Comput-, predict- and decidability in symbolic dynamical systems

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Algorithmic questions in dynamical systems

LRI, Université Paris Sud

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Two types of models of computation:

- **Functional models**, e.g. recursive functions, λ-calculus;
- **Dynamical models**, e.g. Turing machines, counter machines, cellular automata.

Classical dynamical models are **symbolic dynamical systems**: continuous transformations of $\mathcal{A}^{\mathbb{N}}$ or $\mathcal{A}^{\mathbb{Z}}$ (or a subset thereof) with the Cantor topology, where $\mathcal{A}$ is a finite alphabet.

Dynamical models of computations

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From here:

Which dynamical systems are models of computation?

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- It is able of "robust" computation, i.e. on random points or subjected to noise.

Definition

The **shift action** is defined as $\sigma(x)_i = x_{i+1}$.

Dynamical models of computation

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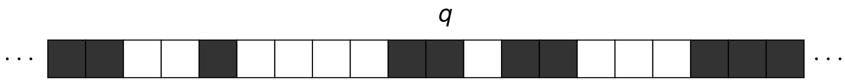
Standard (moving-tape) **Turing machine**

Given two finite set of states Q and symbols $\mathcal{A}$ and a decision function

$$\delta : Q \times \mathcal{A} \mapsto Q \times \mathcal{A} \times \{-1, 0, 1\},$$

For $q, x \in Q \times \mathcal{A}^{\mathbb{Z}}$, $TM(q, x)$ is determined as follows:

- The new state is $\delta_1(q, x_0)$;
- The new configuration is obtained by replacing x_0 by $\delta_2(q, x_0)$ and shifting by $\delta_3(q, x_0)$.



$$\delta(q, \blacksquare) = (q', \square, +1)$$

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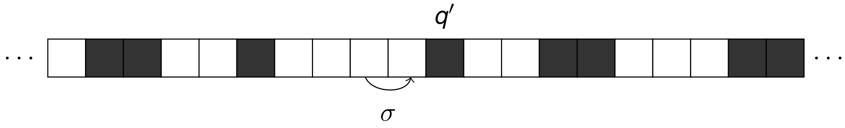

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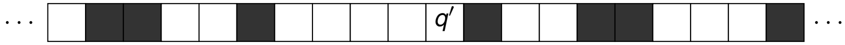
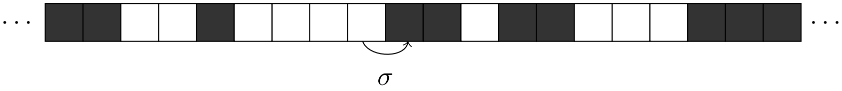


Cellular automata

A cellular automaton is a **continuous, σ-commuting** transformation of $\mathcal{A}^{\mathbb{Z}}$.

Equivalently, it can be defined by a local rule over a finite window:



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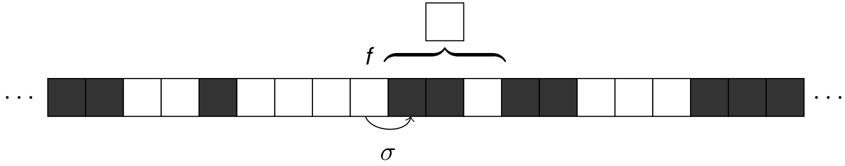

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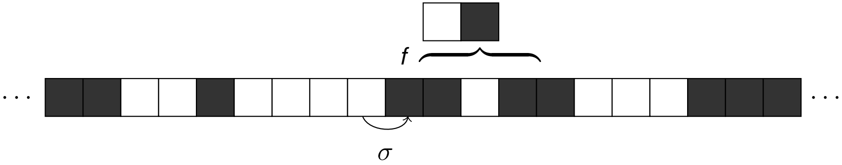

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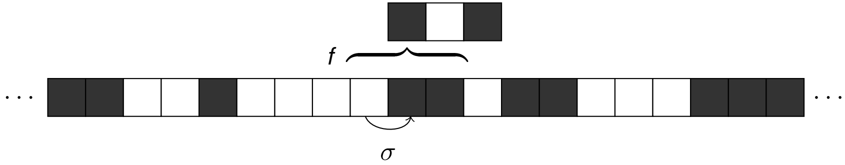

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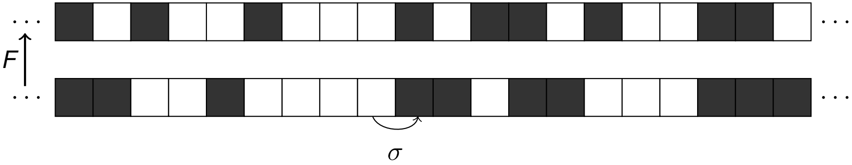

Halting problem

Input An finite word;

Output Does the Turing machine eventually halt on this input?

Theorem (Turing 36)

The halting problem for Turing machines is undecidable.

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Any nontrivial property on the set of input words on which the Turing machine halts is undecidable.

Point-to-set reachability problem for Turing machines

Input An finite point x;

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The point-to-set reachability problem on finite points for Turing machines is undecidable.

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The point-to-set reachability problem on finite points for Turing machines is Σ_1^0-complete.

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Input/output discussion

Universal system The input is a description of the initial point

Universal class The input is a description of the system + the initial point.

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Choice of the input family

- Models of computation work on a **countable** family of states (usual choices: **finite, σ-periodic, almost σ-periodic**);
- Choosing an arbitrary countable family may lead to absurd notions (example later).

Definition (Davis 56)

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Theorem (Cook 00)

The point-to-point reachability problem on almost periodic points for **Rule 110** is Σ_1^0-complete.

(point-to-set and Rice-style theorem follow)

Can we generalize this intuition to an arbitrary countable dense family (Hemmerling 02)?

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Counterexample (Durand, Róka 99)

Take the full shift $(\{0,1\}^{\mathbb{Z}}, \sigma)$ and the set of computable points as input states. Fix:

$$\{x_n = 1^n 0^t 1^\infty : \text{the } n\text{-th TM stops in } t \text{ steps}\} \quad (\text{possibly } t = \infty)$$

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Another problem

The infinite precision on the choice of the initial input might be unphysical and lead to non-robust undecidability results.

A definition in the spirit of set-to-set reachability.

Universality à la Delvenne - Kurka - Blondel

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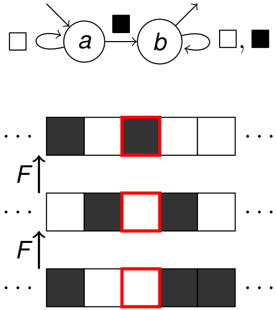

Model-checking problem (on the trace)

Input A finite automaton labelled by elements of $\mathcal{A}$;

Output Does the automaton reach a halting state for some trajectory?

Equivalently: is the intersection of the trace language w/ some regular language empty?

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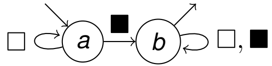

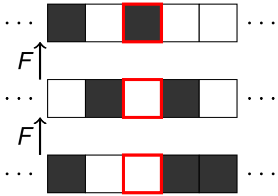

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Definition (Delvenne, Kurka, Blondel 05)

A system is Turing-universal if its model-checking problem is Σ_1^0-complete.

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A problem is **P**-complete if any problem in **P** can be reduced to it using logarithmic space.

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Proposition

The bounded-time point-to-set reachability problem for Turing machines:

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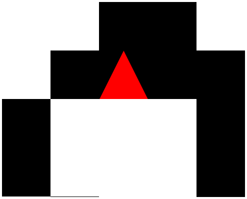
Theorem (Neary, Woods 06)

The bounded-time, point-to-set reachability problem for almost periodic input points in the **rule 110 cellular automaton** is **P**-complete.

A **Turmite** is a two-dimensonal Turing machine on the alphabet $\mathcal{A} = \mathbb{Z}/n\mathbb{Z}$ defined by a rule $\mathcal{A} \to \{R, L\}$.

At each step:

- it increases x_0 by one;
- it shifts the configuration by $(-1, 0)$;
- it rotates the configuration by $\pm 90^o$ (depending on the rule).



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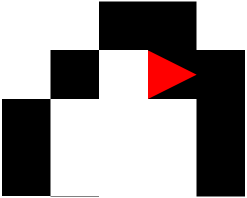

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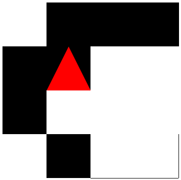

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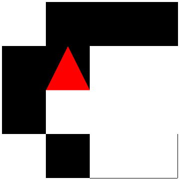

Theorem (Maldonado, Gajardo, H., Moreira 18)

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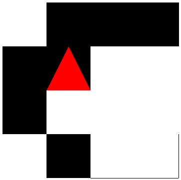

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Conjecture (Langton)

For the rule RL (at least), the unbounded-time reachability problem on finite points is decidable.

Definition?

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Corollary (Rice-style theorem)

Input: A finite point x

Output: Does $F^t(x)$ has a limit point that satisfy property $\mathcal{P}$?

is **undecidable** for any property $\mathcal{P}$.

Definition?

The probability measures that can be reached as the single limit of $\mu \circ TM^{-t}$ for some Turing machine *TM* and initial uniform Bernoulli measure μ are the Δ_2^0-computable measures.

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Theorem (H., Sablik 14; Delacourt, H. 16)

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(+ set version and Rice-style theorem)

The **topological entropy** of a Turing machine is defined as

$$\lim_{n\to\infty} \frac{\log \#\mathcal{L}_n}{n},$$

where $\mathcal{L}_n$ is the set of n-letter words appearing in its trace.

Cautionary tale III

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Approximating the entropy of Turing machines:

Input: A Turing machine TM and a precision $n \in \mathbb{N}$

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Theorem (Jeandel 13)

The entropy of **one-tape** Turing machines is computable.

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- Many topics left uncovered: limit set / attractor point of view, dynamical properties, robustness to noise, other universalities. . .

Questions and remarks are appreciated!