Final Exam Correction

Exercise 1

1. For $t \geq 0$, $\mathbb{E}[|\Pi_t|] \leq e^{-mt}\, \mathbb{E}\left[\sum_{u \in \mathcal{N}_t} |X_u(t)|\right]$

$$\overset{\text{many-to-one lemma}}{=} \mathbb{E}[|B_t|] < \infty. \quad \text{So } \Pi_t \in L^1.$$

Then $\mathbb{E}[\Pi_t] = \mathbb{E}[B_t] = 0$.

2. Let $t, s \geq 0$.

$$\Pi_{t+s} = e^{-m(t+s)} \sum_{u \in \mathcal{N}_s} \sum_{\substack{v \in \mathcal{N}_{t+s} \\ v \geq u}} X_v(t+s)$$

$$= e^{-ms} \sum_{u \in \mathcal{N}_s} \left(e^{-mt} \sum_{\substack{v \in \mathcal{N}_{t+s} \\ v \geq u}} \left(X_v(t+s) - X_u(s)\right) + X_u(s)\, e^{-mt} \sum_{\substack{v \in \mathcal{N}_{t+s} \\ v \geq u}} 1 \right)$$

For $u \in \mathcal{N}_s$, we set $\mathcal{N}_t^{u,s} = \{v \in \mathcal{N}_{t+s} : v \geq u\}$ and, for $v \in \mathcal{N}_t^{u,s}$, $X_v^{u,s}(t) = X_v(t+s) - X_u(s)$.

Then $\Pi_{t+s} = e^{-ms} \sum_{u \in \mathcal{N}_s} \left(\underbrace{e^{-mt} \sum_{v \in \mathcal{N}_t^{u,s}} X_v^{u,s}(t)}_{=:\, \Pi_t^{u,s}} + X_u(s)\, \underbrace{e^{-mt} \# \mathcal{N}_t^{u,s}}_{=:\, W_t^{u,s}} \right)$

By the branching property, conditionally on $\mathcal{F}_s$, the processes $(X_v^u(t), v \in \mathcal{N}_t^u)_{t \geq 0}$ for $u \in \mathcal{N}_s$ are independent and have the same law as a BBM.

So, conditionally on $\mathcal{F}_s$, the processes $(\Pi_t^{u,s}, W_t^{u,s})_{t \geq 0}$ for $u \in \mathcal{N}_s$ are independent and have the same law $(\Pi_t, W_t)_{t \geq 0}$.

3. First note that, for $t \geq 0$, $\Pi_t \in L^1$ and Π_t is $\mathcal{F}_t$-measurable.

Let $t, s \geq 0$, by question 2:

So $\mathbb{E}[\Pi_{t+s} | \mathcal{F}_s] = e^{-ms} \sum_{u \in \mathcal{N}_s} \left(\underbrace{\mathbb{E}[\Pi_t^{u,s} | \mathcal{F}_s]}_{= \mathbb{E}[\Pi_t] = 0 \text{ by quest 1.}} + X_u(s)\, \underbrace{\mathbb{E}[W_t^{u,s} | \mathcal{F}_s]}_{= \mathbb{E}[W_t] = 1} \right)$

So $(\Pi_t)_{t \geq 0}$ is an $(\mathcal{F}_t)$-martingale.

4. Since $(M_t)_{t\geq 0}$ is a martingale it is enough to prove it is bounded in L^2.

$$E[M_t^2] = e^{-2mt}\Big(E\Big[\sum_{u\in\mathcal{N}_t} X_u(t)^2\Big] + E\Big[\sum_{\substack{u,v\in\mathcal{N}_t\\ u\neq v}} X_u(t)\,X_v(t)\Big]\Big)$$

many-to-one ... many-to-two

$$= e^{-2mt}\Big(e^{mt}\underbrace{E[B_t^2]}_{=t} + E[L(L-1)]\int_0^t e^{2mt-ms}\underbrace{E[B_t^{1,s}B_t^{2,s}]}_{=s}\,ds\Big)$$

$=s$ (the variance of the common part to the two paths)

$$= \underbrace{e^{-mt}\,t}_{\text{fonction bornée de } t} + E[L(L-1)]\underbrace{\int_0^t e^{-ms}\,s\,ds}_{\leq \int_0^\infty e^{-ms}\,s\,ds<\infty}$$

Thus $(M_t)_{t\geq 0}$ is bounded in L^2, hence it converges a.s. and in L^2.

5. Let $h>0$. On $\{\tau_\emptyset > h\}$ the decomposition of quest. 2 becomes, for $t\geq 0$,

$$M_{t+h} = e^{-mh}\left(M_t^{\emptyset,h} + X_\emptyset(h)\,W_t^{\emptyset,h}\right).$$

Moreover, by quest. 2 again, $(M_t^{\emptyset,h})_{t\geq 0}$ and $(W_t^{\emptyset,h})_{t\geq 0}$ have almost sure limits $M_\infty^{\emptyset,h}$ and $W_\infty^{\emptyset,h}$ and, conditionally on $\mathcal{F}_h$, $(M_\infty^{\emptyset,h}, W_\infty^{\emptyset,h})$ has the same distribution as (M_∞, W_∞).

Then, letting $t\to\infty$ in the equality above we get, a.s.,

$$M_\infty = e^{-mh}\left(M_\infty^{\emptyset,h} + X_\emptyset(h)\,W_\infty^{\emptyset,h}\right).$$

6. Let $t\geq 0$. On the event $\{\tau<\infty\}$ which has probability 1, we decompose

$$M_{\tau_\emptyset+t} = e^{-m(\tau_\emptyset+t)}\sum_{i=1}^{L_\emptyset}\sum_{\substack{u\in\mathcal{N}_{\tau_\emptyset+t}\\ u\geq i}} X_u(\tau_\emptyset+t)$$

$$= e^{-m\tau_\emptyset}\sum_{i=1}^{L_\emptyset}\Big(\underbrace{e^{-mt}\sum_{\substack{u\in\mathcal{N}_{\tau_\emptyset+t}\\ u\geq i}}\big(X_u(\tau_\emptyset+t) - X_\emptyset(\tau_\emptyset)\big)}_{=:M_t^i} + X_\emptyset(\tau_\emptyset)\,e^{-mt}\underbrace{\sum_{\substack{u\in\mathcal{N}_{\tau_\emptyset+t}\\ u\geq i}} 1}_{=:W_t^i}\Big)$$

By the branching property, conditionally on $\mathcal{F}_{\tau_\emptyset}$, the processes $(M_t^i, W_t^i)_{t\geq 0}$ for $1\leq i\leq L_\emptyset$ are independent and have the same law $(M_t, W_t)_{t\geq 0}$.

Writing M_∞^i and W_∞^i for the a.s. limits as $t\to\infty$ we get the result.

7. Let $\alpha, \beta \in \mathbb{R}$ and $h > 0$.

$$\varphi(\alpha,\beta) = \mathbb{E}\left[e^{i\alpha\Pi_\infty + i\beta W_\infty} \mathbb{1}_{\tau_\phi \le h}\right] + \mathbb{E}\left[e^{i\alpha\Pi_\infty + i\beta W_\infty} \mathbb{1}_{\tau_\phi > h}\right]$$

<u>1st term</u>: On $\tau_\phi \le h$, we write (by quest. 6 and the hint)

$$i\alpha\Pi_\infty + i\beta W_\infty = i e^{-m\tau_\phi} \sum_{j=1}^{L_\phi} \left(\alpha \Pi_\infty^j + (\alpha X_\phi(\tau_\phi) + \beta) W_\infty^j\right)$$

Then $\mathbb{E}\left[e^{i\alpha\Pi_\infty + i\beta W_\infty} \mathbb{1}_{\tau_\phi \le h} \middle| \mathcal{F}_{\tau_\phi}\right]$ conditionally on $\mathcal{F}_{\tau_\phi}$, $(\Pi_\infty^j, W_\infty^j)_{1 \le j \le L_\phi}$ are independent

$$= \mathbb{1}_{\tau_\phi \le h} \prod_{j=1}^{L_\phi} \mathbb{E}\left[e^{i\alpha e^{-m\tau_\phi}\Pi_\infty^j + i(\alpha X_\phi(\tau_\phi)+\beta)e^{-m\tau_\phi} W_\infty^j} \middle| \mathcal{F}_{\tau_\phi}\right]$$

$$= \mathbb{1}_{\tau_\phi \le h} \prod_{j=1}^{L_\phi} \varphi\left(\alpha e^{-m\tau_\phi}, (\alpha X_\phi(\tau_\phi)+\beta)e^{-m\tau_\phi}\right)$$

and they have the same Law as (Π_∞, W_∞)

So $\mathbb{E}\left[e^{i\alpha\Pi_\infty + i\beta W_\infty^\circ} \mathbb{1}_{\tau_\phi \le h}\right] = \mathbb{E}\left[\mathbb{1}_{\tau_\phi \le h} \varphi\left(\alpha e^{-m\tau_\phi}, (\alpha X_\phi(\tau_\phi)+\beta)e^{-m\tau_\phi}\right)^{L_\phi}\right]$

integrate with respect to L_ϕ first

$$\overset{\downarrow}{=} \mathbb{E}\left[\mathbb{1}_{\tau_\phi \le h} f\left(\varphi\left(\alpha e^{-m\tau_\phi}, (\alpha \underbrace{X_\phi(\tau_\phi)} + \beta)e^{-m\tau_\phi}\right)\right)\right]$$

we can replace by B_{τ_ϕ} with $B \perp\!\!\!\perp \tau_\phi$

$$= \int_0^h \mathbb{E}\left[f\left(\varphi\left(\alpha e^{-ms}, (\alpha B_s + \beta)e^{-ms}\right)\right)\right] e^{-s}\, ds$$

<u>2nd term</u>: On $\tau_\phi > h$, we use decomposition of question 5 and the hint to get

$$\mathbb{E}\left[e^{i\alpha\Pi_\infty + i\beta W_\infty^\circ} \mathbb{1}_{\tau_\phi > h}\right] = \mathbb{E}\left[e^{i\alpha e^{-mh}\Pi_\infty^{\phi,h} + i(\beta + \alpha X_\phi(h))e^{-mh} W_\infty^{\phi,h}} \mathbb{1}_{\tau_\phi > h}\right]$$

take $\mathbb{E}[\,\cdot \mid \mathcal{F}_h]$ first

$$\overset{\downarrow}{=} \mathbb{E}\left[\varphi\left(\alpha e^{-mh}, (\beta + \alpha X_\phi(h))e^{-mh}\right) \mathbb{1}_{\tau_\phi > h}\right]$$

$$= e^{-h}\, \mathbb{E}\left[\varphi\left(\alpha e^{-mh}, (\beta + \alpha B_h)e^{-mh}\right)\right].$$

<u>Conclusion</u>:
$$\varphi(\alpha,\beta) = \int_0^h \mathbb{E}\left[f\left(\varphi\left(\alpha e^{-ms}, (\alpha B_s + \beta)e^{-ms}\right)\right)\right] e^{-s}\, ds + e^{-h}\, \mathbb{E}\left[\varphi\left(\alpha e^{-mh}, (\beta + \alpha B_h)e^{-mh}\right)\right].$$

8. φ is $\mathcal{C}^2$ on $\mathbb{R}^2$ because $T_\infty, W_\infty \in L^2$ and takes values in $\{z \in \mathbb{C} : |z| \leq 1\}$.

- f is continuous on $\{z \in \mathbb{C} : |z| \leq 1\}$ so $s \geq 0 \longmapsto e^{-s} f(\varphi(e^{-ms}\alpha, e^{-ms}(\alpha B_s + \beta)))$ is continuous and dominated by 1. By continuity under the expectation, we deduce that $s \longmapsto e^{-s}\mathbb{E}[f(\varphi(e^{-ms}\alpha, e^{-ms}(\alpha B_s + \beta)))]$ is continuous.
Therefore $\frac{1}{h}\int_0^h e^{-s}\mathbb{E}[f(\varphi(e^{-ms}\alpha, e^{-ms}(\alpha B_s+\beta)))]\,ds \xrightarrow[h\to 0]{} f(\varphi(\alpha,\beta))$.

- Now set $F(t,x) = e^{-t}\varphi(e^{-mt}\alpha, e^{-mt}(\alpha x + \beta))$.
We aim at proving
$$\frac{1}{h}\Big(\mathbb{E}[F(h,B_h)] - \varphi(\alpha,\beta)\Big) \xrightarrow[h\to 0]{} \frac{\alpha^2}{2}\frac{\partial^2\varphi}{\partial\beta^2}(\alpha,\beta) - m\alpha\frac{\partial\varphi}{\partial\alpha}(\alpha,\beta) - m\beta\frac{\partial\varphi}{\partial\beta}(\alpha,\beta) - \varphi(\alpha,\beta)$$
Note that F is $\mathcal{C}^2$ on $[0,\infty)\times\mathbb{R}$ with bounded derivatives (because so is φ).
By Itô's formula,
$$dF(t,B_t) = \frac{\partial F}{\partial x}(t,B_t)\,dB_t + \Big(\frac{\partial F}{\partial t}(t,B_t) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,B_t)\Big)dt$$
so
$$\mathbb{E}[F(h,B_h)] - \underbrace{F(0,0)}_{=\varphi(\alpha,\beta)} = \underbrace{\mathbb{E}\Big[\int_0^h \frac{\partial F}{\partial x}(t,B_t)\,dB_t\Big]}_{=0 \text{ (martingale part)}} + \underbrace{\mathbb{E}\Big[\int_0^h\Big(\frac{\partial F}{\partial t}(t,B_t) + \frac{1}{2}\frac{\partial^2F}{\partial x^2}(t,B_t)\Big)dt\Big]}$$
by Fubini-Lebesgue (the integrand is bounded)

So
$$\frac{1}{h}\Big(\mathbb{E}[F(h,B_h)] - \varphi(\alpha,\beta)\Big) = \frac{1}{h}\int_0^h \mathbb{E}\Big[\underbrace{\frac{\partial F}{\partial t}(t,B_t) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,B_t)}\Big]dt$$
continuous function of t and bounded deterministically so the expectation is also continuous.
$$\xrightarrow[h\to 0]{} \mathbb{E}\Big[\frac{\partial F}{\partial t}(0,0) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(0,0)\Big]$$
$$= -\varphi(\alpha,\beta) - m\alpha\frac{\partial\varphi}{\partial\alpha}(\alpha,\beta) - m\beta\frac{\partial\varphi}{\partial\beta}(\alpha,\beta) + \frac{\alpha^2}{2}\frac{\partial^2\varphi}{\partial\beta^2}(\alpha,\beta)$$

- We multiply the formula from question 6 by $\frac{1}{h}$, then let $h\to 0$ and obtain by the two previous points:
$$f(\varphi(\alpha,\beta)) - \varphi(\alpha,\beta) - m\alpha\frac{\partial\varphi}{\partial\alpha}(\alpha,\beta) - m\beta\frac{\partial\varphi}{\partial\beta}(\alpha,\beta) + \frac{\alpha^2}{2}\frac{\partial^2\varphi}{\partial\beta^2}(\alpha,\beta) = 0.$$

Exercise 2

1.a. Let $K>0$ and $t\geq 0$.

- T_K is a stopping time:

$$\{T_K > t\} = \{\forall s\in[0,t],\ \forall u\in \mathcal{N}_s,\ X_u(s)\leq \lambda_c s + K\}$$ continuity of trajectories

$$= \bigcap_{s\in[0,t]\cap\mathbb{Q}} \underbrace{\{\forall u\in\mathcal{N}_s,\ X_u(s)\leq \lambda_c s+K\}}_{\in\mathcal{F}_s\subset\mathcal{F}_t}$$

$\in \mathcal{F}_t$ so $\{T_K\leq t\}\in\mathcal{F}_t$

- $T_K\wedge t$ is a bounded stopping time and $(W_t^{\lambda_c})_{t\geq 1}$ is a mean 1 martingale, so, by the optional stopping theorem, $\mathbb{E}[W^{\lambda_c}_{T_K\wedge t}] = 1$.

So $1 = \mathbb{E}[\underbrace{W^{\lambda_c}_{T_K}}\ \mathbb{1}_{T_K\leq t}] + \underbrace{\mathbb{E}[W_t^{\lambda_c}\ \mathbb{1}_{T_K>t}]}_{\geq 0}$

$$\geq \exp(\lambda_c \Pi_{T_K} - \lambda_c^2 T_K)$$
$$= \exp(\lambda_c(\lambda_c T_K + K) - \lambda_c^2 T_K) \quad \text{by definition of } T_K$$
$$= \exp(\lambda_c K)$$

So $1\geq e^{\lambda_c K}\,\mathbb{P}(T_K\leq t)$ and therefore $\mathbb{P}(T_K\leq t)\leq e^{-\lambda_c K}$.

1.b) $E_K^c = \{T_K<\infty\} = \bigcup_{n\geq 1}\uparrow \{T_K\leq n\}$

so $\mathbb{P}(E_K^c) = \lim_{n\to\infty}\uparrow \mathbb{P}(T_K\leq n) \leq e^{-\lambda_c K}$.

2.a. Let $\lambda\geq 0$, $K>0$, $t\geq 0$.

$$\mathbb{P}\left(\max_{s\in[0,t]} B_s - \lambda s\leq K,\ \max_{s\in[t,t+1]} B_s-\lambda s\geq K\right)$$

$$= \sum_{i\geq 1}\mathbb{P}\left(\max_{s\in[0,t]} B_s-\lambda s\leq K,\ \underbrace{\max_{s\in[t,t+1]} B_s-\lambda s\geq K}, \ B_t-\lambda t - K\in[-i,-(i-1)]\right)$$

$$\Leftrightarrow \max_{s\in[t,t+1]} B_s - B_t - \underbrace{\lambda(s-t)}_{\geq 0} \geq \underbrace{K+\lambda t - B_t}_{\geq (i-1) \text{ on the event considered}}$$

$$\leq \sum_{i\geq 1} \mathbb{P}\Big(\max_{s\in[0,t]} B_s - \lambda s \leq K,\ B_t - \lambda t - K \in [-i, -(i-1)],\ \underbrace{\max_{s\in[t,t+1]} B_s - B_t \geq i-1}_{\text{independent of } (B_r)_{r\leq t} \text{ and } \overset{(d)}{=} \max_{s\in[0,1]} B_s}\Big)$$

$$= \sum_{i\geq 1} \mathbb{P}\Big(\max_{s\in[0,t]} B_s - \lambda s \leq K,\ B_t - \lambda t - K \in [-i, -(i-1)]\Big)\ \underbrace{\mathbb{P}\Big(\max_{s\in[0,1]} B_s \geq i-1\Big)}$$

$$= \mathbb{P}(|B_1| \geq i-1)$$

$$= 2\,\mathbb{P}(B_1 \geq i-1)$$

$$\leq e^{-(i-1)^2/2}$$

$$\leq \sum_{i\geq 1} \underbrace{e^{-(i-1)^2/2}}\ \mathbb{P}\Big(\max_{s\in[0,t]} B_s - \lambda s \leq K,\ B_t - \lambda t - K \in [-i, -(i-1)]\Big).$$

↳ so there was a mistake in the statement of the question, this is larger than $e^{-(i-1)^2/2}$ but still enough for next question

2.b) For any $i \geq 1$, by Girsanov theorem,

$$\mathbb{P}\Big(\max_{s\in[0,t]} B_s - \lambda s \leq K,\ B_t - \lambda t - K \in [-i, -(i-1)]\Big)$$

$$= \mathbb{E}\Big[e^{-\lambda B_t - \frac{\lambda^2}{2}t}\ \mathbb{1}_{\max_{s\in[0,t]} B_s \leq K,\ B_t \in [K-i, K-i+1]}\Big]$$

$$\leq e^{-\lambda(K-i) - \frac{\lambda^2}{2}t}\ \underbrace{\mathbb{P}\Big(\max_{s\in[0,t]} B_s \leq K,\ B_t \geq K-i\Big)}_{\leq \frac{Ki^2}{t^{3/2}}}$$

So

$$\mathbb{P}\Big(\max_{s\in[0,t]} B_s - \lambda s \leq K,\ \max_{s\in[t,t+1]} B_s - \lambda s \geq K\Big)$$

$$\leq \sum_{i\geq 1} e^{-(i-1)^2/2} \times e^{-\lambda K + \lambda i - \frac{\lambda^2}{2}t}\ \frac{Ki^2}{t^{3/2}}$$

$$= K e^{-\lambda K}\ \frac{e^{-\lambda^2 t/2}}{t^{3/2}}\ \underbrace{\sum_{i\geq 1} i^2 e^{-(i-1)^2/2 + \lambda i}}$$

$$= \sum_{i\geq 1} i^2 e^{-i^2/2 + (\lambda+1)i - 1/2} < \infty$$

because the term $-\frac{i^2}{2}$ dominates in the exponential

$$\leq C K e^{-\lambda K}\ \frac{e^{-\lambda^2 t/2}}{t^{3/2}}.$$

3.a. Let $K>0$. We decompose

$$H_K = \sum_{j\geq 0} \#\{u\in T : \exists t\in[j,j+1),\, u\in\mathcal{N}_t,\, X_u(t)=\lambda_c t+K,\, \forall s<t,\, X_u(s)\leq \lambda_c s+K\}$$

Consider $u\in T$ such that there exists $t\in[j,j+1)$ such that $u\in\mathcal{N}_t$, $X_u(t)=\lambda_c t+K$ and $\forall s<t$, $X_u(s)<\lambda_c s+K$.

This particle u has a.s. at least one descendant $v\in\mathcal{N}_{j+1}$ because $\mathbb{P}(L=0)=0$.

This particle v satisfies $\max_{s\in[0,j]} X_v(s)-\lambda_c s = \max_{s\in[0,j]} X_u(s)-\lambda_c s \leq K$

and $\max_{s\in[j,j+1]} X_v(s)-\lambda_c s \geq X_u(t)-\lambda_c t = K$

Moreover for u' with the same property as u but $u'\neq u$, descendants of u and u' at time $j+1$ are disjoint sets. So

$$H_K \leq \sum_{j\geq 0} \#\{v\in\mathcal{N}_{j+1} : \max_{s\in[0,j]} X_v(s)-\lambda_c s\leq K,\, \max_{s\in[j,j+1]} X_v(s)-\lambda_c s\geq K\}$$

$$= \sum_{j\geq 0} \sum_{v\in\mathcal{N}_{j+1}} \mathbb{1}_{\max_{s\in[0,j]} X_v(s)-\lambda_c s\leq K}\, \mathbb{1}_{\max_{s\in[j,j+1]} X_v(s)-\lambda_c s\geq K}$$

There was an index mistake in the statement of the question.

3.b. $$\mathbb{E}[H_K] \leq \sum_{j\geq 0} \mathbb{E}\Big[\sum_{v\in\mathcal{N}_{j+1}} \mathbb{1}_{\max_{s\in[0,j]} X_v(s)-\lambda_c s\leq K}\, \mathbb{1}_{\max_{s\in[j,j+1]} X_v(s)-\lambda_c s\geq K}\Big]$$

$$= \sum_{j\geq 0} e^{m(j+1)}\, \underbrace{\mathbb{P}\Big(\max_{s\in[0,j]} B_s-\lambda_c s\leq K,\, \max_{s\in[j,j+1]} B_s-\lambda_c s\geq K\Big)}$$

if $j\geq 1$, by 2.b., this is $\leq CKe^{-\lambda_c K}\dfrac{e^{-\lambda_c^2 j/2}}{j^{3/2}} = CKe^{-\lambda_c K}\dfrac{e^{-mj}}{j^{3/2}}$

if $j=0$, this equals $\mathbb{P}\Big(\max_{s\in[0,1]} B_s-\lambda_c s\geq K\Big) \leq \mathbb{P}\Big(\max_{s\in[0,1]} B_s\geq K\Big)$

$= \mathbb{P}(|B_1|\geq K)$

$\leq e^{-K^2/2}$ as in 2.a.

So $$\mathbb{E}[H_K] \leq e^m e^{-K^2/2} + \sum_{j\geq 1} e^m \times CKe^{-\lambda_c K}\frac{1}{j^{3/2}}$$

$$\leq C'\left(e^{-K^2/2}+Ke^{-\lambda_c K}\right) \text{ with } C' = e^m \times \max\Big(1, C\sum_{j\geq 1}\frac{1}{j^{3/2}}\Big).$$