Lecture 12: Derivative martingale and convergence of the maximum

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<u>Proof of the theorem (1st part)</u>: We prove that $Z_t \xrightarrow[t\to\infty]{a.s.} Z_\infty \geq 0$.

Recall $E_L = \{\forall s \geq 0, \forall u \in \mathcal{N}(s), X_u(s) \leq \lambda_c s + L\}$ and $\mathbb{P}(E_L) \xrightarrow[L\to\infty]{} 1$. Moreover, the family of events $(E_L)_{L>0}$ is increasing and $\mathbb{P}(E_L) \longrightarrow 1$ so $\mathbb{P}\left(\bigcup_{L>0} E_L\right) = 1$.

We are going to prove a.s. convergence of Z_t on E_L for any $L>0$.

On E_L, $Z_t^{(L)} = \sum_{u\in\mathcal{N}_t} (\lambda_c t + L - X_u(t)) e^{\lambda_c(X_u(t) - \lambda_c t)}$

$= Z_t + L W_t^{\lambda_c}$.

Recall $W_t^{\lambda_c} = \sum_{u\in\mathcal{N}_t} e^{\lambda_c X_u(t) - (m + \frac{\lambda_c^2}{2})t}$

$\lambda_c^2 = 2m$ $\Rightarrow$ $= \sum_{u\in\mathcal{N}_t} e^{\lambda_c(X_u(t) - \lambda_c t)}$

But we know that $Z_t^{(L)} \xrightarrow{a.s.} Z_\infty^{(L)}$ and $W_t^{\lambda_c} \xrightarrow{a.s.} 0$

so, on E_L, $Z_t = Z_t^{(L)} - L W_t^{\lambda_c} \xrightarrow{a.s.} Z_\infty^{(L)} \geq 0$

Since $\mathbb{P}\left(\bigcup_{L>0} E_L\right) = 1$, $(Z_t)_{t\geq 0}$ converges a.s. on the full probability space to some limit $Z_\infty \geq 0$ (which satisfies $\forall L>0$, on E_L, $Z_\infty = Z_\infty^{(L)}$ a.s.). ∎

<u>Remark</u>: $Z_\infty^{(L)}$ is nondecreasing in L, so one can write $Z_\infty = \lim_{L\to\infty} \uparrow Z_\infty^{(L)}$.

Now it remains to prove that $Z_\infty > 0$ a.s. on survival. We start by proving that $Z_\infty^{(L)}$ is non trivial.

<u>Lemma</u>: For any $L>0$, $(Z_t^{(L)})_{t\geq 0}$ converges in L^2 to $Z_\infty^{(L)}$.

<u>Proof</u>: It is enough to prove that $(Z_t^{(L)})_{t\geq 0}$ is bounded in L^2.

Write $f_t^{(L)}(x) = (\lambda_c t + L - x) e^{\lambda_c(x - \lambda_c t)}$

We first decompose the square as follows:

$$(Z_t^{(L)})^2 = \sum_{u\in\mathcal{N}_t} f_t^{(L)}(X_u(t))^2 \mathbb{1}_{\max_{s\in[0,t]} X_u(s) - \lambda_c s \leq L}$$

$$+ \sum_{\substack{u,v\in\mathcal{N}_t \\ u\neq v}} f_t^{(L)}(X_u(t)) \mathbb{1}_{\max_{s\in[0,t]} X_u(s) - \lambda_c s \leq L} f_t^{(L)}(X_v(t)) \mathbb{1}_{\max_{s\in[0,t]} X_v(s) - \lambda_c s \leq L}$$

We now bound the expectation of each of these terms by a constant (depending on L).

- For the 1st term, by the many-to-one, we get

$$\mathbb{E}\Big[\sum_{u\in\mathcal{N}_t} f_t^{(L)}(X_u(t))^2 \mathbb{1}_{\max_{s\in[0,t]} X_u(s)-\lambda_c s\le L}\Big] = e^{mt}\,\mathbb{E}\Big[f_t^{(L)}(B_t)^2 \mathbb{1}_{\max_{s\in[0,t]} B_s-\lambda_c s\le L}\Big]$$

$$\le e^{mt}\,\mathbb{E}\Big[(\lambda_c t+L-B_t)^2 e^{2\lambda_c(B_t-\lambda_c t)} \mathbb{1}_{\max_{s\in[0,t]} B_s-\lambda_c s\le L}\Big]$$

$$= \cancel{e^{mt}}\,\mathbb{E}\Big[e^{-\lambda_c B_t-\cancel{\frac{\lambda_c^2}{2}t}}(L-B_t)^2 e^{2\lambda_c B_t}\mathbb{1}_{\max_{s\in[0,t]} B_s\le L}\Big] \quad \Big\}\ \text{Girsanov}$$

$$= \mathbb{E}\Big[(L-B_t)^2 e^{\lambda_c B_t}\mathbb{1}_{\max_{s\in[0,t]} B_s\le L}\Big]$$

$$\le \sum_{k\ge1} k^2 e^{\lambda_c(L-k+1)} \underbrace{\mathbb{P}\big(B_t\in[L-k,L-k+1],\ \max_{s\in[0,t]} B_s\le L\big)}$$

$$\le \Big(\frac{L}{t^{3/2}}\wedge1\Big)e^{\lambda_c(L+1)}\underbrace{\sum_{k\ge1}k^4e^{-\lambda_c k}}_{<\infty}$$

$$\le \mathbb{P}\big(B_t\ge L-k,\ \max B_s\le L\big)\le \frac{Lk^2}{t^{3/2}}\wedge 1 \ \text{ by Corollary 2 Lecture 6}$$

$$\le \Big(\frac{L}{t^{3/2}}\wedge1\Big)k^2$$

So this is bounded.

- We now deal with the 2^{nd} term. By the many-to-two lemma,

$$\mathbb{E}\Big[\sum_{\substack{u,v\in\mathcal{N}_t\\ u\ne v}} f_t^{(L)}(X_u(t))\mathbb{1}_{\max_{s\in[0,t]} X_u(s)-\lambda_c s\le L}\, f_t^{(L)}(X_v(t))\mathbb{1}_{\max_{s\in[0,t]} X_v(s)-\lambda_c s\le L}\Big]$$

$$= \mathbb{E}[L(L-1)]\int_0^t e^{2mt-mr}\,\mathbb{E}\Big[f_t^{(L)}(B_t^{1,r})\mathbb{1}_{\max_{s\in[0,t]} B_s^{1,r}-\lambda_c s\le L}\, f_t^{(L)}(B_t^{2,r})\mathbb{1}_{\max_{s\in[0,t]} B_s^{2,r}-\lambda_c s\le L}\Big]dr$$

We first focus on the expectation in the integral:

$$\mathbb{E}\Big[f_t^{(L)}(B_t^{1,r})\mathbb{1}_{\max_{s\in[0,t]} B_s^{1,r}-\lambda_c s\le L}\, f_t^{(L)}(B_t^{2,r})\mathbb{1}_{\max_{s\in[0,t]} B_s^{2,r}-\lambda_c s\le L}\Big]$$

$$= \mathbb{E}\Big[\mathbb{1}_{\max_{s\in[0,r]} B_s^0-\lambda_c s\le L}\, f_t^{(L)}(B_r^0+B_{t-r}^1)\mathbb{1}_{\max_{s\in[0,t-r]} B_r^0+B_s^1-\lambda_c(r+s)\le L}\, f_t^{(L)}(B_r^0+B_{t-r}^2)\mathbb{1}_{\max_{s\in[0,t-r]} B_r^0+B_s^2-\lambda_c(r+s)\le L}\Big]$$

$$= \mathbb{E}\Big[\mathbb{1}_{\max_{s\in[0,r]} B_s^0-\lambda_c s\le L}\,\varphi(B_r^0)^2\Big]$$

with $\varphi(x) = \mathbb{E}\Big[f_t^{(L)}(x+B_{t-r}^1)\mathbb{1}_{\max_{s\in[0,t-r]} x+B_s^1-\lambda_c(r+s)\le L}\Big]$

$$= \mathbb{E}\Big[(\lambda_c t+L-B_{t-r}-x)e^{\lambda_c(B_{t-r}+x-\lambda_c t)}\mathbb{1}_{\max_{s\in[0,t-r]} B_s+x-\lambda_c r-\lambda_c s\le L}\Big]$$

$$= \mathbb{E}_{x-\lambda_c r}\Big[(\lambda_c(t-r)+L-B_{t-r})e^{\lambda_c(B_{t-r}-\lambda_c(t-r))}\mathbb{1}_{\max_{s\in[0,t-r]} B_s-\lambda_c s\le L}\Big] \quad \Big\}\ \text{martingale (1.b.)}$$

$$= e^{-\frac{\lambda_c^2}{2}(t-r)}\times\mathbb{E}_{x-\lambda_c r}\Big[(\lambda_c\times0+L-B_0)e^{\lambda_c B_0-\frac{\lambda_c^2}{2}\times0}\mathbb{1}_{\max_{s\in[0,0]} B_s-\lambda_c s\le L}\Big]$$

$$= e^{-\frac{\lambda_c^2}{2}(t-r)}(\lambda_c r+L-x)e^{\lambda_c(x-\lambda_c r)}\mathbb{1}_{x-\lambda_c r\le L}$$

$$= e^{-\lambda_c^2(t-r)}\underbrace{\mathbb{E}\Big[\mathbb{1}_{\max_{s\in[0,r]} B_s-\lambda_c s\le L}(\lambda_c r+L-B_r)^2e^{2\lambda_c(B_r-\lambda_c r)}\Big]}$$

$$\le e^{-mr}\Big(\frac{L}{r^{3/2}}\wedge1\Big)C \quad \text{with } C \text{ depending on } L \text{ and } \lambda_c \text{ by the calculation for the } 1^{\text{st}} \text{ term}$$

$$\le Ce^{-2mt+mr}\Big(\frac{L}{r^{3/2}}\wedge1\Big)$$

So finally we get

$$\mathbb{E}\Big[\sum_{\substack{u,v\in\mathcal{N}_t\\ u\neq v}} f_t^{(L)}(X_u(t))\,\mathbb{1}_{\max_{s\in[0,t]} X_u(s)-\lambda_c s\le L}\; f_t^{(L)}(X_v(t))\,\mathbb{1}_{\max_{s\in[0,t]} X_v(s)-\lambda_c s\le L}\Big]$$

$$\le \mathbb{E}[L(L-1)]\int_0^t C\left(\frac{L}{r^{3/2}}\wedge 1\right)dr \le C'.$$

So $(Z_t^{(L)})_{t\ge0}$ is bounded in L^2. ∎

<u>Proof of the theorem (2^{nd} part)</u>: We prove that $Z_\infty>0$ a.s. on survival.

By the previous Lemma, $(Z_t^{(L)})_{t\ge 0}$ converges in L^1 so $\mathbb{E}[Z_\infty^{(L)}]=\mathbb{E}[Z_0^{(L)}]=L$ and therefore $\mathbb{P}(Z_\infty^{(L)}>0)>0$.

Since $Z_\infty=\lim_{L\to\infty}\uparrow Z_\infty^{(L)}$ (see Remark above) we get $\mathbb{P}(Z_\infty>0)>0$.

We have $\{Z_\infty>0\}\subset\{\text{survival}\}$ so it is enough to prove $\mathbb{P}(Z_\infty>0)=\mathbb{P}(\text{survival})$.

Recall the function $f(s)=\mathbb{E}[s^L]$ has two fixed points, 1 and $\mathbb{P}(\text{extinction})$, so it is enough to prove $\mathbb{P}(Z_\infty=0)$ is a fixed point of f.

We decompose Z_∞ at the first branching time: on $\{\tau_\emptyset\le t\}$, we have

$$Z_t=\sum_{i=1}^{L_\emptyset}\sum_{\substack{u\in\mathcal{N}_t\\ u\ge i}}(\lambda_c t-X_u(t))\,e^{\lambda_c(X_u(t)-\lambda_c t)}$$

$$\lambda_c t-X_u(t)=\lambda_c\tau_\emptyset-X_\emptyset(\tau_\emptyset)+\lambda_c(t-\tau_\emptyset)-(X_u(t)-X_\emptyset(\tau_\emptyset))$$

$$=(\lambda_c\tau_\emptyset-X_\emptyset(\tau_\emptyset))\,e^{\lambda_c(X_\emptyset(\tau_\emptyset)-\lambda_c\tau_\emptyset)}\sum_{i=1}^{L_\emptyset}\underbrace{\sum_{\substack{u\in\mathcal{N}_t\\ u\ge i}} e^{\lambda_c(X_u(t)-X_\emptyset(\tau_\emptyset)-\lambda_c(t-\tau_\emptyset))}}$$

this is the critical additive martingale at time $t-\tau_\emptyset$ of the BBM rooted at i: tends to 0 a.s. as $t\to\infty$.

$$+\,e^{\lambda_c(X_\emptyset(\tau_\emptyset)-\lambda_c\tau_\emptyset)}\sum_{i=1}^{L_\emptyset}\underbrace{\sum_{\substack{u\in\mathcal{N}_t\\ u\ge i}}\big(\lambda_c(t-\tau_\emptyset)-(X_u(t)-X_\emptyset(\tau_\emptyset))\big)\,e^{\lambda_c(X_u(t)-X_\emptyset(\tau_\emptyset)-\lambda_c(t-\tau_\emptyset))}}$$

this is the derivative martingale at time $t-\tau_\emptyset$ of the BBM rooted at i: tends to Z_∞^i a.s. as $t\to\infty$ where $(Z_\infty^i)_{i\ge1}$ are iid $\overset{(d)}{=}Z_\infty$ and indep. of $\mathcal{F}_{\tau_\emptyset}$

So letting $t\to\infty$, we get $Z_\infty=e^{\lambda_c(X_\emptyset(\tau_\emptyset)-\lambda_c\tau_\emptyset)}\sum_{i=1}^{L_\emptyset}Z_\infty^i$

in particular
$$\mathbb{P}(Z_\infty=0)=\mathbb{P}(\forall i\in\{1,\dots,L_\emptyset\},\ Z_\infty^i=0)$$
$$=\mathbb{E}\big[\mathbb{P}(Z_\infty=0)^{L_\emptyset}\big]$$
$$=f(\mathbb{P}(Z_\infty=0))\quad\text{so we are done!}$$ ∎

<u>Remark</u>: We have seen in the proof that $\mathbb{E}[Z_\infty^{(L)}] = L$, but also in the previous remark that $Z_\infty = \lim_{L\to\infty} \uparrow Z_\infty^{(L)}$. By the monotone convergence theorem, we get $\mathbb{E}[Z_\infty] = \lim_{L\to\infty} \mathbb{E}[Z_\infty^{(L)}] = +\infty$.

III.8) Probabilistic description of the limit of the maximum (Not covered in class)

The limit Z_∞ is the right quantity to describe various quantities of the BBM in the direction of the max and in particular the limit of $M_t - m_t$:

<u>Theorem (Lalley-Sellke 1987)</u>: $M_t - m_t \xrightarrow{(d)} \frac{1}{\lambda_c} \log Z_\infty + G$ where G is a Gumbel$(c, \frac{1}{\lambda_c})$ r.v. independent of G, for some $c \in \mathbb{R}$.

More precisely, we have almost surely

$$\lim_{s\to\infty} \lim_{t\to\infty} \mathbb{P}(M_t - m_t \leq x \mid \mathcal{F}_s) = \mathbb{P}\left(\frac{1}{\lambda_c} \log Z_\infty + G \leq x \mid Z_\infty\right)$$

by definition of the Gumbel

$$= \exp\left(-Z_\infty e^{-\lambda_c(x-c)}\right)$$

<u>Remark</u>: Note that in this convergence, M_t is centered but not normalized so it feels the effect of the following behaviors, which play a role in the limit in distribution:

- If the first particle goes very high or has a large number of children, this has an ever-lasting positive influence on $M_t - m_t$.
- If the highest particle at time $t-1$ goes up, then $M_t - m_t$ will be larger than $M_{t-1} - m_{t-1}$.

So the limit distribution is influenced by events happening at the beginning but also at larger times (in particular at times close to t).

The convergence $\lim_{s\to\infty} \lim_{t\to\infty} \mathbb{P}(M_t - m_t \leq x \mid \mathcal{F}_s) = \mathbb{P}\left(\frac{1}{\lambda_c} \log Z_\infty + G \leq x \mid Z_\infty\right)$ shows that Z_∞ contains the information of the influence of the beginning of the BBM (things happening before some large fixed s when $t \to \infty$) whereas G corresponds to later fluctuations.

Proof: We take the result of Bramson in 1983 for granted and show how to deduce this theorem from it. Recall he proved that, for any $x \in \mathbb{R}$,

$$\mathbb{P}(\Pi_t - m_t \leq x) \xrightarrow[t\to\infty]{} w(x) \quad \text{where } w \text{ is a } \mathcal{C}^2 \text{ function satisfying}$$

$$1 - w(x) \sim C_w x e^{-\lambda_c x} \quad \text{as } x \to \infty.$$

Write $u(t,x) = \mathbb{P}(\Pi_t \leq x)$.

The previous convergence can be rewritten as $u(t, m_t + x) \xrightarrow[t\to\infty]{} w(x)$.

By monotonicity and continuity of w, this implies that, if $x_t \xrightarrow[t\to\infty]{} x$, then

$$u(t, m_t + x_t) \xrightarrow[t\to\infty]{} w(x).$$

Let $t \geq s \geq 0$.

$$\{\Pi_t \leq x\} = \left\{ \forall v \in \mathcal{N}_s, \max_{\substack{u \in \mathcal{N}_t \\ u \geq v}} X_u(t) \leq x \right\}$$

$$= \bigcap_{v \in \mathcal{N}_s} \Bigg\{ \underbrace{\max_{\substack{u \in \mathcal{N}_t \\ u \geq v}} \big(X_u(t) - X_v(s)\big)} \leq x - X_v(s) \Bigg\}$$

by the branching property, conditionally on $\mathcal{F}_s$, these r.v. for $v \in \mathcal{F}_s$ are independent of each other and have the same law as Π_{t-s}

So $\mathbb{P}(\Pi_t \leq x \mid \mathcal{F}_s) = \prod_{v \in \mathcal{N}_s} u(t-s, x - X_v(s))$

and

$$\mathbb{P}(\Pi_t - m_t \leq x \mid \mathcal{F}_s) = \prod_{v \in \mathcal{N}_s} u(t-s, x + m_t - X_v(s))$$

$$= \prod_{v \in \mathcal{N}_s} u\big(t-s, m_{t-s} + x + \underbrace{m_t - m_{t-s}}_{= \lambda_c s - \frac{3}{2\sqrt{2}} \log \frac{t}{t-s} \xrightarrow[t\to\infty]{} \lambda_c s} - X_v(s)\big)$$

If $x_t \to x$ then $u(t, m_t + x_t) \xrightarrow[t\to\infty]{} w(x)$

$$\xrightarrow[t\to\infty]{\text{a.s.}} \prod_{v \in \mathcal{N}_s} w(x + \lambda_c s - X_v(s)).$$

$$= \exp\Big(\sum_{v \in \mathcal{N}_s} \log w(x + \lambda_c s - X_v(s)) \Big).$$

We have seen in Lecture 7 that $\lambda_c s - \Pi_s \xrightarrow[s\to\infty]{\text{a.s.}} \infty$

so $\lambda_c s - X_v(s) \xrightarrow[s\to\infty]{} \infty$ uniformly in $v \in \mathcal{N}_s$, a.s.

and so $\log w(x + \lambda_c s - X_v(s)) = -(1+o(1))\, C_w\, (x + \lambda_c s - X_v(s))\, e^{-\lambda_c(x + \lambda_c s - X_v(s))}$

where $o(1)$ holds as $s \to \infty$, uniformly in $v \in \mathcal{N}_s$, a.s.

We get $\mathbb{P}\left(M_t - m_t \leq x \mid \mathcal{F}_s\right) = \exp\left(-(1+o(1))\, C_\omega e^{-\lambda_c x} \underbrace{\sum_{v \in \mathcal{N}_s} \left(x + \lambda_c s - X_v(s)\right) e^{\lambda_c X_v(s) - \lambda_c^2 s}}_{= x W_s + Z_s \xrightarrow[s\to\infty]{a.s.} Z_\infty}\right)$

$$\xrightarrow[s\to\infty]{a.s.} \exp\left(-C_\omega e^{-\lambda_c x} Z_\infty\right).$$

So the constant c in the statement is given by $c = \frac{1}{\lambda_c} \log C_\omega$. ∎