17(2):417–433, 1997.

[Kůr03] Petr Kůrka. Cellular automata with vanishing particles. *Fund. Inform.*, 58(3-4):203–221, 2003.

[Lin84] Douglas A. Lind. Applications of ergodic theory and sofic systems to cellular automata. *Phys. D*, 10(1-2):36–44, 1984. Cellular automata (Los Alamos, N.M., 1983).

[MM98] Alejandro Maass and Servet Martínez. On Cesàro limit distribution of a class of permutative cellular automata. *J. Statist. Phys.*, 90(1-2):435–452, 1998.

[PY02] Marcus Pivato and Reem Yassawi. Limit measures for affine cellular automata. *Ergodic Theory Dynam. Systems*, 22(4):1269–1287, 2002.

[PY04] Marcus Pivato and Reem Yassawi. Limit measures for affine cellular automata. II. *Ergodic Theory Dynam. Systems*, 24(6):1961–1980, 2004.

[Sab06] Mathieu Sablik. Directional dynamic for cellular automata: A sensitivity to initial condition approach. *Prépublication de l'UMPA*, September 2006.

[Ver04] A. M. Vershik. The Kantorovich metric: the initial history and little-known applications. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 312(Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 11):69–85, 311, 2004.

LATP, Aix-Marseille Université, 39, rue F. Joliot Curie,13453 Marseille Cedex 13, France
E-mail address: sablik@latp.univ-mrs.fr
URL: http://www.latp.univ-mrs.fr/∼sablik/