Lecture 9 : Two-speed BBM

24/01/2025

IV/ Two-speed branching Brownian motion

IV. 1) Definition and main result

The <u>two-speed BBM</u> is defined in terms of three new parameters :

$$b \in (0,1),\ \sigma_1, \sigma_2 > 0 \text{ such that } \sigma_1^2 b + \sigma_2^2 (1-b) = 1.$$

<u>Informal definition</u>: For a given time horizon $t>0$, we consider a BBM where particles move according to a Brownian motion with variance σ_1^2 on $[0, bt]$ and with variance σ_2^2 on $[bt, t]$.

<u>Remark</u>: If $(B_s)_{s\geq 0}$ is a standard Brownian motion, then $(\sigma B_s)_{s\geq 0}$ and $(B_{\sigma^2 s})_{s \geq 0}$ are Brownian motions with variance σ^2.

<u>Formal definition</u>: We start with the marked tree $(\mathcal{T}, (\sigma_u)_{u\in\mathcal{T}}, (Y_u)_{u\in\mathcal{T}})$ defined as before : in $\mathcal{T}$ the reproduction occurs according to the law of L and given $\mathcal{T}$, $(\sigma_u)_{u\in\mathcal{T}}$ are iid $\mathrm{Exp}(1)$ and $(Y_u)_{u\in\mathcal{T}}$ are iid Brownian motions (variance 1).
The novelty appears here : for a time horizon $t>0$, we define $X_u^t(s)$ as follows

$$\forall s\in[0,t],\ \forall u \in \mathcal{N}_s,\ \text{if } s \leq bt,\ X_u^t(s) = \begin{cases} \sigma_1 Y_\phi(s) & \text{if } u = \phi, \\ X^t_{p(u)}(b_u -) + \sigma_1 Y_u(s-b_u) & \text{if } u \neq \phi. \end{cases}$$

$$\text{if } s > bt,\ X_u^t(s) = \begin{cases} X_u^t(bt) + \sigma_2\left(Y_u(s-b_u) - Y_u(bt-b_u)\right) & \text{if } b_u \leq bt, \\ X^t_{p(u)}(b_u -) + \sigma_2 Y_u(s-b_u) & \text{if } b_u > bt. \end{cases}$$

As before, for $u\in\mathcal{N}_t$ and $s\in[0,t]$, we write $X_u^t(s) = X_v^t(s)$ where v is the unique ancestor of u alive at time s.

<u>Remark</u>: The definition is not consistent for all times anymore : for each time horizon $t>0$, we have a different process. Our interest will be to describe the maximum at time t and see the influence of the time-inhomogeneity.

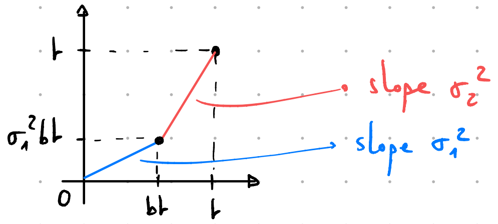
<u>Lemma (Many-to-one)</u>: Let $F : \mathcal{C}([0,t]) \longrightarrow \mathbb{R}$ be measurable and either ≥ 0 or bounded.

Then $\mathbb{E}\left[\sum_{u\in\mathcal{N}_t} F((X_u^t(s))_{s\in[0,t]})\right] = e^{mt}\, \mathbb{E}\left[F((\overline{B}_s^t)_{s\in[0,t]})\right]$, where $(\overline{B}^t_s)_{s\in[0,t]}$ is a Brownian motion with variance σ_1^2 on $[0,bt]$ and σ_2^2 on $[bt,t]$, i.e.

$$\overline{B}^t_s = \begin{cases} \sigma_1 B_s & \text{for } s\in[0,bt] \\ \sigma_1 B_{bt} + \sigma_2(B_s - B_{bt}) & \text{for } s \in (bt, t] \end{cases} \quad \text{with } B \text{ a standard BM.}$$

<u>Remark</u>: Another useful way to define $(\bar{B}^t_s)_{s\in[0,t]}$ with the same distribution is

$$\bar{B}^t_s = B_{\varphi(s)} \quad \text{where} \quad \varphi(s) = \begin{cases} \sigma_1^2 s & \text{if } s \le bt \\ \sigma_1^2 bt + \sigma_2^2 (s-bt) & \text{if } s \ge bt \end{cases} \quad \text{(check it!)}.$$



<u>Proof</u>: Same as for the usual BBM: First condition on $(\mathcal{T}, (\sigma_u)_{u\in\mathcal{T}})$:

$$\mathbb{E}\Big[\sum_{u\in\mathcal{N}_t} F\big((X^t_u(s))_{s\in[0,t]}\big) \,\Big|\, \mathcal{T},(\sigma_u)_{u\in\mathcal{T}}\Big] = \sum_{u\in\mathcal{N}_t} \mathbb{E}\Big[F\big(\underbrace{(X^t_u(s))_{s\in[0,t]}}_{\text{given } (\mathcal{T},(\sigma_u)_{u\in\mathcal{T}}),\text{ this is distributed as } (\tilde{B}^t_s)_{s\in[0,t]}}\big) \,\Big|\, \mathcal{T},(\sigma_u)_{u\in\mathcal{T}}\Big]$$

$$= \#\mathcal{N}_t \times \mathbb{E}\Big[F\big((\tilde{B}^t_s)_{s\in[0,t]}\big)\Big]$$

and then we use that $\mathbb{E}[\#\mathcal{N}_t] = e^{mt}$. ■

<u>Remark</u>: The assumption $\sigma_1^2 b + \sigma_2^2(1-b) = 1$ implies that $\bar{B}^t_t \overset{(d)}{=} B_t$. Indeed:

$$\mathrm{Var}(\bar{B}^t_t) = \mathrm{Var}\big(\sigma_1 \underbrace{B_{bt} + \sigma_2 (B_t - B_{bt})}_{\text{independent}}\big) = \sigma_1^2 \mathrm{Var}(B_{bt}) + \sigma_2^2 \mathrm{Var}(B_t - B_{bt}) = \sigma_1^2 bt + \sigma_2^2 (1-b)t = t.$$

So given $(\mathcal{T},(\sigma_u)_{u\in\mathcal{T}})$, the variables $(X^t_u(t))_{u\in\mathcal{N}_t}$ are $\mathcal{N}(0,t)$-distributed, as in the standard BBM: this makes the comparison more meaningful.

<u>Lemma (Many-to-two)</u>: Let $F: \mathcal{C}([0,t]) \longrightarrow \mathbb{R}$ be measurable and either ≥ 0 or bounded.

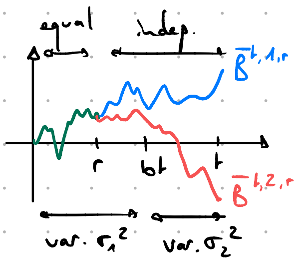
Then

$$\mathbb{E}\Big[\sum_{\substack{u,v\in\mathcal{N}_t \\ u\ne v}} F\big((X^t_u(s))_{s\in[0,t]}\big) F\big((X^t_v(s))_{s\in[0,t]}\big)\Big]$$

$$= \mathbb{E}[L(L-1)] \int_0^t e^{2mt - mr}\, \mathbb{E}\Big[F\big((\bar{B}^{t,1,r}_s)_{s\in[0,t]}\big) F\big((\bar{B}^{t,2,r}_s)_{s\in[0,t]}\big)\Big]\, dr$$

where for $i\in\{1,2\}$, $\bar{B}^{t,i,r}_s = \begin{cases} \bar{B}^{t,0}_s & \text{if } s \le r \\ \bar{B}^{t,0}_r + \bar{B}^{t,i}_{s-r} & \text{if } s > r \end{cases}$,

with $\bar{B}^{t,0}, \bar{B}^{t,1}, \bar{B}^{t,2}$ independent with the same law as $\bar{B}^t$.



<u>Proof</u>: Same as for the usual BBM: first note that conditionally on $(\mathcal{T},(\sigma_u)_{u\in\mathcal{T}})$, for $u\ne v\in\mathcal{N}_t$, the law of $(X^t_u(s), X_v(s))_{s\in[0,t]}$ is the same as the law of $(\bar{B}^{t,1,d_{u\wedge v}}_s, \bar{B}^{t,2,d_{u\wedge v}}_s)_{s\in[0,t]}$ (recall $d_{u\wedge v}$ is the death time

of $u \wedge v$ the most recent common ancestor of u and v).

This shows that

$$\mathbb{E}\Big[\sum_{\substack{u,v\in\mathcal{N}_t\\ u\neq v}} F((X_u^t(s))_{s\in[0,t]})\, F((X_v^t(s))_{s\in[0,t]}) \,\Big|\, \mathcal{T}, (\sigma_u)_{u\in\mathcal{T}}\Big] = \sum_{\substack{u,v\in\mathcal{N}_t\\ u\neq v}} G(d_{u\wedge v})$$

where $G(r) = \mathbb{E}\Big[F((\bar{B}_s^{t,1,r})_{s\in[0,t]})\, F((\bar{B}_s^{t,2,r})_{s\in[0,t]})\Big]$.

Then, we have seen: $\mathbb{E}\Big[\sum_{\substack{u,v\in\mathcal{N}_t\\ u\neq v}} G(d_{u\wedge v})\Big] = \mathbb{E}[L(L-1)] \int_0^t e^{2mt-mr} G(r)\,dr$. ∎

Our goal in this chapter is describe the behavior of $\overline{M}_t = \max_{u\in\mathcal{N}_t} X_u^t(t)$.

Theorem (Fang-Zeitouni 2012, Bovier-Hartung 2014):

On the survival event, $\overline{M}_t = \overline{m}_t + O_{\mathbb{P}}(1)$ as $t\to\infty$ where

$$\overline{m}_t = \begin{cases} \tilde{m}_t := \lambda_c t - \frac{1}{2\lambda_c}\log t & \text{if } \sigma_1 < \sigma_2, \quad \Rightarrow \sigma_1 < 1 < \sigma_2 \\ m_t := \lambda_c t - \frac{3}{2\lambda_c}\log t & \text{if } \sigma_1 = \sigma_2, \quad \Rightarrow \sigma_1 = 1 = \sigma_2 \\ \lambda_c(b\sigma_1 + (1-b)\sigma_2)t - \frac{3(\sigma_1+\sigma_2)}{2\lambda_c}\log t & \text{if } \sigma_1 > \sigma_2. \quad \Rightarrow \sigma_1 > 1 > \sigma_2 \end{cases}$$

Remarks:

- The case $\sigma_1 = \sigma_2$, which implies $\sigma_1 = \sigma_2 = 1$, is the standard case. We have not fully proved this result ($O(\log\log t)$ instead of $O(1)$ in the upper bound).
- In the case $\sigma_1 < \sigma_2$, we recover the same behavior $\tilde{m}_t$ as in the case of e^{mt} independent Brownian motions. So the correlation structure plays no significant role here!
- In the case $\sigma_1 > \sigma_2$, even the first order differs: the speed is strictly smaller than λ_c because $b\sigma_1 + (1-b)\sigma_2 < 1$ if $b\in(0,1)$ and $\sigma_1 \neq \sigma_2$ (check it!) So huge impact of the correlation structure!

  When $\sigma_1 \downarrow 1$ (so $\sigma_2 \uparrow 1$ simultaneously), this speed converges to λ_c, but the coefficient for the $\log t$ term approaches $-\frac{6}{2\lambda_c}\log t$, so there is a discontinuity for this coefficient on both sides.

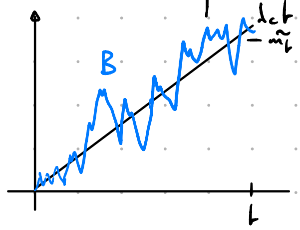
IV.2) Some heuristics

Recall the maximum of e^{mt} independent copies of $\bar{B}^t_t$ behaves like $\tilde{m}_t + O_P(1)$, with $\tilde{m}_t = \lambda_c t - \frac{1}{2\lambda_c}\log t$ (we use here that $\bar{B}^t_t \overset{(d)}{=} B_t$).

First question: Typically, where is at time bt a particles ending above $\tilde{m}_t$?

Recall that $(\bar{B}^t_s)_{s\in[0,t]} = (B_{\varphi(s)})_{s\in[0,t]}$, which gives in particular
$(\bar{B}^t_{bt}, \bar{B}^t_t) \overset{(d)}{=} (B_{\sigma_1^2 bt}, B_t)$.

But we have seen that, when $B_t \geq \tilde{m}_t$, $(B_s)_{s\in[0,t]}$ typically behaves as a Brownian motion with drift λ_c (by Girsanov) so we expect to have $B_{\sigma_1^2 bt} = \lambda_c \sigma_1^2 bt + O(\sqrt{t})$.



Exercise 1: Check this property by proving the following results

1.) Prove that $\mathbb{P}(\bar{B}^t_t \geq \tilde{m}_t) \sim \frac{1}{\lambda_c\sqrt{2\pi t}} e^{-\tilde{m}_t^2/2t}$ as $t \longrightarrow \infty$.

2.) Fix $K>0$ and let $I_{t,K} = [\lambda_c\sigma_1^2 bt - K\sqrt{t}, \lambda_c \sigma_1^2 bt + K\sqrt{t}]$. Prove that
$$\mathbb{P}(\bar{B}^t_t \geq \tilde{m}_t, \bar{B}^t_{bt} \in I_{t,K}) \sim \frac{1}{\lambda_c\sqrt{2\pi t}} e^{-\tilde{m}_t^2/2t} \int_{-K/\sqrt{\sigma_1^2 b\sigma_2^2(1-b)}}^{K/\sqrt{\sigma_1^2 b \sigma_2^2(1-b)}} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx \quad \text{as } t\longrightarrow\infty.$$

3.) For any $\varepsilon>0$, prove that there exists $K>0$ such that
$$\liminf_{t\longrightarrow\infty} \mathbb{P}(\bar{B}^t_{bt} \in I_{t,K} \mid \bar{B}^t_t \geq \tilde{m}_t) \geq 1-\varepsilon.$$

Solution to the exercise:

1.) Already done before (consequence of the tail of $\mathcal{N}(0,1)$).

2.) We have $(\bar{B}^t_{bt}, \bar{B}^t_t) = (Z_1, Z_1+Z_2)$ with $Z_1 \overset{(d)}{=} \mathcal{N}(0,\sigma_1^2 bt)$ and $Z_2 \overset{(d)}{=} \mathcal{N}(0,\sigma_2^2(1-b)t)$, so integrating first w.r.t. Z_1 we get:
$$\mathbb{P}(\bar{B}^t_t \geq \tilde{m}_t, \bar{B}^t_{bt} \in I_t) = \int_{I_t} \mathbb{P}(Z_2 \geq \tilde{m}_t - x) \frac{e^{-x^2/2\sigma_1^2 bt}}{\sqrt{2\pi\sigma_1^2 bt}}\,dx$$

Then, uniformly in $x\in I_t$, we have

$$\mathbb{P}(Z_2 \geq \tilde{m}_t - x) = \mathbb{P}\left(\mathcal{N}(0,1) \geq \frac{\tilde{m}_t - x}{\sqrt{\sigma_2^2(1-b)t}}\right)$$

$$1-\sigma_1^2 b = \sigma_2^2(1-b)$$

$$\frac{\tilde{m}_t - x}{\sqrt{\sigma_2^2(1-b)t}} \sim \frac{\lambda_c(1-\sigma_1^2 b)t}{\sqrt{\sigma_2^2(1-b)t}} \sim \lambda_c\sqrt{\sigma_2^2(1-b)t} \longrightarrow \infty$$ (so we can use Gaussian tail)

$$\sim \frac{1}{\lambda_c\sqrt{2\pi\sigma_2^2(1-b)t}}\exp\left(-\frac{(\tilde{m}_t - x)^2}{2\sigma_2^2(1-b)t}\right)$$

So $$\mathbb{P}\left(\bar{B}^t_t \geq \tilde{m}_t,\ \bar{B}^t_{bt} \in I_t\right) \sim \frac{1}{\lambda_c 2\pi t}\frac{1}{\sqrt{\sigma_1^2 b\sigma_2^2(1-b)}}\int_{I_t}\exp\Big(\underbrace{-\frac{(\tilde{m}_t - x)^2}{2\sigma_2^2(1-b)t} - \frac{x^2}{2\sigma_1^2 bt}}_{=?}\Big)dx$$

$$\frac{(\tilde{m}_t - x)^2}{2\sigma_2^2(1-b)t} + \frac{x^2}{2\sigma_1^2 bt} = \frac{1}{2t}\left[x^2\underbrace{\left(\frac{1}{\sigma_2^2(1-b)} + \frac{1}{\sigma_1^2 b}\right)}_{=\frac{1}{\sigma_1^2 b\sigma_2^2(1-b)}} + \frac{\tilde{m}_t^2}{\sigma_2^2(1-b)} - \frac{2\tilde{m}_t x}{\sigma_2^2(1-b)}\right]$$

because $\sigma_1^2 b + \sigma_2^2(1-b) = 1$

$$= \frac{1}{2t\sigma_2^2(1-b)}\left[\frac{x^2}{\sigma_1^2 b} - 2\tilde{m}_t x + \tilde{m}_t^2\right]$$

$$= \frac{1}{2t\sigma_2^2(1-b)}\Big[\frac{(x-\sigma_1^2 b\tilde{m}_t)^2}{\sigma_1^2 b} \underbrace{-\sigma_1^2 b\tilde{m}_t^2 + \tilde{m}_t^2}_{=\sigma_2^2(1-b)\tilde{m}_t^2}\Big]$$

$$= \frac{(x-\sigma_1^2 b\tilde{m}_t)^2}{2t\sigma_1^2 b\sigma_2^2(1-b)} + \frac{\tilde{m}_t^2}{2t}$$

we see here that I_t is appropriately centered!

$$\mathbb{P}\left(\bar{B}^t_t \geq \tilde{m}_t,\ \bar{B}^t_{bt} \in I_t\right) \sim \frac{1}{\lambda_c 2\pi t}\frac{1}{\sqrt{\sigma_1^2 b\sigma_2^2(1-b)}}e^{-\tilde{m}_t^2/2t}\underbrace{\int_{I_t}\exp\left(-\frac{(x-\sigma_1^2 b\tilde{m}_t)^2}{2t\sigma_1^2 b\sigma_2^2(1-b)}\right)dx}$$

set $y = \frac{x-\sigma_1^2 b\tilde{m}_t}{\sqrt{\sigma_1^2 b\sigma_2^2(1-b)t}} \longrightarrow = \int_{-K/\sqrt{\sigma_1^2 b\sigma_2^2(1-b)}}^{K/\sqrt{\sigma_1^2 b\sigma_2^2(1-b)}}\exp\left(-\frac{y^2}{2}\right)\sqrt{\sigma_1^2 b\sigma_2^2(1-b)t}\,dy$

3.| Simply choose K large enough such that $\int_{-K/\sqrt{\sigma_1^2 b\sigma_2^2(1-b)}}^{K/\sqrt{\sigma_1^2 b\sigma_2^2(1-b)}}\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx \geq 1-\varepsilon$.

We now come back to heuristics:

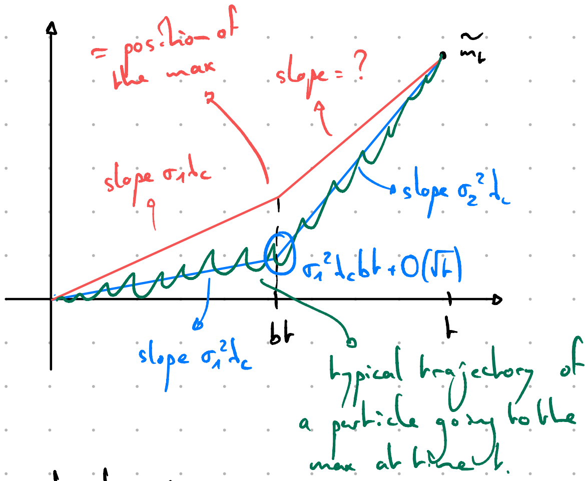
Case $\sigma_1 = \sigma_2 = 1$:

Recall briefly the heuristics we have already seen.

A trajectory going to the top at time t cannot be anywhere in $I_t = [\lambda_c bt - K\sqrt{t}, \lambda_c bt + K\sqrt{t}]$ at time bt, because the maximum at time bt is smaller than $\lambda_c bt$. Applying the same reasoning at other intermediate times $s \in [0,t]$, we see that the trajectory going to the top has to stay below the line $s \mapsto \lambda_c s + L$. The cost of this results in a different coefficient for the logarithmic correction of the maximum.

Case $\sigma_1 < 1 < \sigma_2$:

A trajectory going to $\tilde m_t$ would typically be at $d_c \sigma_1^2 bt + O(\sqrt t)$ if behaving without the constraints due to the correlations.

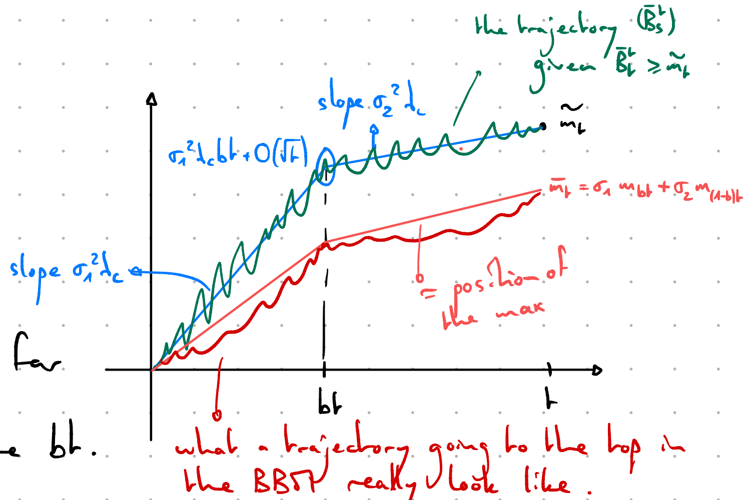


But at time bt, the maximum is at $\sigma_1 m_{bt} \simeq \sigma_1 d_c bt$ (we use here that up to time bt, our BBM has constant variance σ_1^2 so it behaves like $\sigma_1 \times$ usual BBM)

This means that the trajectory going to the top is far from the maximum of the BBM at intermediate times (for example, at time bt, it is at $\sigma_1^2 d_c bt + O(\sqrt t) \ll \sigma_1 d_c bt$ because $\sigma_1 < 1$).

So the constraint appearing in the case $\sigma_1 = \sigma_2$ is not here, and this explains that we find the same behavior as in the iid case.

Case $\sigma_1 > 1 > \sigma_2$:



Now $\sigma_1 > 1$ so $\sigma_1^2 d_c bt \gg \sigma_1 d_c bt$.

So the trajectory of $(\bar B_s^t)$ given $\bar B_t^t \geq \tilde m_t$ is completely impossible in the BBM: it goes far above the position of the maximum at time bt.

The constraint is much stronger than in the usual BBM (need to change path at a linear order) which explains that the speed of the maximum is strictly smaller.

Moreover it suggests that the best strategy to go as high as possible at time t is to go as high as possible at time bt, which is $\sigma_1 m_{bt}$. From there the highest particle gives birth to a BBM with variance σ_2^2 of length $(1-b)t$ so its maximum is $\sigma_2 m_{(1-b)t}$. This explains the formula for m_t, which equals $\sigma_1 m_{bt} + \sigma_2 m_{(1-b)t} + O(1)$ in that case.

IV.3) The case $\sigma_1 > \sigma_2$

IV.3.1) Lower bound

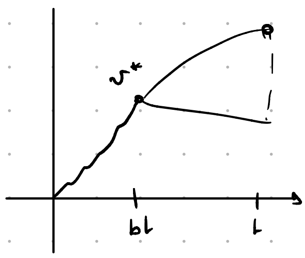
We prove it in the case $P(L=0)=0$ (so $P(\text{survival})=1$).

Let $\varepsilon>0$.

By the lower bound for usual BBM, there exists $y>0$ such that, for t large enough $P(M_t \geq \overline{m}_t - y) \geq 1-\varepsilon$.

Now let v^* denote the highest particle at time bt. We have

$$\overline{M}_t \geq \max_{\substack{u\in\mathcal{N}_t\\ u\geq v^*}} X_u^t(t) = \underbrace{X^t_{v^*}(bt)}_{\overset{(d)}{=}\ \sigma_1 M_{bt}} + \underbrace{\max_{\substack{u\in\mathcal{N}_t\\ u\geq v^*}} X_u^t(t) - X_{v^*}(bt)}_{\substack{\text{by the branching property}\\ \text{this is indep. of } X_{v^*}(bt)\\ \text{and } \overset{(d)}{=}\ \sigma_2 M_{t-bt}}}$$



So $$P\Big(\overline{M}_t \geq \sigma_1(m_{bt}-y)+\sigma_2(m_{(1-b)t}-y)\Big)$$
$$\geq P\Big(X^t_{v^*}(bt)\geq \sigma_1(m_{bt}-y),\ \max_{\substack{u\in\mathcal{N}_t\\ u\geq v^*}} X_u^t(t)-X_{v^*}(bt)\geq \sigma_2(m_{(1-b)t}-y)\Big)$$
$$= P\big(M_{bt}\geq m_{bt}-y\big)\times P\big(M_{t-bt}\geq m_{(1-b)t}-y\big)$$
$$\geq (1-\varepsilon)^2 \quad \text{for } t \text{ large enough.}$$

This proves the lower bound, because $\overline{m}_t = \sigma_1 m_{bt} + \sigma_2 m_{(1-b)t} + \frac{3\sigma_1}{2\lambda_c}\log b + \frac{3\sigma_2}{2\lambda_c}\log(1-b)$

so with $z := (\sigma_1+\sigma_2)y + \frac{3\sigma_1}{2\lambda_c}\log b + \frac{3\sigma_2}{2\lambda_c}\log(1-b)$, we proved:

$$P(\overline{M}_t \geq \overline{m}_t - z) \geq (1-\varepsilon)^2 \quad \text{for } t \text{ large enough.} \qquad \blacksquare$$

Exercise 2: Prove it in the case $P(L=0)>0$.