Projection Theorems using Effective Dimension

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Fractal Geometry

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[1] Wikipedia.org

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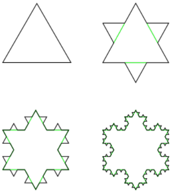

1

- von Koch snowflake is small with respect to area (zero area)
- Yet it is large with respect to length (infinite length)

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Fractal Geometry and Fractal Dimensions

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  - packing dimension

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  - The fractal dimension of a line is 1, the dimension of a plane is 2, etc.
  - The Hausdorff (and packing) dimension of the von Koch snowflake is $\frac{\ln 4}{\ln 3}$.
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- Fractal geometry has become important in a number of different fields.
- Fractal geometry uses techniques from many areas of mathematics.
  - Combinatorics, classical geometry, Fourier analysis,...

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Kolmogorov Complexity

Definition

Fix a universal Turing machine U. Let u be a finite binary string. The *Kolmogorov complexity of u* is

$$K(u) = \min\{|\pi| \mid \pi \in \{0,1\}^*, \text{ and } U(\pi) = u\}.$$

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$$K(u) = \min\{|\pi| \mid \pi \in \{0,1\}^*, \text{ and } U(\pi) = u\}.$$

- The choice of universal TM is irrelevant.
- Can be extended to $\mathbb{N}$ and $\mathbb{Q}$ in a natural way.
- Can be relativized to an oracle $A \subseteq \mathbb{N}$, written as $K^A(u)$.

Kolmogorov Complexity in Euclidean Space

Definition

Let $n, r \in \mathbb{N}$, and $x \in \mathbb{R}^n$. The *Kolmogorov complexity of* x *at precision* r is

$$K_r(x) = \min\{K(q) \mid q \in B_{2^{-r}}(x) \cap \mathbb{Q}^n\},$$

where $B_{2^{-r}}(x)$ is the ball of radius 2^{-r} around x.

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For our purposes today, we may define

$$K_r(x) = K(u),$$

where $u = x{\restriction}r$ is the first nr bits in the binary representation of x.

Effective Dimensions of Points

Definition (Mayordomo '03)

Let $n \in \mathbb{N}$, and $x \in \mathbb{R}^n$.
The *(effective Hausdorff) dimension of* x is

$$\dim(x) = \liminf_{r \to \infty} \frac{K_r(x)}{r}.$$

Definition (Athreya et al. '07, Lutz and Mayordomo '08)

Let $n \in \mathbb{N}$, and $x \in \mathbb{R}^n$. The *(effective) strong dimension of* x is

$$\text{Dim}(x) = \limsup_{r \to \infty} \frac{K_r(x)}{r}.$$

The effective dimensions of a point x measure the density of algorithmic information in x.

The Point-to-Set Principle

Theorem (J. Lutz and N. Lutz, '16)

For every set $E \subseteq \mathbb{R}^n$,

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in S} \dim^A(x), \text{ and}$$

$$\dim_P(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in S} \mathrm{Dim}^A(x).$$

- The Hausdorff dimension of a *set* is characterized by the dimension of the *points* in the set.
- Allows us to use computability to answer questions in fractal geometry.

Some Successes

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- Neil Lutz showed that the intersection bound holds for *every* subsets $A, B \subseteq \mathbb{R}^n$ holds. For every A, B and almost every point z,

$$\dim_H(A \cap (B + z)) \leq \max\{0, \dim_H(A \times B) - n.$$

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$$\dim_H(A \cap (B + z)) \leq \max\{0, \dim_H(A \times B) - n.$$
- N. Lutz and S. improved theorem of Molter and Rela on the lower bounds on the Hausdorff dimensions of Furstenberg sets.

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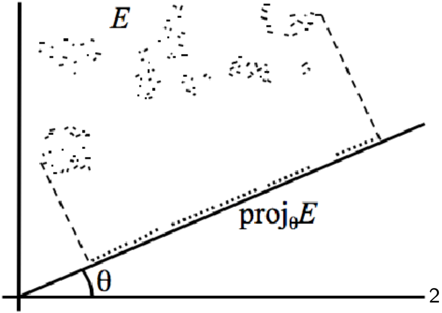
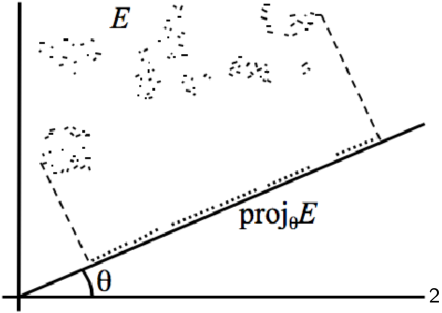
Orthogonal Projections



[2] Kenneth Falconer, Sixty years of fractal projections

Orthogonal Projections



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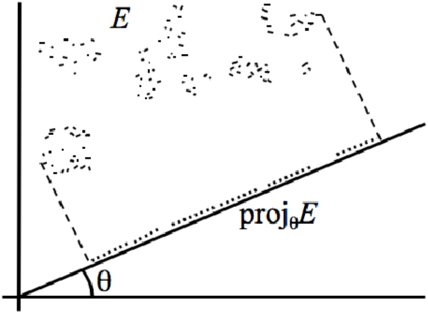

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$$\dim_H(\text{proj}_\theta E) \leq \min\{\dim_H(E), 1\}.$$

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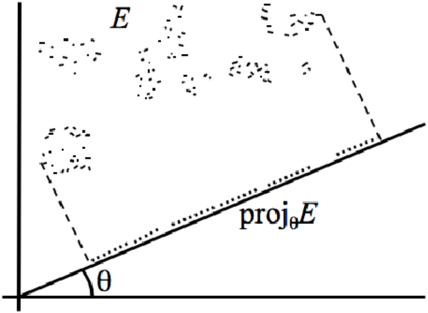

If E is big, is it true that $\text{proj}_\theta E$ is big?

- A projection is Lipschitz continuous, so

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- Known that there are sets E such that this inequality is strict.

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Marstrands Projection Theorem

Theorem (Marstrand '54)

Let $E \subseteq \mathbb{R}^2$ *be an analytic set with* $\dim_H(E) = s$*. Then for almost every* $\theta \in (0, 2\pi)$,

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- Mattila generalized this to arbitrary n.
- This theorem is now recognized as a fundamental theorem of fractal geometry.
- Active area of research investigating the projections specific classes of fractal sets.
- Davies has shown that, assuming the Continuum Hypothesis, there are sets for which this theorem does not hold.

Our Results

We use algorithmic information theory to reprove Marstrands theorem, and prove two new results on the fractal dimension of projections.

Theorem (N. Lutz and S. '17)

Let $E \subseteq \mathbb{R}^2$ *be any set with* $\dim_H(E) = \dim_P(E) = s$. *Then for almost every* $\theta \in (0, 2\pi)$,

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Overview of Proof

Our goal is to give lower bounds on the fractal dimension of the projection of a set E onto the line at angle θ.

- We will first focus on the *effective* dimension of projected *points*.
  - We will use the point-to-set principle to connect this to our goal.

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- We will first focus on the *effective* dimension of projected *points*.
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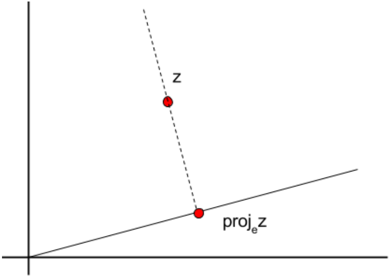
- To prove lower bounds of the effective dimension of a point, we will prove lower bounds of the *complexity* of the point *at every precision*.

Overview of Proof

Our goal is to give lower bounds on the fractal dimension of the projection of a set E onto the line at angle θ.

- We will first focus on the *effective* dimension of projected *points*.
  - We will use the point-to-set principle to connect this to our goal.
- To prove lower bounds of the effective dimension of a point, we will prove lower bounds of the *complexity* of the point *at every precision*.
- It will suffice to show that, for sufficiently nice $z \in \mathbb{R}^n$ and angle θ,
$$K_r^{\theta}(z) \leq K_r(\text{proj}_{\theta}(z)).$$

Lower Bounds on Complexity of Projections



Our goal is to show that we can compute our original point z given the projected point $\text{proj}_\theta(z)$,

$$K_r^\theta(z) \leq K_r(\text{proj}_\theta(z)).$$

- How can decide which z is the correct point? There are infinitely many of them.

Lower Bounds on Complexity of Projections

Suppose that the following conditions are satisfied.

1. The complexity of z, $K_r(z)$, is small.
2. For every point w such that $\text{proj}_\theta(z) = \text{proj}_\theta(w)$, either
   - the complexity of w, $K_r(w)$, is large, or
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Then we can compute (an approximation of) z given (an approximation of) the projected point $\text{proj}_\theta(z)$, with some small number of bits, i.e.

$$K_r^\theta(z) \lessapprox K_r(\text{proj}_\theta(z)).$$

Bridging Theorem

Theorem

Let $z \in \mathbb{R}^2$, $\theta \in (0, 2\pi)$, $A \subseteq \mathbb{N}$, and $r \in \mathbb{N}$. Assume the following are satisfied.

1. *For every $t \leq r$, $K^z_t(\theta) \geq t - O(\log(t))$.*
2. *$K^{A,\theta}_r(z) \gtrapprox K_r(z)$.*

Then,

$$K^{A,\theta}_r(\text{proj}_\theta(z)) \gtrapprox K_r(z)\,.$$

Intuitively, this theorem states that if

- the complexity of θ is maximal, and
- the oracle A and angle θ do not affect the complexity of z,

then we can ensure that the sufficient conditions of the previous slide are satisfied.

New Proof of Marstrands Theorem

Theorem (Marstrand '54)

Let $E \subseteq \mathbb{R}^2$ be an analytic set with $\dim_H(E) = s$. Then for almost every $\theta \in (0, 2\pi)$,

$$\dim_H(\text{proj}_\theta E) = \min\{s, 1\}.$$

By the point to set principle, it suffices to show that, for almost every θ, for every oracle $A \subseteq \mathbb{N}$, and every $\epsilon > 0$, there is a point $z \in E$ such that

$$\dim^A(\text{proj}_\theta z) \geq \min\{s, 1\} - \epsilon.$$

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- To use the bridging theorem, we want to pick a θ which has maximal complexity.
- Then, for any $A \subseteq \mathbb{N}$ and $\epsilon > 0$, we need to pick a z so that (A, θ) does not affect the complexity of z.
  - This is the tricky part.

Using Restricted Point-to-Set Principle

Theorem (Hitchcock '03)

Let $E \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{N}$ be such that E is a Σ^0_2 set relative to B. Then

$$\dim_H(E) = \sup_{x \in E} \dim^B(x) \, .$$

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- This restricted version allows us to eliminate a quantifier (the choice of oracle).
- Standard arguments show that if E is analytic, then there is a subset $F \subseteq E$ such that
    - $\dim_H(F) = \dim_H(E)$, and
    - For some oracle B, F is Σ^0_2 relative to A.

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    - $\dim_H(F) = \dim_H(E)$, and
    - For some oracle B, F is Σ^0_2 relative to A.
- For any such F, and any θ, $\text{proj}_\theta F$ is Σ^0_2 relative to (B, θ).

New Proof of Marstrands Theorem

Let $F \subseteq E$ as in the previous slide, and $B \subseteq \mathbb{N}$ such that F is Σ^2_0 relative to B. It suffices to show that, for almost every θ and every $\epsilon > 0$, there is a point $z \in E$ such that

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- *First* pick $z_1, z_2, \ldots$: Using the point to set principle, choose z_n such that $\dim^B(z_n) \geq s - 1/n$.

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  - For every n, $\dim^{B,z_n}(\theta) = 1$ (standard argument), and
  - For every n and r, $K^{B,\theta}_r(z_n) = K^B_r(z_n)$ (by a theorem of Calude and Zimand).

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    - For every n and r, $K^{B,\theta}_r(z_n) = K^B_r(z_n)$ (by a theorem of Calude and Zimand).
- Then the conditions of our bridging theorem are satisfied for all sufficiently large r, and therefore

$$\begin{aligned}\dim^{B,\theta}(\text{proj}_\theta(z_n)) &= \liminf_{r\to\infty} \frac{K^{B,\theta}_r(\text{proj}_\theta(z_n))}{r} \\ &\geq \liminf_{r\to\infty} \frac{K^B_r(z_n)}{r}\end{aligned}$$

Projections of Non-analytic Sets

Theorem (N. Lutz and S. '17)

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- To use the bridging theorem, we want to pick a θ such that $\dim(\theta) = 1$.
  - Almost every θ satisfies this property.
- Then, for any $A \subseteq \mathbb{N}$ and $\epsilon > 0$, we need to pick a z such that (A, θ) does not affect the complexity of z.
  - The assumption that $\dim_H(E) = \dim_P(E)$ allows us to do this without needing the existence of nice subsets of E.

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- Pick a θ which has maximal complexity.
- Then, for any $A \subseteq \mathbb{N}$ and $\epsilon > 0$, we need to pick a z so that (A, θ) does not affect the complexity of z, $K_r(z)$, *for infinitely many r*.
    - This follows by the point to set principle.

The End

Thank you!