On the Lebesgue measure of the Feigenbaum Julia set

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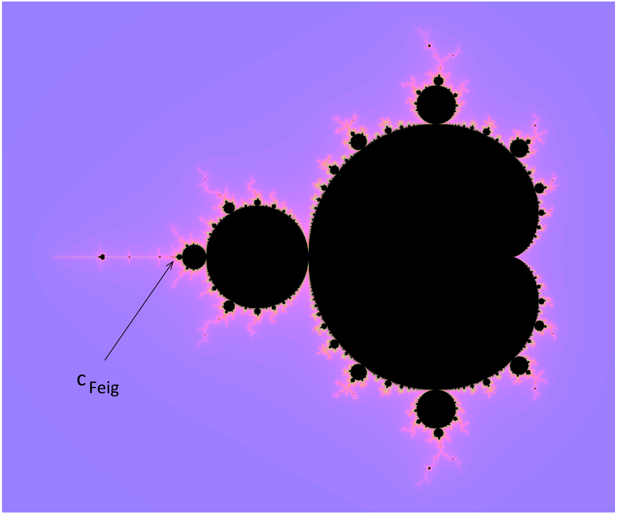

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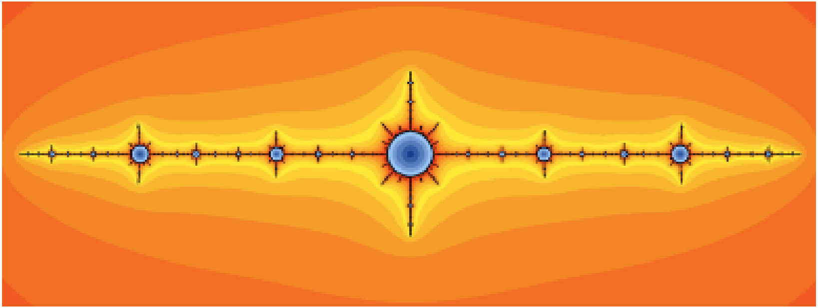

Figure: The airplane map $p(z) = z^2 + c$, $c \approx -1.755$.

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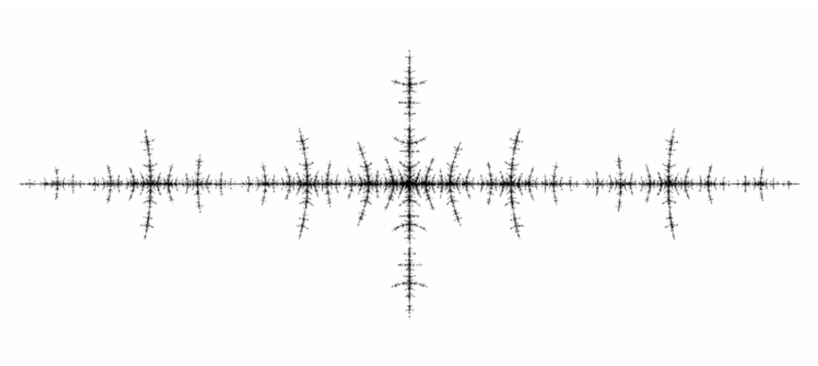

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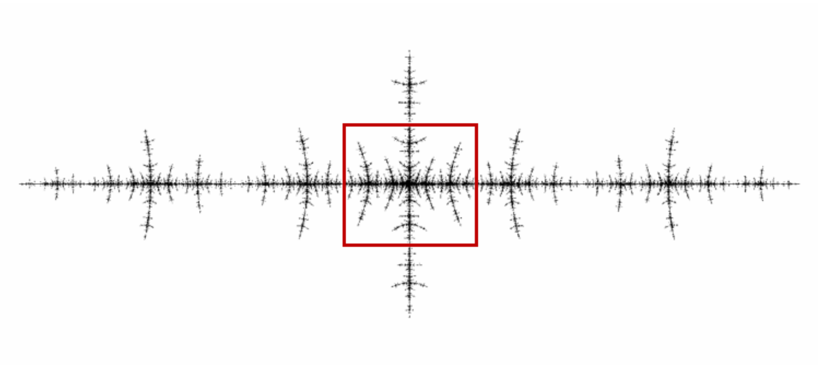

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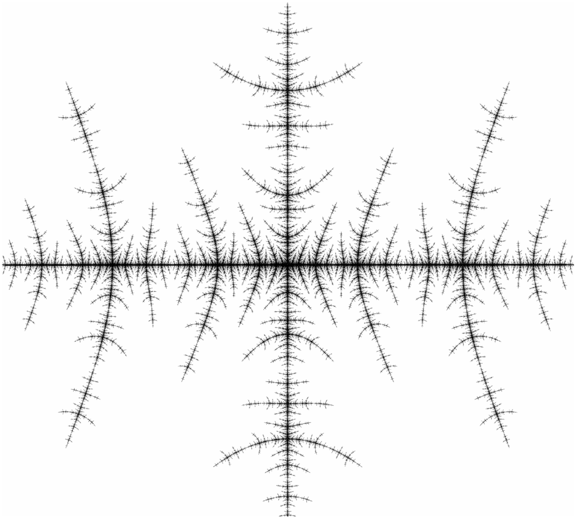

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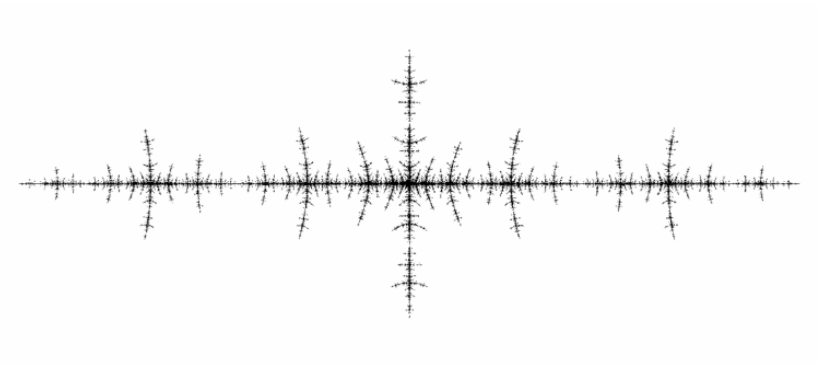

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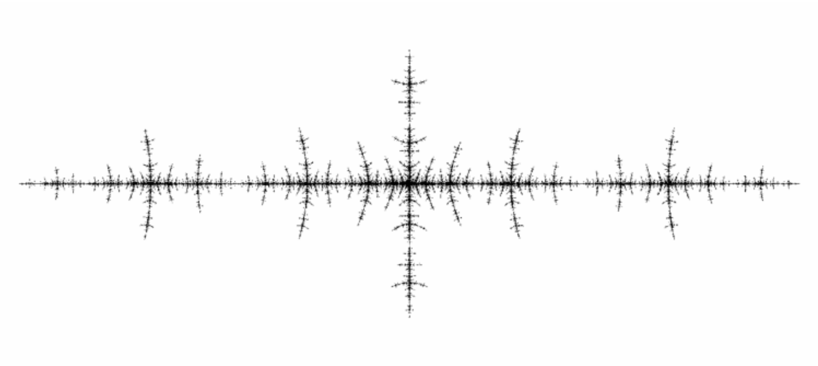

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Theorem (D.-Sutherland)

The Julia set of f_{Feig} has Hausdorff dimension less than two (and hence its Lebesgue measure is zero).

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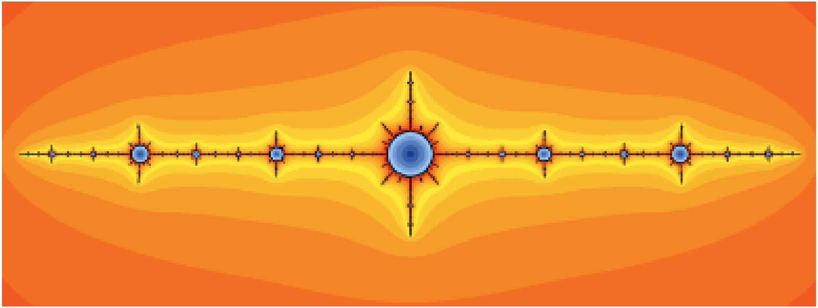

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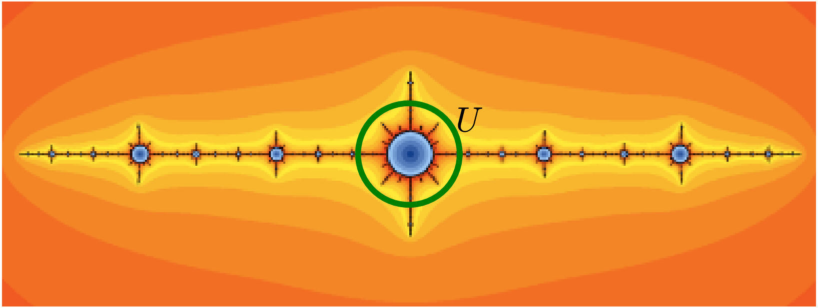

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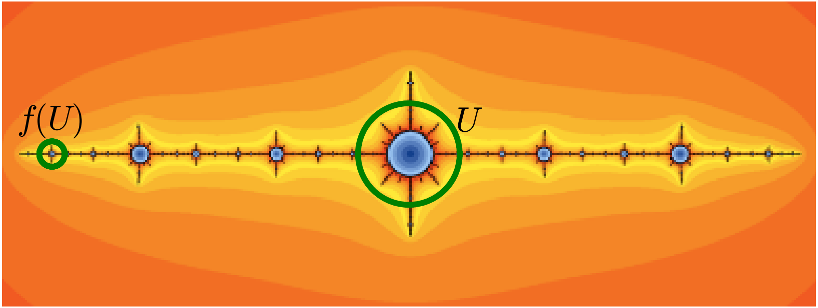

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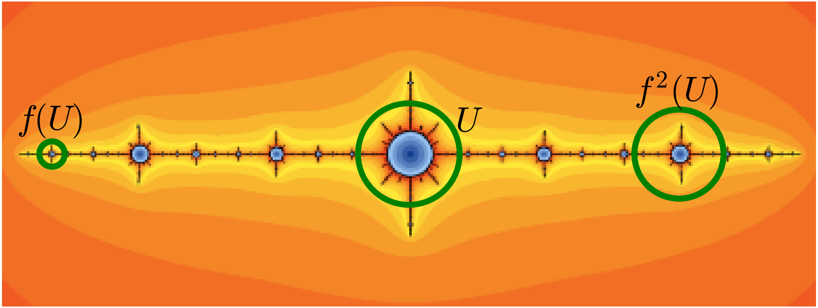

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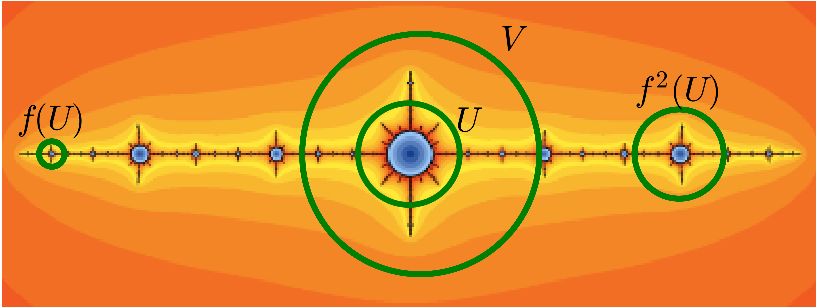

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Definition

A Feigenbaum map is an infinitely renormalizable quadratic-like map with bounded combinatorics and a priori bounds.

Nice domains

Denote by f_n the n-th prerenormalization of f, by $\mathcal{J}_n$ the Julia set of f_n and by $\mathcal{O}(f)$ the critical orbit of f.

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Avila and Lyubich constructed domains $U^n \subset V^n$ (called *nice domains*) for which

- $f_n(U^n) = V^n$;
- $U^n \supset \mathcal{J}_n \cap \mathcal{O}(f)$;
- $V^{n+1} \subset U^n$;
- $f^k(\partial V^n) \cap V^n = \emptyset$ for all n, k;
- $A^n = V^n \setminus U^n$ is “far” from $\mathcal{O}(f)$;
- $\mathrm{area}(A^n) \asymp \mathrm{area}(U^n) \asymp \mathrm{diam}(U^n)^2 \asymp \mathrm{diam}(V^n)^2$.

Escaping and returning set

For each $n \in \mathbb{N}$, let X_n be the set of points in U^0 that land in V^n under some iterate of f, and let Y_n be the set of points in A^n that never return to V^n under iterates of f. Introduce the quantities

$$\eta_n = \frac{\mathrm{area}(X_n)}{\mathrm{area}(U^0)}, \quad \xi_n = \frac{\mathrm{area}(Y_n)}{\mathrm{area}(A^n)}.$$

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Let f be a periodic point of renormalization ($\mathcal{R}^p f = f$ for some p). Then exactly one of the following is true:

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- $\inf \eta_n > 0$, ξ_n *converges to* 0 *exponentially fast, and* $\mathrm{area}(\mathcal{J}_f) > 0$ *(Black Hole case).*

An approach to prove $\dim_{\mathrm{H}}(\mathcal{J}_{\mathrm{Feig}}) < 2$.

One can construct a number $C > 0$ (depending on the geometry of A^n and the critical orbit $\mathcal{O}(f)$) such that if $\eta_n/\xi_n < C$ for some n then $\dim_{\mathrm{H}}(\mathcal{J}_{\mathrm{Feig}}) < 2$. But ...

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need a different approach.

The structure of F.

The Cvitanović-Feigenbaum equation:

$$\begin{cases} F(z) & = & -\frac{1}{\lambda}F^2(\lambda z), \\ F(0) & = & 1, \\ F(z) & = & H(z^2), \end{cases}$$

with $H^{-1}(z)$ univalent in $\mathbb{C} \setminus ((-\infty, -\frac{1}{\lambda}] \cup [\frac{1}{\lambda^2}, \infty))$, where $\frac{1}{\lambda} = 2.5029\ldots$ is one of the Feigenbaum constants.

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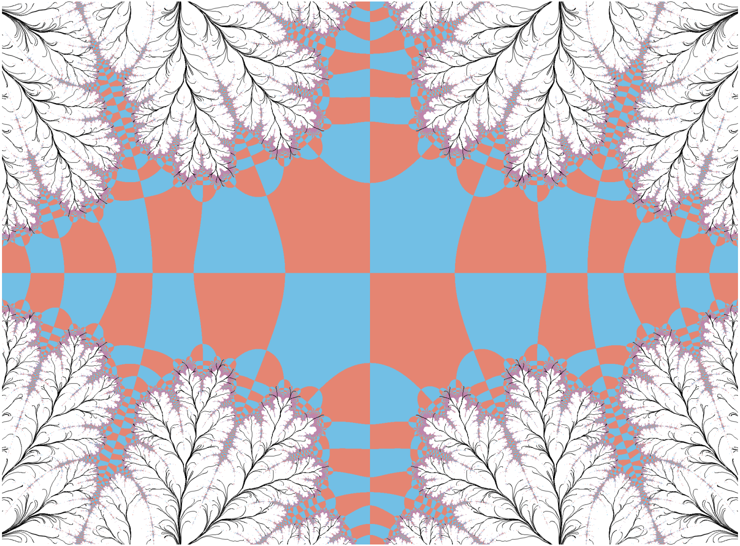
Theorem (H. Epstein, X. Buff)

All critical points of $\hat{F}$ are simple. The critical values of $\hat{F}$ are contained in real axis. Moreover, $\hat{F}$ is a ramified covering.

Partition of $\hat{W}$



Central tiles

Denote by $P_{\mathrm{I}}, P_{\mathrm{II}}, P_{\mathrm{III}}$ and P_{IV} the connected components of $\hat{F}^{-1}(\mathbb{H}_{\pm})$ containing 0 on the boundary. Set

$$W = \mathrm{Int}(\overline{P_{\mathrm{I}} \cup P_{\mathrm{II}} \cup P_{\mathrm{III}} \cup P_{\mathrm{IV}}}).$$

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For $n \in \mathbb{N}$ and any set A let $A^{(n)} = \lambda^n A$ and denote by $F_n = F^{2^n}|_{W^{(n)}}$ the n-th pre-renormalization of F.

The (new) returning sets

$$\tilde{X}_n = \{z \in W^{(1)} : F^k(z) \in W^{(n)} \ \text{ for some } \ k\}, \ \tilde{\eta}_n = \frac{\mathrm{area}(\tilde{X}_n)}{\mathrm{area}(W^{(1)})}.$$

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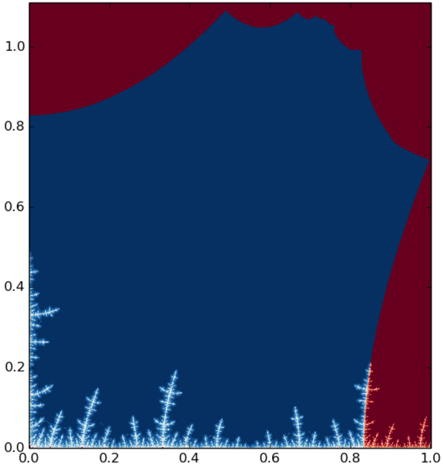

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Using Avila-Lyubich trichotomy we obtain:

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Idea to prove $\tilde{\eta}_n \to 0$: construct recursive estimates of the form

$$\tilde{\eta}_{n+m} \leqslant C_{n,m}\tilde{\eta}_n\tilde{\eta}_m.$$

Copies of central tiles

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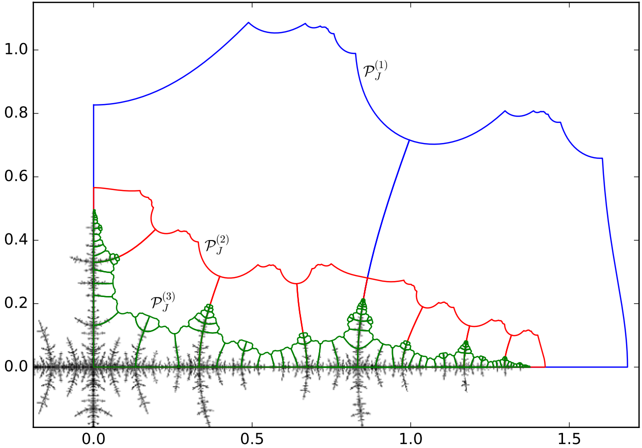

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A copy Q of $P_J^{(m)}$ under F^k is *separated* if there exists $0 \leqslant i < k$ with $F^i(Q) \subset W^{(m)}$ and $F^i(Q) \cap \mathcal{J}_F^{(n-1)} = \emptyset$ for maximal such i.

Comparing the returning sets

Fix $n, m \in \mathbb{N}$. Let $\mathfrak{P}$ and $\mathfrak{S}$ be the collection of all primitive and separated copies of $P_J^{(m)}$, where J is any quadrant. Modulo zero measure one has:

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$$\tilde{X}_{n+m} = \bigcup_{Q \in \mathfrak{P} \cup \mathfrak{S}} X_Q.$$

Koebe space

Proposition

Let T be a primitive or a separated copy of $P_J^{(m)}$ under F^k with $m \geq 2$. Then the inverse branch $\phi : P_J^{(m)} \to T$ of F^k analytically continues to a univalent map on $\operatorname{sign}(P_J^{(m)})\, \lambda^m \mathbb{C}_\lambda$ where

$$\mathbb{C}_\lambda = \mathbb{C} \setminus \left((-\infty, -\tfrac{1}{\lambda}] \cup [\tfrac{F(\lambda)}{\lambda^2}, \infty) \right).$$

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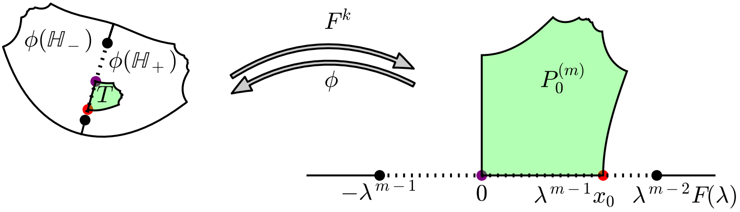
$$\mathbb{C}_\lambda = \mathbb{C} \setminus \left((-\infty, -\tfrac{1}{\lambda}] \cup [\tfrac{F(\lambda)}{\lambda^2}, \infty) \right).$$



Koebe distortion

We construct numbers $M(A)$ such that

Corollary

Let A, B be two measurable subsets of P_J of positive measure and let T be a primitive or a separated copy of $P_L^{(m)}$ under F^k for some $k \geqslant 0$ and $m \geq 2$. Then

$$\frac{\mathrm{area}(F^{-k}(B^{(m)}) \cap T)}{\mathrm{area}(F^{-k}(A^{(m)}) \cap T)} \leq M(A)\mathrm{area}(B).$$

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$$\Sigma_{n,m} = P_{\mathsf{I}}^{(n)} \setminus (\lambda^{n-1}\tilde{X}_{m+1} \cup \bigcup_{Q \in \mathfrak{S}} Q),$$
$$M_{n,m} = M((\lambda^{-n}\Sigma_{n,m}) \cap P_{\mathsf{I}}).$$

Recursive estimates

Using the identities

$$\mathrm{area}(\tilde{X}_n) = \sum_{Q \in \mathfrak{P}} \mathrm{area}(Q),$$
$$\mathrm{area}(\tilde{X}_{n+m}) = \sum_{Q \in \mathfrak{P} \cup \mathfrak{S}} \mathrm{area}(X_Q),$$

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Proposition

For every $n \geqslant 2$ and $m \geqslant 1$, one has

$$\tilde{\eta}_{n+m} \leqslant M_{n,m} \mathrm{area}(P_I)\, \tilde{\eta}_n\, \tilde{\eta}_{m+1}.$$

Results of computations

Using rigorous computer estimates we prove:

$$M_6 = \lim M_{6,m} < 9.4, \qquad \tilde{\eta}_6 = \frac{\mathrm{area}(\tilde{X}_6 \cap P_{\mathsf{I}}^{(1)})}{\mathrm{area}(P_{\mathsf{I}}^{(1)})} < \frac{0.09}{\mathrm{area}(P_{\mathsf{I}})}.$$

We obtain $\tilde{\eta}_6 M_6 \mathrm{area}(P_{\mathsf{I}}) < 0.846 < 1$, so $\mathcal{J}_F$ has Hausdorff dimension less than 2.

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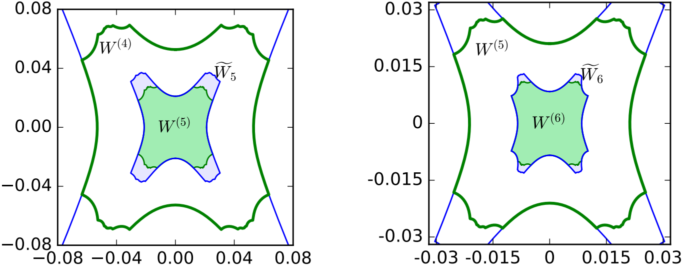

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Lemma

Let D be a disk in the complement of V_2 and let D_0 be a connected component of $F^{-k}(D)$ for any $k \geqslant 0$. Then for $n \geqslant 3$, either $D_0 \cap W^{(n)} = \emptyset$ or $D_0 \subset \widetilde{W}^{(n)}$.

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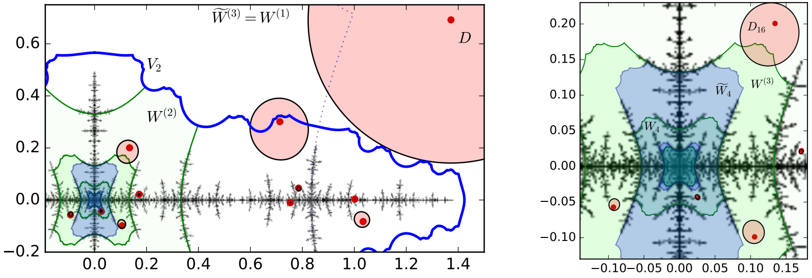

Figure: While the preimage labeled D_{16} partially intersects $W^{(3)}$, it lies completely inside $\widetilde{W}_3 = W^{(1)}$.

Thank you!