Lecture 5: Starting the study of extremal particles 08/01/2025

III/ Extremal particles of the BBM

In this chapter, we study the asymptotic behavior of $M_t := \max_{u \in \mathcal{N}(t)} X_u(t)$.

We always assume that $E[L^2] < \infty$.

III.1) First order

We have seen that $\frac{M_t}{t} \xrightarrow{P} \lambda_c := \sqrt{2m}$ (when $P(L=0)=0$, i.e. no extinction).

We prove here the following stronger result:

Theorem: $\frac{M_t}{t} \xrightarrow[t\to\infty]{a.s.} \lambda_c = \sqrt{2m}$ on the survival event.

Recall the definition of the additive martingale: $W_t^\lambda := \sum_{u \in \mathcal{N}(t)} e^{\lambda X_u(t) - (\frac{\lambda^2}{2}+m)t}$.

Proof:

① Upper bound: Keeping only the highest particle, we get $W_t^\lambda \geq e^{\lambda M_t - (\frac{\lambda^2}{2}+m)t}$.

With $\lambda = \sqrt{2m}$, it yields $W_t^{\lambda_c} \geq \exp(\sqrt{2m}(M_t - \sqrt{2m}\,t))$.

But $W_t^{\lambda_c} \longrightarrow W_\infty^{\lambda_c} < \infty$ a.s. so $\limsup_{t\to\infty} M_t - \sqrt{2m}\,t < \infty$ a.s.

This proves $\limsup_{t\to\infty} \frac{M_t}{t} \leq \sqrt{2m}$ a.s.

(Remark: Using that $W_\infty^{\sqrt{2m}} = 0$ a.s. we can even deduce $\lim_{t\to\infty} M_t - \sqrt{2m}\,t = -\infty$ a.s.)

② Lower bound: Consider $\lambda \in (0, \lambda_c)$ and $\varepsilon > 0$ such that $\lambda + \varepsilon < \lambda_c$. Then

$$W_t^{\lambda+\varepsilon} = \sum_{u\in\mathcal{N}(t)} e^{\underbrace{(\lambda+\varepsilon)X_u(t)}_{\leq \lambda X_u(t) + \varepsilon M_t} - (\frac{(\lambda+\varepsilon)^2}{2}+m)t} \leq e^{\varepsilon M_t} e^{\frac{(\lambda^2 - (\lambda+\varepsilon)^2)}{2}t} W_t^\lambda = e^{\varepsilon(M_t - \lambda t)} e^{-\frac{\varepsilon^2}{2}t} W_t^\lambda$$

On the survival event, $W_t^{\lambda+\varepsilon} \longrightarrow W_\infty^{\lambda+\varepsilon} > 0$ a.s. and $e^{-\frac{\varepsilon^2}{2}t} W_t^\lambda \longrightarrow 0$ a.s.

so $e^{\varepsilon(M_t - \lambda t)} \longrightarrow \infty$ a.s., which implies $\liminf_{t\to\infty} \frac{M_t}{t} \geq \lambda$ a.s.

This concludes the proof by letting $\lambda \to \lambda_c = \sqrt{2m}$.

(Remark: A key idea behind this proof is that if $M_t \leq \lambda t$ then $W_t^{\lambda+\varepsilon}$ should be small.
Indeed W_t^λ is mainly supported by particles with a position $\lambda t + O(\sqrt{t})$: note that

$$E\Big[\sum_{u\in\mathcal{N}_t} e^{\lambda X_u(t) - (m+\frac{\lambda^2}{2})t} \mathbb{1}_{X_u(t) \notin [\lambda t - K\sqrt{t}, \lambda t + K\sqrt{t}]}\Big] = E\Big[e^{\lambda B_t - \frac{\lambda^2}{2}t} \mathbb{1}_{B_t \notin [\lambda t - K\sqrt{t}, \lambda t + K\sqrt{t}]}\Big]$$ by the many-to-one

$$= P(B_t \notin [-K\sqrt{t}, K\sqrt{t}])$$ by Girsanov (see next lecture for a proof!)

$$= P(B_1 \in [-K, K]) \xrightarrow[K\to\infty]{} 0 \quad)$$

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Exercise 1 : A proof without additive martingales

Recall that $N_t(at) := \sum_{u \in \mathcal{N}_t} \mathbb{1}_{X_u(t) \geq at}$ and $\mathbb{E}[N_t(at)] \underset{t\to\infty}{\sim} \frac{e^{mt - \frac{a^2}{2}t}}{a\sqrt{2\pi t}}$ for any $a>0$.

1.) Upper bound along a subsequence

1.a.) Let $a > \sqrt{2m}$. Prove that $N_k(ak) \underset{k\to\infty}{\longrightarrow} 0$ a.s., where $k\to\infty$ along integers.

1.b.) Deduce that $\limsup_{k\to\infty} \frac{M_k}{k} \leq \sqrt{2m}$ a.s., where $k\to\infty$ along integers.

2.) Lower bound along a subsequence (harder!)

2.a.) Let $a \in (0, \sqrt{2m})$. Prove that there exist $s>0$ such that $\mathbb{E}[N_s(as)] > 1$.

2.b.) Let $\tilde{\mathcal{N}}_0 = \mathcal{N}_0$ and by induction $\tilde{\mathcal{N}}_{(k+1)s} = \left\{ u \in \mathcal{N}_{(k+1)s} : \begin{array}{l} X_u((k+1)s) - X_u(ks) \geq as \\ \text{and } \exists v \in \tilde{\mathcal{N}}_{ks},\ u \geq v \end{array} \right\}$.

Let $p := \mathbb{P}(\forall k \in \mathbb{N}, \tilde{\mathcal{N}}_{ks} \neq \emptyset)$. Prove that $p>0$.

(Hint: Note that $(\#\tilde{\mathcal{N}}_{ks})_{k\geq 0}$ is a Galton-Watson process)

2.c.) Prove that $\mathbb{P}(\forall k \geq 0, M_{ks} \geq aks) \geq p$.

2.d.) Deduce that, for any $\varepsilon>0$, there exists k_0 and $C>0$ such that

$$\mathbb{P}(\forall k \geq k_0, M_{ks} \geq aks - C \mid \text{survival}) \geq 1-\varepsilon.$$

Hint: Use the same argument as in Lecture 3 for the lower bound on $\max_{u\in\mathcal{N}_t} X_u(t)$.

It was done there with assumption $\mathbb{P}(L=0)=0$ but note that without

this assumption, we have: on the survival event, $N_t \underset{t\to\infty}{\longrightarrow} \infty$ a.s.

Note: This has now been done in Lecture 7 in a context more similar to here.

2.e.) Conclude that $\liminf_{k\to\infty} \frac{M_{ks}}{ks} \geq a$ a.s. on the survival event.

3.) Filling the gaps

3.a.) Let $s, \varepsilon > 0$. Show that $\mathbb{E}\left[\#\{u \in \mathcal{N}_t : \exists r \in [s-t,t], |X_u(t) - X_u(r)| > \varepsilon t\}\right] = O\left(e^{mt - \frac{(\varepsilon t)^2}{2s}}\right)$.

(Hint: You can use that $\sup_{r\in[0,s]} B_r \overset{(d)}{=} -\inf_{r\in[0,s]} B_r \overset{(d)}{=} |B_s|$)

3.b.) Deduce that a.s., for k large enough, $\forall u \in \mathcal{N}_{ks}, \forall r \in [(k-1)s, ks], |X_u(ks) - X_u(r)| \leq \varepsilon ks$.

3.c.) Conclude.

III.2) A reference model: the i.i.d. case

To see the influence of the tree structure on the maximal position at time t, we compare it to the case we would consider $\lfloor e^{mt} \rfloor$ particles with independent Brownian trajectories of length t.

Let $(B_t^i)_{t \geq 0}$ for $i \geq 1$ be iid Brownian motions.

We compare $(X_u(t), u \in \mathcal{N}(t))$ with $(B_t^i, \ i \in \{1, \dots, \lfloor e^{mt} \rfloor\})$.

Remark: We have the same result for the many-to-one (up to the integer part):

$$\mathbb{E}\left[\sum_{i=1}^{\lfloor e^{mt} \rfloor} F\left((B_s^i)_{s \in [0,t]}\right)\right] = \lfloor e^{mt} \rfloor \, \mathbb{E}\left[F\left((B_s)_{s \in [0,t]}\right)\right].$$

Correlations can only be seen at the level of many-to-two: here we have

$$\mathbb{E}\Big[\sum_{\substack{i,j=1 \\ i \neq j}}^{\lfloor e^{mt} \rfloor} F\left((B_s^i)_{s \in [0,t]}, (B_s^j)_{s \in [0,t]}\right)\Big] = \lfloor e^{mt} \rfloor \left(\lfloor e^{mt} \rfloor - 1\right) \mathbb{E}\Big[F\big(\underbrace{(B_s^1)_{s \in [0,t]}, (B_s^2)_{s \in [0,t]}}_{\text{independent}}\big)\Big]$$

which is different from the BBM case.

Our goal is to compare the maximal position in the BBM and the i.i.d. cases.

Let $\tilde{M}_t = \max_{1 \leq i \leq \lfloor e^{mt} \rfloor} B_t^i$.

Exercise 2: Prove that $\frac{\tilde{M}_t}{t} \xrightarrow{a.s.} \lambda_c = \sqrt{2m}$

The first order is the same! We prove here the more precise expansion:

Theorem: $\tilde{M}_t - \lambda_c t + \frac{1}{2\lambda_c} \log t \xrightarrow[t \to \infty]{(d)} \mathrm{Gumbel}\left(-\frac{1}{\lambda_c} \log(\lambda_c \sqrt{2\pi}), \frac{1}{\lambda_c}\right)$,

where $\mathrm{Gumbel}(c,b)$ for $c \in \mathbb{R}$ and $b > 0$ has cumulative distribution function $x \in \mathbb{R} \longmapsto \exp(-e^{-(x-c)/b})$.

Remark: When looking at $\max(X_1, \dots, X_n)$ as $n \to \infty$ where $(X_i)_{i \geq 1}$ are iid random variables, there are three possible families of limiting distributions (after proper recentering): Gumbel, Fréchet and Weibull distributions.

Remark: Note that the tail of $G \sim \mathrm{Gumbel}(c,b)$ is asymmetric: we have

$$\mathbb{P}(G \geq x) \sim e^{-(x-c)/b} \quad \text{and} \quad \mathbb{P}(X \leq -x) \sim e^{-e^{(x+c)/b}} \quad \text{as } x \to \infty$$

so the left tail is much thinner (double exponential) than the right tail (exponential). This is not surprising for the limit of a maximum: having a larger maximum only requires one r.v. to be large whereas having a smaller one requires all r.v. to be small enough.

<u>Proof:</u> We prove convergence of the cumulative distribution function.

Fix $y \in \mathbb{R}$. Write $x_t = \lambda_c t - \frac{1}{2\lambda_c}\log t + y$.

$$\mathbb{P}\left(\tilde{\Pi}_t - \lambda_c t + \frac{1}{2\lambda_c}\log t \leq y\right) = \mathbb{P}(\tilde{\Pi}_t \leq x_t) = \mathbb{P}(B_t \leq x_t)^{\lfloor e^{mt}\rfloor} = \left(1 - \mathbb{P}(B_t > x_t)\right)^{\lfloor e^{mt}\rfloor}$$

Then $\mathbb{P}(B_t > x_t) = \mathbb{P}\left(B_1 > \frac{x_t}{\sqrt{t}}\right) \sim \frac{\sqrt{t}}{\sqrt{2\pi}\, x_t} e^{-x_t^2/2t}$ as $t \to \infty$ $\left(\text{because } \frac{x_t}{\sqrt{t}} \longrightarrow \infty\right)$

$$\sim \frac{1}{\sqrt{2\pi t}\,\lambda_c} \exp\left(-\frac{1}{2t}\left(\underbrace{\lambda_c^2}_{=2m} t^2 + \underbrace{\frac{1}{4\lambda_c^2}(\log t)^2}_{\text{negligible}} + \underbrace{y^2}_{\text{negligible}} - t\log t + 2\lambda_c t y - \underbrace{\frac{y}{\lambda_c}\log t}_{\text{negligible}}\right)\right)$$

$$\sim \frac{1}{\sqrt{2\pi \cancel{t}}\,\lambda_c} \exp\left(-mt + \cancel{\frac{1}{2}\log t} - \lambda_c y\right)$$

$$\sim \frac{e^{-\lambda_c y}}{\lambda_c\sqrt{2\pi}} e^{-mt}$$

$$\mathbb{P}\left(\tilde{\Pi}_t - \lambda_c t + \frac{1}{2\lambda_c}\log t \leq y\right) = \exp\left(\underbrace{\lfloor e^{mt}\rfloor}_{\sim e^{mt}} \times \underbrace{\log\left(1 - \mathbb{P}(B_t > x_t)\right)}_{\sim -\mathbb{P}(B_t > x_t)}\right) \xrightarrow[t \to \infty]{} \exp\left(-\frac{e^{-\lambda_c y}}{\lambda_c\sqrt{2\pi}}\right). \quad \blacksquare$$

<u>Exercise 3:</u> Recall N_t is the number of particles in the BBM at time t and assume the Brownian motions B^i for $i \geq 1$ are independent of L.

Assume $\mathbb{P}(L=0) = 0$ so that there is a.s. survival.

Recall $e^{-mt} N_t \xrightarrow{a.s.} W_\infty^\circ \in (0,\infty)$. Consider now $\hat{\Pi}_t := \max_{1 \leq i \leq N_t} B_t^i$.

Prove that $\hat{\Pi}_t - \lambda_c t + \frac{1}{2\lambda_c}\log t \xrightarrow[t\to\infty]{(d)} \frac{1}{\lambda_c}\log W_\infty^\circ + G$,

where G is $\mathrm{Gumbel}\left(-\frac{1}{\lambda_c}\log(\lambda_c\sqrt{2\pi}), \frac{1}{\lambda_c}\right)$ distributed and independent of W_∞°.

Remark: Another reference model would be the case of fully correlated particles: at time t, it consists of $\lfloor e^{mt} \rfloor$ particles following the same Brownian motion of length t. In that case the maximum is simply the position of one Brownian motion, which is of order $\sqrt{t}$ (much smaller than $d_c t$).

Next goal: Get the second order term in the expansion of Π_t for the BBM.
Spoiler: Logarithmic correction with a different constant.

Note that we have the following general result telling us that more correlations implies a smaller maximum (if the variances are the same!)

Slepian's lemma: Let $n \geq 1$. Let $(X_1, \dots, X_n)$ and $(Y_1, \dots, Y_n)$ be centered Gaussian vectors. If $\forall i$, $\mathbb{E}[X_i^2] = \mathbb{E}[Y_i^2]$ and $\forall i,j$, $\mathbb{E}[X_i X_j] \leq \mathbb{E}[Y_i Y_j]$, then $\max(Y_1, \dots, Y_n)$ is stochastically dominated by $\max(X_1, \dots, X_n)$.

(We say that Z is stochastically dominated by Z' if $\forall x \in \mathbb{R}$, $P(Z \leq x) \geq P(Z' \leq x)$.)

Proof: We first prove that if $f \in \mathcal{C}^2(\mathbb{R}^n)$ satisfies $\forall i \neq j$, $\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$ and has bounded second derivatives, then $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ where $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$.
For this let $Z(t) := \sqrt{t}\, X + \sqrt{1-t}\, Y$ where we assume w.l.o.g. that X and Y are independent. Then $Z(0) = Y$ and $Z(1) = X$ so it is enough to prove that $\frac{d}{dt} \mathbb{E}[f(Z(t))] \leq 0$. (This technique is called Gaussian interpolation).

$$\frac{d}{dt} \mathbb{E}[f(Z(t))] = \mathbb{E}\left[\frac{d}{dt} \left(f(\sqrt{t}\, X + \sqrt{1-t}\, Y) \right) \right]$$

$$= \sum_{i=1}^{n} \mathbb{E}\left[\left(\frac{X_i}{2\sqrt{t}} - \frac{Y_i}{2\sqrt{1-t}} \right) \frac{\partial f}{\partial x_i}(Z(t)) \right]$$

$$= \sum_{i=1}^{n} \left(\frac{1}{2\sqrt{t}} \underbrace{\mathbb{E}\left[X_i \frac{\partial f}{\partial x_i}(Z(t)) \right]}_{= \sum_{j=1}^{n} \mathbb{E}[X_i X_j]\, \mathbb{E}\left[\sqrt{t} \frac{\partial^2 f}{\partial x_j \partial x_i}(Z(t)) \right]} - \frac{1}{2\sqrt{1-t}} \underbrace{\mathbb{E}\left[Y_i \frac{\partial f}{\partial x_i}(Z(t)) \right]}_{= \sum_{j=1}^{n} \mathbb{E}[Y_i Y_j]\, \mathbb{E}\left[\sqrt{1-t} \frac{\partial^2 f}{\partial x_j \partial x_i}(Z(t)) \right]} \right)$$

by Gaussian integration by part (see below)

$$= \frac{1}{2} \sum_{i,j=1}^{n} \underbrace{\left(\mathbb{E}[X_i X_j] - \mathbb{E}[Y_i Y_j] \right)}_{\begin{cases} \leq 0 & \text{if } i \neq j \\ = 0 & \text{if } i = j \end{cases}} \underbrace{\mathbb{E}\left[\frac{\partial^2 f}{\partial x_j \partial x_i} (Z(t)) \right]}_{\geq 0 \text{ if } i \neq j}$$

$$\underline{\leq 0}.$$

This proves the first claim.

Now to prove the lemma, consider $x \in \mathbb{R}$. We aim at showing

$$\mathbb{P}(\max(X_1, \dots, X_n) \leq x) \leq \mathbb{P}(\max(Y_1, \dots, Y_n) \leq x)$$

$$\Longleftrightarrow \mathbb{E}\left[\prod_{i=1}^{n} \mathbb{1}_{(-\infty, x]}(X_i) \right] \leq \mathbb{E}\left[\prod_{i=1}^{n} \mathbb{1}_{(-\infty, x]}(Y_i) \right]. \quad (*)$$

Let $h_k : \mathbb{R} \longrightarrow [0,1]$ be $\mathcal{C}^2$ non-increasing functions such that $h_k \xrightarrow[k \to \infty]{} \mathbb{1}_{(-\infty, x]}$.

Then $f_k : x \in \mathbb{R}^n \longmapsto \prod_{i=1}^{n} h_k(x_i)$ satisfies the assumptions of the claim so $\mathbb{E}[f_k(X)] \leq \mathbb{E}[f_k(Y)]$, which gives $(*)$ by letting $k \to \infty$. ∎

<u>Lemma (Gaussian integration by part)</u>: Let $n \geq 1$ and $g \in \mathcal{C}^1(\mathbb{R}^n)$ such that ∇g is bounded. Let $X = (X_1, \dots, X_n)$ be a centered Gaussian vector. Then, for any $i \in \{1, \dots, n\}$, $\mathbb{E}[X_i \, g(X)] = \sum_{j=1}^{n} \mathbb{E}[X_i X_j] \, \mathbb{E}\left[\frac{\partial g}{\partial x_j}(X) \right]$.

<u>Exercise 4</u> (Proof)

1. Prove the result for $n = 1$ using the usual integration by parts.
2. Fix $i \in \{1, \dots, n\}$. Prove that there exist a Gaussian vector $Z = (Z_1, \dots, Z_n)$ independent of X_i such that for all $j \in \{1, \dots, n\}$, $X_j = \mathbb{E}[X_i X_j] X_i + Z_j$.
3. Conclude.

<u>Remark</u>: To use Slepian's lemma to compare two models, we need exactly the same number of variables, so it cannot be used directly to say $\Pi_t \overset{(st)}{\leq} \tilde{\Pi}_t$, where "$\overset{(st)}{\leq}$" means stochastically dominated, but we can get $\Pi_t \overset{(st)}{\leq} \hat{\Pi}_t$ by applying Slepian's lemma conditionally on the tree (recall $\hat{\Pi}_t$ is defined in exercise 3).