# Final exam

## February 18, 9:00–12:00

**Instructions:** All electronic equipment is forbidden. The only authorized material is a handwritten double-sided A4 sheet. Exercises and questions can be treated in any order.

**Total:** 25 points

**Reminder:** If $(B_t)_{t\geq 0}$ denotes a Brownian motion, the following holds: for any $x \geq 0$,

$$\mathbb{P}(B_1 \geq x) \leq \frac{e^{-x^2/2}}{2} \qquad \text{and} \qquad \mathbb{P}(B_1 \geq x) \underset{x\to\infty}{\sim} \frac{e^{-x^2/2}}{x\sqrt{2\pi}},$$

and, for any $a, y \geq 0$ and $t > 0$,

$$\mathbb{P}\left(\max_{s\in[0,t]} B_s \leq a\right) \leq \frac{a}{\sqrt{t}} \qquad \text{and} \qquad \mathbb{P}\left(\max_{s\in[0,t]} B_s \leq a, B_t \geq a-y\right) \leq \frac{ay^2}{t^{3/2}}.$$

Finally, recall that $\max_{s\in[0,t]} B_s$ has the same distribution as $|B_t|$.

Throughout the exam, we consider a standard BBM whose reproduction law, represented by a random variable $L$, has a finite second moment.

**Exercice 1.** (14 points) For $t \geq 0$, we define

$$M_t = e^{-mt} \sum_{u\in\mathcal{N}_t} X_u(t).$$

Recall that $W_t = e^{-mt}\#\mathcal{N}_t$, for $t \geq 0$, is a $(\mathcal{F}_t)_{t\geq 0}$-martingale converging a.s. and in $L^2$ to a limit $W_\infty$. Note that we omit the superscript 0 in the notation $W_t$.

1. (2 pts) Prove that $M_t \in L^1$ for every $t \geq 0$ and calculate $\mathbb{E}[M_t]$.
2. (1 pt) Prove that, for any $t, s \geq 0$,
$$M_{t+s} = e^{-ms} \sum_{u\in\mathcal{N}_s} (M_t^{u,s} + X_u(s) W_t^{u,s}),$$
where, conditionally on $\mathcal{F}_s$, the processes $(M_t^{u,s}, W_t^{u,s})_{t\geq 0}$ for $u \in \mathcal{N}_s$ are independent and have the same distribution as $(M_t, W_t)_{t\geq 0}$.
3. (1 pt) Prove that $(M_t)_{t\geq 0}$ is a $(\mathcal{F}_t)_{t\geq 0}$-martingale.
4. (2 pts) Prove that $(M_t)_{t\geq 0}$ converges a.s. and in $L^2$ towards a limit $M_\infty$.
5. (1 pt) Let $h > 0$. Prove that, on $\{\tau_\emptyset > h\}$, a.s.
$$M_\infty = e^{-mh}\left(M_\infty^{\emptyset,h} + X_\emptyset(h) W_\infty^{\emptyset,h}\right),$$
where, conditionally on $\mathcal{F}_h$, $(M_\infty^{\emptyset,h}, W_\infty^{\emptyset,h})$ has the same distribution as $(M_\infty, W_\infty)$.
6. (2 pts) Prove that a.s.
$$M_\infty = e^{-m\tau_\emptyset} \sum_{j=1}^{L_\emptyset} \left(M_\infty^j + X_\emptyset(\tau_\emptyset) W_\infty^j\right),$$
where, conditionally on $\mathcal{F}_{\tau_\emptyset}$, $(M_\infty^j, W_\infty^j)$ for $1 \leq j \leq L_\emptyset$ are independent and have the same distribution as $(M_\infty, W_\infty)$.

7. (2 pts) Let $f(s) = \mathbb{E}[s^L]$ be the probability generating function of the offspring distribution, which is a continuous function on the closed unit disk of the complex plane. We define $\varphi$ to be the characteristic function of the pair $(M_\infty, W_\infty)$, i.e., for $\alpha, \beta \in \mathbb{R}$, we set $\varphi(\alpha, \beta) = \mathbb{E}[\exp(i\alpha M_\infty + i\beta W_\infty)]$. Prove that for every $\alpha, \beta \in \mathbb{R}$ and $h > 0$, we have

$$\varphi(\alpha, \beta) = \int_0^h e^{-s}\mathbb{E}[f(\varphi(e^{-ms}\alpha, e^{-ms}(\alpha B_s + \beta)))]\,\mathrm{d}s + e^{-h}\mathbb{E}[\varphi(e^{-mh}\alpha, e^{-mh}(\alpha B_h + \beta))].$$

**Hint:** In addition to decompositions of $M_\infty$ seen above, you can also use the following decompositions of $W_\infty$ that have been seen in class. Firstly, a.s. $W_\infty = e^{-m\tau_\emptyset}\sum_{j=1}^{L_\emptyset} W_\infty^j$. Secondly, on $\{\tau_\emptyset > h\}$, a.s. $W_\infty = e^{-mh}W_\infty^{\emptyset,h}$.

8. (3 pts) By letting $h \to 0$ in the previous equality, prove that $\varphi$ solves the following differential equation on $\mathbb{R}^2$:

$$\frac{\alpha^2}{2}\frac{\partial^2\varphi}{\partial\beta^2} - m\alpha\frac{\partial\varphi}{\partial\alpha} - m\beta\frac{\partial\varphi}{\partial\beta} + f(\varphi) - \varphi = 0.$$

**Remark:** Justify passages to the limit.

**Hint:** You can use the following fact: if $(X, Y)$ is a couple of $L^2$ real random variables, then its characteristic function is a $C^2$ function on $\mathbb{R}^2$ and its derivatives are obtained through differentiation under the expectation.

**Exercice 2.** (11 points) Recall that $\lambda_c = \sqrt{2m}$ and that the critical additive martingale is defined by $W_t^{\lambda_c} = \sum_{u\in\mathcal{N}_t} e^{\lambda_c X_u(t) - \lambda_c^2 t}$.

1. (a) (2 pts) For $K > 0$, we define $T_K = \inf\{s \geq 0 : \exists u \in \mathcal{N}_s, X_u(s) > \lambda_c s + K\}$. Prove that $T_K$ is a stopping time and that, for any $t \geq 0$,

$$\mathbb{P}(T_K \leq t) \leq e^{-\lambda_c K}.$$

**Hint:** Apply the optional stopping theorem with the critical additive martingale and a suitably chosen stopping time.

(b) (1 pt) Recall that we define the event $E_K = \{\forall s \geq 0, \forall u \in \mathcal{N}_s, X_u(s) \leq \lambda_c s + K\}$. Deduce that, for any $K > 0$,

$$\mathbb{P}(E_K^c) \leq e^{-\lambda_c K}.$$

The remainder of the exercise can be solved independently of the first part.

2. Let $\lambda \geq 0$.

(a) (2 pts) Prove that for any $K > 0$ and $t > 0$,

$$\mathbb{P}\Big(\max_{s\in[0,t]} B_s - \lambda s \leq K, \max_{s\in[t,t+1]} B_s - \lambda s \geq K\Big) \leq \sum_{i\geq 1} e^{-(i+\lambda)^2/2}\mathbb{P}\Big(\max_{s\in[0,t]} B_s - \lambda s < K, B_t - \lambda t - K \in [-i, -(i-1)]\Big).$$

(b) (2 pts) Deduce that there exists $C > 0$ such that for any $K > 0$ and $t > 0$,

$$\mathbb{P}\Big(\max_{s\in[0,t]} B_s - \lambda s \leq K, \max_{s\in[t,t+1]} B_s - \lambda s \geq K\Big) \leq CKe^{-\lambda K}\frac{e^{-\lambda^2 t/2}}{t^{3/2}}.$$

3. Let $K > 0$. We write $H_K$ for the total number of particles hitting the line $t \mapsto \lambda_c t + K$ for the first time in their full lineage, that is

$$H_K = \#\{u \in T : \exists t \geq 0, u \in \mathcal{N}_t, X_u(t) = \lambda_c t + K \text{ and } \forall s < t, X_u(s) < \lambda_c s + K\},$$

where $T$ denotes the genealogical tree in the BBM. Assume $\mathbb{P}(L = 0) = 0$.

(a) (2 pts) Prove that, for any $K > 0$, almost surely

$$H_K \leq \sum_{j\geq 0}\sum_{v\in\mathcal{N}_j} \mathbb{1}_{\max_{s\in[0,j]} X_v(s) - \lambda_c s \leq K,\, \max_{s\in[j,j+1]} X_v(s) - \lambda_c s \geq K}.$$

(b) (2 pts) Deduce that there exists $C > 0$ such that, for any $K > 0$,

$$\mathbb{E}[H_K] \leq C\Big(Ke^{-\lambda_c K} + e^{-K^2/2}\Big).$$